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FRACTIONAL MULTITERM GRONWALL LEMMA FOR FUNCTIONS OF TWO VARIABLES AND ITS APPLICATION TO FRACTIONAL HIGHER-ORDER SYSTEMS OF HYPERBOLIC TYPE

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Abstract. We study a fractional (in Riemann-Liouville sense) partial differential problem of higher order in the space of fractionally absolutely continuous functions of two variables. We obtain existence of a solution in a non-Lipschitz case applying the Schaeffer fixed point theorem and a fractional multiterm Gronwall lemma for functions of two variables to a nonlinear integral equation.

1. Introduction

We consider the general nonlinear equation with Riemann-Liouville partial derivatives

$$\begin{aligned}
&D^{\alpha,\beta}_{x,y}z(x,y)\\
&= f\Big(x,y,z(x,y),D^{\alpha-1}_x z(x,y),\dots,D^{\alpha-(k-1)}_x z(x,y),\\
&\quad D^{\beta-1}_y z(x,y),\dots,D^{\beta-(l-1)}_y z(x,y),D^{\alpha-1,\beta-1}_{x,y}z(x,y),\dots,D^{\alpha-(k-1),\beta-(l-1)}_{x,y}z(x,y),\\
&\quad D^{\alpha-1,\beta_1}_{x,y}z(x,y),\dots,D^{\alpha-(k-1),\beta_t}_{x,y}z(x,y),D^{\alpha_1,\beta-1}_{x,y}z(x,y),\dots,D^{\alpha_s,\beta-(l-1)}_{x,y}z(x,y),\\
&\quad D^{\alpha_1,\beta_1}_{x,y}z(x,y),\dots,D^{\alpha_s,\beta_t}_{x,y}z(x,y),D^{\alpha_1}_x z(x,y),\dots,D^{\alpha_s}_x z(x,y),\\
&\quad D^{\beta_1}_y z(x,y),\dots,D^{\beta_t}_y z(x,y)\Big)
\end{aligned}\tag{1.1}$$

for $(x,y)\in P=[0,1]\times[0,1]$ a.e., with initial conditions (below, by the notation $i=\overline{1,m}$ we mean that $i=1,\dots,m$)

$$\begin{gathered}
\frac{\partial^{k+j}}{\partial x^k\partial y^j}(I^{k-\alpha,l-\beta}_{x,y}z)(x,0)=\nu^j(x),\quad j=\overline{0,l-1},\ x\in I=[0,1]\text{ a.e.}\\
\frac{\partial^{i+l}}{\partial x^i\partial y^l}(I^{k-\alpha,l-\beta}_{x,y}z)(0,y)=\mu^i(y),\quad i=\overline{0,k-1},\ y\in I\text{ a.e.}\\
\frac{\partial^{i+j}}{\partial x^i\partial y^j}(I^{k-\alpha,l-\beta}_{x,y}z)(0,0)=e^{i,j},\quad i=\overline{0,k-1},\ j=\overline{0,l-1}.
\end{gathered}\tag{1.2}$$

Here $k-1<\alpha\le k$, $l-1<\beta\le l$ ($k,l\in\mathbb{N}$), $0<\alpha_p<\alpha-(k-1)$ for $p=\overline{1,m}$, $0<\beta_q<\beta-(l-1)$ for $q=\overline{1,n}$ ($m,n\in\mathbb{N}$), $f:P\times\mathbb{R}^r\to\mathbb{R}$ ($r=(k+m)(l+n)$), $\mu^i\in L^1(I)$, $\nu^j\in L^1(I)$, $e^{i,j}\in\mathbb{R}$ for $i=\overline{0,k-1}$, $j=\overline{0,l-1}$ (if $k=1$, then the derivatives of orders involving $\alpha-i$ with $i=\overline{1,k-1}$ do not appear on the right-hand side of the equation and, similarly, if $l=1$, then the derivatives of orders involving $\beta-j$ with $j=\overline{1,l-1}$ do not appear on the right-hand side of the equation).

We obtain a theorem on the existence of a solution to problem (1.1)-(1.2) assuming only that f is of Caratheodory type and satisfies a growth condition (Theorem 4.1). The method of the proof involves the replacing of the differential Cauchy problem with an integral equation and is based on

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the Schaefer fixed point theorem (see Appendix) and a fractional Gronwall lemma derived in the paper (Lemma 3.2). This lemma is a two-dimensional counterpart of a multiterm Gronwall lemma for functions of one variable proved in [3]. It can be also viewed as an extension of a fractional Gronwall lemma for functions of two variables derived in [4].

We study problem (1.1)-(1.2) in the space of fractionally absolutely continuous functions of two variables that has been introduced and investigated in [5]. The collection of partial derivatives appearing on the right-hand side of the equation (1.1) is a consequence of a general imbedding theorem for spaces of fractionally absolutely continuous functions of two variables (see Theorem 5.1). Due to the noncompactness of single integral operators and compactness assumption in the Schaefer fixed point theorem, the right-hand side of (1.1) does not contain derivatives involving the differentiation of order α in x and differentiation of order β in y. Such a case is discussed in Remark 4.2. The form of boundary conditions is connected with Corollary 5.5. The equivalence of differential problem (1.1) -(1.2) and integral equation (4.1) follows from the imbeddings theorems presented in the Appendix.

The distinguishing features of the problem are its generality and minimal assumptions. To the best our knowledge, the presented results have not been obtained by other authors. Let us only mention that in the case of α, $\beta \in (0,1)$ a particular case of the system (4.3) (see Remark 4.2), namely, the system

$$D^{\alpha,\beta}_{x,y} z(x,y) = f\Big(x, y, z(x,y), D^{\alpha}_{x} z(x,y), D^{\beta}_{y} z(x,y)\Big) \tag{1.3}$$

for $(x,y) \in P$ a.e., with initial conditions

$$\begin{aligned} I^{1-\alpha,1-\beta}_{a+,x,c+,y} z(x,0) &= \gamma(x), \quad x \in I \\ I^{1-\alpha,1-\beta}_{a+,x,c+y} z(0,y) &= \delta(y), \quad y \in I, \end{aligned} \tag{1.4}$$

is investigated in paper [6] where, among other things, existence of a unique solution has been obtained with the aid of the Banach contraction principle under Lipschitz condition. Problem (1.3)-(1.4) is a fractional version of the following hyperbolic system (also called Goursat-Darboux system or continuous Fornasini-Marchesini system)

$$\frac{\partial^2 z}{\partial x \partial y}(x,y) = f\Big(x, y, z(x,y), \frac{\partial z}{\partial x} z(x,y), \frac{\partial z}{\partial y} z(x,y)\Big)$$

for $(x,y) \in P$ a.e., with initial conditions

$$\begin{aligned} z(x,c) &= \gamma(x), \quad x \in [a,b] \\ z(a,y) &= \delta(y), \quad y \in [c,d] \end{aligned}$$

that can be applied to describe a gas absorption process (cf. [15, 2, 10, 11]). In [4], the relative weak compactness of the set of solutions to a control system of type (1.3) has been proved. The existence and uniqueness of a solution, as well as the Ulam-Hyers type stability for system (1.3) but with fractional partial derivatives in Caputo sense is studied in [8].

This article consists of four parts. In the first of them - Preliminaries - we introduce the function spaces used in the paper. In the second part, we prove a generalization of the fractional Gronwall lemma for functions of two variables obtained in [4]. In the main, third part of the paper, we prove the existence theorem for problem (1.3)-(1.4) via existence of a solution to an integral equation. We also comment (see Remark 4.2) the existence of a unique solution to a more general Cauchy problem and give the examples of the investigated problems. In the Appendix, first, we recall the imbeddings results for the spaces of fractionally absolutely continuous functions of two variables, that have been derived in [5]. Next, the Schaefer fixed point theorem is formulated.

Due to the technical nature of the work, to facilitate the reader's understanding of the text some calculations and formulas, although they could have been written down in a shorter form, were sometimes given in a longer form.

2. Preliminaries

Details of the results presented in this section (including the proofs) are contained in the paper [5].

2.1. **Spaces AC_x^α.** Let $L^1 = L^1(P,\mathbb{R})$ be the classical space of Lebesgue integrable functions on $P = [0,1] \times [0,1]$ and $\alpha > 0$. Consider an operator (see [13], [6])

$$I_x^\alpha : L^1 \ni \varphi \mapsto \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\varphi(s,y)}{(x-s)^{1-\alpha}} ds \in L^1,$$

Additionally, we put $I_x^0 \varphi = \varphi$ for $\varphi \in L^1$. $I_x^\alpha \varphi$ is named the single partial left-sided integral of φ of order α with respect to x.

Definition 2.1. By $AC_x^k = AC_x^k(P,\mathbb{R})$ ($k \in \mathbb{N}$) we denote the set of functions $z : P \to \mathbb{R}$ of the form

$$z(x,y) = \frac{\mu_0(y)}{\Gamma(1)} x^0 + \cdots + \frac{\mu_{k-1}(y)}{\Gamma(k)} x^{k-1} + I_x^k \varphi(x,y) \tag{2.1}$$

for $(x,y) \in P$ a.e., where $\mu_i \in L^1(I) = L^1(I,\mathbb{R})$ for $i = \overline{0,k-1}$ (here $I = [0,1]$), $\varphi \in L^1$. Additionally, we adopt $AC_x^0 = L^1$.

Using the Cauchy formula

$$I_x^k \varphi(x,y) = \int_0^x \int_0^{x_2} \cdots \int_0^{x_k} \varphi(s,y)\, ds\, dx_k \ldots dx_2$$

one can show that a function z of the form (2.1) has the single partial derivatives $\frac{\partial^i z}{\partial x^i}$ a.e. on P for $1 \le i \le k$. In particular,

$$\frac{\partial^k z}{\partial x^k}(x,y) = \varphi(x,y).$$

Now, we are in a position to define a single partial Riemann-Liouville derivative of any order of a function of two variables.

Definition 2.2. We say that a function $z : P \to \mathbb{R}$ has a single partial left-sided derivative $D_x^\alpha z$ of order $\alpha \in (k-1,k]$ ($k \in \mathbb{N}$) with respect to x if

$$I_x^{k-\alpha} z \in AC_x^k,$$

i.e. (see (2.1))

$$I_x^{k-\alpha} z(x,y) = \frac{\mu_0(y)}{\Gamma(1)} x^0 + \cdots + \frac{\mu_{k-1}(y)}{\Gamma(k)} x^{k-1} + I_x^k \varphi(x,y)$$

for $(x,y) \in P$ a.e. In such a case, we put

$$D_x^\alpha z(x,y) := \frac{\partial^k}{\partial x^k} I_x^{k-\alpha} z(x,y) = \varphi(x,y)$$

for $(x,y) \in P$ a.e.

We have the following theorem characterizing functions z possessing the derivative $D_x^\alpha z$.

Theorem 2.3. *Let $\alpha \in (k-1,k]$ ($k \in \mathbb{N}$). A function $z : P \to \mathbb{R}$ has the fractional partial derivative $D_x^\alpha z$ if and only if z is of the form*

$$z(x,y) = \frac{\mu_0(y)}{\Gamma(1-(k-\alpha))} x^{-(k-\alpha)} + \cdots + \frac{\mu_{k-1}(y)}{\Gamma(k-(k-\alpha))} x^{k-1-(k-\alpha)} + I_x^\alpha \varphi(x,y) \tag{2.2}$$

for $(x,y) \in P$ a.e., where $\mu_i \in L^1(I)$ for $i = \overline{0,k-1}$, $\varphi \in L^1$. In such a case,

$$D_x^\alpha z = \varphi$$

a.e. on P. The representation (2.2) *is unique.*

By AC_x^α we denote the set of all functions z of the form (2.2).

2.2. **Spaces $AC^{\alpha,\beta}_{x,y}$.** Let $(\alpha,\beta)\in(k-1,k]\times(l-1,l]$ ($k,\,l\in\mathbb{N}$). Let us consider an operator (see [6, 13])

$$I^{\alpha,\beta}_{x,y}:L^1(P)\ni\varphi\mapsto\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^y\frac{\varphi(s,t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}}\,ds\,dt\in L^1(P).$$

It is called the mixed left-sided integral of φ of order (α,β). Additionally, we put

$$I^{0,0}_{x,y}\varphi=\varphi,\quad I^{\alpha,0}_{x,y}\varphi=I^{\alpha}_{x}\varphi,\quad I^{0,\beta}_{x,y}\varphi=I^{\beta}_{y}\varphi$$

for $\varphi\in L^1(P)$.

Definition 2.4. By $AC^{k,l}_{x,y}=AC^{k,l}_{x,y}(P,\mathbb{R})$ ($k,l\in\mathbb{N}$) we denote the set of all functions $z:P\to\mathbb{R}$ of the form

$$\begin{aligned}z(x,y)&=I^{k,l}_{x,y}\varphi(x,y)+I^{l}_{y}\Big(\mu_0(y)\frac{x^0}{\Gamma(1)}+\dots+\mu_{k-1}(y)\frac{x^{k-1}}{\Gamma(k)}\Big)\\&\quad+I^{k}_{x}\left(\nu_0(x)\frac{y^0}{\Gamma(1)}+\dots+\nu_{l-1}(x)\frac{y^{l-1}}{\Gamma(l)}\right)+\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}e_{i,j}\frac{x^i}{\Gamma(i+1)}\frac{y^j}{\Gamma(j+1)}\end{aligned}\tag{2.3}$$

for $(x,y)\in P$ a.e., where $\varphi\in L^1$, $\mu_i,\nu_j\in L^1(I)$, $e_{i,j}\in\mathbb{R}$. We also put, $AC^{k,0}_{x,y}=AC^{k}_{x}$, $AC^{0,l}_{x,y}=AC^{l}_{y}$, $AC^{0,0}_{x,y}=L^1$.

One can show (see [16]) that a function $z\in AC^{k,l}_{x,y}$ possesses derivatives $\frac{\partial^{i+j}z}{\partial x^i\partial y^j}$ a.e. on P for all (i,j) such that $0\le i\le k$, $0\le j\le l$. In particular,

$$\frac{\partial^{k+l}z}{\partial x^k\partial y^l}(x,y)=\varphi(x,y)$$

for $(x,y)\in P$ a.e.

Now, we can define a mixed partial Riemann-Liouville derivative of any order of a function of two variables.

Definition 2.5. Let $(\alpha,\beta)\in(k-1,k]\times(l-1,l]$ ($k,\,l\in\mathbb{N}$). We say that a function $z\in L^1(P)$ has a mixed left-sided partial derivative $D^{\alpha,\beta}_{x,y}z$ of order (α,β) if

$$I^{k-\alpha,l-\beta}_{x,y}z\in AC^{k,l}_{x,y},$$

i.e. (see (2.3))

$$\begin{aligned}I^{k-\alpha,l-\beta}_{x,y}z(x,y)&=I^{k,l}_{x,y}\varphi(x,y)+I^{l}_{y}\Big(\mu_0(y)\frac{x^0}{\Gamma(1)}+\dots+\mu_{k-1}(y)\frac{x^{k-1}}{\Gamma(k)}\Big)\\&\quad+I^{k}_{x}\Big(\nu_0(x)\frac{y^0}{\Gamma(1)}+\dots+\nu_{l-1}(x)\frac{y^{l-1}}{\Gamma(l)}\Big)+\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}e_{i,j}\frac{x^i}{\Gamma(i+1)}\frac{y^j}{\Gamma(j+1)}\end{aligned}$$

for $(x,y)\in P$ a.e. In such a case, we put

$$D^{\alpha,\beta}_{x,y}z:=\frac{\partial^{k+l}}{\partial x^k\partial y^l}I^{k-\alpha,l-\beta}_{x,y}z=\varphi.$$

Additionally, by $D^{\alpha,0}_{x,y}z$, $D^{0,\beta}_{x,y}z$ we shall denote the derivatives $D^{\alpha}_{x}z$, $D^{\beta}_{y}z$, respectively.

We have very useful characterization of functions z possessing the derivative $D^{\alpha,\beta}_{x,y}z$.

Theorem 2.6. *A function $z\in L^1(P)$ has the derivative $D^{\alpha,\beta}_{x,y}z$ if and only if z is of the form*

$$\begin{aligned}z(x,y)&=I^{\alpha,\beta}_{x,y}\varphi(x,y)+I^{\beta}_{y}\big(\frac{\mu_0(y)}{\Gamma(1-(k-\alpha))}x^{-(k-\alpha)}+\dots+\frac{\mu_{k-1}(y)}{\Gamma(k-(k-\alpha))}x^{k-1-(k-\alpha)}\big)\\&\quad+I^{\alpha}_{x}\big(\frac{\nu_0(x)}{\Gamma(1-(l-\beta))}y^{-(l-\beta)}+\dots+\frac{\nu_{l-1}(x)}{\Gamma(l-(l-\beta))}y^{l-1-(l-\beta)}\big)\\&\quad+\sum_{i=0}^{k-1}\sum_{j=0}^{l-1}e_{i,j}\frac{x^{i-(k-\alpha)}}{\Gamma(i+1-(k-\alpha))}\frac{y^{j-(l-\beta)}}{\Gamma(j+1-(l-\beta))}\end{aligned}\tag{2.4}$$

for $(x,y) \in P$ a.e., where $\varphi \in L^1(P)$, μ_j, $v_i \in L^1(I)$, $e_{i,j} \in \mathbb{R}$. In such a case,

$$D^{\alpha,\beta}_{x,y} z(x,y) = \varphi(x,y),$$
$$D^{\alpha}_{x} z(x,y) = I^{\beta}_{y}\varphi(x,y) + \frac{\nu_0(x)}{\Gamma(1-(l-\beta))} y^{-(l-\beta)} + \dots + \frac{\nu_{l-1}(x)}{\Gamma(l-(l-\beta))} y^{l-1-(l-\beta)},$$
$$D^{\beta}_{y} z(x,y) = I^{\alpha}_{x}\varphi(x,y) + \frac{\mu_0(y)}{\Gamma(1-(k-\alpha))} x^{-(k-\alpha)} + \dots + \frac{\mu_{k-1}(y)}{\Gamma(k-(k-\alpha))} x^{k-1-(k-\alpha)}$$

for $(x,y) \in P$ a.e. The representation (2.4) is unique.

By $AC^{\alpha,\beta}_{x,y}$ where $(\alpha,\beta) \in (k-1,k] \times (l-1,l]$ (k, $l \in \mathbb{N}$), we denote the set of all functions of the form (2.4). Additionally, by $AC^{\alpha,0}_{x,y}$, $AC^{0,\beta}_{x,y}$ we denote the sets AC^{α}_{x}, AC^{β}_{y}, respectively.

3. Gronwall lemma

In this section, we prove a two-dimensional counterpart of the fractional multiterm Gronwall lemma for functions of one variable that has been obtained in [3]. We shall use it in the last section to prove the existence of a solution to an integral equation and, consequently, the Cauchy problem (1.1)-(1.2).

Proposition 3.1. *Let the constants $a_1, \dots, a_p > 0$ ($p \in \mathbb{N}$) and multiindexes $\gamma_1 = (\alpha_1, \beta_1), \dots, \gamma_p = (\alpha_p, \beta_p)$ with $\alpha_i > 0$, $\beta_i > 0$ ($i = \overline{1,p}$) be given and assume that*

$$\alpha_1 = \min\{\alpha_i;\ i = \overline{1,p}\}, \quad \beta_1 = \min\{\beta_i;\ i = \overline{1,p}\}.$$

If the functions $a, u \in L^1(P, \mathbb{R}^+_0)$ are such that

$$u(x,y) \le a(x,y) + \sum_{i=1}^{p} a_i \int_0^x \int_0^y \frac{u(s,t)}{(x-s)^{1-\alpha_i}(y-t)^{1-\beta_i}}\, ds\, dt \tag{3.1}$$

for $(x,y) \in P$ a.e., then

$$\begin{aligned} &u(x,y)\\ &\le a(x,y) + \sum_{\substack{n\in\mathbb{N},\\ n\alpha_1<1 \text{ and } n\beta_1<1}} G^n p^n I^{n\gamma_1}_{x,y}(\Psi_a(\cdot,\cdot) + E I^1_y \Phi_a(\cdot,\cdot) + E I^1_x \Xi_a(\cdot,\cdot) + E^2 I^{1,1}_{x,y} a(\cdot,\cdot))(x,y)\\ &\quad + \sum_{\substack{n\in\mathbb{N},\\ n\alpha_1\ge 1 \text{ and } n\beta_1\ge 1}} \frac{G^n p^n}{\Gamma(n\gamma_1)} I^{1,1}_{x,y}(\Psi_a(\cdot,\cdot) + E\Phi_a(\cdot,\cdot) + E\Xi_a(\cdot,\cdot) + E^2 a(\cdot,\cdot))(x,y) \end{aligned}$$

for $(x,y) \in P$ a.e., where

$$G = \max\{a_1\Gamma(\alpha_1)\Gamma(\beta_1), \dots, a_p\Gamma(\alpha_p)\Gamma(\beta_p)\},$$
$$E = \frac{1}{\min\{\Gamma(s);\ s \ge 1\}},$$
$$\begin{aligned}\Psi_v(x,y) = \max\big\{&I^{(\gamma_2-\gamma_1)i_2+\dots+(\gamma_p-\gamma_1)i_p}_{x,y} v(x,y); i_2,\dots,i_p \in \mathbb{N}_0,\\ &(\alpha_2-\alpha_1)i_2+\dots+(\alpha_p-\alpha_1)i_p < 1,\ (\beta_2-\beta_1)i_2+\dots+(\beta_p-\beta_1)i_p < 1\big\},\end{aligned}$$
$$\begin{aligned}\Phi_v(x,y) = \max\{&I^{(\alpha_2-\alpha_1)i_2+\dots+(\alpha_p-\alpha_1)i_p}_{x} v(x,y);\ i_2,\dots,i_p \in \mathbb{N}_0,\\ &(\alpha_2-\alpha_1)i_2+\dots+(\alpha_p-\alpha_1)i_p < 1\},\end{aligned}$$
$$\begin{aligned}\Xi_v(x,y) = \max\{&I^{(\beta_2-\beta_1)i_2+\dots+(\beta_p-\beta_1)i_p}_{y} v(x,y);\ i_2,\dots,i_p \in \mathbb{N}_0,\\ &(\beta_2-\beta_1)i_2+\dots+(\beta_p-\beta_1)i_p < 1\}\end{aligned}$$

for $(x,y) \in P$ a.e., $v \in L^1(P)$.

Proof. We denote

$$B^1 : L^1(P) \ni \varphi \mapsto a_1\Gamma(\alpha_1)\Gamma(\beta_1) I^{\gamma_1}_{x,y}\varphi \in L^1(P),$$
$$\dots$$

$$B^p : L^1(P) \ni \varphi \mapsto a_p\Gamma(\alpha_p)\Gamma(\beta_p) I^{\gamma_p}_{x,y}\varphi \in L^1(P).$$

Assumption (3.1) can be written in the form

$$u(x,y) \le a(x,y) + (B_1 + \dots + B_p)u(x,y).$$

In the same way as in [3, proof of a Gronwall lemma], we see that, for any $n \in \mathbb{N}_0$,

$$u(x,y) \le \sum_{i=0}^{n} (B_1 + \dots + B_p)^i a(x,y) + (B_1 + \dots + B_p)^{n+1} u(x,y)$$

and

$$(B_1 + \dots + B_p)^n u(x,y) \le G^n I^{n\gamma_1}_{x,y} (I^0_{x,y} + I^{\delta_2}_{x,y} + \dots + I^{\delta_p}_{x,y})^n u(x,y)$$

for $(x,y) \in P$ a.e., where $I^0_{x,y} : L^1(P) \to L^1(P)$ is the identity operator and $\delta_2 = \gamma_2 - \gamma_1, \dots, \delta_p = \gamma_p - \gamma_1$. In consequence, denoting

$$\begin{aligned}
N &= \{(i_1, \dots, i_p) \in \mathbb{N}_0 \times \dots \times \mathbb{N}_0;\ i_1 + \dots + i_p = n\},\\
N_{<,<} &= \big\{(i_1, \dots, i_p) \in N;\ (\alpha_2 - \alpha_1)i_2 + \dots + (\alpha_p - \alpha_1)i_p < 1\\
&\qquad \text{and } (\beta_2 - \beta_1)i_2 + \dots + (\beta_p - \beta_1)i_p < 1\big\},\\
N_{<,\ge} &= \big\{(i_1, \dots, i_p) \in N;\ (\alpha_2 - \alpha_1)i_2 + \dots + (\alpha_p - \alpha_1)i_p < 1\\
&\qquad \text{and } (\beta_2 - \beta_1)i_2 + \dots + (\beta_p - \beta_1)i_p \ge 1\big\},\\
N_{\ge,<} &= \big\{(i_1, \dots, i_p) \in N;\ (\alpha_2 - \alpha_1)i_2 + \dots + (\alpha_p - \alpha_1)i_p \ge 1\\
&\qquad \text{and } (\beta_2 - \beta_1)i_2 + \dots + (\beta_p - \beta_1)i_p < 1\},\\
N_{\ge,\ge} &= \big\{(i_1, \dots, i_p) \in N;\ (\alpha_2 - \alpha_1)i_2 + \dots + (\alpha_p - \alpha_1)i_p \ge 1\\
&\qquad \text{and } (\beta_2 - \beta_1)i_2 + \dots + (\beta_p - \beta_1)i_p \ge 1\},
\end{aligned}$$

$$\begin{aligned}
(B_1 + \dots + B_p)^n u(x,y) &\le G^n I^{n\gamma_1}_{x,y} \Big(\sum_{(i_1,\dots,i_p)\in N} \frac{n!}{i_1! \cdots _p!} I^{0\cdot i_1 + \delta_2 i_2 + \dots + \delta_p i_p}_{x,y} u \Big)(x,y)\\
&= G^n I^{n\gamma_1}_{x,y} \Big(\sum_{(i_1,\dots,i_p)\in N_{<,<}} \frac{n!}{i_1! \cdots _p!} I^{0\cdot i_1 + \delta_2 i_2 + \dots + \delta_p i_p}_{x,y} u\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{<,\ge}} \frac{n!}{i_1! \cdots _p!} I^{0\cdot i_1 + \delta_2 i_2 + \dots + \delta_p i_p}_{x,y} u\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{\ge,<}} \frac{n!}{i_1! \cdots _p!} I^{0\cdot i_1 + \delta_2 i_2 + \dots + \delta_p i_p}_{x,y} u\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{\ge,\ge}} \frac{n!}{i_1! \cdots _p!} I^{0\cdot i_1 + \delta_2 i_2 + \dots + \delta_p i_p}_{x,y} u \Big)(x,y)
\end{aligned}$$

for $(x,y) \in P$ a.e. So,

$$\begin{aligned}
&(B_1 + \dots + B_p)^n u(x,y)\\
&\le G^n I^{n\gamma_1}_{x,y} \big(p^n \Psi_u(\cdot,\cdot)\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{<,\ge}} \frac{n!}{i_1! \cdots _p!} I_y^{(\beta_2-\beta_1)i_1 + \dots + (\beta_p - \beta_1)i_p} I_x^{(\alpha_2-\alpha_1)i_1 + \dots + (\alpha_p - \alpha_1)i_p} u(\cdot,\cdot)\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{\ge,<}} \frac{n!}{i_1! \cdots _p!} I_x^{(\alpha_2-\alpha_1)i_1 + \dots + (\alpha_p - \alpha_1)i_p} I_y^{(\beta_2-\beta_1)i_1 + \dots + (\beta_p - \beta_1)i_p} u(\cdot,\cdot)\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{\ge,\ge}} \frac{n!}{i_1! \cdots _p!} I_{x,y}^{(\alpha_2-\alpha_1)i_1 + \dots + (\alpha_p - \alpha_1)i_p, (\beta_2-\beta_1)i_1 + \dots + (\beta_p - \beta_1)i_p} u(\cdot,\cdot)\big)(x,y).
\end{aligned}$$

Thus,

$$(B_1 + \dots + B_p)^n u(x,y)$$

$$
\begin{aligned}
&\le G^n I_{x,y}^{n\gamma_1}\Big(p^n\Psi_u(\cdot,\cdot) + \sum_{(i_1,\dots,i_p)\in N_{<,\ge}} \frac{n!}{i_1!\cdots p!} I_y^{(\beta_2-\beta_1)i_1+\dots+(\beta_p-\beta_1)i_p}\Phi_u(\cdot,\cdot)\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{\ge,<}} \frac{n!}{i_1!\cdots p!} I_x^{(\alpha_2-\alpha_1)i_1+\dots+(\alpha_p-\alpha_1)i_p}\Phi_u(\cdot,\cdot)\\
&\quad + \sum_{(i_1,\dots,i_p)\in N_{\ge,\ge}} \frac{n!}{i_1!\cdots p!} I_{x,y}^{(\alpha_2-\alpha_1)i_1+\dots+(\alpha_p-\alpha_1)i_p,(\beta_2-\beta_1)i_1+\dots+(\beta_p-\beta_1)i_p} u(\cdot,\cdot)\Big)(x,y).
\end{aligned}
$$

In consequence,

$$
(B_1+\dots+B_p)^n u(x,y) \le G^n p^n I_{x,y}^{n\gamma_1}\big(\Psi_u(\cdot,\cdot) + E I_y^1\Phi_u(\cdot,\cdot) + E I_y^1 \Xi_u(\cdot,\cdot) + E^2 I_{x,y}^{1,1} u(\cdot,\cdot)\big)(x,y).
$$

Therefore,

$$
\begin{aligned}
(B_1+\dots+B_p)^n u(x,y) \le \frac{G^n p^n}{\Gamma(n\alpha_1)\Gamma(n\beta_1)}\Big(&\int_0^x\int_0^y \Psi_u(s,t)\,ds\,dt + E\int_0^x\int_0^y \Phi_u(s,t)\,ds\,dt\\
&+ E\int_0^x\int_0^y \Xi_u(s,t)\,ds\,dt + E^2\int_0^x\int_0^y u(s,t)\,ds\,dt\Big)
\end{aligned}
$$

for sufficiently large $n\in\mathbb{N}$ (such that $n\alpha_1\ge 1$ and $n\beta_1\ge 1$) and $(x,y)\in P$ a.e.

Clearly, the nonnegative series given by the right-hand side of the above inequality is convergent for any $(x,y)\in P$. It implies the pointwise (a.e. on P) convergence of the series

$$
\sum_{n=0}^{\infty}(B_1+\dots+B_p)^n u(x,y).
$$

So,

$$
\begin{aligned}
&u(x,y)\\
&\le \lim_{n\to\infty}\Big(\sum_{i=0}^{n}(B_1+\dots+B_p)^i a(x,y) + (B_1+\dots+B_p)^{n+1}u(x,y)\Big)\\
&= \sum_{n=0}^{\infty}(B_1+\dots+B_p)^n a(x,y) = a(x,y)\\
&\quad + \sum_{\substack{n\in\mathbb{N},\\ n\alpha_1<1 \text{ or } n\beta_1<1}} (B_1+\dots+B_p)^n a(x,y) + \sum_{\substack{n\in\mathbb{N},\\ n\alpha_1\ge 1 \text{ and } n\beta_1\ge 1}} (B_1+\dots+B_p)^n a(x,y)\\
&\le a(x,y) + \sum_{\substack{n\in\mathbb{N},\\ n\alpha_1<1 \text{ or } n\beta_1<1}} G^n p^n I_{x,y}^{n\gamma_1}\Big(\Psi_a(\cdot,\cdot) + E I_y^1\Phi_a(\cdot,\cdot) + E I_y^1\Xi_a(\cdot,\cdot) + E^2 I_{x,y}^{1,1}a(\cdot,\cdot)\Big)(x,y)\\
&\quad + \sum_{\substack{n\in\mathbb{N},\\ n\alpha_1\ge 1 \text{ and } n\beta_1\ge 1}} \frac{G^n p^n}{\Gamma(n\alpha_1)\Gamma(n\beta_1)}\Big(I_{x,y}^{1,1}\Psi_a(x,y) + E I_{x,y}^{1,1}\Phi_a(x,y)\\
&\quad + E I_{x,y}^{1,1}\Xi_a(x,y) + E^2 I_{x,y}^{1,1} a(x,y)\Big)
\end{aligned} \tag{3.2}
$$

for $(x,y)\in P$ a.e. □

Since the first sum of the right-hand side in (3.2) is finite and series $\sum_{n=1}^{\infty}\frac{G^n p^n}{\Gamma(n\alpha_1)\Gamma(n\beta_1)}$ is convergent, the right-hand side in (3.2) is a function belonging to $L^1(P)$ which does not depend on $u\in L^1(P)$. In other words, under assumptions of Proposition (3.1), there exists a function $\sigma\in L^1(P)$ not depending on u and such that

$$
u(x,y)\le\sigma(x,y)
$$

for $(x,y)\in P$ a.e. Thus, we have the following fractional Gronwall lemma for functions of two variables.

Lemma 3.2. *Let the constants $a_1,\dots,a_p>0$ ($p\in\mathbb{N}$) and multiindexes $\gamma_1=(\alpha_1,\beta_1),\dots,\gamma_p=(\alpha_p,\beta_p)$ with $\alpha_i>0$, $\beta_i>0$ ($i=\overline{1,p}$) be given. If the functions $a,u\in L^1(P,\mathbb{R}_0^+)$ satisfy inequality (3.1), then there exists a function $\sigma\in L^1(P)$ not depending on $u\in L^1(P)$ and such that*

$$u(x,y)\le\sigma(x,y)$$

for $(x,y)\in P$ a.e.

Proof. Just increase the right side of (3.1) by adding the term $a_0\int_0^x\int_0^y\frac{u(s,t)}{(x-s)^{1-\alpha_0}(y-t)^{1-\beta_0}}\,ds\,dt$ with $a_0=1$,

$$\alpha_0=\min\{\alpha_i;\ i=\overline{1,p}\},\quad \beta_0=\min\{\beta_i;\ i=\overline{1,p}\}$$

and use Proposition 3.1. □

4. Cauchy problem

Let us consider the Cauchy problem (1.1)-(1.2). As we said in the Introduction, because of the noncompactness of single integral operators of arbitrary (positive integer and fractional) order and compactness assumption in the Schaefer fixed point theorem it is important that right-hand side of (1.1) contains neither derivatives of orders involving order α with respect to x nor derivatives of orders involving order β with respect to y.

By a solution to this Cauchy problem we mean a function $z\in AC_{x,y}^{\alpha,\beta}$. It is easy to see that a solution of problem (1.1) -(1.2) exists if and only if there exists a solution $\varphi\in L^1(P)$ to the integral equation

$$\begin{aligned}
\varphi(x,y)=f\Big(&x,y,I_{x,y}^{\alpha,\beta}\varphi(x,y)+\lambda_1(x,y),\\
&I_{x,y}^{1,\beta}\varphi(x,y)+\lambda_{2,1}(x,y),\dots,I_{x,y}^{k-1,\beta}\varphi(x,y)+\lambda_{2,k-1}(x,y),\\
&I_{x,y}^{\alpha,1}\varphi(x,y)+\lambda_{3,1}(x,y),\dots,I_{x,y}^{\alpha,l-1}\varphi(x,y)+\lambda_{3,l-1}(x,y),\\
&I_{x,y}^{1,1}\varphi(x,y)+\lambda_{4,1,1}(x,y),\dots,I_{x,y}^{k-1,l-1}\varphi(x,y)+\lambda_{4,k-1,l-1}(x,y),\\
&I_{x,y}^{1,\beta-\beta_1}\varphi(x,y)+\lambda_{5,1,1}(x,y),\dots,I_{x,y}^{k-1,\beta-\beta_n}\varphi(x,y)+\lambda_{5,k-1,n}(x,y),\\
&I_{x,y}^{\alpha-\alpha_1,1}\varphi(x,y)+\lambda_{6,1,1}(x,y),\dots,I_{x,y}^{\alpha-\alpha_m,l-1}\varphi(x,y)+\lambda_{6,l-1,m}(x,y),\\
&I_{x,y}^{\alpha-\alpha_1,\beta-\beta_1}\varphi(x,y)+\lambda_{7,1,1}(x,y),\dots,I_{x,y}^{\alpha-\alpha_p,\beta-\beta_n}\varphi(x,y)+\lambda_{7,m,n}(x,y),\\
&I_{x,y}^{\alpha-\alpha_1,\beta}\varphi(x,y)+\lambda_{8,1}(x,y),\dots,I_{x,y}^{\alpha-\alpha_m,\beta}\varphi(x,y)+\lambda_{8,m}(x,y),\\
&I_{x,y}^{\alpha,\beta-\beta_1}\varphi(x,y)+\lambda_{9,1}(x,y),\dots,I_{x,y}^{\alpha,\beta-\beta_n}\varphi(x,y)+\lambda_{9,n}(x,y)\Big)
\end{aligned}\tag{4.1}$$

where λs are the appropriate functions belonging to $L^1(P)$ and depending on the initial data v^j, μ^i, $e^{i,j}$ (see Appendix - Theorems 5.2, 5.3, 5.4).

Let us assume that

(A1) the function $f=f(x,y,\lambda_1,\dots,\lambda_r):P\times\mathbb{R}^r\to\mathbb{R}$ is of Caratheodory type in $(x,y)\in P$, $(\lambda_1,\dots,\lambda_r)\in\mathbb{R}^r$ and

$$|f(x,y,\lambda_1,\dots,\lambda_r)|\le d|(\lambda_1,\dots,\lambda_r)|+b(x,y)$$

for $(x,y)\in P$ a.e., $(\lambda_1,\dots,\lambda_r)\in\mathbb{R}^r$, where $d>0$ and $b\in L^1(P,\mathbb{R}_0^+)$.

Consider an operator $T:L^1(P)\to L^1(P)$ defined by the right-hand side of the above integral equation. It can be written as the following superposition

$$T=\mathbb{F}\circ\Lambda$$

where $\mathbb{F}:L^1(P,\mathbb{R}^n)\to L^1(P)$ is the superposition operator defined by f, $\Lambda:L^1(P)\to L^1(P,\mathbb{R}^r)$ is of the form

$$\begin{aligned}
\Lambda\varphi(x,y)=\Big(&I_{x,y}^{\alpha,\beta}\varphi(x,y)+\lambda_1(x,y),\\
&I_{x,y}^{1,\beta}\varphi(x,y)+\lambda_{2,1}(x,y),\dots,I_{x,y}^{k-1,\beta}\varphi(x,y)+\lambda_{2,k-1}(x,y),\\
&I_{x,y}^{\alpha,1}\varphi(x,y)+\lambda_{3,1}(x,y),\dots,I_{x,y}^{\alpha,l-1}\varphi(x,y)+\lambda_{3,l-1}(x,y),
\end{aligned}$$

$$
\begin{aligned}
&I_{x,y}^{1,1}\varphi(x,y)+\lambda_{4,1,1}(x,y),\dots,I_{x,y}^{k-1,l-1}\varphi(x,y)+\lambda_{4,k-1,l-1}(x,y),\\
&I_{x,y}^{1,\beta-\beta_1}\varphi(x,y)+\lambda_{5,1,1}(x,y),\dots,I_{x,y}^{k-1,\beta-\beta_n}\varphi(x,y)+\lambda_{5,k-1,n}(x,y),\\
&I_{x,y}^{\alpha-\alpha_1,1}\varphi(x,y)+\lambda_{6,1,1}(x,y),\dots,I_{x,y}^{\alpha-\alpha_m,l-1}\varphi(x,y)+\lambda_{6,l-1,m}(x,y),\\
&I_{x,y}^{\alpha-\alpha_1,\beta-\beta_1}\varphi(x,y)+\lambda_{7,1,1}(x,y),\dots,I_{x,y}^{\alpha-\alpha_p,\beta-\beta_n}\varphi(x,y)+\lambda_{7,m,n}(x,y),\\
&I_{x,y}^{\alpha-\alpha_1,\beta}\varphi(x,y)+\lambda_{8,1}(x,y),\dots,I_{x,y}^{\alpha-\alpha_m,\beta}\varphi(x,y)+\lambda_{8,m}(x,y),\\
&I_{x,y}^{\alpha,\beta-\beta_1}\varphi(x,y)+\lambda_{9,1}(x,y),\dots,I_{x,y}^{\alpha,\beta-\beta_n}\varphi(x,y)+\lambda_{9,n}(x,y)\Big)
\end{aligned}
$$

for $\varphi \in L^1(P)$. From [12, Lemma 1.1] the compactness of Λ follows. It means, in view of the continuity of $\mathbb{F}$ (see the assumption (A2) and Krasnoselskii theorem [9, Part I.2, Theorem 2.1]) that $T : L^1(P) \to L^1(P)$ is compact. Moreover, the set

$$\{\varphi \in L^1(P);\ \varphi = \lambda T\varphi \text{ for some } 0 < \lambda < 1\} \tag{4.2}$$

is bounded in $L^1(P)$. Indeed, for any φ belonging to the above set, we have

$$
\begin{aligned}
|\varphi(x,y)| &\le |\lambda T\varphi(x,y)|\\
&\le d\Big(|I_{x,y}^{\alpha,\beta}\varphi(x,y)|+\sum_{i=1}^{k-1}|I_{x,y}^{i,\beta}\varphi(x,y)|+\sum_{j=1}^{l-1}|I_{x,y}^{\alpha,j}\varphi(x,y)|\\
&\quad+\sum_{i=1}^{k-1}\sum_{j=1}^{l-1}|I_{x,y}^{i,j}\varphi(x,y)|+\sum_{i=1}^{k-1}\sum_{q=1}^{n}|I_{x,y}^{i,\beta-\beta_q}\varphi(x,y)|\\
&\quad+\sum_{p=1}^{m}\sum_{j=1}^{l-1}|I_{x,y}^{\alpha-\alpha_p,j}\varphi(x,y)|+\sum_{p=1}^{m}\sum_{q=1}^{n}|I_{x,y}^{\alpha-\alpha_p,\beta-\beta_q}\varphi(x,y)|\\
&\quad+\sum_{p=1}^{m}|I_{x,y}^{\alpha-\alpha_p,\beta}\varphi(x,y)|+\sum_{q=1}^{n}|I_{x,y}^{\alpha,\beta-\beta_q}\varphi(x,y)|\Big)+B(x,y)
\end{aligned}
$$

for $(x,y)\in P$ a.e., where

$$
\begin{aligned}
B(x,y)&=d\Big(|\lambda_1(x,y)|+\sum_{i=1}^{k-1}|\lambda_{2,i}(x,y)|+\sum_{j=1}^{l-1}|\lambda_{3,j}(x,y)|\\
&\quad+\sum_{i=1}^{k-1}\sum_{j=1}^{l-1}|\lambda_{4,i,j}(x,y)|+\sum_{i=1}^{k-1}\sum_{q=1}^{n}|\lambda_{5,i,q}(x,y)|+\sum_{p=1}^{m}\sum_{j=1}^{l-1}|\lambda_{6,j,p}(x,y)|\\
&\quad+\sum_{p=1}^{m}\sum_{q=1}^{n}|\lambda_{7,p,q}(x,y)|+\sum_{p=1}^{m}|\lambda_{8,p}(x,y)|+\sum_{q=1}^{n}|\lambda_{9,q}(x,y)|\Big)+b(x,y)
\end{aligned}
$$

for $(x,y)\in P$ a.e. From the lemma 3.2 it follows that there exists a function $\sigma\in L^1(P)$ such that

$$|\varphi(x,y)|\le\sigma(x,y)$$

for $(x,y)\in P$ a.e. It means that the set (4.2) is bounded. Thus, from Schaefer fixed point theorem follows that T has a fixed point. Consequently, we have the following result.

Theorem 4.1. *If the function f satisfies condition* (A1)*, then problem* (1.1)-(1.2) *has at least one solution in $AC_{x,y}^{\alpha,\beta}$.*

Remark 4.2. Let us consider a more general equation

$$
\begin{aligned}
&D^{\alpha,\beta}_{x,y} z(x,y)\\
&= f\Big(x,y,z(x,y), D^{\alpha}_{x}(x,y), D^{\beta}_{y} z(x,y),\\
&\quad D^{\alpha-1}_{x} z(x,y),\dots, D^{\alpha-(k-1)}_{x} z(x,y), D^{\beta-1}_{y} z(x,y),\dots, D^{\beta-(l-1)}_{y} z(x,y),\\
&\quad D^{\alpha-1,\beta}_{x,y} z(x,y),\dots, D^{\alpha-(k-1),\beta}_{x,y} z(x,y), D^{\alpha,\beta-1}_{x,y} z(x,y),\dots, D^{\alpha,\beta-(l-1)}_{x,y} z(x,y),\\
&\quad D^{\alpha-1,\beta-1}_{x,y} z(x,y),\dots, D^{\alpha-(k-1),\beta-(l-1)}_{x,y} z(x,y),\\
&\quad D^{\alpha-0,\beta_1}_{x,y} z(x,y),\dots, D^{\alpha-(k-1),\beta_n}_{x,y} z(x,y), D^{\alpha_1,\beta-0}_{x,y} z(x,y),\dots, D^{\alpha_m,\beta-(l-1)}_{x,y} z(x,y),\\
&\quad D^{\alpha_1,\beta_1}_{x,y} z(x,y),\dots, D^{\alpha_m,\beta_n}_{x,y} z(x,y), D^{\alpha_1}_{x} z(x,y),\dots, D^{\alpha_m}_{x} z(x,y),\\
&\quad D^{\beta_1}_{y} z(x,y),\dots, D^{\beta_n}_{y} z(x,y)\Big)
\end{aligned}
\tag{4.3}
$$

for $(x,y) \in P = [0,1]\times[0,1]$ a.e., with the initial conditions (1.2). Here $k-1 < \alpha \le k$, $l-1 < \beta \le l$ $(k,l \in \mathbb{N})$, $0 < \alpha_p < \alpha-(k-1)$ for $p = \overline{1,m}$, $0 < \beta_q < \beta-(l-1)$ for $q = \overline{1,n}$ $(m,n \in \mathbb{N})$, $f = f(x,y,\lambda) : P \times \mathbb{R}^r \to \mathbb{R}$ with $r = kl + k + l + mn + m + n$, μ^i for $i = \overline{0,k-1}$ and ν^j for $j = \overline{0,l-1}$ belong to $L^1(I)$, $e^{i,j} \in \mathbb{R}$ for $i = \overline{0,k-1}$, $j = \overline{0,l-1}$ (if $k = 1$, then the derivatives of orders involving the orders $\alpha - i$ with $i = \overline{1,k-1}$ do not appear on the right-hand side of the system and, similarly, if $l = 1$, then the derivatives of orders involving the orders $\beta - j$ with $j = \overline{1,l-1}$ do not appear on the right-hand side of the system). Assuming that

(A2) the function $f = f(x,y,\lambda_1,\dots,\lambda_r) : P \times \mathbb{R}^r \to \mathbb{R}$ is Lebesgue measurable in $(x,y) \in P$, lipschitzian in $(\lambda_1,\dots,\lambda_r) \in \mathbb{R}^r$ and the function
$$f(x,y,0,\dots,0) : P \times \mathbb{R}^r \to \mathbb{R},$$
belongs to $L^1(P)$

one can prove (in a standard way, using Banach contraction principle) existence of a unique solution in the space $AC^{\alpha,\beta}_{x,y}$ to problem (4.3)-(1.2).

Example 4.3. From Theorem 4.1, the existence of a solution in the space $AC^{1,1}_{x,y}$ to the equation

$$D^{1,1}_{x,y} z(x,y) = f(x,y,z(x,y), D^{3/4,2/3}_{x,y} z(x,y), D^{3/4}_{x} z(x,y), D^{2/3}_{y} z(x,y))$$

for $(x,y) \in P$ *a.e.*, with initial conditions

$$
\begin{gathered}
\frac{\partial}{\partial x} z(x,0) = \nu^0(x), \quad x \in I \text{ a.e.}\\
\frac{\partial}{\partial y} z(0,y) = \mu^0(y), \quad y \in I \text{ a.e.}\\
z(0,0) = e^{0,0},
\end{gathered}
$$

where $f : P \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the assumptions (A1) and ν^0, $\mu^0 \in L^1(I)$, $e^{0,0} \in \mathbb{R}$, follows.

Example 4.4. From Theorem 4.1, the existence of a solution in the space $AC^{5/4,1/3}_{x,y}$ to the equation

$$D^{5/4,1/3}_{x,y} z(x,y) = f\Big(x,y,z(x,y), D^{1/4}_{x} z(x,y), D^{1/4,1/6}_{x,y} z(x,y), D^{1/5,1/6}_{x,y} z(x,y), D^{1/5}_{x} z(x,y), D^{1/6}_{y} z(x,y)\Big)$$

for $(x,y) \in P$ *a.e.*, with initial conditions

$$
\begin{gathered}
\frac{\partial^2}{\partial x^2} (I^{3/4,2/3}_{x,y} z)(x,0) = \nu^0(x), \quad x \in I \text{ a.e.}\\
\frac{\partial}{\partial y} (I^{3/4,2/3}_{x,y} z)(0,y) = \mu^0(y), \; y \in I \text{ a.e.}\\
\frac{\partial^2}{\partial x \partial y} (I^{3/4,2/3}_{x,y} z)(0,y) = \mu^1(y), \; y \in I \text{ a.e.}
\end{gathered}
$$

$$I^{3/4,2/3}_{x,y} z(0,0) = e^{0,0}$$
$$\frac{\partial}{\partial x}(I^{3/4,2/3}_{x,y} z)(0,0) = e^{1,0},$$

where $f : P \times \mathbb{R}^6 \to \mathbb{R}$ satisfies assumption (A1) and ν^0, μ^0, $\mu^1 \in L^1(I)$, $e^{0,0}$, $e^{1,0} \in \mathbb{R}$, follows.

Example 4.5. From Remark 4.2, the existence of a unique solution in the space $AC^{1,1}_{x,y}$ to the equation

$$D^{1,1}_{x,y} z(x,y) = f(x,y,z(x,y), D^1_x(x,y), D^1_y z(x,y), D^{3/4,2/3}_{x,y} z(x,y), D^{3/4}_x z(x,y), D^{2/3}_y z(x,y))$$

for $(x,y) \in P$ a.e., with initial conditions

$$\frac{\partial}{\partial x} z(x,0) = \nu^0(x), \ x \in I \text{ a.e.}$$
$$\frac{\partial}{\partial y} z(0,y) = \mu^0(y), \ y \in I \text{ a.e.}$$
$$z(0,0) = e^{0,0},$$

where $f : P \times \mathbb{R}^6 \to \mathbb{R}$ satisfies assumptions (A2) and ν^0, $\mu^0 \in L^1(I)$, $e^{0,0} \in \mathbb{R}$, follows.

Example 4.6. From Remark 4.2, the existence of a unique solution in the space $AC^{5/4,1/3}_{x,y}$ to the following Cauchy problem follows

$$\begin{aligned} &D^{5/4,1/3}_{x,y} z(x,y) \\ &= f\Big(x,y,z(x,y), D^{5/4}_x(x,y), D^{1/3}_y z(x,y), D^{1/4}_x z(x,y), D^{1/4,1/3}_{x,y} z(x,y), \\ &\quad D^{5/4,1/6}_{x,y} z(x,y), D^{1/4,1/6}_{x,y} z(x,y), D^{1/5,1/3}_{x,y} z(x,y), D^{1/5,1/6}_{x,y} z(x,y), D^{1/5}_x z(x,y), D^{1/6}_y z(x,y)\Big), \end{aligned}$$

for $(x,y) \in P$ a.e., with initial conditions

$$\frac{\partial^2}{\partial x^2}(I^{3/4,2/3}_{x,y} z)(x,0) = \nu^0(x), \quad x \in I \text{ a.e.}$$
$$\frac{\partial}{\partial y}(I^{3/4,2/3}_{x,y} z)(0,y) = \mu^0(y), \quad y \in I \text{ a.e.}$$
$$\frac{\partial^2}{\partial x \partial y}(I^{3/4,2/3}_{x,y} z)(0,y) = \mu^1(y), \quad y \in I \text{ a.e.}$$
$$I^{3/4,2/3}_{x,y} z(0,0) = e^{0,0}$$
$$\frac{\partial}{\partial x}(I^{3/4,2/3}_{x,y} z)(0,0) = e^{1,0}$$

where $f : P \times \mathbb{R}^8 \to \mathbb{R}$ satisfies the assumption (A2) and ν^0, μ^0, $\mu^1 \in L^1(I)$, $e^{0,0}$, $e^{1,0} \in \mathbb{R}$, follows.

5. Appendix

5.1. **Imbeddings for $AC^{\alpha,\beta}_{x,y}$.** Details of the results presented in this section (including the proofs) are contained in the paper [5]. We have the following general result on imbeddings of the spaces of fractionally absolutely continuous functions of two variables.

Theorem 5.1. *If $0 < \alpha_2 \le \alpha_1$ and $0 < \beta_2 \le \beta_1$, then*

$$AC^{\alpha_1,\beta_1}_{x,y} \subset AC^{\alpha_2,\beta_2}_{x,y}$$

if and only if

$$(\alpha_2 \in (0,1] \text{ and } \alpha_1 \in [\alpha_2,1] \cup [1+\alpha_2,2] \cup \dots) \quad \text{or}$$
$$(\alpha_2 \in (1,\infty) \text{ and } \alpha_1 = \alpha_2, 1+\alpha_2, \ 2+\alpha_2, \ \dots)$$

and

$$(\beta_2 \in (0,1] \text{ and } \beta_1 \in [\beta_2,1] \cup [1+\beta_2,2] \cup \dots) \quad \text{or}$$
$$(\beta_2 \in (1,\infty) \text{ and } \beta_1 = \beta_2, 1+\beta_2, \ 2+\beta_2, \dots).$$

Theorem 5.2. *If $(\alpha,\beta)\in(k-1,k]\times(l-1,l]$ ($k,l\in\mathbb{N}$) and $z\in AC^{\alpha,\beta}_{x,y}$ is of the form (2.4), then, for $i=\overline{1,k-1}$, $j=\overline{1,l-1}$ with $k\geq 2$, $l\geq 2$, we have*

$$\begin{aligned}
D^{\alpha-i,\beta-j}_{x,y}z(x,y) &= I^{i,j}_{x,y}\varphi(x,y)+I^{j}_{y}\Big(\mu_{k-i}(y)\frac{x^0}{\Gamma(1)}+\dots+\mu_{k-1}(y)\frac{x^{k-1-(k-i)}}{\Gamma(k-1-(k-i)+1)}\Big)\\
&\quad+I^{i}_{x}\Big(\nu_{l-j}(x)\frac{y^0}{\Gamma(1)}+\dots+\nu_{l-1}(x)\frac{y^{l-1-(l-j)}}{\Gamma(l-1-(l-j)+1)}\Big)\\
&\quad+\sum_{s=0}^{k-1-(k-i)}\sum_{t=0}^{l-1-(l-j)}e_{k-i+s,l-j+t}\frac{x^s}{\Gamma(s+1)}\frac{y^t}{\Gamma(t+1)},\\
D^{\alpha-i,\beta}_{x,y}z(x,y) &= I^{i}_{x}\varphi(x,y)+\mu_{k-i}(y)\frac{x^0}{\Gamma(1)}+\dots+\mu_{k-1}(y)\frac{x^{k-1-(k-i)}}{\Gamma(k-1-(k-i)+1)},\\
D^{\alpha-i,0}_{x,y}z(x,y) &= I^{i,\beta}_{x,y}\varphi(x,y)+\frac{I^{\beta}_{y}\mu_{k-i}(y)}{\Gamma(1)}x^0\dots+\frac{I^{\beta}_{y}\mu_{k-1}(y)}{\Gamma(i)}x^{i-1}\\
&\quad+I^{i}_{x}\big(\frac{\nu_0(x)}{\Gamma(1-(l-\beta))}y^{-(l-\beta)}+\dots+\frac{\nu_{l-1}(x)}{\Gamma(l-(l-\beta))}y^{l-1-(l-\beta)}\big)\\
&\quad+\sum_{s=k-i}^{k-1}\sum_{t=0}^{l-1}e_{s,t}\frac{x^{s-(k-i)}}{\Gamma(s-(k-i)+1)}\frac{y^{t-(l-\beta)}}{\Gamma(t-(l-\beta)+1)},\\
D^{\alpha,\beta-j}_{x,y}z(x,y) &= I^{j}_{y}\varphi(x,y)+\nu_{l-j}(x)\frac{y^0}{\Gamma(1)}+\dots+\nu_{l-1}(x)\frac{y^{l-1-(l-j)}}{\Gamma(l-1-(l-j)+1)},\\
D^{0,\beta-j}_{x,y}z(x,y) &= I^{\alpha,j}_{x,y}\varphi(x,y)+\frac{I^{\alpha}_{x}v_{l-j}(x)}{\Gamma(1)}y^0\dots+\frac{I^{\alpha}_{x}v_{l-1}(x)}{\Gamma(j)}y^{j-1}\\
&\quad+I^{j}_{y}\big(\frac{\mu_0(y)}{\Gamma(1-(k-\alpha))}x^{-(k-\alpha)}+\dots+\frac{\mu_{k-1}(y)}{\Gamma(k-(k-\alpha))}x^{k-1-(k-\alpha)}\big)\\
&\quad+\sum_{s=0-i}^{k-1}\sum_{t=l-j}^{l-1}e_{s,t}\frac{x^{s-(k-\alpha)}}{\Gamma(s-(k-\alpha)+1)}\frac{y^{t-(l-j)}}{\Gamma(t-(l-j)+1)}
\end{aligned}$$

for $(x,y)\in P$ a.e.

Theorem 5.3. *If $\alpha_1\in(k_1-1,k_1]$, $\beta_1\in(l_1-1,l_1]$, $\alpha_2\in(0,\alpha_1-k_1+1)$, $\beta_2\in(0,\beta_1-l_1+1)$ (k_1, l_1, k_2, $l_2\in\mathbb{N}$) and $z\in AC^{\alpha_1,\beta_1}_{x,y}$ is of the form*

$$\begin{aligned}
z(x,y) &= I^{\alpha_1,\beta_1}_{x,y}\varphi(x,y)+I^{\beta_1}_{y}\big(\frac{\mu_0(y)}{\Gamma(1-(k_1-\alpha_1))}x^{-(k_1-\alpha_1)}\\
&\quad+\dots+\frac{\mu_{k_1-1}(y)}{\Gamma(k_1-(k_1-\alpha_1))}x^{k_1-1-(k_1-\alpha_1)}\big)\\
&\quad+I^{\alpha_1}_{x}\big(\frac{\nu_0(x)}{\Gamma(1-(l_1-\beta_1))}y^{-(l_1-\beta_1)}+\dots+\frac{\nu_{l_1-1}(x)}{\Gamma(l_1-(l_1-\beta_1))}y^{l_1-1-(l_1-\beta_1)}\big)\\
&\quad+\sum_{i=0}^{k_1-1}\sum_{j=0}^{l_1-1}e_{i,j}\frac{x^{i-(k_1-\alpha_1)}y^{j-(l_1-\beta_1)}}{\Gamma(i+1-(k_1-\alpha_1))\Gamma(j+1-(l_1-\beta_1))},
\end{aligned}\tag{5.1}$$

then $z\in I^{\alpha_2,\beta_2}_{x,y}(L^1(P))\subset AC^{\alpha_2,\beta_2}_{x,y}$, i.e.

$$z(x,y)=I^{\alpha_2,\beta_2}_{x,y}\overline{\varphi}(x,y),\ (x,y)\in P\quad\text{a.e.},$$

where $\overline{\varphi}\in L^1(P)$ is of the form

$$\begin{aligned}
\overline{\varphi}(x,y) &= I^{\alpha_1-\alpha_2,-\beta_2+\beta_1}_{x,y}\varphi(x,y)+\frac{x^{-\alpha_2-(k_1-\alpha_1)}}{\Gamma(1-\alpha_2-(k_1-\alpha_1))}I^{\beta_1-\beta_2}_{y}\mu_0(y)\\
&\quad+\dots+\frac{x^{-\alpha_2+k_1-1-(k_1-\alpha_1)}}{\Gamma(1-\alpha_2+k_1-1-(k_1-\alpha_1))}I^{\beta_1-\beta_2}_{y}\mu_{k_1-1}(y)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{y^{-\beta_2-(l_1-\beta_1)}}{\Gamma(1-\beta_2-(l_1-\beta_1))} I_x^{\alpha_1-\alpha_2}\nu_0(x) \\
&+ \dots + \frac{y^{-\beta_2+l_1-1-(l_1-\beta_1)}}{\Gamma(1-\beta_2+l_1-1-(l_1-\beta_1))} I_x^{\alpha_1-\alpha_2}\nu_{l_1-1}(x) \\
&+ \sum_{i=0}^{k_1-1}\sum_{j=0}^{l_1-1} e_{i,j} \frac{x^{-\alpha_2+i-(k_1-\alpha_1)}y^{-\beta_2+j-(l_1-\beta_1)}}{\Gamma(1-\alpha_2+i-(k_1-\alpha_1))\Gamma(1-\beta_2+j-(l_1-\beta_1))}
\end{aligned}
$$

for $(x, y) \in P$ a.e.

Theorem 5.4. *If $\alpha_1 \in (k_1 - 1, k_1]$, $\beta_1 \in (l_1 - 1, l_1]$ (k_1, $l_1 \in \mathbb{N}$) and $z \in AC^{\alpha_1,\beta_1}_{x,y}$ is of the form (5.1), then*

- *$z \in AC^{\alpha_1-i,\beta_2}_{x,y}$ for $i = \overline{0, k_1 - 1}$, $\beta_2 \in (0, \beta_1 - l_1 + 1)$ and*

$$
\begin{aligned}
D^{\alpha_1-i,\beta_2}_{x,y} z(x,y) &= I^{i,\beta_1-\beta_2}_{x,y}\varphi(x,y) + I_x^i\nu_0(x)\frac{y^{1-\beta_2-(l_1-\beta_1)-1}}{\Gamma(1-\beta_2-(l_1-\beta_1))} \\
&\quad + \dots + I_x^i\nu_{l_1-1}(x)\frac{y^{l_1-\beta_2-(l_1-\beta_1)-1}}{\Gamma(l_1-\beta_2-(l_1-\beta_1))} \\
&\quad + \sum_{s=k-i}^{k_1-1}\sum_{j=1}^{l_1-1} e_{s,j}\frac{x^{s-(k-i)}}{\Gamma(s+1-(k-i))}\frac{y^{j-\beta_2-(l_1-\beta_1)}}{\Gamma(j-\beta_2-(l_1-\beta_1)+1)})
\end{aligned}
$$

for $(x, y) \in P$ a.e.

- *$z \in AC^{\alpha_2,\beta_1-j}_{x,y}$ for $j = \overline{0, l_1 - 1}$, $\alpha_2 \in (0, \alpha_1 - k_1 + 1)$ and*

$$
\begin{aligned}
D^{\alpha_2,\beta_1-j}_{x,y} z(x,y) &= I^{\alpha_1-\alpha_2,j}_{x,y}\varphi(x,y) + I_y^j\mu_0(y)\frac{x^{1-\alpha_2-(k_1-\alpha_1)-1}}{\Gamma(1-\alpha_2-(k_1-\alpha_1))} \\
&\quad + \dots + I_y^j\mu_{k_1-1}(y)\frac{x^{k_1-\alpha_2-(k_1-\alpha_1)-1}}{\Gamma k_1-\alpha_2-(k_1-\alpha_1))} \\
&\quad + \sum_{i=1}^{k_1-1}\sum_{t=l_1-j}^{l_1-1} e_{i,t}\frac{x^{i-\alpha_2-(k_1-\alpha_1)}}{\Gamma(i-\alpha_2-(k_1-\alpha_1)+1)}\frac{y^{t-(l_1-j)}}{\Gamma(t+1-(l_1-j))})
\end{aligned}
$$

for $(x, y) \in P$ a.e.

- *$z \in AC^{0,\beta_2}_{x,y} = AC^{\beta_2}_y$ for $\beta_2 \in (0, \beta_1 - l_1 + 1)$ and*

$$
\begin{aligned}
D^{\beta_2}_y z(x,y) = D^{0,\beta_2}_{x,y} z(x,y) &= \frac{I_x^{\alpha_1}\nu_0(x) + \sum_{i=0}^{k_1-1} e_{i,0}\frac{x^{i-(k_1-\alpha_1)}}{\Gamma(i+1-(k_1-\alpha_1))}}{\Gamma(1-\beta_2-(l_1-\beta_1))} y^{-\beta_2-(l_1-\beta_1)} \\
&\quad + \dots + \frac{I_x^{\alpha_1}\nu_{l_1-1}(x) + \sum_{i=0}^{k_1-1} e_{i,l_1-1}\frac{x^{i-(k_1-\alpha_1)}}{\Gamma(i+1-(k_1-\alpha_1))}}{\Gamma(1-\beta_2-(l_1-\beta_1)+l_1-1)} y^{-\beta_2+l_1-1-(l_1-\beta_1)} \\
&\quad + I_y^{-\beta_2+\beta_1}(I_x^{\alpha_1}\varphi(x,y) + \frac{\mu_0(y)}{\Gamma(1-(k_1-\alpha_1))}x^{-(k_1-\alpha_1)} \\
&\quad + \dots + \frac{\mu_{k_1-1}(y)}{\Gamma(k_1-(k_1-\alpha_1))}x^{k_1-1-(k_1-\alpha_1)})(x,y)
\end{aligned}
$$

for $(x, y) \in P$ a.e.

- *$z \in AC^{\alpha_2,0}_{x,y} = AC^{\alpha_2}_x$ for $\alpha_2 \in (0, \alpha_1 - k_1 + 1)$ and*

$$
\begin{aligned}
D^{\alpha_2}_x z(x,y) = D^{\alpha_2,0}_{x,y} z(x,y) &= \frac{I_y^{\beta_1}\mu_0(y) + \sum_{j=0}^{l_1-1} e_{0,j}\frac{y^{j-(l_1-\beta_1)}}{\Gamma(j+1-(l_1-\beta_1))}}{\Gamma(1-\alpha_2-(k_1-\alpha_1))} x^{-\alpha_2-(k_1-\alpha_1)} \\
&\quad + \dots + \frac{I_y^{\beta_1}\mu_{k_1-1}(y) + \sum_{j=0}^{l_1-1} e_{k_1-1,j}\frac{y^{j-(l_1-\beta_1)}}{\Gamma(j+1-(l_1-\beta_1))}}{\Gamma(1-\alpha_2-(k_1-\alpha_1)+k_1-1)} x^{-\alpha_2+k_1-1-(k_1-\alpha_1)} \\
&\quad + I_x^{-\alpha_2+\alpha_1}(I_y^{\beta_1}\varphi(x,y) + \frac{\nu_0(x)}{\Gamma(1-(l_1-\beta_1))}y^{-(l_1-\beta_1)}
\end{aligned}
$$

$$+\cdots+\frac{\nu_{l_1-1}(y)}{\Gamma(l_1-(l_1-\beta_1))}y^{l_1-1-(l_1-\beta_1)})(x,y)$$

for $(x,y)\in P$ *a.e.*

From Theorem 5.2 the following corollary follows.

Corollary 5.5. *If* $(\alpha,\beta)\in(k-1,k]\times(l-1,l]$ *(*$k,l\in\mathbb{N}$*) and* $z\in AC^{\alpha,\beta}_{x,y}$ *is of the form* (2.4)*, then, for* $i=\overline{0,k-1}$*,* $j=\overline{0,l-1}$*,*

$$\nu_j(x)=\frac{\partial^{k+j}}{\partial x^k\partial y^j}(I^{k-\alpha,l-\beta}_{x,y}z)(x,0),\ x\in I\ a.e.,$$
$$\mu_i(y)=\frac{\partial^{i+l}}{\partial x^i\partial y^l}(I^{k-\alpha,l-\beta}_{x,y}z)(0,y),\ y\in I\ a.e.,$$
$$e_{i,j}=\frac{\partial^{i+j}}{\partial x^i\partial y^j}(I^{k-\alpha,l-\beta}_{x,y}z)(0,0).$$

5.2. **Schaefer fixed point theorem.** Let X and Y be normed spaces. We say that an operator $T:X\to Y$ is *compact* (see [1]) if it is continuous and for every nonempty bounded set $C\subset X$, the set $T(C)$ is relatively compact (for such set its closure is compact or, equivalently, which is contained in a compact set). From a general Schaefer theorem (see [14, Theorem 4.3.2]) the following particular one follows

Theorem 5.6. *Let* B *be a normed space, and* $T:B\to B$ *a compact operator. If the set*

$$\{\varphi\in B;\ \varphi=\lambda T\varphi\ \textit{for some}\ 0<\lambda<1\}$$

is bounded, then T *has a fixed point.*

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