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INFINITELY MANY SOLUTIONS TO CRITICAL PROBLEMS INVOLVING THE p-GRUSHIN OPERATOR

PAOLO MALANCHINI, GIOVANNI MOLICA BISCI, SIMONE SECCHI

Abstract. We consider a critical problem in a bounded domain involving the p-Grushin operator. After a truncation argument, we obtain infinitely many solutions to our problem via Krasnoselskii's genus, extending a previous result of García Azorero and Peral Alonso in [13] to the p-Grushin operator. A central part of our analysis is the verification of the Palais-Smale condition of the associated functional under a certain level.

1. Introduction

In this note we consider the boundary-value problem

$$\begin{gathered} -\Delta_\alpha^p u = \lambda |u|^{q-2}u + |u|^{p_\alpha^*-2}u \qquad \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0$, $q \in (1, p)$, Δ_α^p is the p-Grushin operator and p_α^* is the Sobolev critical exponent in this framework defined in (2.1).

For the reader's convenience, we briefly recall the definition of the p-Grushin operator Δ_α^p. If we split the Euclidean space $\mathbb{R}^N$ as $\mathbb{R}^m \times \mathbb{R}^\ell$, where $m \geq 1$, $\ell \geq 1$ satisfy $m + \ell = N$, a generic point $z \in \mathbb{R}^N$ can be written as

$$z = (x, y) = (x_1, \dots, x_m, y_1, \dots, y_\ell),$$

where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^\ell$. We fix a nonnegative real parameter α, and we define the Grushin gradient $\nabla_\alpha = (X_1, \dots, X_N)$ as the system of vector fields

$$X_i = \frac{\partial}{\partial x_i},\ i = 1, \dots, m, \quad X_{m+j} = |x|^\alpha \frac{\partial}{\partial y_j},\ j = 1, \dots, \ell.$$

The positive integer

$$N_\alpha \coloneqq m + (1 + \alpha)\ell$$

is the *homogeneous dimension* associated to the decomposition $N = m + \ell$.

For each $p \in (1, N_\alpha)$ and for each differentiable function $u \colon \mathbb{R}^N \to \mathbb{R}$, we define the *Grushin p-Laplace Operator* (p-Grushin for short) by

$$\Delta_\alpha^p u \coloneqq \sum_{i=1}^N X_i(|\nabla_\alpha u|^{p-2} X_i u).$$

If $\alpha = 0$ the operator Δ_α^p reduces to the p-Laplace operator Δ_p, while for $p = 2$ it coincides with the Baouendi–Grushin operator

$$\Delta_\alpha = \Delta_x + |x|^{2\alpha}\Delta_y,$$

where Δ_x and Δ_y are the Laplace operators in the variables x and y, respectively. This degenerate operator was introduced by Baouendi [5] and by Grushin [15]. A crucial property of the Baouendi-Grushin operator is that it is not uniformly elliptic in $\mathbb{R}^N$, since it is degenerate on the subspace $\Sigma = \{0\} \times \mathbb{R}^\ell$. Going back to our problem (1.1), when Ω is disjoint from Σ, the Grushin operator

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reduces to a uniformly elliptic operator, and its theory is classical. For this reason, when dealing with the (p-)Grushin operator, it is usually assumed that

$$\Omega \cap \Sigma \neq \emptyset. \tag{1.2}$$

It seems that Bieske and Gong [7] were the first to treat a p-Laplace equation in Grushin-type spaces, while fundamental solutions for the p-Laplace equation on a class of Grushin vector spaces were studied by Bieske in [6].

More recent works, such as those by Huang and Yang [17] and Huang, Ma, and Wang [18], investigate the existence, uniqueness, and regularity of solutions to the p-Laplace equation involving Grushin-type operators. Moreover, in [26], the authors prove a Liouville-type theorem for stable solutions of weighted p-Grushin equations. In [4] a sharp remainder formula for the Poincaré inequality associated with p-Grushin vector fields was established. Finally, in the very recent [9] existence and positivity of solutions for critical problems involving the p-Grushin operator with a subcritial perturbation are studied.

The study of critical problems started with the celebrated paper [8], where Brézis and Nirenberg proved the existence of positive solutions for the problem

$$\begin{gathered} -\Delta u = \lambda u + |u|^{2^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, depending on the values of the positive parameter λ. Here $N \geq 3$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent.

Analogous results for the p-Laplace operator have been obtained by García Azorero and Peral Alonso [13]. They considered the problem

$$\begin{gathered} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with $p^* = Np/(N-p)$. We can gather their main results as follows: see [13, Theorems 3.2, 3.3, 4.5].

Theorem 1.1. (i) *If $q \in (p, p^*)$, there exists $\lambda_0 > 0$ such that* (1.3) *has a nontrivial solution for all $\lambda \geq \lambda_0$.*
(ii) *If $q \in (\max\{p, p^* - p/(p-1)\}, p^*)$, there exists a nontrivial solution of* (1.3) *for all $\lambda > 0$.*
(iii) *If $q \in (1, p)$, there exists $\bar\lambda > 0$ such that, for $0 < \lambda < \bar\lambda$, problem* (1.3) *admits infinitely many solutions.*

Problem (1.1) with $p = 2$ was investigated in [1], where the authors extend the Brézis-Nirenberg result in the Grushin setting. Critical problems associated with the Grushin operator have been extensively studied. See [12] for a multiplicity result in the case $p = 2$. We refer, among others, to the recent papers [19, 21], which address existence and multiplicity results for critical Grushin problems with a Hardy term and for the Grushin-Choquard equation, respectively.

A multiplicity result for problem (1.1) with $q = p$ was proved in [22], extending to all $\alpha > 0$ the result of [23] for the p-Laplace operator. Gandal, Loiudice and Tyagi recently studied problem (1.1) with $p \in (q, p_\alpha^*)$ in [11], extending the result of García Azorero and Peral Alonso to the p-Grushin framework. Among other results, the authors established the counterparts of statements (i) and (ii) in Theorem 1.1 for problem (1.1).

In this article we prove the existence of infinitely many solutions to (1.1), extending Theorem 1.1 (iii) to the p-Grushin framework and concluding the study of critical problems for the p-Grushin operator started in [11].

We will look for solutions of (1.1) in the Sobolev space $\mathring{W}^{1,p}_\alpha(\Omega)$ defined as the completion of $C^1_c(\Omega)$ with respect to the norm

$$\|u\|_{\alpha,p} = \Big(\int_\Omega |\nabla_\alpha u|^p \,\mathrm{d}z \Big)^{1/p},$$

whose details are given in Section 2.1.

We say that a function $u \in \mathring{W}^{1,p}_\alpha(\Omega)$ is a weak solution of (1.1) if [1]

$$\int_\Omega |\nabla_\alpha u|^{p-2}\nabla_\alpha u \nabla_\alpha \varphi \, \mathrm{d}z = \lambda \int_\Omega |u|^{q-2}u\varphi \, \mathrm{d}z + \int_\Omega |u|^{p^*_\alpha - 2}u\varphi \, \mathrm{d}z,$$

for all $\varphi \in \mathring{W}^{1,p}_\alpha(\Omega)$, and so weak solutions of (1.1) coincide with critical points of the functional $I_\lambda \colon \mathring{W}^{1,p}_\alpha(\Omega) \to \mathbb{R}$ defined by

$$I_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla_\alpha u|^p \, \mathrm{d}z - \frac{\lambda}{q}\int_\Omega |u|^q \, \mathrm{d}z - \frac{1}{p^*_\alpha}\int_\Omega |u|^{p^*_\alpha} \, \mathrm{d}z. \tag{1.4}$$

We can state our main result.

Theorem 1.2. *Let $q \in (1,p)$ and let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain satisfying* (1.2)*. Then there exists $\bar{\lambda} > 0$ such that for all $\lambda \in (0, \bar{\lambda})$, problem* (1.1) *has infinitely many weak solutions in $\mathring{W}^{1,p}_\alpha(\Omega)$.*

The article is organized as follows. In Section 2, we introduce the functional framework and recall some preliminary results, including a concentration–compactness theorem, the main properties of Krasnoselskii's genus, and a classical deformation lemma. In Section 3, we establish the Palais-Smale condition for the functional I_λ under a suitable energy level. Finally, in Section 4, we prove the existence of infinitely many solutions to problem (1.1), thus completing the proof of Theorem 1.2.

2. Preliminaries

2.1. **Functional setting.** Fix a bounded domain $\Omega \subset \mathbb{R}^N$ and a number $p \in (1,\infty)$. The Sobolev space $\mathring{W}^{1,p}_\alpha(\Omega)$ is defined as the completion of $C^1_c(\Omega)$ with respect to the norm

$$\|u\|_{\alpha,p} = \Big(\int_\Omega |\nabla_\alpha u|^p \, \mathrm{d}z\Big)^{1/p}.$$

Specially, the space $\mathring{W}^{1,2}_\alpha(\Omega) = \mathscr{H}_\alpha(\Omega)$ is a Hilbert space endowed with the inner product

$$\langle u, v\rangle_\alpha = \int_\Omega \nabla_\alpha u \nabla_\alpha v \, \mathrm{d}z.$$

The following embedding result was proved in [20, Proposition 3.2 and Theorem 3.3], see also [17, Corollary 2.11].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then the embedding*

$$\mathring{W}^{1,p}_\alpha(\Omega) \hookrightarrow L^r(\Omega)$$

is continuous for every $r \in [1, p^*_\alpha]$ and compact for every $r \in [1, p^*_\alpha)$, where p^*_α is the critical Sobolev exponent for the p-Grushin operator defined as

$$p^*_\alpha = \frac{pN_\alpha}{N_\alpha - p}. \tag{2.1}$$

We may thus define the best constant of the Sobolev embedding $\mathring{W}^{1,p}_\alpha(\Omega) \hookrightarrow L^{p^*_\alpha}(\Omega)$:

$$S := \inf_{u \in \mathring{W}^{1,p}_\alpha(\Omega),\, u \neq 0} \frac{\int_\Omega |\nabla_\alpha u|^p \ \mathrm{d}z}{\big(\int_\Omega |u|^{p^*_\alpha} \ \mathrm{d}z\big)^{p/p^*_\alpha}}. \tag{2.2}$$

For $z = (x,y) \in \mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^\ell$ let

$$d(z) = \Big(|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2\Big)^{\frac{1}{2(\alpha+1)}},$$

be the homogeneous distance associated with the Grushin geometry. We indicate with

$$B_R(z) = \{\xi \in \mathbb{R}^N : d(z - \xi) < R\}$$

the d-ball of center $z \in \mathbb{R}^N$ and radius $R > 0$. A direct computation shows that $|B_R(z)| = |B_1(z)|R^{N_\alpha}$, where $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}^N$.

[1] The symbol $\nabla_\alpha u \nabla_\alpha \varphi$ is a shorthand for $\nabla_\alpha u \cdot \nabla_\alpha \varphi$.

2.2. **Some tools.** We will need a concentration-compactness result for the p-Grushin operator proved in [22, Theorem 3.3]. The space of all finite signed Radon measures on $\Omega \subset \mathbb{R}^N$ is denoted by $\mathcal{M}(\Omega,\mathbb{R})$. A sequence of measures $(\mu_n) \subset \mathcal{M}(\Omega,\mathbb{R})$ converges tightly to a measure μ, written as $\mu_n \overset{*}{\rightharpoonup} \mu$, if

$$\int_\Omega f \,\mathrm{d}\mu_n \to \int_\Omega f \,\mathrm{d}\mu \quad \text{for all } f \in C_b(\Omega), \tag{2.3}$$

where $C_b(\Omega)$ is the space of the bounded, continuous functions on Ω. On the other hand, $(\mu_n) \subset \mathcal{M}(\Omega,\mathbb{R})$ is said to converge weakly to μ, written as $\mu_n \rightharpoonup \mu$, if (2.3) holds for all $f \in C_0(\Omega)$, where $C_0(\Omega)$ is the space of the continuous functions that vanish at infinity.

Theorem 2.2 ([22, Theorem 3.3])**.** *Suppose that $(u_n) \subset \mathring{W}^{1,p}_\alpha(\Omega)$ satisfies $u_n \rightharpoonup u$ in $\mathring{W}^{1,p}_\alpha(\Omega)$, and $|\nabla_\alpha u_n| \rightharpoonup \mu$, $|u_n|^{p^*_\alpha} \overset{*}{\rightharpoonup} \nu$ for some $u \in \mathring{W}^{1,p}_\alpha(\Omega)$ and μ, ν are bounded non-negative measures on Ω. Then there exist an at most countable set $\mathcal{A}$, a family $[z_j]_{j\in\mathcal{A}}$ of distinct points in Ω and two families of positive numbers $[\mu_j]_{j\in\mathcal{A}}$, $[\nu_j]_{j\in\mathcal{A}}$ such that*

$$\nu = |u|^{p^*_\alpha} + \sum_{j\in\mathcal{A}} \nu_j \delta_{z_j}, \quad \mu \geq |\nabla_\alpha u|^p + \sum_{j\in\mathcal{A}} \mu_j \delta_{z_j}, \quad \mu_j \geq S\nu_j^{p/p^*_\alpha} \quad \text{for all } j \in \mathcal{A},$$

where δ_z indicates the Dirac delta concentrated at $z \in \mathbb{R}^N$.

We now recall the notion of Krasnoselskii's genus. Let X be a Banach space and $\mathcal{E}$ be the class of the closed and symmetric with respect to the origin subsets of $X \setminus \{0\}$.

The genus of $A \in \mathcal{E}$ is the smallest positive integer k such that there exists an odd and continuous map $\phi\colon A \to \mathbb{R}^k \setminus \{0\}$ and it is denoted by $\gamma(A)$. If there is no such map, we set $\gamma(A) = +\infty$. Finally, we define $\gamma(\emptyset) = 0$.

The main properties of the genus are collected in the following proposition, see e.g. [3, Lemma 1.2]. For $A \in \mathcal{E}$ and $\delta > 0$, let $N_\delta(A)$ denote a uniform δ-neighborhood of A, i.e.

$$N_\delta(A) = \{x \in X : \operatorname{dist}(x, A) \leq \delta\}.$$

Proposition 2.3. *Let A, $B \in \mathcal{E}$.*

(i) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(ii) *If there exists an odd homeomorphism between A and B, then $\gamma(A) = \gamma(B)$.*
(iii) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
(iv) *If $\gamma(B) < \infty$, $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$.*
(v) *$\gamma(\mathbb{S}^{k-1}) = k$, where $\mathbb{S}^{k-1}$ is the k dimensional sphere in $\mathbb{R}^k$.*
(vi) *If A is compact then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $N_\delta(A) \in \mathcal{E}$ and $\gamma(A) = \gamma(N_\delta(A))$.*

We will need the following deformation lemma, see [3, Lemma 1.3]. Given $I \in C^1(X,\mathbb{R})$, for any $c, d \in \mathbb{R}$, we denote

$$K_c = \{u \in X : I'(u) = 0 \text{ and } I(u) = c\}, \quad I^d = \{u \in X : I(u) \leq d\}.$$

Lemma 2.4. *Suppose $I \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition at l evel $c \in \mathbb{R}$ and let U be any neighborhood of K_c. Then there exist $\eta_t(x) = \eta(t,x) \in C([0,1] \times X, X)$ and constants $\varepsilon_0 > \varepsilon > 0$ such that*

(i) *$\eta_0(x) = x$ for all $x \in X$.*
(ii) *$\eta_t(x) = x$ for $x \notin I^{-1}([c-\varepsilon_0, c+\varepsilon_0])$ and all $t \in [0,1]$.*
(iii) *η_t is a homeomorphism of X into X for all $t \in [0,1]$.*
(iv) *$I(\eta_t(x)) \leq I(x)$ for all $x \in X$ and $t \in [0,1]$.*
(v) *$\eta_1(I^{c+\varepsilon} \setminus U) \subset I^{c-\varepsilon}$.*
(vi) *If $K_c = \emptyset$, then $\eta_1(I^{c+\varepsilon}) \subset I^{c-\varepsilon}$.*
(vii) *If I is even, then η_t is odd for any $t \in [0,1]$.*

Remark 2.5. By [3, Remark 1.4], if $c < 0$ we may choose $\varepsilon_0 < -c$.

3. Palais-Smale condition

To prove Theorem 1.2, we need the functional I_λ defined in (1.4) to satisfy the Palais-Smale condition under a certain level.

Lemma 3.1. *I_λ satisfies the Palais-Smale condition $(PS)_c$ for all $c < S^{N_\alpha/p}/N_\alpha - h\lambda^{\frac{p_\alpha^*}{p_\alpha^*-q}}$, where*

$$h = N_\alpha^{\frac{q}{p_\alpha^*-q}} |\Omega| \Big(\frac{1}{q} - \frac{1}{p}\Big)^{\frac{p_\alpha^*}{p_\alpha^*-q}} \Big[\Big(\frac{q}{p_\alpha^*}\Big)^{\frac{q}{p_\alpha^*-q}} - \Big(\frac{q}{p_\alpha^*}\Big)^{\frac{p_\alpha^*}{p_\alpha^*-q}}\Big] \geq 0. \tag{3.1}$$

Proof. Pick any $c < S^{N_\alpha/p}/N_\alpha - h\lambda^{\frac{p_\alpha^*}{p_\alpha^*-q}}$. Consider a sequence $(u_n) \subset \mathring{W}_\alpha^{1,p}(\Omega)$ such that

$$I_\lambda(u_n) = \frac{1}{p}\|u_n\|_{\alpha,p}^p - \frac{\lambda}{q}\|u_n\|_q^q - \frac{1}{p_\alpha^*}\|u_n\|_{p_\alpha^*}^{p_\alpha^*} = c + o(1), \tag{3.2}$$

$$I_\lambda'(u_n) = o(1) \quad \text{in } W_\alpha^{-1,p'}(\Omega) := (\mathring{W}_\alpha^{1,p}(\Omega))^*. \tag{3.3}$$

Here and in the following, $o(1)$ stands for a term that vanishes as $n \to \infty$. First of all, we claim that

$$\text{the sequence } (u_n) \text{ is bounded in } \mathring{W}_\alpha^{1,p}(\Omega). \tag{3.4}$$

From (3.3) it follows that

$$\|u_n\|_{\alpha,p}^p - \lambda\|u_n\|_q^q - \|u_n\|_{p_\alpha^*}^{p_\alpha^*} = o(1)\|u_n\|_{\alpha,p}$$

and inserting this identity into (3.2) gives

$$\frac{1}{p}\|u_n\|_{\alpha,p}^p - \frac{\lambda}{q}\|u_n\|_q^q - \frac{1}{p_\alpha^*}\|u_n\|_{\alpha,p}^p + \frac{\lambda}{p_\alpha^*}\|u_n\|_q^q = c + o(1) + o(1)\|u_n\|_{\alpha,p},$$

that is

$$\frac{1}{N_\alpha}\|u_n\|_{\alpha,p}^p = o(1)\|u_n\|_{\alpha,p} + \lambda\Big(\frac{1}{q} - \frac{1}{p_\alpha^*}\Big)\|u_n\|_q^q + c + o(1). \tag{3.5}$$

Observe that $\frac{1}{q} - \frac{1}{p_\alpha^*} > 0$ since $q < p$. Moreover, by the continuous embedding $\mathring{W}_\alpha^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ of Proposition 2.1 and (3.5) we obtain

$$\|u_n\|_{\alpha,p}^p \leq c_1 + c_2\|u_n\|_{\alpha,p} + c_3\|u_n\|_{\alpha,p}^q,$$

where c_1, c_2 and c_3 are positive constants not depending on n. Therefore, since $q < p$, (3.4) is proved.

So, there exist a sequence, still denoted by (u_n) and $u \in \mathring{W}_\alpha^{1,p}(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } \mathring{W}_\alpha^{1,p}(\Omega), \\ u_n \to u \quad \text{in } L^r(\Omega), \text{ for any } r \in [1, p_\alpha^*), \\ u_n \to u \quad \text{a.e. in } \Omega. \end{gathered} \tag{3.6}$$

Now, by Theorem 2.2 there exist two non-negative bounded measures μ, ν in Ω, an at most countable family of points $[z_j]_{j\in\mathcal{A}}$ and positive numbers $[\mu_j]_{j\in\mathcal{A}}$, $[\nu_j]_{j\in\mathcal{A}}$ such that

$$|\nabla_\alpha u_n|^p \overset{*}{\rightharpoonup} \mu, \quad |u_n|^{p_\alpha^*} \overset{*}{\rightharpoonup} \nu, \tag{3.7}$$

$$\nu = |u|^{p_\alpha^*} + \sum_{j\in\mathcal{A}} \nu_j\delta_{z_j}, \quad \mu \geq |\nabla_\alpha u|^p + S\sum_{j\in\mathcal{A}} \mu_j\delta_{z_j}. \tag{3.8}$$

Pick $z_j \in \Omega$ for some fixed $j \in \mathcal{A}$. Let us consider a cut-off function, that is $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi \equiv 1 \text{ on } B_\varepsilon(z_j), \quad \varphi \equiv 0 \text{ on } \mathbb{R}^N \setminus B_{2\varepsilon}(z_j), \quad |\nabla_\alpha\varphi| \leq \frac{d}{\varepsilon}, \tag{3.9}$$

for a positive constant d. The existence of a cut-off function for general Lipschitz vector fields, as observed in [16, p. 1850], was proved in [10, 14]. Since the sequence $(u_n\varphi)$ is bounded in $\mathring{W}_\alpha^{1,p}(\Omega)$, by (3.3) we obtain that $\langle I_\lambda'(u_n), \varphi u_n\rangle = 0$. So, passing to the limit as $n \to \infty$ by (3.6) and (3.7)

$$\int_\Omega \varphi\,\mathrm{d}\mu + \lim_{n\to\infty}\int_\Omega u_n|\nabla_\alpha u_n|^{p-2}\nabla_\alpha u_n\nabla_\alpha\varphi\,\mathrm{d}z - \lambda\int_\Omega |u|^q\varphi\,\mathrm{d}z - \int_\Omega \varphi\,\mathrm{d}\nu = 0. \tag{3.10}$$

Now, by (3.6)–(3.9),

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} \Big| \int_\Omega u_n |\nabla_\alpha u_n|^{p-2} \nabla_\alpha u_n \nabla_\alpha \varphi \,\mathrm{d}z \Big| \le \lim_{n\to\infty} \int_\Omega |u_n|\, |\nabla_\alpha u_n|^{p-1} |\nabla_\alpha \varphi| \,\mathrm{d}z \\
&\le \frac{d}{\varepsilon} \lim_{n\to\infty} \int_{B_{2\varepsilon}(z_j)} |u_n| |\nabla_\alpha u_n|^{p-1} \,\mathrm{d}z \\
&\le \lim_{n\to\infty} \frac{d}{\varepsilon} \Big(\int_{B_{2\varepsilon}(z_j)} |\nabla_\alpha u_n|^p \,\mathrm{d}z \Big)^{\frac{p-1}{p}} \Big(\int_{B_{2\varepsilon}(z_j)} |u_n|^p \,\mathrm{d}z \Big)^{1/p} \\
&= \frac{d}{\varepsilon} \mu(B_{2\varepsilon}(z_j))^{\frac{p-1}{p}} \Big(\int_{B_{2\varepsilon}(z_j)} |u|^p \,\mathrm{d}z \Big)^{1/p} \\
&\le \frac{d}{\varepsilon} \mu(B_{2\varepsilon}(z_j))^{\frac{p-1}{p}} \Big(\int_{B_{2\varepsilon}(z_j)} |u|^{p_\alpha^*} \,\mathrm{d}z \Big)^{1/p_\alpha^*} |B_{2\varepsilon}(z_j)|^{1/N_\alpha} \\
&\le \tilde{c} \mu(B_{2\varepsilon}(z_j))^{\frac{p-1}{p}} \Big(\int_{B_{2\varepsilon}(z_j)} |u|^{p_\alpha^*} \,\mathrm{d}z \Big)^{1/p_\alpha^*} \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}
$$

where $\tilde{c}$ is a constant not depending on ε, since $|B_{2\varepsilon}(z_j)| \sim \varepsilon^{N_\alpha}$.

Then by (3.10),

$$
0 = \lim_{\varepsilon\to 0} \Big(\int_\Omega \varphi \,\mathrm{d}\mu - \lambda \int_\Omega |u|^q \varphi \,\mathrm{d}z - \int_\Omega \varphi \,\mathrm{d}\mu \Big) = \mu_j - \nu_j. \tag{3.11}
$$

By Theorem 2.2, $\mu_j \ge S \nu_j^{p/p_\alpha^*}$, and by (3.11) $\nu_j \ge S \nu_j^{p/p_\alpha^*}$. That is either $\nu_j = 0$, or

$$
\nu_j \ge S^{\frac{1}{1-\frac{p}{p_\alpha^*}}} = S^{N_\alpha/p}. \tag{3.12}
$$

Note that, in particular, we can conclude that the family $\mathcal{A}$ is bounded, since ν is a finite measure. The aim now is to prove that (3.12) is not admissible.

Let us assume that there exists $j_0 \in \mathcal{A}$ such that $\nu_{j_0} \neq 0$, i.e. $\nu_{j_0} \ge S^{N_\alpha/p}$. Now

$$
\begin{aligned}
c &= \lim_{n\to\infty} I_\lambda(u_n) \\
&= \lim_{n\to\infty} \Big(I_\lambda(u_n) - \frac{1}{p} \langle I_\lambda'(u_n), u_n \rangle \Big) \\
&= \lambda \Big(\frac{1}{p} - \frac{1}{q} \Big) \int_\Omega |u|^q \,\mathrm{d}z + \frac{1}{N_\alpha} \int_\Omega |u|^{p_\alpha^*} \,\mathrm{d}z + \frac{1}{N_\alpha} \sum_{j\in\mathcal{A}} \nu_j \\
&\ge \lambda \Big(\frac{1}{p} - \frac{1}{q} \Big) \int_\Omega |u|^q \,\mathrm{d}z + \frac{1}{N_\alpha} \int_\Omega |u|^{p_\alpha^*} \,\mathrm{d}z + \frac{1}{N_\alpha} S^{N_\alpha/p}
\end{aligned}
$$

and by Hölder's inequality

$$
c \ge \frac{1}{N_\alpha} S^{N_\alpha/p} + \frac{1}{N_\alpha} \int_\Omega |u|^{p_\alpha^*} \,\mathrm{d}z - \lambda \Big(\frac{1}{q} - \frac{1}{p} \Big) |\Omega|^{(p_\alpha^*-q)/p_\alpha^*} \Big(\int_\Omega |u|^{p_\alpha^*} \,\mathrm{d}z \Big)^{q/p_\alpha^*}. \tag{3.13}
$$

Now, Let us consider the function $f\colon \mathbb{R} \to \mathbb{R}$, defined as $f(x) := A x^{p_\alpha^*} - \lambda B x^q$, with $A = 1/N_\alpha$ and $B = (\frac{1}{q} - \frac{1}{p}) |\Omega|^{(p_\alpha^*-q)/p_\alpha^*}$. This function attains its absolute minimum (for $x > 0$) at the point $x_0 = (\lambda B q / p_\alpha^* A)^{\frac{1}{p_\alpha^*-q}}$. That is, by a straightforward calculation,

$$
f(x) \ge f(x_0) = -h \lambda^{\frac{p_\alpha^*}{p_\alpha^*-q}},
$$

where h is given in (3.1). By (3.13), $c \ge S^{N_\alpha/p}/N_\alpha - h\lambda^{\frac{p_\alpha^*}{p_\alpha^*-q}}$, which is in contradiction with the hypothesis on c.

This implies that $\nu_j = 0$ for all $j \in \mathcal{A}$, and so $\lim_{n\to\infty} \int_\Omega |u_n|^{p_\alpha^*} \,\mathrm{d}z = \int_\Omega |u|^{p_\alpha^*} \,\mathrm{d}z$, and thus

$$
u_n \to u \quad \text{strongly in } L^{p_\alpha^*}(\Omega). \tag{3.14}
$$

Now, to conclude with the strong convergence of u_n to u in $\mathring{W}^{1,p}_\alpha(\Omega)$ we proceed as in [22, Lemma 6.1]. Let $A_p\colon \mathring{W}^{1,p}_\alpha(\Omega) \to W^{-1,p'}_\alpha(\Omega)$ be the continuous operator defined by

$$\langle A_p(u), v\rangle \coloneqq \int_\Omega |\nabla_\alpha u|^{p-2}\nabla_\alpha u \nabla_\alpha v \,\mathrm{d}z.$$

Straightforward computations show that the sequence $(A_p(u_n))$ is a Cauchy sequence in $W^{-1,p'}_\alpha(\Omega)$. In fact $A_p(u_n) = I'_\lambda(u_n) + \lambda|u_n|^{q-2}u_n + |u_n|^{p^*_\alpha-2}u_n$ and the claim follows from (3.14) and the Palais-Smale condition. Now, a direct application of [25, Eq (2.2)] implies

$$\langle A_p(u_n) - A_p(u_m), u_n - u_m\rangle \geq \begin{cases} c_1\|u_n - u_m\|^p_{\alpha,p} & \text{if } p > 2,\\ c_2 M^{p-2}\|u_n - u_m\|^2_{\alpha,p} & \text{if } 1 < p \leq 2,\end{cases}$$

where $c_1 = c_1(N, \alpha, p)$, $c_2 = c_2(N, \alpha, p, \Omega)$ and $M = \max[\|u_n\|_{\alpha,p}, \|u_m\|_{\alpha,p}]$. So

$$\|u_n - u_m\|_{\alpha,p} \leq \begin{cases} c_1^{\frac{1}{1-p}}\|A_p(u_n) - A_p(u_m)\|^{\frac{1}{p-1}}_{W^{-1,p'}_\alpha(\Omega)} & \text{if } p > 2,\\ c_2^{-1}M^{2-p}\|A_p(u_n) - A_p(u_m)\|_{W^{-1,p'}_\alpha(\Omega)} & \text{if } 1 < p \leq 2,\end{cases}$$

so we deduce that $u_n \to u$ strongly in $\mathring{W}^{1,p}_\alpha(\Omega)$. □

4. Existence of infinitely many solutions

Observe that, because of the presence of the critical term, the functional I_λ is neither coercive nor bounded from below. Therefore, we use a truncation argument to tackle the issue posed by the critical term. By the Sobolev embedding in Proposition 2.1, we obtain

$$I_\lambda(u) \geq \frac{1}{p}\int_\Omega |\nabla_\alpha u|^p \,\mathrm{d}z - \frac{1}{p^*_\alpha S^{p^*_\alpha/p}}\Big(\int_\Omega |\nabla_\alpha u|^p\Big)^{p^*_\alpha/p} - \frac{\lambda}{q}C_q\Big(\int_\Omega |\nabla_\alpha u|^p\Big)^{q/p}, \tag{4.1}$$

for all $u \in \mathring{W}^{1,p}_\alpha(\Omega)$, where S is defined in (2.2) and C_q is the constant arising from the continuous embedding $\mathring{W}^{1,p}_\alpha(\Omega) \hookrightarrow L^q(\Omega)$, namely $C_q > 0$ is such that

$$\|u\|_q \leq C_q^{1/q}\|u\|_{\alpha,p} \quad \text{for all } u \in \mathring{W}^{1,p}_\alpha(\Omega). \tag{4.2}$$

Defining the function $g_\lambda\colon [0, +\infty) \to \mathbb{R}$ as

$$g_\lambda(x) \coloneqq \frac{1}{p}x^p - \frac{1}{p^*_\alpha S^{p^*_\alpha/p}}x^{p^*_\alpha} - \frac{\lambda}{q}C_q x^q$$

from (4.1) we obtain

$$I_\lambda(u) \geq g_\lambda(\|u\|_{\alpha,p}).$$

Let us fix a sufficiently small $\hat{x} > 0$ such that

$$\frac{1}{p}\hat{x}^p - \frac{1}{p^*_\alpha S^{p^*_\alpha/p}}\hat{x}^{p^*_\alpha} > 0,$$

and choose λ_0 such that

$$0 < \lambda_0 < \frac{q}{C_q\hat{x}^q}\Big(\frac{1}{p}\hat{x}^p - \frac{1}{p^*_\alpha S^{p^*_\alpha/p}}\hat{x}^{p^*_\alpha}\Big). \tag{4.3}$$

So, it holds that $g_{\lambda_0}(\hat{x}) > 0$. Let us now define

$$x_1 \coloneqq \max\{x \in (0, \hat{x}) \,:\, g_{\lambda_0}(x) \leq 0\}. \tag{4.4}$$

Since $q < p$, we have that $g_{\lambda_0}(x) < 0$ as x approaches 0. Therefore we can conclude that $g_{\lambda_0}(x_1) = 0$, as in Figure 1.

Let us now consider a smooth function $\Phi\colon [0, +\infty) \to [0, 1]$ such that

$$\Phi(x) = \begin{cases} 1 & \text{if } x \leq x_1,\\ 0 & \text{if } x \geq \hat{x}\end{cases}$$

and, for any $u \in \mathring{W}^{1,p}_\alpha(\Omega)$, we consider the truncated functional

$$J_\lambda(u) \coloneqq \frac{1}{p}\int_\Omega |\nabla_\alpha u|^p \,\mathrm{d}z - \frac{1}{p^*_\alpha S^{p^*_\alpha/p}}\Big(\int_\Omega |u|^{p^*_\alpha}\,\mathrm{d}z\Big)\Phi(\|u\|_{\alpha,p}) - \frac{\lambda}{q}\int_\Omega |u|^q \,\mathrm{d}z.$$

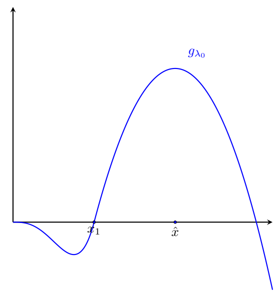

FIGURE 1.

It is readily seen that J_λ is bounded from below and coercive. Other relevant properties of J_λ are listed below.

Lemma 4.1. *There exists $\bar{\lambda}$ such that for all $\lambda \in (0, \bar{\lambda})$ the following hold.*

(i) *If $J_\lambda(u) \leq 0$ then $\|u\|_{\alpha,p} < x_1$ and $I_\lambda(u) = J_\lambda(u)$ in a neighborhood of u.*
(ii) *The functional J_λ satisfies the Palais-Smale condition for all $c < 0$.*

Proof. (i) Let $J_\lambda(u) \leq 0$. We distinguish two cases. If $\|u\|_{\alpha,p} \geq \hat{x}$,

$$J_\lambda(u) \geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{\lambda C_q}{q}\|u\|_{\alpha,p}^q = h_\lambda(\|u\|_{\alpha,p}),$$

where C_q is defined in (4.2) and $h_\lambda\colon [0,+\infty) \to \mathbb{R}$ is defined as

$$h_\lambda(x) := \frac{1}{p}x^p - \frac{\lambda C_q}{q}x^q.$$

Observe that the nontrivial zero of h_λ is given by $\tilde{x} = (\frac{\lambda p C_q}{q})^{\frac{1}{p-q}}$ and, since $q < p$, h_λ is positive and increasing for $x > \tilde{x}$. If we choose $\lambda < \tilde{\lambda}$, where

$$\tilde{\lambda} = \frac{q}{pC_q}\hat{x}^{p-q}$$

then $\hat{x} > \tilde{x}$ and so

$$0 \geq J_\lambda(u) \geq h_\lambda(\|u\|_{\alpha,p}) > h_\lambda(\tilde{x}) = 0,$$

which is a contradiction.

If $\|u\|_{\alpha,p} < \hat{x}$, since $\Phi \leq 1$, $J_\lambda(u) \geq g_\lambda(\|u\|_{\alpha,p})$ and so, choosing $\lambda < \lambda_0$, where λ_0 is defined in (4.3),

$$0 \geq J_\lambda(u) \geq g_\lambda(\|u\|_{\alpha,p}) > g_{\lambda_0}(\|u\|_{\alpha,p}),$$

since the function $\lambda \mapsto g_\lambda(\cdot)$ is decreasing. Now, by the definition of x_1, it follows that $\|u\|_{\alpha,p} < x_1$ (see also again Figure 1), that is $\Phi(\|u\|_{\alpha,p}) = 1$ and I_λ and J_λ coincide. Finally, by the continuity of the functional J_λ, there exists a neighborhood U of u such that $J_\lambda(u) < 0$ for all $u \in U$. Therefore, for any $u \in U$, it follows that $I_\lambda(u) = J_\lambda(u)$.

(ii) Let $c < 0$ and $(u_n) \subset \mathring{W}^{1,p}_\alpha(\Omega)$ a sequence such that $J_\lambda(u_n) \to c$ and $J_\lambda'(u) \to 0$ as $n \to \infty$. By the previous statement we obtain that $I_\lambda(u_n) = J_\lambda(u_n)$ for sufficiently large $n \in \mathbb{N}$. Since J_λ is coercive, the sequence (u_n) is bounded in $\mathring{W}^{1,p}_\alpha(\Omega)$. Choose λ_* small such that

$$\frac{S^{N_\alpha/p}}{N_\alpha} - h\lambda_*^{\frac{p_\alpha^*}{p_\alpha^*-q}} \geq 0,$$

where h is defined in (3.1). For $\lambda < \lambda_*$, then $c < \frac{S^{N_\alpha/p}}{N_\alpha} - h\lambda_*^{\frac{p_\alpha^*}{p_\alpha^*-q}}$ and Lemma 3.1 gives that J_λ satisfies the Palais-Smale condition at level c.

Combining the above results, the choice of $\bar{\lambda} = \min\{\tilde{\lambda}, \lambda_0, \lambda_*\}$ gives that (i) and (ii) are satisfied for all $\lambda \in (0, \bar{\lambda})$. $\square$

Before proving Theorem 1.2, we need some properties of the sublevels of the functional J_λ. Then we can construct the minimax levels associated to the functional. We refer to Section 2.2 for the details and the notation regarding Krasnoselskii's genus.

For simplicity, for each $\varepsilon > 0$ we set

$$J_\lambda^{-\varepsilon} := \{u \in \mathring{W}^{1,p}_\alpha(\Omega) : J_\lambda(u) \le -\varepsilon\}.$$

Lemma 4.2. *For each $n \in \mathbb{N}$ there exists $\varepsilon = \varepsilon(n)$ such that*

$$\gamma(J_\lambda^{-\varepsilon}) \ge n.$$

Proof. Fix $n \in \mathbb{N}$, and let E_n be a n-dimensional subspace of $\mathring{W}^{1,p}_\alpha(\Omega)$. So, $\|\cdot\|_{\alpha,p}$ and $\|\cdot\|_q$ are equivalent norms on E_n, that is there exists a constant $C_n > 0$ such that

$$C_n^{1/q}\|u\|_{\alpha,p} \le \|u\|_q, \quad \forall u \in E_n. \tag{4.5}$$

Recall that x_1 was defined in (4.4). For any $u \in E_n$ with $\|u\|_{\alpha,p} \le x_1$ we obtain and from (4.5) that

$$I_\lambda(u) = J_\lambda(u) \le \frac{1}{p}\|u\|^p_{\alpha,p} - \frac{\lambda C_n}{q}\|u\|^q_{\alpha,p}. \tag{4.6}$$

Let $S_\eta := \{u \in E_n : \|u\|_{\alpha,p} = \eta\}$, where η is such that

$$0 < \eta < \min\Big\{x_1, \Big(\frac{\lambda C_n p}{q}\Big)^{\frac{1}{p-q}}\Big\}. \tag{4.7}$$

So, by (4.6)-(4.7), for each $u \in S_\eta$,

$$J_\lambda(u) \le \eta^q\Big(\frac{1}{p}\eta^{p-q} - \frac{\lambda C_n}{q}\Big) < 0$$

and by the compactness of S_η we can choose $\varepsilon = \varepsilon(n) > 0$ such that $J_\lambda(u) < -\varepsilon$ for all $u \in S_\eta$. So we have $S_\eta \subset J_\lambda^{-\varepsilon}$ and by (i), (ii) and (v) of Proposition 2.3 we obtain

$$\gamma(J_\lambda^{-\varepsilon}) \ge \gamma(S_\eta) = n,$$

concluding the proof. □

For any $k \in \mathbb{N}$ we now define the min-max values c_k as

$$c_k := \inf_{A \in \mathcal{E}_k} \sup_{u \in A} J_\lambda(u),$$

where $\mathcal{E}_k = \{A \in \mathcal{E} : \gamma(A) \ge k\}$. Note that trivially follows that $c_k \le c_{k+1}$ for all $k \in \mathbb{N}$.

Lemma 4.3. *$c_k \in (-\infty, 0)$ for every $\lambda > 0$ and $k \in \mathbb{N}$.*

Proof. Fix $\lambda > 0$ and $k \in \mathbb{N}$. Since J_λ is bounded from below we have that $c_k > -\infty$.

Note that $J_\lambda^{-\varepsilon}$ is closed and symmetric, moreover $0 \notin J_\lambda^{-\varepsilon}$ since $J_\lambda(0) = 0$. From Lemma 4.2, there exists $\varepsilon > 0$ such that $\gamma(J_\lambda^{-\varepsilon}) \ge k$, so $J_\lambda^{-\varepsilon} \in \mathcal{E}_k$ and $\sup_{u \in J_\lambda^{-\varepsilon}} J_\lambda(u) \le -\varepsilon$. Then

$$c_k = \inf_{A \in \mathcal{E}_k} \sup_{u \in A} J_\lambda(u) \le \sup_{u \in J_\lambda^{-\varepsilon}} J_\lambda(u) \le -\varepsilon < 0,$$

concluding the proof. □

Lemma 4.4. *Let $k \in \mathbb{N}$ and $\lambda \in (0, \bar\lambda)$, where $\bar\lambda$ is as in Lemma 4.1. If $c = c_k = c_{k+1} = \dots = c_{k+r}$ for some $r \in \mathbb{N}$, then $\gamma(K_c) \ge r + 1$.*

Proof. Let $\lambda \in (0, \bar\lambda)$ and $k \in \mathbb{N}$. By Lemma 4.3 it follows that $c = c_k = c_{k+1} = \dots c_{k+r} < 0$. We can conclude, thanks to Lemma 4.1, that J_λ satisfies the Palais-Smale condition at level c, and so the set K_c is compact.

Now, suppose that $\gamma(K_c) \le r$. Therefore, from Proposition 2.3 (vi), there exists $\delta > 0$ and a neighborhood $N_\delta(K_c)$ of K_c such that $\gamma(N_\delta(K_c)) = \gamma(K_c) \le r$.

From (v) and (vii) of Lemma 2.4 and from Remark 2.5, there exist $\varepsilon \in (0, -c)$ and an odd homeomorphism $\eta \colon \mathring{W}^{1,p}_\alpha(\Omega) \to \mathring{W}^{1,p}_\alpha(\Omega)$ such that

$$\eta(J_\lambda^{c+\varepsilon} \setminus N_\delta(K_c)) \subset J_\lambda^{c-\varepsilon}. \tag{4.8}$$

Now, by definition

$$c = c_{k+r} = \inf_{A \in \mathcal{E}_{k+r}} \sup_{u \in A} J_\lambda(u)$$

and so there exists $A \in \mathcal{E}_{k+r}$ such that

$$\sup_{u \in A} J_\lambda(u) < c + \varepsilon,$$

implying that $A \subset J_\lambda^{c+\varepsilon}$. Therefore, we obtain by (4.8) that

$$\eta(A \setminus N_\delta(K_c)) \leq \eta(J_\lambda^{c+\varepsilon} \setminus N_\delta(K_c)) \subset J_\lambda^{c-\varepsilon}. \tag{4.9}$$

On the other hand, using (i) and (iii) of Proposition 2.3, we obtain

$$\gamma(\eta(\overline{A \setminus N_\delta(K_c)})) \geq \gamma(\overline{A \setminus N_\delta(K_c)}) \geq \gamma(A) - \gamma(N_\delta(K_c)) \geq k,$$

since $A \in \mathcal{E}_{k+r}$ and $\gamma(N_\delta(K_n)) \leq r.$

Therefore, we have $\eta(\overline{A \setminus N_\delta(K_c)}) \in \mathcal{E}_k$. From the definition of c_k, we obtain

$$\sup_{u \in \eta(\overline{A \setminus N_\delta(K_c)})} J_\lambda(u) \geq c_k = c,$$

which is a contradiction with (4.9). Therefore, $\gamma(K_c) \geq r + 1$. □

We can now conclude the proof of our main result.

Proof of Theorem 1.2. Let $\lambda \in (0, \bar{\lambda})$, where $\bar{\lambda}$ is as in Lemma 4.1. From Lemma 4.3, it follows that $c_k < 0$ and Lemma 4.1 (ii) guarantees that J_λ satisfies the Palais-Smale condition at level c_k. Therefore, we can conclude that c_k is a critical value of J_λ for every k, see e.g. [2, Proposition 10.8]. Finally, by Lemma 4.1 (i), the negative values c_k are critical points of I_λ. Now we distinguish two cases.

(i) If $-\infty < c_1 < c_2 < \dots < c_k < c_{k+1} < \dots$ for every $k \in \mathbb{N}$, then I_λ possesses infinitely many critical points.

(ii) If there exist $k, r \in \mathbb{N}$ such that $c_k = c_{k+1} = \dots = c_{k+r} = c$, by Lemma 4.4 we obtain $\gamma(K_c) \geq r+1 > 1$. Then K_c has infinitely many distinct points (see e.g. [24, Remark 7.3]) and by Lemma 4.1 (ii), we obtain infinitely many negative critical values for I_λ.

Hence, in both cases, problem (1.1) has infinitely many solutions for all $\lambda \in (0, \bar{\lambda})$. □

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Paolo Malanchini
Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano, Bicocca, via Roberto Cozzi 55, 20125, Milano, Italy
Email address: p.malanchini@campus.unimib.it

Giovanni Molica Bisci
Department of Human Sciences and Promotion of Quality of Life, San Raffaele University, via di Val Cannuta 247, I-00166 Roma, Italy
Email address: giovanni.molicabisci@uniroma5.it

Simone Secchi
Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano, Bicocca, via Roberto Cozzi 55, 20125, Milano, Italy
Email address: simone.secchi@unimib.it