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SYSTEMS OF KIRCHHOFF QUASILINEAR SCHRÖDINGER EQUATIONS WITH HARDY POTENTIAL AND CRITICAL CHOQUARD-TYPE NONLINEARITY

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Abstract. In this article, we study the existence of solutions for a system of Kirchhoff quasilinear Schrödinger equations involving Hardy potential and critical Choquard-type nonlinearity with parameters. We first establish the concentration compactness principle for our problem. Then, we prove the existence of at least n pairs of solutions for different exponent of subcritical perturbation by employing the symmetric mountain pass theorem, Krasnoselskii's genus theory and $\mathbb{Z}_2$-symmetric version of mountain pass theorem.

1. Introduction and main results

In this article, we study a system of Kirchhoff quasilinear Schrödinger equation involving Hardy potential and critical Choquard nonlinearity with a parameter,

$$
\begin{aligned}
-\mathcal{M}(u)-a\eta_1\frac{|u|^{2(2\alpha-1)}u}{|x|^2}&=\frac{\gamma r}{s+r}g(x)|u|^{r-2}u|v|^s+\Big(\int_{\mathbb{R}^N}\frac{|u|^{2\alpha 2^*_\mu}}{|x-y|^\mu}dy\Big)|u|^{2\alpha 2^*_\mu-2}u,\quad \text{in } \mathbb{R}^N\setminus\{0\},\\
-\mathcal{M}(v)-a\eta_2\frac{|v|^{2(2\alpha-1)}v}{|x|^2}&=\frac{\gamma s}{s+r}g(x)|u|^{r}|v|^{s-2}v+\Big(\int_{\mathbb{R}^N}\frac{|v|^{2\alpha 2^*_\mu}}{|x-y|^\mu}dy\Big)|v|^{2\alpha 2^*_\mu-2}v,\quad \text{in } \mathbb{R}^N\setminus\{0\},
\end{aligned}
\tag{1.1}
$$

where

$$
N\geq 3,\quad \mathcal{M}(u)=\Big(a+b\Big(\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)^{\theta-1}\Big)\Delta u+a\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u
$$

with $a>0$, $b\geq 0$, $0<\mu<N$, $\alpha>\frac{\theta}{2}$, $1\leq\theta<2\alpha 2^*_\mu$, $0\leq\eta_1,\eta_2<\frac{(N-2)^2}{4}$ and $2\alpha<r+s<2\alpha 2^*$. Here $2^*=\frac{2N}{N-2}$ is the Sobolev critical exponent and $2^*_\mu:=\frac{(2N-\mu)}{N-2}$ is the critical exponent with respect to the Hardy-Littlewood-Sobolev inequality, $\gamma>0$ is a real parameter and $g(\geq 0)\in L^{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}(\mathbb{R}^N)$.

If $\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u=\Delta(|v|^{2\alpha})|v|^{2\alpha-2}v=0$ and $\eta_1=\eta_2=0$, then system (1.1) reduces to a standard nonlocal problem, which is closely associated with the stationary form of a model introduced by Kirchhoff [18]. In particular, this model is represented by the equation

$$
\kappa\frac{\partial^2u}{\partial t^2}-\Big(\frac{\kappa_0}{k}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big)\frac{\partial^2u}{\partial x^2}=0,\tag{1.2}
$$

where κ,κ_0,k,E,L are constants. This model describes the classical D'Alembert wave equation by considering the effect of the changes in the length of the string during the vibrations.

Solutions of (1.1) for $\alpha=1$, are related to standing-wave solutions for the quasilinear Schrödinger equation

$$
-i\partial_tz=-\Delta z+V(x)\Delta z-s(|z|^2)z-\delta\Delta g(|z|^2)g'(|z|^2)z\quad x\in\mathbb{R}^N,\tag{1.3}
$$

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where δ is a real constant, $V(x)$ represents a given potential, s and g are real-valued functions. The study of (1.3) is motivated by its appearance in various physical phenomena. For example, when $g(s) = s$, equation (1.3) describes the time progression of the condensate wave function in a superfluid film, often referred to as the superfluid film equation in fluid mechanics, as described by Kurihara [19]. In the case where $g(s) = \sqrt{(1+s)}$, equation (1.3) was used as a model of the self-channeling of a high-intensity ultra short laser in matter as discussed in [4, 32] and references therein. One of the key difficulties in studying quasilinear Schrödinger problem, with the non-homogeneous term $\Delta(|u|^{2\alpha})|u|^{2\alpha-2}u$, is that there is no suitable space where the energy functional is well defined. Various approaches have been employed to overcome this challenge, including change of variables [26], minimizations [30] and the Nehari or Pohozaev manifold [33]. As far as we know, there are only a few papers concerning this operator for the parameter α; see [15, 21, 38].

On the other hand, problem (1.1) contains a nonlocal nonlinearity known as Choquard type nonlinearity. The stationary nonlinear Choquard or nonlinear Schrödinger-Newton equation

$$-\Delta u + V(x)u = (\mathcal{I}_\mu * |u|^2)u + \lambda f(x,u) \text{ in } \mathbb{R}^N, \tag{1.4}$$

was first introduced by Pekar [28] in 1954, to describe the quantum mechanics of a polaron at rest. Choquard derived equation (1.4) to study the electron trapped in its hole. Later, the same equation was used by Lieb and Penrose as an approximation of the Hartree-Fock theory and as a model of the self gravitational collapse of a quantum mechanical wave function, see [3, 29]. Moroz and Van Schaftingen [27] showed the existence, qualitative properties and decay estimates of ground state solutions. For a deeper understanding of the Choquard equation, we refer the interested readers to [22, 35] and references therein.

Now, we state the Hardy inequality, also called the localized inverse square potential. The inequality was initially introduced by the British Mathematician Hardy in 1920; see [12].

Proposition 1.1. *For $u \in D^{1,2}(\mathbb{R}^N)$ and $N \geq 3$ the subsequent inequality is established,*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{1.5}$$

The constant $\frac{4}{(N-2)^2}$ is optimal, but the inequality is not attained in the Sobolev space $D^{1,2}(\mathbb{R}^N)$. This non-attainability reflects the criticality of the potential $|x|^{-2}$, which lies on the threshold of the Hardy–Sobolev embedding.

From a physical point of view, equation (1.1), which involve Hardy-type potential, arise naturally in a various context, including the linearization of combustion models [1], quantum cosmology [2], and molecular physics [20]. In recent years, many researchers studied the elliptic problems involving the Hardy potential, critical Sobolev exponent and obtained several important results (see [6, 7, 14]). In particular, the elliptic problem

$$\begin{gathered} -\Delta u - \mu \frac{u}{|x|^2} = a|u|^{2^*-2}u + \lambda u, \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.6}$$

studied by Jannelli [14] to demonstrate the existence of positive solution. Later, Cao and Peng proved sign-changing solutions to (1.6) in [7]. Later, T. Shang and R. Liang in [34], proved the existence of infinitely many nontrivial solutions for the following quasilinear Schrödinger equation involving Hardy potential,

$$-\Delta u - \mu \frac{u}{|x|^2} + V(x)u - \Delta(|u|^2)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.7}$$

where $V(x)$ is a positive potential and nonlinearity $f(x,u)$ is subcritical. For instance, we provide few papers involving Hardy potential see [13, 37, 39] and references therein.

Another driving force for elliptic system involving the Hardy potential has been seldom studied, only few papers are available (see [17, 36]). Kang [17], applied the global compactness framework of the variational method, along with the analysis of Palais-Smale sequences, to prove the existence of infinitely many solutions to the system. Wang [36] used the concentration compactness principle and Kajikiya's new version of the symmetric mountain pass lemma to obtain infinitely many solutions for the system. Later, Massar, Hamydy, and Tsouli [25], studied the following nonlocal Schrödinger system

$$
\begin{gathered}
-\Big(a+b\int_\Omega |\Delta u|^2dx\Big)\Delta u - a\Delta(|u|^2)u = F_u(x,u,v)+\eta\frac{m}{m+n}|u|^{2m-2}u|v|^{2n} \quad \text{in } \Omega,\\
-\Big(a+b\int_\Omega |\Delta v|^2dx\Big)\Delta v - a\Delta(|v|^2)v = F_u(x,u,v)+\eta\frac{n}{m+n}|u|^{2m}|v|^{2n-2}v \quad \text{in } \Omega,\\
u=v=0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{1.8}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $a,\ b > 0$, $m,\ n > 1$ with $m+n = 2^*$, $\nabla F = (F_u, F_v)$ is the gradient of a C^1 function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ with respect to (u, v). They demonstrate the existence and multiplicity of nontrivial solutions to the problem (1.8) by employing the concentration compactness principle together with genus theory.

Inspired by the aforementioned papers, there is almost no existing research on the existence and multiplicity of solutions for quasilinear Schrödinger Kirchhoff system involving the Hardy potential with critical Choquard nonlinearity. However, we face two main difficulties, in which one arises from the lack of a suitable working space in which the associated energy functional is not well defined due to this standard critical point theory cannot be applied directly. To overcome this issue, we use an appropriate change of variables, in the spirit of [8]. The other one is lack of compactness due to the presence of the critical Choquard nonlinearity and the Hardy potential. For this, we first establish the concentration compactness principle for the system to resolve this issue. Then, together with the new version of the symmetric mountain pass theorem due to Kajikiya [16] we obtain the existence and multiplicity of the solutions for the problem (1.1).

Throughout this paper we assume that the following:

$$
\begin{gathered}
a>0, \quad b\geq 0, \quad \alpha>\frac{\theta}{2}, \quad 0<\mu<\min\{4,N\},\\
0\leq \eta=\max\{\eta_1,\eta_2\}<\frac{(N-2)^2}{4}, \quad N\geq 3, \quad 1\leq\theta<2\alpha 2^*_\mu.
\end{gathered}
\tag{1.9}
$$

Now the main results can be described as follows.

Theorem 1.2. *Suppose $2\alpha < r+s < 4\alpha$and (1.9) hold. Let $\Omega := \{x \in \mathbb{R}^N \ :\ g(x) > 0\}$ be an open subset of $\mathbb{R}^N$ such that $0 < |\Omega| < \infty$. Then there exists $\gamma^* > 0$ such that for any $\gamma \in (0, \gamma^*)$, problem (1.1) has a sequence of nontrivial weak solutions in $\mathcal{D}$.*

Theorem 1.3. *Suppose $r + s = 4\alpha$ and (1.9) hold. Then there exists a positive constant $\hat{a}$ such that for each $a > \hat{a}$ and for each $\gamma \in \Big(0, \big(aS - \frac{4Sa\eta}{(N-2)^2}\big)\|g\|^{-1}_{\frac{2^*}{2^*-2}}\Big)$, problem (1.1) has at least n pairs of nontrivial weak solutions in $\mathcal{D}$.*

Theorem 1.4. *Suppose $4\alpha < r+s < 2\alpha 2^*$ and (1.9) hold. Then for all $\gamma > 0$, problem (1.1) has at least n pairs of nontrivial weak solutions in $\mathcal{D}$.*

This article is organized as follows: In Section 2, we introduce some of the basic definitions relevant to our important result. Section 3, is devoted to the proof of the second compactness principle at origin and infinity for Hardy term and Choquard type nonlinearity, it is helpful in demonstrating the Palais-Smale condition at some special levels. Finally, in Section 4, we prove the main results.

2. Preliminaries

In this section, we present several definitions. After that, we give some preliminary results and a variational framework. Firstly, we recall the definition of the Sobolev space

$$
D^{1,2}(\mathbb{R}^N) = \big\{u \in L^{2^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2\,dx < \infty\big\},
$$

with the norm

$$
\|u\| = \|u\|_{D^{1,2}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\Big)^{1/2}.
$$

Set $\mathcal{D} = D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$. Then $\mathcal{D}$ is a Hilbert space with respect to the inner product

$$\langle (u_1, v_1), (u_2, v_2)\rangle = \int_{\mathbb{R}^N} (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2)\, dx \quad \forall (u_1, v_1), (u_2, v_2) \in \mathcal{D}$$

and equipped with the norm

$$\|(u, v)\|_{\mathcal{D}} = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\, dx\Big)^{1/2}.$$

We denote the dual of $\mathcal{D}$ by $\mathcal{D}^*$ and the dual paring between $\mathcal{D}$ and its dual $\mathcal{D}^*$ is denoted by $\langle \cdot, \cdot\rangle$. Now, we recall the well known Hardy-Littlewood- Sobolev inequality see [22, Theorem 4.3].

Proposition 2.1 (Hardy-Littlewood-Sobolev inequality). *Assume t, $r > 1$, $\mu > 0$, and $\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$, where t' and r' denote the Hölder conjugate of t and r, respectively. Then there exists a constant $C(N, \mu, t, r)$ independent of f, g such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x - y|^\mu} \mathrm{d}x\mathrm{d}y \le C(N, \mu, t, r)\|f\|_{L^t(\mathbb{R}^N)} \|g\|_{L^r(\mathbb{R}^N)}. \tag{2.1}$$

If $t = r = \frac{2N}{2N-\mu}$, then

$$C(t, r, \mu, N) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \Big\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{\mu}{2})} \Big\}^{-1+\frac{\mu}{N}}.$$

The equality holds in (2.1), if and only if $f = (constant)g$ and

$$g(x) = A(\gamma^2 + |x - m|^2)^{\frac{2N-\mu}{2}}$$

for some $A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}$ and $m \in \mathbb{R}^N$.

Let $f = g = |u|^q$ then the following Hardy-Littlewood-Sobolev inequality (2.1), integral is $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu}\, dx\, dy$ is well-defined if $|u|^q \in L^r(\mathbb{R}^N)$ for some $r > 1$ satisfying $\frac{2}{r} + \frac{\mu}{N} = 2$. For $u \in D^{1,2}(\mathbb{R}^N)$, the Sobolev embedding theorem, yield $2 \le qr \le \frac{2N}{N-2}$. Thus

$$2_{*\mu} = \frac{2N - \mu}{N} \le q \le \frac{2N - \mu}{N - 2} = 2^*_\mu.$$

Now, $2_{*\mu} = \frac{2N-\mu}{N}$ is the lower critical exponent and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of Hardy-Littlewood-Sobolev inequality. Moreover, we define

$$\|(u, v)\|_\mu = \Big(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu} + |v(x)|^{2^*_\mu} |v(y)|^{2^*_\mu}}{|x - y|^\mu}\, dx\, dy\Big)^{\frac{1}{2.2^*_\mu}},$$

which turns out to be a norm on $L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$. By the Hardy-Littlewood-Sobolev inequality, for all $(u, v) \in \mathcal{D}$, one can see that

$$\|(u, v)\|^2_\mu \le (2C(N, \mu))^{1/2^*_\mu} S^{-1} \|(u, v)\|^2$$

then

$$S_\mu \ge \frac{S}{(2C(N, \mu))^{\frac{1}{2^*_\mu}}} > 0,$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ which is defined as

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \Big\{ \int_{\mathbb{R}^N} |\nabla u|^2\, dx : \|u\|_{2^*} = 1\Big\}, \tag{2.2}$$

$$S_\mu := \inf_{(u,v)\in\mathcal{D}\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\, dx}{\Big(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu} + |v(x)|^{2^*_\mu} |v(y)|^{2^*_\mu}}{|x-y|^\mu}\, dx\, dy\Big)^{1/2^*_\mu}}, \tag{2.3}$$

$$S_{\eta,\mu} := \inf_{(u,v)\in\mathcal{D}\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - \eta \frac{|u|^2}{|x|^2})\, dx + \int_{\mathbb{R}^N} \big(|\nabla v|^2 - \eta \frac{|v|^2}{|x|^2}\big)\, dx}{\Big(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu} + |v(x)|^{2^*_\mu} |v(y)|^{2^*_\mu})}{|x-y|^\mu}\, dx\, dy\Big)^{1/2^*_\mu}}. \tag{2.4}$$

According to Guo and Tang [11], we know the existence of $S_{\eta,\mu}$. Above equations (2.3) and (2.4), we have $S_{\eta,\mu} \le S_\mu$. The energy functional for the equation (1.1) is given by

$$\begin{aligned}
\mathcal{J}_\gamma(u) &= \frac{a}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^\theta + \Big(\int_{\mathbb{R}^N} |\nabla v|^2 dx\Big)^\theta\Big] \\
&\quad - \frac{\gamma}{r+s}\int_{\mathbb{R}^N} g(x)|u|^r|v|^s\,dx - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N} \frac{|u(x)|^{4\alpha}}{|x|^2}dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N} \frac{|v(x)|^{4\alpha}}{|x|^2}dx \\
&\quad + \frac{a}{4\alpha}\int_{\mathbb{R}^N} (|\nabla(|u|^{2\alpha})|^2 + |\nabla(|v|^{2\alpha})|^2)\,dx \\
&\quad - \frac{1}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2\alpha 2^*_\mu}|u(y)|^{2\alpha 2^*_\mu} + |v(x)|^{2\alpha 2^*_\mu}|v(y)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dx\,dy.
\end{aligned} \tag{2.5}$$

We observe that functional is not well-defined in $\mathcal{D}$. To overcome this difficulty, we apply the following change of variables which was developed by Colin and Jeanjean in [8], where $f : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{gathered}
f'(s) = \frac{1}{\big(1 + 2\alpha|f(s)|^{2(2\alpha-1)}\big)^{1/2}} \quad \text{in } [0,\infty), \\
f(s) = -f(-s) \quad \text{in } (-\infty, 0].
\end{gathered} \tag{2.6}$$

The above function $f(s)$ and its derivative satisfy the following properties and the proof of these properties can be found in [8, 38].

Lemma 2.2. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible.*
(2) *$f(0) = 0$;*
(3) *$0 < f'(s) \le 1$ for all $s \in \mathbb{R}$.*
(4) *$\frac{1}{2}f(s) \le \alpha s f'(s) \le \alpha f(s)$ for all $s > 0$.*
(5) *$|f(s)| \le |s|$ for all $s \in \mathbb{R}$;*
(6) *$|f(s)| \le (2\alpha)^{1/(4\alpha)}|s|^{1/2\alpha}$ for all $s \in \mathbb{R}$.*
(7) *$\lim_{s\to+\infty} \frac{f(s)}{s^{\frac{1}{2\alpha}}} = (2\alpha)^{\frac{1}{4\alpha}}$.*
(8) *$|f(s)| \ge f(1)|s|$ for $|s| \le 1$ and $|f(s)| \ge f(1)|s|^{1/2\alpha}$ for $|s| \ge 1$.*
(9) *$f''(s) < 0$ when $s > 0$ and $f''(s) > 0$ when $s < 0$.*
(10) *$\lim_{s\to 0} \frac{f(s)}{s} = 1$.*
(11) *$|f(s)|^{2\alpha-1}|f'(s)| < \frac{1}{\sqrt{2\alpha}}$ for all $s \in \mathbb{R}$.*

Thus, after the change of variables, $f(w)$ and $v = f(z)$, we can define a new energy functional $\mathcal{I}_\gamma : \mathcal{D} \to \mathbb{R}$ as

$$\begin{aligned}
\mathcal{I}_\gamma(w,z) &= \frac{a}{2}\int_{\mathbb{R}^N} (|\nabla w|^2 + |\nabla z|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N} |\nabla f(w)|^2 dx\Big)^\theta + \Big(\int_{\mathbb{R}^N} |\nabla f(z)|^2 dx\Big)^\theta\Big] \\
&\quad - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N} \frac{|f(w)|^{4\alpha}}{|x|^2}dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N} \frac{|f(z)|^{4\alpha}}{|x|^2}dx - \frac{\gamma}{r+s}\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s\,dx \\
&\quad - \frac{1}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}dy\,dx.
\end{aligned} \tag{2.7}$$

Then $\mathcal{I}_\gamma$ is well defined on $\mathcal{D}$. Indeed, Lemma 2.2-(6) and (1.5) yield

$$\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w)|^2}{|x|^2}dx \le 2\alpha\int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2}dx \le 2\alpha\Big(\frac{2}{N-2}\Big)^2\int_{\mathbb{R}^N} |\nabla w|^2 dx. \tag{2.8}$$

By Hölder inequality, Lemma 2.2-(6) and (2.2), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s\,dx \\
&\leq \|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\Big(\int_{\mathbb{R}^N}|f(w)|^{2\alpha 2^*}\,dx\Big)^{\frac{r}{2\alpha 2^*}}\Big(\int_{\mathbb{R}^N}|f(z)|^{2\alpha 2^*}\,dx\Big)^{\frac{s}{2\alpha 2^*}} \\
&\leq (2\alpha)^{\frac{r+s}{2\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\Big(\int_{\mathbb{R}^N}|w|^{2^*}dx\Big)^{\frac{r}{2\alpha 2^*}}\Big(\int_{\mathbb{R}^N}|z|^{2^*}dx\Big)^{\frac{s}{2\alpha 2^*}} \\
&\leq \big(2\alpha S^{-1}\big)^{\frac{r+s}{4\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\Big(\int_{\mathbb{R}^N}|\nabla w|^2dx\Big)^{\frac{r}{4\alpha}}\Big(\int_{\mathbb{R}^N}|\nabla z|^2dx\Big)^{\frac{s}{4\alpha}} \\
&\leq \big(2\alpha S^{-1}\big)^{\frac{r+s}{4\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\|(w,z)\|^{\frac{r+s}{2\alpha}}.
\end{aligned}
\tag{2.9}
$$

Also applying the Lemma 2.2(6), (2.1) and (2.2), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(w(y))|^{2\alpha 2^*_\mu}|f(w(x))|^{2\alpha 2^*_\mu}+|f(z(y))|^{2\alpha 2^*_\mu}|f(z(x))|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\
&\leq (2\alpha)^{2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(y)|^{2^*_\mu}|w(x)|^{2^*_\mu}+|z(y)|^{2^*_\mu}|z(x)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\
&\leq (2\alpha)^{2^*_\mu}C(N,\mu)(\|w\|_{2^*}^{22^*_\mu}+\|z\|_{2^*}^{22^*_\mu}) \\
&\leq (2\alpha)^{2^*_\mu}C(N,\mu)S^{-2^*_\mu}(\|w\|^{22^*_\mu}+\|z\|^{22^*_\mu}) \\
&\leq (2\alpha)^{2^*_\mu}C(N,\mu)2S^{-2^*_\mu}\|(w,z)\|^{22^*_\mu} \\
&\leq (2\alpha)^{2^*_\mu}S_\mu^{-2^*_\mu}\|(w,z)\|^{22^*_\mu}.
\end{aligned}
\tag{2.10}
$$

Note that if $(w,z)\in\mathcal{D}$ is a nontrivial critical point of the functional $\mathcal{I}_\gamma$, then for every $(\phi,\psi)\in\mathcal{D}$, we have $\langle\mathcal{I}'_\gamma(w,z),(\phi,\psi)\rangle=0$ i.e.

$$
\begin{aligned}
&a\int_{\mathbb{R}^N}(\nabla w\nabla\phi+\nabla z\nabla\psi)\,dx+b\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2dx\Big)^{\theta-1}\int_{\mathbb{R}^N}|f'(w)|^2\nabla w\nabla\phi+f'(w)f''(w)|\nabla w|^2\phi\,dx \\
&+b\Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2dx\Big)^{\theta-1}\int_{\mathbb{R}^N}|f'(z)|^2\nabla z\nabla\psi+f'(z)f''(z)|\nabla z|^2\psi\,dx \\
&-a\eta_1\int_{\mathbb{R}^N}\frac{|f(w)|^{2(2\alpha-1)}}{|x|^2}f(w)f'(w)\phi\,dx-a\eta_2\int_{\mathbb{R}^N}\frac{|f(z)|^{2(2\alpha-1)}}{|x|^2}f(z)f'(z)\psi\,dx \\
&-\gamma\frac{r}{r+s}\int_{\mathbb{R}^N}g(x)|f(z)|^s|f(w)|^{r-2}f(w)f'(w)\phi\,dx \\
&-\gamma\frac{s}{r+s}\int_{\mathbb{R}^N}g(x)|f(w)|^r|f(z)|^{s-2}f(z)f'(z)\psi\,dx \\
&-\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|f(w)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}dy\Big)|f(w)|^{2\alpha 2^*_\mu-2}f(w)f'(w)\phi\,dx \\
&-\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}dy\Big)|f(z)|^{2\alpha 2^*_\mu-2}f(z)f'(z)\psi\,dx=0.
\end{aligned}
$$

This implies (w,z) is a weak solution of the system

$$
\begin{aligned}
&-a\Delta w-b\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2\,dx\Big)^{\theta-1}\left(f'(w)f''(w)|\nabla w|^2+|f'(w)|^2\Delta w\right)\\
&-a\eta_1\frac{|f(w)|^{2(2\alpha-1)}}{|x|^2}f(w)f'(w)\\
&=\gamma\frac{r}{r+s}g(x)|f(z)|^s|f(w)|^{r-2}f(w)f'(w)+\Big(\int_{\mathbb{R}^N}\frac{|f(w)|^{2\alpha 2_\mu^*}}{|x-y|^\mu}dy\Big)|f(w)|^{2\alpha 2_\mu^*-2}f(w)f'(w).\\
&\quad -a\Delta z-b\Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2\,dx\Big)^{\theta-1}\left(f'(z)f''(z)|\nabla z|^2+|f'(z)|^2\Delta z\right)\\
&\quad -a\eta_2\frac{|f(z)|^{2(2\alpha-1)}}{|x|^2}f(z)f'(z)\\
&=\gamma\frac{s}{r+s}g(x)|f(w)|^r|f(z)|^{s-2}f(z)f'(z)+\Big(\int_{\mathbb{R}^N}\frac{|f(z)|^{2\alpha 2_\mu^*}}{|x-y|^\mu}dy\Big)|f(z)|^{2\alpha 2_\mu^*-2}f(z)f'(z).
\end{aligned}
\tag{2.11}
$$

Observe that the transformed problem (2.11) is equivalent to the given problem (1.1), which takes $(u,v)=(f(w),f(z))$ as its solution. Thus, our goal is to find the solutions of (2.11) is related to (1.1).

3. Compactness result

In this section, we first recall the concentration compactness principle related to critical elliptic system studied by Kang [17] and for Choquard term in [10]. This is based on Lions [23, 24]. After that, we prove the concentration compactness Lemma for the Hardy term, which plays an important role in proving the (PS) condition.

Lemma 3.1. *Let $\{(w_n,z_n)\}_n$ be a bounded sequence in $\mathcal{D}$ converging weakly and almost everywhere to some (w,z) and $|\nabla w_n|^2+|\nabla z_n|^2\rightharpoonup\omega$ weakly in the sense of measure. Moreover, assume that*

$$
\Big(\int_{\mathbb{R}^N}\frac{|w_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|w_n(x)|^{2_\mu^*}+\Big(\int_{\mathbb{R}^N}\frac{|z_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|z_n(x)|^{2_\mu^*}\rightharpoonup\zeta,
$$

in the sense of measure where ω and ζ are bounded nonnegative Radon measures on $\mathbb{R}^N$ and define

$$
\omega_\infty:=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\ge R}(|\nabla w_n|^2+|\nabla z_n|^2)\,dx,
$$

$$
\zeta_\infty:=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\ge R}\Big[\Big(\int_{\mathbb{R}^N}\frac{|w_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|w_n(x)|^{2_\mu^*}+\Big(\int_{\mathbb{R}^N}\frac{|z_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|z_n(x)|^{2_\mu^*}\Big]\,dx.
$$

Then there exists a countable set J of distinct points $\{x_j\}_{j\in J}\subset\mathbb{R}^N$ and nonnegative numbers $\{\omega_j\}_{j\in J}$, $\{\zeta_j\}_{j\in J}$ such that

$$
\zeta=\Big(\int_{\mathbb{R}^N}\frac{|w(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|w(x)|^{2_\mu^*}+\Big(\int_{\mathbb{R}^N}\frac{|z(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|z(x)|^{2_\mu^*}+\sum_{j\in J}\zeta_j\delta_{x_j},
$$

$$
\omega\ge|\nabla w|^2+|\nabla z|^2+\sum_{j\in J}\omega_j\delta_{x_j},
$$

where δ_x is the Dirac mass of mass 1 concentrated at $x\in\mathbb{R}^N$ and for all $j\in J$

$$
S_\mu\zeta_j^{1/2_\mu^*}\le\omega_j.
$$

For the energy at infinity, we have

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\,dx:=\omega(\mathbb{R}^N)+\omega_\infty,
$$

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^N}\Big[\Big(\int_{\mathbb{R}^N}\frac{|w_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)|w_n(x)|^{2_\mu^*}+\Big(\int_{\mathbb{R}^N}\frac{|z_n(y)|^{2_\mu^*}}{|x-y|^\mu}dy\Big)\ |z_n(x)|^{2_\mu^*}\Big]dx:=\zeta(\mathbb{R}^N)+\zeta_\infty,
$$

$$S_\mu \zeta_\infty^{1/2^*_\mu} \le \omega_\infty\Big(\int_{\mathbb{R}^N} d\omega + \omega_\infty\Big).$$

Lemma 3.2. *Consider $\{(w_n, z_n)\}$ be a bounded sequence which is converge weakly to (w, z) in $\mathcal{D}$, f is continuous with $f(0) = 0$. Assume that $|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(z_n)|^2 \rightharpoonup \omega$, $\frac{|f^{2\alpha}(w_n)|^2}{|x|^2} + \frac{|f^{2\alpha}(z_n)|^2}{|x|^2} \rightharpoonup \nu$ and*

$$\Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(w_n)|^{2^*_\mu} + \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(z_n)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(z_n)|^{2^*_\mu} \rightharpoonup \zeta$$

in the sense of measure and ω, ζ, ν are bounded nonnegative Radon measures on $\mathbb{R}^N$ and define

$$\begin{gathered}
\omega_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} (|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(z_n)|^2)\, dx, \\
\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} \Big(\frac{|f^{2\alpha}(w_n)|^2}{|x|^2} + \frac{|f^{2\alpha}(z_n)|^2}{|x|^2}\Big)\, dx, \\
\zeta_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\ge R} \Big[\Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(w_n)|^{2^*_\mu} \\
+ \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(z_n)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(z_n)|^{2^*_\mu}\Big] dx.
\end{gathered}$$

Then there exists a countable set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^N \setminus \{0\}$ and nonnegative numbers $\{\omega_j\}_{j\in J}$, $\{\zeta_j\}_{j\in J}, \omega_0, \zeta_0$, and ν_0 such that

$$\begin{gathered}
\zeta = \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(w)|^{2^*_\mu} + \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(z)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(z)|^{2^*_\mu} + \sum_{j\in J} \zeta_j \delta_{x_j} + \zeta_0\delta_0, \\
\omega \ge |\nabla f^{2\alpha}(w)|^2 + |\nabla f^{2\alpha}(z)|^2 + \sum_{j\in J} \omega_j \delta_{x_j} + \omega_0\delta_0, \\
\nu \ge \frac{|f^{2\alpha}(w)|^2}{|x|^2} + \frac{|f^{2\alpha}(z)|^2}{|x|^2} + \nu_0\delta_0,
\end{gathered}$$

and

$$S_{\eta,\mu} \zeta_0^{1/2^*_\mu} \le \omega_0 - \eta\nu_0, \tag{3.1}$$

where δ_x is the Dirac mass of mass 1 concentrated at $x \in \mathbb{R}^N$. For the concentration at infinity, we have

$$\begin{gathered}
\limsup_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(w_n)|^2)\, dx := \omega(\mathbb{R}^N) + \omega_\infty, \\
\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big(\frac{|f^{2\alpha}(w_n)|^2}{|x|^2} + \frac{|f^{2\alpha}(z_n)|^2}{|x|^2}\Big)\, dx := \nu(\mathbb{R}^N) + \nu_\infty, \\
\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big[\Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(w_n)|^{2^*_\mu} + \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(z_n)|^{2^*_\mu}}{|x-y|^\mu} dy\Big)|f^{2\alpha}(z_n)|^{2^*_\mu}\Big] dx \\
:= \zeta(\mathbb{R}^N) + \zeta_\infty,
\end{gathered}$$

and

$$S_{\eta,\mu} \zeta_\infty^{1/2^*_\mu} \le \omega_\infty - \eta\nu_\infty. \tag{3.2}$$

Proof. Suppose $\{(f^{2\alpha}(s_n), f^{2\alpha}(t_n))\} = \{(f^{2\alpha}(w_n) - f^{2\alpha}(w), f^{2\alpha}(z_n) - f^{2\alpha}(z))\}$. Then the sequence $\{(f^{2\alpha}(s_n), f^{2\alpha}(t_n))\}_n$ converges weakly to $(0, 0)$ in $\mathcal{D}$ and $\{(f^{2\alpha}(s_n), f^{2\alpha}(t_n)\}_n \to (0, 0)$ pointwise a.e. in $\mathbb{R}^N \times \mathbb{R}^N$ as every bounded sequence $\{(f^{2\alpha}(w_n), f^{2\alpha}(z_n))\}_n$ converges weakly to $\{(f^{2\alpha}(w), f^{2\alpha}(z))\}$ in $\mathcal{D}$. By Lemma 3.1, we have

$$\begin{gathered}
|\nabla f^{2\alpha}(s_n)|^2 + |\nabla f^{2\alpha}(t_n)|^2 \rightharpoonup \tilde{\omega} := \omega - (|\nabla f^{2\alpha}(w)|^2 + |\nabla f^{2\alpha}(z)|^2), \\
\frac{|f^{2\alpha}(s_n)|^2}{|x|^2} + \frac{|f^{2\alpha}(t_n)|^2}{|x|^2} \rightharpoonup \tilde{\nu} := \nu - \Big(\frac{|f^{2\alpha}(w)|^2}{|x|^2} + \frac{|f^{2\alpha}(z)|^2}{|x|^2}\Big),
\end{gathered}$$

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(s_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(s_n)|^{2^*_\mu}+\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(t_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(t_n)|^{2^*_\mu}\rightharpoonup\tilde\zeta\\
&:=\zeta-\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(w)|^{2^*_\mu}-\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(z)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(z)|^{2^*_\mu}.
\end{aligned}
$$

Fix $\Phi\in C_c^\infty(\mathbb{R}^N)$ such that $0\le\Phi\le 1$, $\Phi(0)=1$ and $\operatorname{supp}\Phi=\overline{B_1(0)}$. For $\epsilon>0$, take $\Phi_\epsilon(x)=\Phi(\frac{x}{\epsilon}), x\in\mathbb{R}^N$. Then (2.4) implies

$$
\begin{aligned}
&S_{\eta,\mu}\Big(\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|\Phi_\epsilon f^{2\alpha}(s_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|\Phi_\epsilon f^{2\alpha}(s_n)|^{2^*_\mu}dx\\
&+\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|\Phi_\epsilon f^{2\alpha}(t_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|\Phi_\epsilon f^{2\alpha}(t_n)|^{2^*_\mu}dx\Big)^{1/2^*_\mu}\\
&\le\int_{\mathbb{R}^N}\left(|\nabla(\Phi_\epsilon f^{2\alpha}(s_n))|^2+|\nabla(\Phi_\epsilon f^{2\alpha}(t_n))|^2\right)dx-\eta\int_{\mathbb{R}^N}\Big(\frac{|\Phi_\epsilon f^{2\alpha}(s_n)|^2}{|x|^2}+\frac{|\Phi_\epsilon f^{2\alpha}(t_n)|^2}{|x|^2}\Big)dx.
\end{aligned}
$$

Using Lemma 3.3 and $(f^{2\alpha}(s_n),f^{2\alpha}(t_n))\to(0,0)$ in $L^2_{\rm loc}(\mathbb{R}^N)\times L^2_{\rm loc}(\mathbb{R}^N)$, yield

$$
\begin{aligned}
&S_{\eta,\mu}\Big(\int_{\mathbb{R}^N}|\Phi_\epsilon|^{22^*_\mu}\Big[\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(s_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(s_n)|^{2^*_\mu}+\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(t_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(t_n)|^{2^*_\mu}\Big]dx\Big)^{1/2^*_\mu}\\
&\le\int_{\mathbb{R}^N}|\Phi_\epsilon|^2\left(|\nabla f^{2\alpha}(s_n)|^2+|\nabla f^{2\alpha}(t_n)|^2\right)dx-\eta\int_{\mathbb{R}^N}|\Phi_\epsilon|^2\Big(\frac{|f^{2\alpha}(s_n)|^2}{|x|^2}+\frac{|f^{2\alpha}(t_n)|^2}{|x|^2}\Big)dx.
\end{aligned}
$$

On passing to the limit as $n\to\infty$, we obtain

$$
S_{\eta,\mu}\Big(\int_{\mathbb{R}^N}|\Phi_\epsilon|^{22^*_\mu}d\tilde\zeta\Big)^{1/2^*_\mu}\le\int_{\mathbb{R}^N}|\Phi_\epsilon|^2d\tilde\omega-\eta\int_{\mathbb{R}^N}|\Phi_\epsilon|^2d\tilde\nu. \tag{3.3}
$$

Taking $\epsilon\to0$, inequality (3.3) yields

$$
S_{\eta,\mu}\zeta_0^{1/2^*_\mu}\le\omega_0-\eta\nu_0.
$$

This concludes the proof of (3.1).

Next, we prove (3.2). For $R>1$, fix $\Phi_R\in C_c^\infty(\mathbb{R}^N)$ such that $0\le\Phi_R(x)\le1$, $\Phi_R(x)=1$ for $|x|>R+1$ and $\Phi_R(x)=0$ for $|x|<R$. Using (2.4) and Lemma 3.1, we have

$$
\begin{aligned}
&S_{\eta,\mu}\Big(\int_{\mathbb{R}^N}|\Phi_R|^{22^*_\mu}\Big[\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(s_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(s_n)|^{2^*_\mu}+\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(t_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(t_n)|^{2^*_\mu}\Big]dx\Big)^{1/2^*_\mu}\\
&\le\int_{\mathbb{R}^N}|\Phi_R|^2\Big(|\nabla f^{2\alpha}(s_n)|^2+|\nabla f^{2\alpha}(t_n)|^2\Big)dx-\eta\int_{\mathbb{R}^N}|\Phi_R|^2\Big(\frac{|f^{2\alpha}(s_n)|^2}{|x|^2}+\frac{|f^{2\alpha}(t_n)|^2}{|x|^2}\Big)dx.
\end{aligned}
$$

On passing to the limit as $n\to\infty$, we obtain

$$
S_{\eta,\mu}\Big(\int_{\mathbb{R}^N}|\Phi_R|^{22^*_\mu}d\tilde\zeta\Big)^{1/2^*_\mu}\le\int_{\mathbb{R}^N}|\Phi_R|^2d\tilde\omega-\eta\int_{\mathbb{R}^N}|\Phi_R|^2d\tilde\nu,
$$

again passing to the limit $R\to\infty$, which implies

$$
S_{\eta,\mu}\zeta_\infty^{1/2^*_\mu}\le\omega_\infty-\eta\nu_\infty. \tag{3.4}
$$

This completes the proof (3.2). □

Lemma 3.3. *Let $\{(s_n,t_n)\}$ be a bounded sequence in $\mathcal{D}$ that converges weakly to $(0,0)$ in $\mathcal{D}$ and f is continuous with $f(0)=0$. Then for every $\alpha>\frac12$ and $\phi\in C_c^\infty(\mathbb{R}^N)$,*

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|\phi f^{2\alpha}(s_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|\phi f^{2\alpha}(s_n)|^{2^*_\mu}dx-\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(s_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|\phi(x)|^{2^*_\mu}|\phi f^{2\alpha}(s_n)|^{2^*_\mu}dx\Big|\\
&+\Big|\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|\phi f^{2\alpha}(t_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|\phi\tilde f^{2\alpha}(t_n)|^{2^*_\mu}dx-\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(t_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|\phi(x)|^{2^*_\mu}|\phi f^{2\alpha}(t_n)|^{2^*_\mu}dx\Big|\\
&\to0,\quad \textit{as } n\to\infty\,.
\end{aligned}
$$

Proof. As $\{(s_n, t_n)\}$ converges weakly to $(0,0)$ in $\mathcal{D}$ and f is continuous function with $f(0)=0$. Therefore, $\{(f^{2\alpha}(s_n), f^{2\alpha}(t_n))\}$ weakly converges to $(0,0)$ in $\mathcal{D}$. Hence, the rest of the proof proceeds similarly to [10, Lemma 2.5]. $\square$

Lemma 3.4. *Suppose* (1.9) *holds and* $2\alpha < r+s < 4\alpha$. *Consider* $\{(w_n, z_n)\}_n$ *any* $(PS)_c$ *sequence for* $\mathcal{I}_\gamma$. *Then* $\{(w_n, z_n)\}$ *is bounded in* $\mathcal{D}$.

Proof. Let $\{(w_n, z_n)\}$ be a $(PS)_c$ sequence in $\mathcal{D}$. Then

$$\mathcal{I}_\gamma(w_n, z_n) \to c \quad \text{and} \quad \mathcal{I}_\gamma^{'}(w_n, z_n) \to 0 \quad \text{in} \quad \mathcal{D}^*, \quad \text{as } n \to +\infty.$$

Choose $\hat{w}_n = (1 + 2\alpha|f(w_n)|^{2(2\alpha-1)})^{1/2} f(w_n) = \frac{f(w_n)}{f'(w_n)}$ and $\hat{z}_n = \frac{f(z_n)}{f'(z_n)}$, we have $(\hat{w}, \hat{z}_n) \in \mathcal{D}$. Then

$$|\nabla \hat{w}_n| = \Big(1 + \frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|,$$
$$|\nabla \hat{z}_n| = \Big(1 + \frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|,$$

and

$$\|(\hat{w}_n, \hat{z}_n)\| \le 2\alpha \|(w_n, z_n)\|. \tag{3.5}$$

This implies $(\hat{w}_n, \hat{z}_n)$ in $\mathcal{D}$. We also have

$$\begin{aligned}
&\langle \mathcal{I}_\gamma'(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle \\
&= a\int_{\mathbb{R}^N} \Big[\Big(1 + \frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|^2 \\
&+ \Big(1 + \frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla z_n|^2\Big]\,dx \\
&+ b\Big[\Big(\int_{\mathbb{R}^N} |\nabla f(w_n)|^2\,dx\Big)^\theta + \Big(\int_{\mathbb{R}^N} |\nabla f(z_n)|^2\,dx\Big)^\theta\Big] - a\eta_1 \int_{\mathbb{R}^N} \frac{|f(w_n)|^{4\alpha}}{|x|^2} dx \\
&- a\eta_2 \int_{\mathbb{R}^N} \frac{|f(z_n)|^{4\alpha}}{|x|^2} dx \\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(w_n)|^{2\alpha 2_\mu^*}|f(w_n)|^{2\alpha 2_\mu^*} + |f(z_n)|^{2\alpha 2_\mu^*}|f(z_n)|^{2\alpha 2_\mu^*}}{|x-y|^\mu} dy\,dx.
\end{aligned} \tag{3.6}$$

Using equations (2.7)-(2.9), (3.6) and combined with $\eta = max(\eta_1, \eta_2)$, we have

$$\begin{aligned}
&c + o_n(1)\|(w_n, z_n)\| \\
&= \mathcal{I}_\gamma(w_n, z_n) - \frac{1}{4\alpha 2_\mu^*}\langle \mathcal{I}_\gamma'(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle \\
&= a\int_{\mathbb{R}^N} \Big[\frac{1}{2} - \frac{1}{4\alpha 2_\mu^*}\Big(1 + \frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla w_n|^2\,dx \\
&\quad - \Big(\frac{a\eta_1}{4\alpha} - \frac{a\eta_1}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^2}{|x|^2} dx \\
&\quad + a\int_{\mathbb{R}^N} \Big[\frac{1}{2} - \frac{1}{4\alpha 2_\mu^*}\Big(1 + \frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla z_n|^2\,dx \\
&\quad - \Big(\frac{a\eta_2}{4\alpha} - \frac{a\eta_2}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(z_n)|^2}{|x|^2} dx \\
&\quad + \Big(\frac{b}{2\theta} - \frac{b}{4\alpha 2_\mu^*}\Big)\Big[\Big(\int_{\mathbb{R}^N} |\nabla f(w_n)|^2\,dx\Big)^\theta + \Big(\int_{\mathbb{R}^N} |\nabla f(z_n)|^2\,dx\Big)^\theta\Big] \\
&\quad - \Big(\frac{\gamma}{r+s} - \frac{\gamma}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s dx
\end{aligned}$$

$$
\begin{aligned}
&\geq a\Big(\frac{1}{2}-\frac{1}{22_\mu^*}\Big)\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\,dx - a\Big(\frac{1}{2}-\frac{1}{22_\mu^*}\Big)\frac{4\eta}{(N-2)^2}\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\,dx\\
&\quad -\Big(\frac{\gamma}{r+s}-\frac{\gamma}{4\alpha 2_\mu^*}\Big)\Big(\frac{2\alpha}{S}\Big)^{\frac{r+s}{4\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\|(w_n,z_n)\|^{\frac{r+s}{2\alpha}}\\
&\geq \Big(\frac{a}{2}-\frac{a}{22_\mu^*}\Big)\Big(\frac{(N-2)^2-4\eta}{(N-2)^2}\Big)\|(w_n,z_n)\|^2\\
&\quad -\Big(\frac{\gamma}{r+s}-\frac{\gamma}{4\alpha 2_\mu^*}\Big)\Big(\frac{2\alpha}{S}\Big)^{\frac{r+s}{4\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\|(w_n,z_n)\|^{\frac{r+s}{2\alpha}},
\end{aligned}
\tag{3.7}
$$

which implies that boundedness of $\{(w_n,z_n)\}$ in $\mathcal{D}$, since $2<22_\mu^*$, $\eta<\frac{(N-2)^2}{4}$ and $2\alpha<r+s<4\alpha<4\alpha 2_\mu^*$. □

Lemma 3.5. *Suppose $r+s=4\alpha$, $0<\gamma<aS\big(1-\frac{4\eta}{(N-2)^2}\big)\|g\|^{-1}_{\frac{2^*}{2^*-2}}$, and (1.9) hold . Then any $(PS)_c$ sequence for $\mathcal{I}_\gamma$ is bounded in $\mathcal{D}$.*

Proof. Let $\{(w_n,z_n)\}$ be a $(PS)_c$ sequence for $\mathcal{I}_\gamma$ at level $c\in\mathbb{R}$. Using a similar calculation as in Lemma 3.4 and substitute $r+s=4\alpha$ into (3.7), we obtain

$$
\begin{aligned}
c+o(1)\|(w_n,z_n)\| &= \mathcal{I}_\gamma(w_n,z_n)-\frac{1}{4\alpha 2_\mu^*}\langle \mathcal{I}_\gamma'(w_n,z_n),(\hat{w}_n,\hat{z}_n)\rangle\\
&\geq \Big(\frac{1}{2}-\frac{1}{22_\mu^*}\Big)\Big(a\Big(1-\frac{4\eta}{(N-2)^2}\Big)-\gamma S^{-1}\|g\|_{\frac{2^*}{2^*-2}}\Big)\|(w_n,z_n)\|^2.
\end{aligned}
$$

Thus, for all $0<\gamma<aS\big(1-\frac{4\eta}{(N-2)^2}\big)\|g\|^{-1}_{\frac{2^*}{2^*-2}}$, the sequence $\{(w_n,z_n)\}$ is a bounded in $\mathcal{D}$. □

Lemma 3.6. *Suppose that (1.9) hold and $4\alpha<r+s<2\alpha 2^*$. Then every $(PS)_c$ sequence for $\mathcal{I}_\gamma$ is bounded in $\mathcal{D}$.*

Proof. Let $\{(w_n,z_n)\}$ be a $(PS)_c$ sequence for $\mathcal{I}_\gamma$ at level $c\in\mathbb{R}$. Then using $\eta=\max(\eta_1,\eta_2)$, (3.6) and (2.8), we deduce that

$$
\begin{aligned}
&c+o(1)\|(w_n,z_n)\|\\
&=\mathcal{I}_\gamma(z_n)-\frac{1}{r+s}\langle \mathcal{I}_\gamma'(w_n,z_n),(\hat{w}_n,\hat{z}_n)\rangle\\
&=\int_{\mathbb{R}^N}\Big[\frac{a}{2}-\frac{a}{r+s}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla w_n|^2+\Big(\frac{b}{2\theta}-\frac{b}{r+s}\Big)\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2dx\Big)^\theta\\
&\quad+\int_{\mathbb{R}^N}\Big[\frac{a}{2}-\frac{a}{r+s}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla z_n|^2+\Big(\frac{b}{2\theta}-\frac{b}{r+s}\Big)\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2dx\Big)^\theta\\
&\quad+\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(w_n)|^{2\alpha 2_\mu^*}|f(w_n)|^{2\alpha 2_\mu^*}}{|x-y|^\mu}\,dy\,dx-\Big(\frac{a\eta_1}{4\alpha}-\frac{a\eta_1}{r+s}\Big)\int_{\mathbb{R}^N}\frac{|f(w_n)|^{4\alpha}}{|x|^2}dx\\
&\quad+\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(z_n)|^{2\alpha 2_\mu^*}|f(z_n)|^{2\alpha 2_\mu^*}}{|x-y|^\mu}\,dy\,dx-\Big(\frac{a\eta_2}{4\alpha}-\frac{a\eta_2}{r+s}\Big)\int_{\mathbb{R}^N}\frac{|f(z_n)|^{4\alpha}}{|x|^2}dx\\
&\geq a\Big(\frac{1}{2}-\frac{2\alpha}{r+s}\Big)\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\,dx-a\Big(\frac{1}{2}-\frac{2\alpha}{r+s}\Big)\frac{4\eta}{(N-2)^2}\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\,dx\\
&\geq a\Big(\frac{1}{2}-\frac{2\alpha}{r+s}\Big)\Big(1-\frac{4\eta}{(N-2)^2}\Big)\|(w_n,z_n)\|^2.
\end{aligned}
$$

Since $\eta<\frac{(N-2)^2}{4}$ and $4\alpha<r+s<2\alpha 2^*<4\alpha 2_\mu^*$, we conclude that $\{(w_n,z_n)\}$ is bounded sequence $\mathcal{D}$. □

Lemma 3.7. *Let $0 < \mu < N$ and f be continuous with $f(0) = 0$. Assume that $\{f^{2\alpha}(w_n)\}$ is a bounded sequence in $L^{2^*}(\mathbb{R}^N)$ such that $f^{2\alpha}(w_n) \to f^{2\alpha}(w)$ a.e. in $\mathbb{R}^N$. Then we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^{2^*_\mu}}{|x-y|^\mu} dy \Big) |f^{2\alpha}(w_n)|^{2^*_\mu} dx \\
&- \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n) - f^{2\alpha}(w)|^{2^*_\mu}}{|x-y|^\mu} dy \Big) |f^{2\alpha}(w_n) - f^{2\alpha}(w)|^{2^*_\mu} dx \\
&\to \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w)|^{2^*_\mu}}{|x-y|^\mu} dy \Big) |f^{2\alpha}(w)|^{2^*_\mu} dx \quad \text{as } n \to \infty.
\end{aligned}$$

The proof of the above lemma follows in a similar manner to that of [9, Lemma 2.2].

Lemma 3.8. *Suppose (1.9) hold and $2\alpha < r + s < 4\alpha$. Let $\{(w_n, z_n)\} \subset \mathcal{D}$ be a Palais-Smale sequence for $\mathcal{I}_\gamma$ and $c < 0$, then there exists $\gamma^* > 0$ such that $\mathcal{I}_\gamma$ satisfies the $(PS)_c$ condition for all $\gamma \in (0, \gamma^*)$*

Proof. Let $\{(w_n, z_n)\} \subset \mathcal{D}$ be a $(PS)_c$-sequence for $\mathcal{I}_\gamma$. Then Lemma 3.4 gives that the sequence $\{(w_n, z_n)\}$ is bounded in $\mathcal{D}$. So, we have $\{(f^{2\alpha}(w_n), f^{2\alpha}(z_n))\}$ is also bounded in $\mathcal{D}$. Therefore, we can assume that up to a subsequence $(w_n, z_n) \rightharpoonup (w, z)$ in $\mathcal{D}$, $(w_n, z_n) \to (w, z)$ in $\mathbb{R}^N \times \mathbb{R}^N$. From $f \in C^\infty(\mathbb{R}^N)$, it follow that $(f^{2\alpha}(w_n), f^{2\alpha}(z_n)) \rightharpoonup (f^{2\alpha}(w), f^{2\alpha}(z))$ in $\mathcal{D}$ and then $(f^{2\alpha}(w_n), f^{2\alpha}(z_n)) \to (f^{2\alpha}(w), f^{2\alpha}(z))$ converges pointwise a.e. in $\mathbb{R}^N \times \mathbb{R}^N$. Thus, we can assume that

$$\begin{gathered}
|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(z_n)|^2 \rightharpoonup \omega, \quad \frac{|f(w_n)|^{4\alpha}}{|x|^2} + \frac{|f(z_n)|^{4\alpha}}{|x|^2} \rightharpoonup \nu, \\
\Big(\int_{\mathbb{R}^N} \frac{|f(w_n)|^{2\alpha 2^*_\mu}}{|x-y|^\mu} dy \Big) |f(w_n)|^{2\alpha 2^*_\mu} + \Big(\int_{\mathbb{R}^N} \frac{|f(z_n)|^{2\alpha 2^*_\mu}}{|x-y|^\mu} dy \Big) |f(z_n)|^{2\alpha 2^*_\mu} \rightharpoonup \zeta,
\end{gathered}$$

weakly in the sense of measure. Using the concentration compactness principle in Lemma 3.1 and 3.2, there exists at most countable set J, of distinct points $\{x_j\}_{j \in J} \subset \mathbb{R}^N \setminus \{0\}$ and families of nonnegative numbers $\{\zeta_j : j \in J\}$ and $\{\omega_j : j \in J\}$ and nonnegative numbers ω_0, ζ_0, ν_0 such that

$$\begin{gathered}
\omega \geq |\nabla f^{2\alpha}(w)|^2 + |\nabla f^{2\alpha}(z)|^2 + \sum_{j \in J} \omega_j \delta_{x_j} + \omega_0 \delta_0, \\
\nu \geq \frac{|f(w)|^{4\alpha}}{|x|^2} + \frac{|f(z)|^{4\alpha}}{|x|^2} + \nu_0 \delta_0, \\
\zeta = \Big(\int_{\mathbb{R}^N} \frac{|f(w)|^{2\alpha 2^*_\mu}}{|x-y|^\mu} \Big) |f(w)|^{2\alpha 2^*_\mu} + \Big(\int_{\mathbb{R}^N} \frac{|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu} \Big) |f(z)|^{2\alpha 2^*_\mu} + \sum_{j \in J} \zeta_j \delta_{x_j} + \zeta_0 \delta_0,
\end{gathered}$$

and

$$S_\mu \zeta_j^{\frac{1}{2^*_\mu}} \leq \omega_j \quad \text{and} \quad S_{\eta,\mu} \zeta_0^{1/2^*_\mu} \leq (\omega_0 - \eta \nu_0), \tag{3.8}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$. Moreover, we can construct a smooth cut-off function $\xi_\epsilon^j \in C_0^\infty(\mathbb{R}^N)$ centered at x_j such that

$$0 \leq \xi_\epsilon^j(x) \leq 1, \quad \xi_\epsilon^j(x) = 1 \quad \text{in } B(x_j, \frac{\epsilon}{2}), \quad \xi_\epsilon^j(x) = 0 \quad \text{in } \mathbb{R}^N \setminus B(x_j, \epsilon), \quad |\nabla \xi_\epsilon^j| \leq \frac{4}{\epsilon},$$

for each $\epsilon > 0$ be small enough. By equation (3.5), $\{(\hat{w}_n\xi_\epsilon^j, \hat{z}_n\xi_\epsilon^j)\}$ is bounded in $\mathcal{D}$. Obviously, $\langle \mathcal{I}'_\gamma(w_n, z_n), \{(\hat{w}_n\xi_\epsilon^j, \hat{z}_n\xi_\epsilon^j)\}\rangle \to 0$ as $n \to \infty$. Therefore,

$$
\begin{aligned}
&-\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(\frac{f(w_n)}{f'(w_n)}\nabla w_n + \frac{f(z_n)}{f'(z_n)}\nabla z_n\Big)\nabla\xi_\epsilon^j\,dx\\
&\quad + b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\Big)^{\theta-1}\int_{\mathbb{R}^N} f'(w_n)f(w_n)\nabla w_n\nabla\xi_\epsilon^j\,dx\\
&\quad + b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\Big)^{\theta-1}\int_{\mathbb{R}^N} f'(z_n)f(z_n)\nabla z_n\nabla\xi_\epsilon^j\,dx\Big]\\
&= \lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|^2\xi_\epsilon^j\,dx\\
&\quad + b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\xi_\epsilon^j\,dx\Big)\\
&\quad + a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla z_n|^2\xi_\epsilon^j\,dx\\
&\quad + b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\xi_\epsilon^j\,dx\Big)\\
&\quad - \gamma\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s\xi_\epsilon^j\,dx - a\eta_1\int_{\mathbb{R}^N}\frac{|f(w_n)|^{4\alpha}}{|x|^2}\xi_\epsilon^j dx - a\eta_2\int_{\mathbb{R}^N}\frac{|f(z_n)|^{4\alpha}}{|x|^2}\xi_\epsilon^j dx.\\
&\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{(|f(w_n)|^{2\alpha 2_\mu^*}|f(w_n)|^{2\alpha 2_\mu^*} + |f(z_n)|^{2\alpha 2_\mu^*}|f(z_n)|^{2\alpha 2_\mu^*})\xi_\epsilon^j}{|x-y|^\mu}\Big)\,dy\,dx\Big].
\end{aligned}
\tag{3.9}
$$

Now Lemma 2.2-(4) and Hölder inequality yield

$$
\begin{aligned}
0 &\le \lim_{\epsilon\to 0}\lim_{n\to\infty}\Big|a\int_{\mathbb{R}^N}\frac{f(w_n)}{f'(w_n)}\nabla w_n\nabla\xi_\epsilon^j\,dx\Big|\\
&\le K\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|w_n\nabla w_n\nabla\xi_\epsilon^j|\,dx\\
&\le K\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[\Big(\int_{\mathbb{R}^N}|\nabla w_n|^2\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|w_n\nabla\xi_\epsilon^j|^2\,dx\Big)^{1/2}\Big]\\
&\le K'\lim_{\epsilon\to 0}\Big[\Big(\int_{B(x_j,\epsilon)}|\nabla\xi_\epsilon^j|^N\,dx\Big)^{\frac{2}{N}}\Big]^{1/2}\Big[\Big(\int_{B(x_j,\epsilon)}|w|^{\frac{2N}{N-2}}\,dx\Big)^{\frac{N-2}{N}}\Big]^{1/2}\\
&\le K''\lim_{\epsilon\to 0}\Big(\int_{B(x_j,\epsilon)}|w|^{2^*}\,dx\Big)^{1/2^*} = 0.
\end{aligned}
\tag{3.10}
$$

Similarly, we have

$$
\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big|a\int_{\mathbb{R}^N}\frac{f(z_n)}{f'(z_n)}\nabla z_n\nabla\xi_\epsilon^j\,dx\Big| = 0. \tag{3.11}
$$

By the definition of ξ_ϵ^j and the boundedness of $\{w_n\}$, $\{z_n\}$, we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[b\Big(\int_{\mathbb{R}^N}\frac{|\nabla w_n|^2}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\,dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}\frac{f(w_n)\nabla w_n\nabla\xi_\epsilon^j}{(1+2\alpha|f(w_n)|^{2(2\alpha-1)})^{\frac12}}\,dx\Big)\Big] = 0,\\
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[b\Big(\int_{\mathbb{R}^N}\frac{|\nabla z_n|^2}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\,dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}\frac{f(z_n)\nabla z_n\nabla\xi_\epsilon^j}{(1+2\alpha|f(z_n)|^{2(2\alpha-1)})^{\frac12}}\,dx\Big)\Big] = 0.
\end{aligned}
\tag{3.12}
$$

One can see easily prove that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s\xi_\epsilon^j\,dx\\
&\le \lim_{\epsilon\to 0}\lim_{n\to\infty}\Big(\int_{B(x_j,\epsilon)}|g(x)\xi_\epsilon^j|^{\frac{2\alpha 2^*-(r+s)}{2\alpha 2^*}}\Big)^{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\Big(\int_{B(x_j,\epsilon)}|f(w_n)|^{2\alpha 2^*}\Big)^{\frac{r}{2\alpha 2^*}}\\
&\quad\times\Big(\int_{B(x_j,\epsilon)}|f(z_n)|^{2\alpha 2^*}\Big)^{\frac{s}{2\alpha 2^*}}\\
&\le K\lim_{\epsilon\to 0}\Big[\Big(\int_{B(x_j,\epsilon)}|w|^{2^*}\Big)^{1/2^*}\Big]^{\frac{r}{2\alpha}}\lim_{\epsilon\to 0}\Big[\Big(\int_{B(x_j,\epsilon)}|z|^{2^*}\Big)^{1/2^*}\Big]^{\frac{s}{2\alpha}}=0.
\end{aligned}\tag{3.13}$$

Also we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\xi_\epsilon^j dx&=\int_{\mathbb{R}^N}\xi_\epsilon^j d\omega\\
&\ge\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w)|^2+|\nabla f^{2\alpha}(z)|^2)\xi_\epsilon^j dx+\omega_j,
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big[\Big(\int_{\mathbb{R}^N}\frac{|f(w_n)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dy\Big)|f(w_n)|^{2\alpha 2^*_\mu}+\Big(\int_{\mathbb{R}^N}\frac{|f(z_n)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}d\Big)|f(z_n)|^{2\alpha 2^*_\mu}\Big]\xi_\epsilon^j\,dx\\
&=\int_{\mathbb{R}^N}\xi_\epsilon^j\,d\zeta\\
&\ge\int_{\mathbb{R}^N}\Big[\Big(\int_{\mathbb{R}^N}\frac{|f(w)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dy\Big)|f(w)|^{2\alpha 2^*_\mu}+\Big(\int_{\mathbb{R}^N}\frac{|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}dy\Big)|f(z)|^{2\alpha 2^*_\mu}\Big]\xi_\epsilon^j dx+\zeta_j.
\end{aligned}\tag{3.15}$$

Concentration at x_j,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|f(w_n)|^{4\alpha}}{|x|^2}\xi_\epsilon^j\,dx\le\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{B}_\epsilon(x_j)}\frac{|f(w_n)|^{4\alpha}}{|x_j-\epsilon|^2}\xi_\epsilon^j\,dx=0.$$

Similarly

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|f(z_n)|^{4\alpha}}{|x|^2}\xi_\epsilon^j\,dx\le\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{B}_\epsilon(x_j)}\frac{|f(z_n)|^{4\alpha}}{|x_j-\epsilon|^2}\xi_\epsilon^j\,dx=0.\tag{3.16}$$

Substitute (3.10)-(3.16) into (3.9), using relation $|\nabla f^{2\alpha}(w)|^2\le 2\alpha|\nabla w|^2$, we have

$$\begin{aligned}
0&=\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|^2\xi_\epsilon^j dx+b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\Big)^{\theta-1}\\
&\quad\times\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\xi_\epsilon^j\,dx\Big)a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla z_n|^2\xi_\epsilon^j dx\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\xi_\epsilon^j\,dx\Big)\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu}+|f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\Big)\xi_\epsilon^j\,dy\,dx\Big]\\
&\ge\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big\{a2\alpha\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\xi_\epsilon^j\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu}+|f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu})\xi_\epsilon^j}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\ge\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big\{a\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\xi_\epsilon^j\,dx\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu}\xi_\epsilon^j}{|x-y|^\mu}\,dx\,dy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu}\xi_\epsilon^j}{|x-y|^\mu}\,dx\,dy\Big\}
\end{aligned}$$

$$\geq \lim_{\epsilon\to 0}\Big\{a\int_{\mathbb{R}^N}\xi_\epsilon^j d\omega - \int_{\mathbb{R}^N}\xi_\epsilon^j d\zeta\Big\}.$$

Using standard theory of Radon measures, we conclude that $\zeta_j \geq a\omega_j$, combining with (3.8), we obtain either $\zeta_j \geq (aS_\mu)^{\frac{2_\mu^*}{2_\mu^*-1}}$ or $\zeta_j = 0$. Next, we show that the initial situation cannot happen. If not, there is a $j\in J$ such that $\zeta_j \geq (aS_\mu)^{\frac{2_\mu^*}{2_\mu^*-1}}$. Then Hölder inequality, (2.2) and the Young's inequality give that

$$\begin{aligned}
&\gamma\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s\,dx\\
&\leq \gamma\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}} S^{-\frac{r+s}{4\alpha}}(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)^{\frac{r+s}{4\alpha}}\\
&\leq \Big(\Big[\Big(\frac12-\frac{1}{22_\mu^*}\Big)\frac{a}{2\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)^{-1}\Big]^{\frac{r+s}{4\alpha}}(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)^{\frac{r+s}{4\alpha}}\Big)\\
&\quad\times\Big(\Big[\Big(\frac12-\frac{1}{22_\mu^*}\Big)\frac{a}{2\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)^{-1}\Big]^{\frac{-(r+s)}{4\alpha}}\gamma\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}} S^{-\frac{r+s}{4\alpha}}\Big)\\
&\leq \Big(\frac12-\frac{1}{22_\mu^*}\Big)\frac{a}{2\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)^{-1}(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)\\
&\quad+\frac{4\alpha-(r+s)}{4\alpha}\Big[\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\frac{2\alpha}{aS}\Big(1-\frac{4\lambda}{(N-2)^2}\Big)^{-1}\Big[\frac12-\frac{1}{22_\mu^*}\Big]^{-1}\Big]^{\frac{r+s}{4\alpha-(r+s)}}\\
&\quad\times\gamma^{\frac{4\alpha}{4\alpha-(r+s)}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}^{\frac{4\alpha}{4\alpha-(r+s)}}.
\end{aligned}\tag{3.17}$$

Now using $\eta=\max(\eta_1,\eta_2)$, relation $|\nabla f^{2\alpha}(w)|^2\leq 2\alpha|\nabla w|^2$, (1.5), and (3.17), we have

$$\begin{aligned}
0>c&=\lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n,z_n)-\frac{1}{4\alpha 2_\mu^*}\langle\mathcal{I}'_\gamma(w_n,z_n),(\hat w_n,\hat z_n)\rangle\Big)\\
&=\lim_{n\to\infty}\Big\{a\int_{\mathbb{R}^N}\Big[\frac12-\frac{1}{4\alpha 2_\mu^*}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla w_n|^2\,dx\\
&\quad+\Big(\frac{1}{2\theta}-\frac{1}{4\alpha 2_\mu^*}\Big)b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\,dx\Big)^\theta\\
&\quad+a\int_{\mathbb{R}^N}\Big[\frac12-\frac{1}{4\alpha 2_\mu^*}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla z_n|^2\,dx\\
&\quad+\Big(\frac{1}{2\theta}-\frac{1}{4\alpha 2_\mu^*}\Big)b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\,dx\Big)^\theta\\
&\quad-\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\gamma\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s\,dx\\
&\quad-\Big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2_\mu^*}\Big)a\eta\int_{\mathbb{R}^N}\Big(\frac{|f^{2\alpha}(w_n)|^2+|f^{2\alpha}(z_n)|^2}{|x|^2}dx\Big)\Big\}\\
&\geq\lim_{n\to\infty}\Big\{\Big(\frac{a}{4\alpha}-\frac{a}{4\alpha 2_\mu^*}\Big)\Big(1-\frac{4\eta}{(N-2)^2}\Big)\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\,dx\\
&\quad-\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\gamma\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s\,dx\Big\}\\
&\geq\Big(\frac{a}{4\alpha}-\frac{a}{4\alpha 2_\mu^*}\Big)\Big(1-\frac{4\eta}{(N-2)^2}\Big)\Big(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2+\sum_{j\in J}\omega_j\Big)\\
&\quad-\Big(\frac{\gamma}{r+s}-\frac{\gamma}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N}g(x)|f(w)|^r|f(z)|^s\,dx
\end{aligned}$$

$$
\begin{aligned}
&\geq a\Big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2^*_\mu}\Big)\Big(1-\frac{4\eta}{(N-2)^2}\Big)\omega_{j_0}-\frac{4\alpha-(r+s)}{4\alpha}\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2^*_\mu}\Big)^{\frac{4\alpha}{4\alpha-(r+s)}}\\
&\quad\times \gamma^{\frac{4\alpha}{4\alpha-(r+s)}}\|g\|^{\frac{4\alpha}{4\alpha-(r+s)}}_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\\
&\quad\Big[\frac{2\alpha}{aS}\Big(1-\frac{4\eta}{(N-2)^2}\Big)^{-1}\Big(\frac{1}{2}-\frac{1}{22^*_\mu}\Big)^{-1}\Big]^{\frac{r+s}{4\alpha-(r+s)}}\\
&\geq a\Big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2^*_\mu}\Big)\Big(1-\frac{4\eta}{(N-2)^2}\Big)S_\mu\zeta_{j_0}^{1/2^*_\mu}-\frac{4\alpha-(r+s)}{4\alpha}\Big(\frac{1}{r+s}-\frac{1}{4\alpha 2^*_\mu}\Big)^{\frac{4\alpha}{4\alpha-(r+s)}}\\
&\quad\times \gamma^{\frac{4\alpha}{4\alpha-(r+s)}}\|g\|^{\frac{4\alpha}{4\alpha-(r+s)}}_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\Big[\frac{2\alpha}{aS}\Big(1-\frac{4\eta}{(N-2)^2}\Big)^{-1}\Big(\frac{1}{2}-\frac{1}{22^*_\mu}\Big)^{-1}\Big]^{\frac{r+s}{4\alpha-(r+s)}}.
\end{aligned}\tag{3.18}
$$

Choose $\gamma_1>0$ small such that for any $\gamma\in(0,\gamma_1)$, the right-hand side of (3.18) is greater than zero, which gives a contradiction. So $\zeta_j=0$ for all $j\in J$ which implies $\omega_j=0$ fo all $j\in J$.

Now, we define a cut-off function $\xi^0_\epsilon(x)\in C_0^\infty(\mathbb{R}^N)$ to evaluate the potential mass concentration at the origin. $0\leq\xi^0_\epsilon(x)\leq 1$, $\xi^0_\epsilon(x)=1$ in $B(x_0,\frac{\epsilon}{2})$, $\xi^0_\epsilon(x)=0$ in $\mathbb{R}^N\setminus B(x_0,\epsilon)$, $|\nabla\xi^0_\epsilon|\leq\frac{4}{\epsilon}$. For each $\epsilon>0$ small enough such that $x_j\notin B_\epsilon(0)$ for all $j\in J$.

Concentration at the origin,

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big[\frac{|f(w_n)|^{4\alpha}}{|x|^2}+\frac{|f(z_n)|^{4\alpha}}{|x|^2}\Big]\xi^0_\epsilon dx\\
&=\int_{\mathbb{R}^N}\xi^0_\epsilon d\nu\geq\int_{\mathbb{R}^N}\Big[\frac{|f(w)|^{4\alpha}}{|x|^2}+\frac{|f(z)|^{4\alpha}}{|x|^2}\Big]\xi^0_\epsilon dx+\nu_0.
\end{aligned}\tag{3.19}
$$

Set $\|(\hat{w}_n,\hat{z}_n)\|\leq 2\alpha\|(w_n,z_n)\|$, then we obtain that $\{(\hat{w}_n,\hat{z}_n)\}$ is bounded in $\mathcal{D}$. Replacing ξ^j_ϵ with ξ^0_ϵ in (3.9) and substitute the (3.10)-(3.15), (3.19) in (3.9) and relation $|\nabla f^{2\alpha}(w)|^2\leq 2\alpha|\nabla w|^2$, $\eta=max(\eta_1,\eta_2)$. Testing $\mathcal{I}'_\gamma(w_n,z_n)$ with $(\hat{w}_n\xi^0_\epsilon,\hat{z}_n\xi^0_\epsilon)$, we obtain $\langle\mathcal{I}'_\gamma(w_n,z_n),(\hat{w}_n\xi^0_\epsilon,\hat{z}_n\xi^0_\epsilon)\rangle\to 0$ as $n\to\infty$, that is

$$
\begin{aligned}
0&=\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|^2\xi^0_\epsilon\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\xi^0_\epsilon\,dx\Big)\\
&\quad\times a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla z_n|^2\xi^0_\epsilon dx\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\xi^0_\epsilon\,dx\Big)-a\eta\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{4\alpha}+|f(z_n)|^{4\alpha})\xi^0_\epsilon}{|x|^2}dx\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu}+|f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\Big)\xi^0_\epsilon\,dy\,dx\Big]\\
&\geq\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big\{a2\alpha\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\xi^0_\epsilon\,dx-a\eta\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{4\alpha}+|f(z_n)|^{4\alpha})\xi^0_\epsilon}{|x|^2}dx\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu}+|f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu})\xi^0_\epsilon}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\geq\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big\{a\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\xi^0_\epsilon\,dx-a\eta\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{4\alpha}+|f(z_n)|^{4\alpha})\xi^0_\epsilon}{|x|^2}dx\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu}+|f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu})\xi^0_\epsilon}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\geq\lim_{\epsilon\to 0}\Big\{a\int_{\mathbb{R}^N}\xi^0_\epsilon d\omega-a\eta\int_{\mathbb{R}^N}\xi^0_\epsilon d\nu-\int_{\mathbb{R}^N}\xi^0_\epsilon d\zeta\Big\}.
\end{aligned}
$$

Using standard theory of Radon measures, we observe that $\zeta_0\geq a(\omega_0-\eta\nu_0)$. Combining this with (3.8), we obtain either $\zeta_0\geq(aS_{\eta,\mu})^{\frac{2^*_\mu}{2^*_\mu-1}}$ or $\zeta_0=0$. Now, we claim that the initial situation is not possible. If not, then we have $\zeta_0 \geq (aS_{\eta,\mu})^{\frac{2^*_\mu}{2^*_\mu-1}}$. Using $\eta = max(\eta_1, \eta_2)$, relation $|\nabla f^{2\alpha}(w)|^2 \leq 2\alpha|\nabla w|^2$, (1.5), and (3.17), we deduce

$$
\begin{aligned}
0 > c &= \lim_{n\to\infty} \Big(\mathcal{I}_\gamma(w_n, z_n) - \frac{1}{4\alpha 2^*_\mu}\langle \mathcal{I}'_\gamma(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle\Big) \\
&\geq \lim_{n\to\infty} \Big\{a\Big(\frac{1}{4\alpha} - \frac{1}{4\alpha 2^*_\mu}\Big)\int_{\mathbb{R}^N} (|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(z_n)|^2)\,dx \\
&\quad - \Big(\frac{1}{r+s} - \frac{1}{4\alpha 2^*_\mu}\Big)\gamma\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s\,dx \\
&\quad - \Big(\frac{a\eta}{4\alpha} - \frac{a\eta}{4\alpha 2^*_\mu}\Big)\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^2 + |f^{2\alpha}(z_n)|^2}{|x|^2}\,dx\Big\} \\
&\geq \Big(\frac{a}{4\alpha} - \frac{a}{4\alpha 2^*_\mu}\Big)\big(\|f^{2\alpha}(w)\|^2 + \|f^{2\alpha}(z)\|^2 + \omega_0\big) - \Big(\frac{\gamma}{r+s} - \frac{\gamma}{4\alpha 2^*_\mu}\Big)\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s dx \\
&\quad - \Big(\frac{a\eta}{4\alpha} - \frac{a\eta}{4\alpha 2^*_\mu}\Big)\int_{\mathbb{R}^N}\Big(\frac{|f^{2\alpha}(w)|^2 + |f^{2\alpha}(z)|^2}{|x|^2}\Big)\,dx - \Big(\frac{a\eta}{4\alpha} - \frac{a\eta}{4\alpha 2^*_\mu}\Big)\nu_0 \\
&\geq \Big(\frac{1}{4\alpha} - \frac{1}{4\alpha 2^*_\mu}\Big)aS_{\eta,\mu}\zeta_0^{1/2^*_\mu} - \frac{4\alpha-(r+s)}{4\alpha}\Big(\frac{1}{r+s} - \frac{1}{4\alpha 2^*_\mu}\Big)^{\frac{4\alpha}{4\alpha-(r+s)}}\gamma^{\frac{4\alpha}{4\alpha-(r+s)}}\|g\|^{\frac{4\alpha}{4\alpha-(r+s)}}_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}} \\
&\quad \Big[\frac{2\alpha}{aS}\Big(1 - \frac{4\eta}{(N-2)^2}\Big)^{-1}\Big(\frac{1}{2} - \frac{1}{22^*_\mu}\Big)^{-1}\Big]^{\frac{r+s}{4\alpha-(r+s)}} \\
&\geq \Big(\frac{1}{4\alpha} - \frac{1}{4\alpha 2^*_\mu}\Big)\Big(aS_{\eta,\mu}\Big)^{\frac{2^*_\mu}{2^*_\mu-1}} - \frac{4\alpha-(r+s)}{4\alpha}\Big(\frac{1}{r+s} - \frac{1}{4\alpha 2^*_\mu}\Big)^{\frac{4\alpha}{4\alpha-(r+s)}}\gamma^{\frac{4\alpha}{4\alpha-(r+s)}}\|g\|^{\frac{4\alpha}{4\alpha-(r+s)}}_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}} \\
&\quad \Big[\frac{2\alpha}{aS}\Big(1 - \frac{4\eta}{(N-2)^2}\Big)^{-1}\Big(\frac{1}{2} - \frac{1}{22^*_\mu}\Big)^{-1}\Big]^{\frac{r+s}{4\alpha-(r+s)}}.
\end{aligned}
\tag{3.20}
$$

Choose $\gamma_2 > 0$ be so small such that for any $\gamma \in (0, \gamma_2)$, the right-hand side of (3.20) is greater than zero, which gives a contradiction. Thus, $\zeta_0 = 0$ which implies $\omega_0 - \eta\nu_0 = 0$

We now introduce a cut-off function $\xi_R(x) \in C_0^\infty(\mathbb{R}^N)$ to determine the potential mass concentration at infinity. Moreover, $0 \leq \xi_R(x) \leq 1$, $\xi_R(x) = 1$ in $B(x, R)$, $\xi_R(x) = 0$ in $\mathbb{R}^N \setminus B(x, R)$, $|\nabla\xi_R| \leq \frac{4}{\epsilon}$. Let

$$
\begin{gathered}
\omega_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R} (|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(z_n)|^2)\,dx, \\
\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geq R} \frac{|f^{2\alpha}(w_n)|^2 + |f^{2\alpha}(z_n)|^2}{|x|^2}dx, \\
\zeta_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\Big[\int_{|x|\geq R}\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(w_n)|^{2^*_\mu}\,dx \\
+ \int_{|x|\geq R}\Big(\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(z_n)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|f^{2\alpha}(z_n)|^{2^*_\mu}\,dx\Big],
\end{gathered}
$$

such that

$$
S_{\eta,\mu}\zeta_\infty^{1/2^*_\mu} \leq (\omega_\infty - \eta\nu_\infty). \tag{3.21}
$$

Using that $\langle \mathcal{I}_\gamma'(w_n,z_n),(\hat{w}_n\xi_R,\hat{z}_n\xi_R\rangle \to 0$ as $n\to\infty$, we have

$$
\begin{aligned}
&-\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(\frac{f(w_n)}{f'(w_n)}\nabla w_n+\frac{f(z_n)}{f'(z_n)}\nabla z_n\Big)\nabla\xi_R\,dx\\
&+b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\Big)^{\theta-1}\int_{\mathbb{R}^N}f'(w_n)f(w_n)\nabla w_n\nabla\xi_R\,dx\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\Big)^{\theta-1}\int_{\mathbb{R}^N}f'(z_n)f(z_n)\nabla z_n\nabla\xi_R\,dx\Big]\\
&=\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|^2\xi_R\,dx\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\xi_R\,dx\Big)\\
&\quad+a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla z_n|^2\xi_R\,dx\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\xi_R\,dx\Big)\\
&\quad-\gamma\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s\xi_R\,dx-a\eta\int_{\mathbb{R}^N}\frac{|f(w_n)|^{4\alpha}}{|x|^2}\xi_R\,dx-a\eta\int_{\mathbb{R}^N}\frac{|f(z_n)|^{4\alpha}}{|x|^2}\xi_R dx\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{(|f(w_n)|^{2\alpha2_\mu^*}|f(w_n)|^{2\alpha2_\mu^*}+|f(z_n)|^{2\alpha2_\mu^*}|f(z_n)|^{2\alpha2_\mu^*})\xi_R}{|x-y|^\mu}\Big)dy\,dx\Big].
\end{aligned}\tag{3.22}
$$

One can see easily verify that

$$
\begin{aligned}
&\lim_{R\to\infty}\lim_{n\to\infty}\Big|a\int_{\mathbb{R}^N}(1+2\alpha|f(w_n)|^{2(2\alpha-1)})^{1/2}f(w_n)\nabla w_n\nabla\xi_R\,dx\Big|=0,\\
&\lim_{R\to\infty}\lim_{n\to\infty}\Big|a\int_{\mathbb{R}^N}(1+2\alpha|f(z_n)|^{2(2\alpha-1)})^{1/2}f(z_n)\nabla z_n\nabla\xi_R\,dx\Big|=0,
\end{aligned}\tag{3.23}
$$

$$
\begin{aligned}
&\lim_{R\to 0}\lim_{n\to\infty}\Big[b\Big(\int_{\mathbb{R}^N}\frac{|\nabla w_n|^2}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\,dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}\frac{f(w_n)\nabla w_n\nabla\xi_R}{(1+2\alpha|f(w_n)|^{2(2\alpha-1)})^{1/2}}\,dx\Big)\Big]=0,\\
&\lim_{R\to 0}\lim_{n\to\infty}\Big[b\Big(\int_{\mathbb{R}^N}\frac{|\nabla z_n|^2}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\,dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}\frac{f(z_n)\nabla z_n\nabla\xi_R}{(1+2\alpha|f(z_n)|^{2(2\alpha-1)})^{1/2}}\,dx\Big)\Big]=0.
\end{aligned}\tag{3.24}
$$

and

$$
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}h(x)|f(w_n)|^r|f(z_n)|^s\xi_R\,dx=0.\tag{3.25}
$$

Using the above relation together with $|\nabla f^{2\alpha}(w)|^2\le 2\alpha|\nabla w|^2$, we deduce that

$$
\begin{aligned}
0&=\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big[a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)|\nabla w_n|^2\xi_R\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\xi_R\,dx\Big)\\
&\quad\times a\int_{\mathbb{R}^N}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)|\nabla z_n|^2\xi_R dx\\
&\quad+b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2dx\Big)^{\theta-1}\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\xi_R\,dx\Big)\\
&\quad-a\eta\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{4\alpha}+|f(z_n)|^{4\alpha})\xi_R}{|x|^2}dx\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big(\frac{|f(w_n)|^{2\alpha2_\mu^*}|f(w_n)|^{2\alpha2_\mu^*}+|f(z_n)|^{2\alpha2_\mu^*}|f(z_n)|^{2\alpha2_\mu^*}}{|x-y|^\mu}\Big)\xi_R\,dy\,dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \lim_{\epsilon\to 0}\lim_{n\to\infty}\Big\{a2\alpha\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla z_n|^2)\xi_R\,dx - a\eta\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{4\alpha}+|f(z_n)|^{4\alpha})\xi_R}{|x|^2}dx\\
&\quad -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{2\alpha 2_\mu^*}|f(w_n)|^{2\alpha 2_\mu^*}+|f(z_n)|^{2\alpha 2_\mu^*}|f(z_n)|^{2\alpha 2_\mu^*})\xi_R}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\geq \lim_{\epsilon\to 0}\lim_{n\to\infty}\Big\{a\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\xi_R\,dx - a\eta\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{4\alpha}+|f(z_n)|^{4\alpha})\xi_R}{|x|^2}dx\\
&\quad -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{2\alpha 2_\mu^*}|f(w_n)|^{2\alpha 2_\mu^*}+|f(z_n)|^{2\alpha 2_\mu^*}|f(z_n)|^{2\alpha 2_\mu^*})\xi_R}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\geq \lim_{\epsilon\to 0}\Big\{a\int_{\mathbb{R}^N}\xi_R\,d\omega - a\eta\int_{\mathbb{R}^N}\xi_R d\nu - \int_{\mathbb{R}^N}\xi_R\,d\zeta\Big\}\\
&\geq a\omega_\infty - a\eta\nu_\infty - \zeta_\infty.
\end{aligned}
$$

Therefore $\zeta_\infty \geq a(\omega_\infty - \eta\nu_\infty)$. Combining this with (3.21), we obtain either $\zeta_\infty \geq (aS_{\eta,\mu})^{\frac{2_\mu^*}{2_\mu^*-1}}$ or $\zeta_\infty = 0$. We assert that the initial scenario is impossible. If not, there is a $\zeta_\infty \geq (aS_{\eta,\mu})^{\frac{2_\mu^*}{2_\mu^*-1}}$. In fact, using relation $|\nabla f^{2\alpha}(w)|^2 \leq 2\alpha|\nabla w|^2$, $\eta = max(\eta_1,\eta_2)$, (1.5), and (3.17), we obtain

$$
\begin{aligned}
0 > c &= \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n,z_n) - \frac{1}{4\alpha 2_\mu^*}\langle \mathcal{I}_\gamma'(w_n,z_n),(\hat{w}_n,\hat{z}_n)\rangle\Big)\\
&= \lim_{n\to\infty}\Big\{a\int_{\mathbb{R}^N}\Big[\frac12 - \frac{1}{4\alpha 2_\mu^*}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla w_n|^2dx\\
&\quad + \Big(\frac{1}{2\theta}-\frac{1}{4\alpha 2_\mu^*}\Big)b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2dx\Big)^\theta\\
&\quad + a\int_{\mathbb{R}^N}\Big[\frac12 - \frac{1}{4\alpha 2_\mu^*}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla z_n|^2dx\\
&\quad + \Big(\frac{1}{2\theta}-\frac{1}{4\alpha 2_\mu^*}\Big)b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2dx\Big)^\theta\\
&\quad + \Big(\frac{1}{4\alpha 2_\mu^*}-\frac{1}{r+s}\Big)\gamma\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s\,dx\\
&\quad - \Big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2_\mu^*}\Big)a\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w_n)|^2+|f^{2\alpha}(z_n)|^2}{|x|^2}dx\Big\}\\
&\geq \lim_{n\to\infty}\Big\{a\Big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2_\mu^*}\Big)\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\,dx\\
&\quad - \Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\gamma\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s\,dx\\
&\quad - \Big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2_\mu^*}\Big)a\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w_n)|^2+|f^{2\alpha}(z_n)|^2}{|x|^2}\,dx\Big\}\\
&\geq \Big\{\frac{a}{2\alpha}\big(\frac12-\frac{1}{22_\mu^*}\big)\big(1-\frac{4\eta}{(N-2)^2}\big)(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2) + \big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2_\mu^*}\big)a(\omega_\infty-\eta\nu_\infty)\\
&\quad - \Big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\Big)\gamma\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s dx\Big\}\\
&\geq \big(\frac{1}{4\alpha}-\frac{1}{4\alpha 2_\mu^*}\big)\big(aS_{\eta,\mu}\big)^{\frac{2_\mu^*}{2_\mu^*-1}} - \frac{4\alpha-(r+s)}{4\alpha}\big(\frac{1}{r+s}-\frac{1}{4\alpha 2_\mu^*}\big)^{\frac{4\alpha}{4\alpha-(r+s)}}\gamma^{\frac{4\alpha}{4\alpha-(r+s)}}\|g\|^{\frac{4\alpha}{4\alpha-(r+s)}}_{\frac{2\alpha 2^*}{2\alpha 2^*-(r+s)}}\\
&\quad \Big[\frac{2\alpha}{aS}\big(1-\frac{4\eta}{(N-2)^2}\big)^{-1}\big(\frac12-\frac{1}{22_\mu^*}\big)^{-1}\Big]^{\frac{r+s}{4\alpha-(r+s)}}.
\end{aligned}
\tag{3.26}
$$

Choose $\gamma_3 > 0$ to be so small such that for any $\gamma \in (0, \gamma_3)$, the right-hand side of (3.26) is greater than zero, which gives a contradiction. Therefore, $\zeta_\infty = 0$ which implies $\omega_\infty - \eta\nu_\infty$, in the above argument. So, put $\gamma^* = min\{\gamma_1, \gamma_2, \gamma_3\}$. Then for any $c < 0$ and $\gamma^* > 0$ we have $\omega_j = 0$ for all $j \in J$, $\omega_0 - \eta\nu_0 = 0$ and $\omega_\infty - \eta\nu_\infty = 0$, for all $\gamma \in (0, \gamma^*)$. Hence, taking as $n \to \infty$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w_n)|^{2^*_\mu}|f^{2\alpha}(w_n)|^{2^*_\mu} + |f^{2\alpha}(z_n)|^{2^*_\mu}|f^{2\alpha}(z_n)|^{2^*_\mu}}{|x-y|^\mu}\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f^{2\alpha}(w)|^{2^*_\mu}|f^{2\alpha}(w)|^{2^*_\mu} + |f^{2\alpha}(z)|^{2^*_\mu}|f^{2\alpha}(z)|^{2^*_\mu}}{|x-y|^\mu},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(x)(|f(w_n)|^r - |f(w)|^r)(|f(z_n)|^s - |f(z)|^s)\, dx\\
&\le \|g\|_{\frac{2\alpha 2^*}{2\alpha 2^* - (r+s)}} \||f(w_n)|^r - |f(w)|^r\|_{\frac{2\alpha 2^*}{r}}\\
&\quad \||f(z_n)|^s - |f(z)|^s\|_{\frac{2\alpha 2^*}{s}}.
\end{aligned}$$

Since $\{(w_n, z_n)\}$ is bounded in $\mathcal{D}$, $\langle \mathcal{I}'_\gamma(w, z), (\hat{w}, \hat{z})\rangle = 0$, using the mean value theorem and Lemma 2.2(11), yields $|f^{2\alpha}(w_n) - f^{2\alpha}(w)|^2 \le 2\alpha|w_n - w|^2$, weak lower semi-continuity of the norm, $\eta = max(\eta_1, \eta_2)$, (1.5), Lemma 3.7 and the Brezis-Lieb Lemma (see [5]) gives that as $n \to \infty$,

$$\begin{aligned}
&o(1)\|(w_n, z_n)\|\\
&= \langle \mathcal{I}'_\gamma(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle\\
&\ge a\int_{\mathbb{R}^N}(|\nabla w_n|^2 + |\nabla z_n|^2)\, dx + a\int_{\mathbb{R}^N}\Big(\frac{2\alpha(2\alpha-1)|f(w)|^{2(2\alpha-1)}}{1+2\alpha|f(w)|^{2(2\alpha-1)}}\Big)|\nabla w|^2\, dx\\
&\quad + b\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2\, dx\Big)^\theta + a\int_{\mathbb{R}^N}\Big(\frac{2\alpha(2\alpha-1)|f(z)|^{2(2\alpha-1)}}{1+2\alpha|f(z)|^{2(2\alpha-1)}}\Big)|\nabla z|^2\, dx\\
&\quad + b\Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2\, dx\Big)^\theta - a\eta\int_{\mathbb{R}^N}\frac{|f(w_n)|^{4\alpha} + |f(z_n)|^{4\alpha}}{|x|^2}\, dx\\
&\quad - \gamma\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^r\, dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu})}{|x-y|^\mu}\, dx\, dy\\
&\ge a\|(w_n, z_n)\|^2 - a\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w_n) - f^{2\alpha}(w)|^2 + |f^{2\alpha}(z_n) - f^{2\alpha}(z)|^2}{|x|^2}\, dx\\
&\quad + a\int_{\mathbb{R}^N}\Big[\Big(\frac{2\alpha(2\alpha-1)|f(w)|^{2(2\alpha-1)}}{1+2\alpha|f(w)|^{2(2\alpha-1)}}\Big)|\nabla w|^2 + \Big(\frac{2\alpha(2\alpha-1)|f(z)|^{2(2\alpha-1)}}{1+2\alpha|f(z)|^{2(2\alpha-1)}}\Big)|\nabla z|^2\Big]\, dx\\
&\quad + b\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2\, dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2\, dx\Big)^\theta\Big] - a\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w)|^2 + |f^{2\alpha}(z)|^2}{|x|^2}\, dx\\
&\quad - \gamma\int_{\mathbb{R}^N} h(x)|f(w)|^r|f(z)|^s dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu})}{|x-y|^\mu}\, dx\, dy\\
&\ge a\|(w_n - w, z_n - z)\|^2 + a\|(w, z)\|^2 - a2\alpha\eta\int_{\mathbb{R}^N}\frac{|w_n - w|^2 + |z_n - z|^2}{|x|^2}\, dx\\
&\quad + a\int_{\mathbb{R}^N}\Big[\Big(\frac{2\alpha(2\alpha-1)|f(w)|^{2(2\alpha-1)}}{1+2\alpha|f(w)|^{2(2\alpha-1)}}\Big)|\nabla w|^2 + \Big(\frac{2\alpha(2\alpha-1)|f(z)|^{2(2\alpha-1)}}{1+2\alpha|f(z)|^{2(2\alpha-1)}}\Big)|\nabla z|^2\Big]\, dx\\
&\quad + b\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\, dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2\, dx\Big)^\theta\Big] - a\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w)|^2 + |f^{2\alpha}(z)|^2}{|x|^2}\, dx\\
&\quad - \gamma\int_{\mathbb{R}^N} h(x)|f(w)|^r|f(z)|^s dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu})}{|x-y|^\mu}\, dx\, dy\\
&\ge a\Big(1 - \frac{8\alpha\eta}{(N-2)^2}\Big)\|(w_n - w, z_n - z)\|^2 + a\|(w, z)\|^2 - a\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w)|^2 + |f^{2\alpha}(z)|^2}{|x|^2}\, dx
\end{aligned}$$

$$
\begin{aligned}
&+ a\int_{\mathbb{R}^N}\Big[\Big(\frac{2\alpha(2\alpha-1)|f(w)|^{2(2\alpha-1)}}{1+2\alpha|f(w)|^{2(2\alpha-1)}}\Big)|\nabla w|^2 + \Big(\frac{2\alpha(2\alpha-1)|f(z)|^{2(2\alpha-1)}}{1+2\alpha|f(z)|^{2(2\alpha-1)}}\Big)|\nabla z|^2\Big]\,dx\\
&+ b\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2\,dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2\,dx\Big)^\theta\Big] - \gamma\int_{\mathbb{R}^N} h(x)|f(w)|^r|f(z)|^s dx\\
&- \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu})}{|x-y|^\mu}\,dx\,dy\\
=\ & a\Big(1-\frac{8\alpha\eta}{(N-2)^2}\Big)\|(w_n-w, z_n-z)\|^2 + o\|(w,z)\|.
\end{aligned}
$$

Thus $\{(w_n, z_n)\}$ converges strongly to (w, z) in $\mathcal{D}$ for all $\eta < \frac{(N-2)^2}{8\alpha}$. This completes the proof. $\square$

Lemma 3.9. *Suppose that $r + s = 4\alpha$ and (1.9) hold. Let $\{(w_n, z_n)\}$ be a $(PS)_c$ sequence for $\mathcal{I}_\gamma$ in $\mathcal{D}$ with $c < c^* := min\{c_1^*,\ c_2^*\}$, where*

$$
c_1^* := \frac{1}{8\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)(aS_\mu)^{\frac{2^*_\mu}{2^*_\mu-1}}\quad c_2^* := \frac{1}{8\alpha}(aS_{\eta,\mu})^{\frac{2^*_\mu}{2^*_\mu-1}}.
$$

Then, $\{(w_n, z_n)\}$ satisfies the $(PS)_c$ condition for all $\gamma \in \big(0, a(1-\frac{4\eta}{(N-2)^2})S\|g\|^{-1}_{\frac{2^*}{2^*-2}}\big)$.

Proof. Let $\{(w_n, z_n)\}$ be a $(PS)_c$ for $\mathcal{I}_\gamma$ for $c < c^*$. Then sequence $\{(w_n, z_n)\}$ is bounded from Lemma 3.5. Substitute $r + s = 4\alpha$ in equation (2.9), we obtain

$$
\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s dx \le S^{-1}\|g\|_{\frac{2^*}{2^*-2}}(\|f^{2\alpha}(w)\|^2 + \|f^{2\alpha}(z)\|^2). \tag{3.27}
$$

Now using the relation $|\nabla f^{2\alpha}(w)|^2 \le 2\alpha|\nabla w|^2$, (1.5), $\eta = \max\{\eta_1, \eta_2\}$, (3.27) and for all

$$
\gamma \in \Big(0, a\big(1-\frac{4\eta}{(N-2)^2}\big)S\|g\|^{-1}_{\frac{2^*}{2^*-2}}\Big),
$$

arguing similarly as in Lemma 3.8, we obtain

$$
\begin{aligned}
c^* > c &= \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n, z_n) - \frac{1}{8\alpha}\langle \mathcal{I}'_\gamma(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle\Big)\\
&= \lim_{n\to\infty}\Big\{a\int_{\mathbb{R}^N}\Big[\frac12 - \frac{1}{8\alpha}\Big(1+\frac{2\alpha(2\alpha-1)|f(w_n)|^{2(2\alpha-1)}}{1+2\alpha|f(w_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla w_n|^2\\
&\quad + \big(\frac{1}{2\theta}-\frac{1}{8\alpha}\big)b\Big(\int_{\mathbb{R}^N}|\nabla f(w_n)|^2\,dx\Big)^\theta\\
&\quad + a\int_{\mathbb{R}^N}\Big[\frac12 - \frac{1}{8\alpha}\Big(1+\frac{2\alpha(2\alpha-1)|f(z_n)|^{2(2\alpha-1)}}{1+2\alpha|f(z_n)|^{2(2\alpha-1)}}\Big)\Big]|\nabla z_n|^2\\
&\quad + \Big(\frac{1}{2\theta}-\frac{1}{8\alpha}\Big)b\Big(\int_{\mathbb{R}^N}|\nabla f(z_n)|^2dx\Big)^\theta - \Big(\frac{1}{4\alpha}-\frac{1}{8\alpha}\Big)a\eta\int_{\mathbb{R}^N}\frac{(|f^{2\alpha}(z_n)|^2 + f^{2\alpha}(z_n)|^2)}{|x|^2}\,dx\\
&\quad + \Big(\frac{1}{8\alpha}-\frac{1}{4\alpha}\Big)\gamma\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s\,dx\\
&\quad + \Big(\frac{1}{8\alpha}-\frac{1}{4\alpha 2^*_\mu}\Big)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w_n)|^{2\alpha 2^*_\mu}|f(w_n)|^{2\alpha 2^*_\mu} + |f(z_n)|^{2\alpha 2^*_\mu}|f(z_n)|^{2\alpha 2^*_\mu})}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\ge \lim_{n\to\infty}\Big\{\frac a4\int_{\mathbb{R}^N}(|\nabla w_n|^2 + |\nabla z_n|^2)\,dx - \frac{a\eta}{8\alpha}\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w_n)|^2 + |f^{2\alpha}(z_n)|^2}{|x|^2}dx\\
&\quad - \frac{\gamma}{8\alpha}\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s\,dx\Big\}\\
&\ge \lim_{n\to\infty}\Big\{\frac{a}{8\alpha}\big(1-\frac{4\eta}{(N-2)^2}\big)\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2 + |\nabla f^{2\alpha}(z_n)|^2)\,dx\\
&\quad - \frac{\gamma}{8\alpha}\int_{\mathbb{R}^N} g(x)|f(w_n)|^r|f(z_n)|^s\,dx\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \Big\{ \frac{a}{8\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)\Big((\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)+\sum_{j\in J}\omega_j\Big) \\
&\quad -\frac{\gamma}{8\alpha}S^{-1}\|g\|_{\frac{2^*}{2^*-2}}(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)\Big\} \\
&\geq \frac{a}{8\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)S_\mu\omega_{j_0}+\frac{1}{8\alpha}\Big[a\Big(1-\frac{4\eta}{(N-2)^2}\Big) \\
&\quad -\gamma S^{-1}\|g\|_{\frac{2^*}{2^*-2}}\Big](\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2) \\
&\geq \frac{a}{8\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)S_\mu\zeta_{j_0}^{1/2^*_\mu}+\frac{1}{8\alpha}\Big[a\Big(1-\frac{4\eta}{(N-2)^2}\Big) \\
&\quad -\gamma S^{-1}\|g\|_{\frac{2^*}{2^*-2}}\Big](\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2) \\
&\geq \frac{1}{8\alpha}\Big(1-\frac{4\eta}{(N-2)^2}\Big)(aS_\mu)^{\frac{2^*_\mu}{2^*_\mu-1}} := c_1^*.
\end{aligned}
$$

For level at the origin, using the same calculation as above,

$$
\begin{aligned}
c^* > c &= \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n,z_n)-\frac{1}{8\alpha}\langle\mathcal{I}'_\gamma(w_n,z_n),(\hat{w}_n,\hat{z}_n)\rangle\Big) \\
&\geq \lim_{n\to\infty}\Big\{\frac{a}{8\alpha}\int_{\mathbb{R}^N}(|\nabla f^{2\alpha}(w_n)|^2+|\nabla f^{2\alpha}(z_n)|^2)\,dx-\frac{a\eta}{8\alpha}\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w_n)|^2+|f^{2\alpha}(z_n)|^2}{|x|^2}dx \\
&\quad -\frac{\gamma}{8\alpha}\int_{\mathbb{R}^N}g(x)|f(w_n)|^r|f(z_n)|^s\,dx\Big\} \\
&\geq \Big\{\frac{a}{8\alpha}\Big((\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)+\omega_0-\eta\int_{\mathbb{R}^N}\frac{|f^{2\alpha}(w)|^2+|f^{2\alpha}(z)|^2}{|x|^2}dx-\eta\nu_0\Big) \\
&\quad -\frac{\gamma}{8\alpha}S^{-1}\|g\|_{\frac{2^*}{2^*-2}}(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)\Big\} \\
&\geq \Big\{\frac{a}{8\alpha}(1-\frac{4\eta}{(N-2)^2})(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2) \\
&\quad +\frac{a}{8\alpha}(\omega_0-\eta\nu_0)-\frac{\gamma}{8S\alpha}\|g\|_{\frac{2^*}{2^*-2}}(\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2)\Big\} \\
&\geq \frac{a}{8\alpha}S_{\eta,\mu}\zeta_0^{1/2^*_\mu}+\frac{1}{8\alpha}\Big[a\left(1-\frac{4\eta}{(N-2)^2}\right)-\gamma S^{-1}\|g\|_{\frac{2^*}{2^*-2}}\Big](\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2) \\
&\geq \frac{1}{8\alpha}\big(aS_{\eta,\mu}\big)^{\frac{2^*_\mu}{2^*_\mu-1}} := c_2^*.
\end{aligned}
$$

Similarly, level at infinity

$$
\begin{aligned}
c^* > c &= \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n,z_n)-\frac{1}{8\alpha}\langle\mathcal{I}'_\gamma(w_n,z_n),(\hat{w}_n,\hat{z}_n)\rangle\Big) \\
&\geq \frac{a}{8\alpha}S_{\eta,\mu}\zeta_\infty^{1/2^*_\mu}+\frac{1}{8\alpha}\Big[a\Big(1-\frac{4\eta}{(N-2)^2}\Big)-\gamma S^{-1}\|g\|_{\frac{2^*}{2^*-2}}\Big](\|f^{2\alpha}(w)\|^2+\|f^{2\alpha}(z)\|^2) \\
&\geq \frac{1}{8\alpha}(aS_{\eta,\mu})^{\frac{2^*_\mu}{2^*_\mu-1}} := c_2^*.
\end{aligned}
$$

Since $c < c^* := \min\{c_1^*, c_2^*\}$ which is absurd. Now, the rest of the proof follows in the same manner as the proof of Lemma 3.8. ☐

Lemma 3.10. *Suppose that* (1.9) *holds and* $4\alpha < r+s < 2\alpha 2^*$. *Let* $\{z_n\}$ *be a* $(PS)_c$ *sequence for* $\mathcal{I}_\gamma$ *in* $\mathcal{D}$ *with* $c < c^{**} := min\{c_1^{**},\ c_2^{**}\}$, *where*

$$
c_1^{**} := \Big(\frac{1}{4\alpha}-\frac{1}{r+s}\Big)\Big(1-\frac{4\eta}{(N-2)^2}\Big)\Big(aS_\mu\Big)^{\frac{2^*_\mu}{2^*_\mu-1}},\quad c_2^{**} := \Big(\frac{1}{4\alpha}-\frac{1}{r+s}\Big)\Big(aS_{\eta,\mu}\Big)^{\frac{2^*_\mu}{2^*_\mu-1}}.
$$

Then $\{(w_n,z_n)\}$ *satisfies* $(PS)_c$ *condition.*

Proof. Let $\{(w_n, z_n)\}$ be a $(PS)_c$ for $\mathcal{I}_\gamma$ for $c < c^{**}$. Then by Lemma 3.6, we have $\{(w_n, z_n)\}$ is bounded. following the same arguments as in Lemma 3.8, we obtain

$$\begin{aligned} c^{**} > c &= \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n, z_n) - \frac{1}{r+s}\langle \mathcal{I}'_\gamma(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle\Big) \\ &\geq \big(\frac{1}{2} - \frac{2\alpha}{r+s}\big)\frac{1}{2\alpha}\big(1 - \frac{4\eta}{(N-2)^2}\big)w_{j_0} \geq \big(\frac{1}{4\alpha} - \frac{1}{r+s}\big)\big(1 - \frac{4\eta}{(N-2)^2}\big)(aS_\mu)^{\frac{2^*_\mu}{2^*_\mu - 1}} := c_1^{**}. \end{aligned}$$

Level at the origin

$$c^{**} > c = \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n, z_n) - \frac{1}{r+s}\langle \mathcal{I}'_\gamma(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle\Big) \geq \Big(\frac{1}{4\alpha} - \frac{1}{r+s}\Big)(aS_{\eta,\mu})^{\frac{2^*_\mu}{2^*_\mu - 1}} := c_2^{**}.$$

Similarly, level at infinity

$$c^{**} > c = \lim_{n\to\infty}\Big(\mathcal{I}_\gamma(w_n, z_n) - \frac{1}{r+s}\langle \mathcal{I}'_\gamma(w_n, z_n), (\hat{w}_n, \hat{z}_n)\rangle\Big) \geq \Big(\frac{1}{4\alpha} - \frac{1}{r+s}\Big)(aS_{\eta,\mu})^{\frac{2^*_\mu}{2^*_\mu - 1}} := c_2^{**}.$$

Since $c < c^{**} := \min\{c_1^{**}, c_2^{**}\}$ which is a contradiction. The rest of the proof proceeds similarly to that of Lemma 3.8. ☐

4. Proof of main results

In this section, we give the proof of Theorems 1.2, 1.3 and 1.4, using the new version of symmetric mountain pass theorem see [16, 31]. Let X be a Banach space and Σ be the class of $X \setminus 0$ which are closed and symmetric with respect to origin. For $A \in \Sigma$, we define the genus $\gamma(A)$ by

$$\gamma(A) := \inf\{N \in \mathbb{N} :\ \text{there exist } \xi \in C(A, \mathbb{R}^n \setminus \{0\}, -\xi(x) = \xi(-x))\}.$$

If there is no mapping ξ as above for any $N \in \mathbb{N}$, then $\gamma(A) = +\infty$. Σ_n denote the family of closed, symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq n$.

The following Theorem is an extension of the symmetric mountain pass lemma due to Kajikiya (see [16, Theorem 1]).

Theorem 4.1. *Let $\mathcal{J} \in C^1(X, \mathbb{R})$ and X be an infinite-dimensional Banach space. Suppose that the following properties hold.*

- (i) *$\mathcal{J}$ is even and bounded from below, $\mathcal{J}(0) = 0$ and The functional $\mathcal{J}$ satisfies the local Palais-Smale condition.*
- (ii) *For each $n \in N$, there exists $B_n \in \Sigma_n$ such that $\sup_{u\in B_n} \mathcal{J}(u) < 0$ for each $n \in \mathbb{N}$.*

Then either one the following 2 items hold:

- (1) *There exist a sequence $\{u_n\}$ such that $\mathcal{J}'(u_n) = 0$, $\mathcal{J}(u_n) < 0$ and $u_n \to 0$*
- (2) *There exist two sequence $\{u_n\}$ and $\{v_n\}$ such that $\mathcal{J}'(u_n) = 0$, $\mathcal{J}(u_n) = 0$, $u_n \neq 0$, $u_n \to 0$, $\mathcal{J}'(v_n) = 0$, $\mathcal{J}(v_n) < 0$ and $\{v_n\}$ converge to non-zero limit.*

Proof of Theorem 1.2. Clearly, it follows that $\mathcal{I}_\gamma$ is even and $\mathcal{I}_\gamma(0) = 0$. Furthermore, Lemma 3.8 assures that $\mathcal{I}_\gamma$ satisfies the $(PS)_c$-condition for all $c < 0$. However, observe that $\mathcal{I}_\gamma$ is not bounded from below in $\mathcal{D}$. To apply Theorem 4.1, we use a truncation technique.

Let $(w, z) \in \mathcal{D}$. Using (2.8)-(2.10), $\eta = max(\eta_1, \eta_2)$ and (2.3), we obtain

$$\begin{aligned} &\mathcal{I}_\gamma(w, z) \\ &= \frac{a}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + |\nabla z|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2 dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2 dx\Big)^\theta\Big] \\ &\quad - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(w)|^{4\alpha}}{|x|^2}\,dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(z)|^{4\alpha}}{|x|^2}\,dx - \frac{\gamma}{r+s}\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s\,dx \\ &\quad - \frac{1}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dx\,dy \\ &\geq \frac{a}{2}\Big(1 - \frac{4\eta}{(N-2)^2}\Big)\|(w,z)\|^2 - \frac{\gamma}{r+s}S^{-\frac{r+s}{4\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^* - (r+s)}}\|(w,z)\|^{\frac{r+s}{2\alpha}} - \frac{1}{4\alpha 2^*_\mu}\big(\frac{2\alpha}{S_\mu}\big)^{2^*_\mu}\|(w,z)\|^{22^*_\mu} \end{aligned}$$

$$\geq K_1\|(w,z)\|^2 - \gamma K_2\|(w,z)\|^{\frac{r+s}{2\alpha}} - K_3\|(w,z)\|^{22^*_\mu} \quad (= \chi(\|(w,z)\|)).$$

where, $\chi : \mathbb{R}^+ \to \mathbb{R}$ is a function define as $\chi(t) = K_1t^2 - \gamma K_2 t^{\frac{r+s}{2\alpha}} - K_3t^{22^*_\mu}$, we choose γ^* sufficiently small such that for any $\gamma \in (0,\gamma^*)$ there exist $0 < t_1 < t_2$ such that $\chi(t) < 0$ for all $t \in (0,t_1)$, $\chi(t) > 0$ in $t \in (t_1,t_2)$ and $\chi(t) < 0$ in $t \in (t_2,\infty)$. Clearly, $\chi(t_1) = 0 = \chi(t_2)$. Subsequently, we choose a non-increasing function $\mathcal{Q} \in C^\infty([0,\infty),[0,1])$ such that

$$\mathcal{Q}(t) = \begin{cases} 1 & \text{if } t \in [0,t_1] \\ 0 & \text{if } t \in [t_2,\infty). \end{cases}$$

and set $\varphi(w,z) := \mathcal{Q}(\|(w,z)\|)$. Now, we can define the truncated functional $\hat{I}_\gamma : \mathcal{D} \to \mathbb{R}$ as

$$\begin{aligned} &\hat{I}_\gamma(w,z) \\ &:= \frac{a}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + |\nabla z|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2dx\Big)^\theta\Big] \\ &\quad - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(w)|^{4\alpha}}{|x|^2}dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(z)|^{4\alpha}}{|x|^2}dx - \varphi(w,z)\frac{\gamma}{r+s}\int_{\mathbb{R}^N}g(x)|f(w)|^r|f(z)|^s\,dx \\ &\quad - \varphi(w,z)\frac{1}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu} + |f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dx\,dy. \end{aligned} \tag{4.1}$$

Then it is easy to verify that $\hat{I}_\gamma(w,z)$ satisfies the following Lemma.

Lemma 4.2. *Let $c < 0$ and $2\alpha < r + s < 4\alpha$. Then*
- (i) *$\hat{I}_\gamma(w,z) \in C^1(X,\mathbb{R})$ and $\hat{I}_\gamma(w,z)$ bounded from below.*
- (ii) *If $\hat{I}_\gamma(w,z) < 0$, then $\|(w,z)\|_X \leq t_1$ and $\hat{I}_\gamma(w,z) = \mathcal{I}_\gamma(w,z)$.*
- (iii) *Let $c < 0$, then there exists γ^* such that for all $\gamma \in (0,\gamma^*)$, $\hat{I}_\gamma(w,z)$ satisfies the Palais-Smale condition.*

Proof. Clearly, (i) and (ii) are immediate. To demonstrate (iii), consider that all the $(PS)_c$ sequence for $\hat{I}_\gamma(w,z)$ with $c < 0$ must be bounded. Similar to the proof of Lemma 3.8, there exists a strong convergent subsequence in $\mathcal{D}$. ☐

For $n \in \mathbb{N}$, we consider n numbers of disjoint open sets denoted by G_j, $j = 1,2,\dots n$ with $\cup_{j=1}^n G_j \subset \Omega$, where $\Omega \neq \emptyset$ is given as in Theorem 1.2. Now we can choose $(w_j,z_j) \in D^{1,2}(\mathbb{R}^N) \cap C_0^\infty(G_j) \setminus \{0\} \times D^{1,2}(\mathbb{R}^N) \cap C_0^\infty(G_j) \setminus \{0\}$, with $\|(w_j,z_j)\| = 1$ for each $j = 1,2,\dots,n$. Set

$$Y_n = \operatorname{span}\{(w_1,z_1),(w_2,z_2),\dots,(w_n,z_n)\}.$$

We now assert that there exists a sufficiently small $0 < \varrho_n < t_1$ such that

$$m_n := \max\{\hat{I}_\gamma(w,z) : (w,z) \in Y_n,\ \|(w,z)\| = \varrho_n\} < 0. \tag{4.2}$$

Suppose that (4.2) does not hold. Then there exists a sequence $\{(w_k,z_k)\} := \{(w_k^n,z_k^n)\}$ in Y_n such that $k \to \infty$,

$$\|(w_k,z_k)\| \to \infty, \quad \text{and} \quad \hat{I}_\gamma(w_k,z_k) \geq 0. \tag{4.3}$$

Let us set $(u_k,v_k) = \frac{(w_k,z_k)}{\|(w_k.z_k)\|}$. Then $(u_k,v_k) \in \mathcal{D}$ and $\|(u_k,v_k)\| = 1$. Since Y_n is finite dimensional, there exists $(u,v) \in Y_n \setminus \{0\}$ such that $(u_k(x),v_k(x)) \to (u(x),v(x))$ strongly with respect to $\|\cdot\|$, and $(u_k(x),v_k(x)) \to (u(x),v(x))$ converges pointwise a.e. in $\mathbb{R}^N$.

As $(u,v) \not\equiv 0$, we obtain $w_k(x) \to \infty$, $z_k(x) \to \infty$ as $k \to \infty$. Thus,

$$\begin{aligned} &\frac{1}{\|(w_k,z_k)\|^{2\theta}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|w_k(x)|^{2^*_\mu}|w_k(y)|^{2^*_\mu} + |z_k(x)|^{2^*_\mu}|z_k(y)|^{2^*_\mu})}{|x-y|^\mu}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_k(x)|^{2^*_\mu-\theta}|w_k(y)|^{2^*_\mu-\theta}}{|x-y|^\mu}|u_k(x)|^\theta|u_k(y)|^\theta\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z_k(x)|^{2^*_\mu-\theta}|z_k(y)|^{2^*_\mu-\theta}}{|x-y|^\mu}|v_k(x)|^\theta|v_k(y)|^\theta\,dx\,dy \\ &\to \infty \quad \text{as } k \to \infty. \end{aligned}$$

Using this together with Lemma 2.2(3),(8), from (4.1), we obtain

$$
\begin{aligned}
\hat{I}_\gamma(w_k,z_k) &\le \frac{a}{2}(\|w_k\|^2+\|z_k\|^2)+\frac{b}{2\theta}(\|w_k\|^{2\theta}+\|z_k\|^{2\theta})\\
&\quad -\frac{(f(1))^{4\alpha 2^*_\mu}}{4\alpha 2^*_\mu}\int_{\mathbb{R}^{2N}}\frac{|w_k(x)|^{2^*_\mu}|w_k(y)|^{2^*_\mu}+|z_k(x)|^{2^*_\mu}|z_k(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\\
&\le \frac{a}{2}\|(w_k,z_k)\|^2+\frac{b}{2\theta}\|(w_k,z_k)\|^{2\theta}\\
&\quad -\frac{(f(1))^{4\alpha 2^*_\mu}}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_k(x)|^{2^*_\mu}|w_k(y)|^{2^*_\mu}+|z_k(x)|^{2^*_\mu}|z_k(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\\
&\le \|(w_k,z_k)\|^{2\theta}\Big(\frac{a}{2}\frac{1}{\|(w_k,z_k)\|^{2\theta-2}}+\frac{b}{2\theta}-\frac{(f(1))^{4\alpha 2^*_\mu}}{4\alpha 2^*_\mu}\frac{1}{\|(w_k,z_k)\|^{2\theta}}\\
&\quad \times\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)|^{2^*_\mu}|w_k(y)|^{2^*_\mu}+|z_k(x)|^{2^*_\mu}|z_k(y)|^{2^*_\mu}}{|x-y|^\mu}\,dx\,dy\Big)\\
&\to -\infty \quad \text{as } k\to\infty
\end{aligned}
$$

which contradicts (4.3). Consequently, the claim is proved. Now we can choose $B_n := \{(w,z)\in Y_n : \|(w,z)\| = \varrho_n\}$. Surely, $\gamma(A_n) = n$ and A_n is closed and symmetric, therefore $A_n \in \Sigma_n$. Furthermore, $\sup_{(w,z)\in A_n}\hat{I}_\gamma(w,z) < 0$ can be found in (4.2). Therefore, all the assumptions in Theorem 4.1 are satisfied by $\hat{I}_\gamma(w_n,z_n) \le 0$ for each $n\in\mathbb{N}$ and $\|(w_n,z_n)\| \to 0$ as $n\to\infty$, $\hat{I}_\gamma$ admits a sequence of critical points $\{(w_n,z_n)\}$ in $\mathcal{D}$ such that $(w_n,z_n)\neq 0$. Consequently, for any $n > n_0$, there exists $n_0 \in \mathbb{N}$ such that $\|z\| < t_1$ for every $n \ge n_0$. This means that $\hat{I}_\gamma(w_n,z_n) = \mathcal{I}_\gamma(w_n,z_n)$. This completes the proof of the theorem. □

The following is a symmetric version of the mountain pass theorem from [31] needed for proving Theorem 1.3.

Theorem 4.3. *Let X be an infinite-dimensional Banach space such that $X = M\oplus N$, where M is finite-dimensional. Let $\mathcal{I}\in C^1(X,\mathbb{R})$ be an even functional such that $\mathcal{I}(0) = 0$ and the subsequent conditions are satisfied:*

1. *there exist positive constants $\epsilon,\rho > 0$ such that $\mathcal{I}(u)\ge\rho$ for all $u\in\partial B_\epsilon(0)\cap N$;*
2. *there exists $c^* > 0$ such that $\mathcal{I}$ satisfies the $(PS)_c$ condition for $c\in(0,c^*)$;*
3. *for any finite dimensional subspace $\tilde{X}\subset X$, there is $R = R(\tilde{X}) > 0$ such that $\mathcal{I}(u)\le 0$ for all $u\in\tilde{X}\setminus B_R(0)$.*

Let $M = \operatorname{span}\{v_1,v_2,\dots,v_n\}$, where M is n-dimensional vector space. Define $R_k = R(M_k)$ and $D_k = B_{R_k}(0)\cap M_k$ and inductively select $v_{k+1}\notin M_k := \operatorname{span}\{v_1,v_2,\dots,v_k\}$ for $k\ge n$. Define

$$
T_k = \{g\in C(D_k,E) : g|_{\partial B_{R_k}(0)}, g \text{ is odd and } g(u) = u, \text{ for all } B_{R_k}(0)\cap M_k\},
$$

and

$$
\Upsilon_j = \{g(\overline{D_k\setminus A}) : h\in T_k, k\ge j, A \text{ is closed and symmetric, and } \gamma(A)\le k-j\}, \tag{4.4}
$$

where $\gamma(A)$ is Krasnoselskii's genus of A. For each $j\in\mathbb{N}$, set

$$
c_j := \inf_{U\in\Upsilon_j}\max_{u\in U}\mathcal{I}(u).
$$

Thus, $0 < c_j \le c_{j+1}$ for $j > n$ and if $j > n$ and $c_j < c^*$, then we conclude that c_j is the critical value of $\mathcal{I}$. Furthermore, if $c_j = c_{j+1} = \dots = c_{j+m} = c < c^*$ for $j > n$, then $\gamma(K_c)\ge m+1$, where

$$
K_c = \{u\in X : \mathcal{I}(u) = c \text{ and } \mathcal{I}'(u) = 0\}.
$$

Proof of Theorem 1.3. We shall apply Theorem 4.3 to $\mathcal{I}_\gamma$. We know that $\mathcal{D}$ is Banach space and $\mathcal{I}_\gamma\in C^1(\mathcal{D},\mathbb{R})$. By (2.7), functional $\mathcal{I}_\gamma$ is even and $\mathcal{I}_\gamma(0,0) = 0$. We divide the proof into the following five steps:

Step 1. We will show that $\mathcal{I}_\gamma$ satisfies hypothesis Theorem 4.3(1) for $(w,z) \in \mathcal{D}$. Using $\eta = \max(\eta_1,\eta_2)$, (2.8)-(2.10), and (2.3), we obtain

$$\begin{aligned}
\mathcal{I}_\gamma(w,z) &= \frac{a}{2}\int_{\mathbb{R}^N}(|\nabla w|^2+|\nabla z|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2dx\Big)^\theta\Big]\\
&\quad - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(w)|^{4\alpha}}{|x|^2}\,dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(z)|^{4\alpha}}{|x|^2}\,dx - \frac{\gamma}{r+s}\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s\,dx\\
&\quad - \frac{1}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(w)|^{2\alpha 2^*_\mu}|f(w)|^{2\alpha 2^*_\mu}+|f(z)|^{2\alpha 2^*_\mu}|f(z)|^{2\alpha 2^*_\mu}}{|x-y|^\mu}\,dx\,dy\\
&\geq \frac{\|(w,z)\|^2}{2}\Big(a - \frac{4a\eta}{(N-2)^2} - \gamma S^{-1}\|g\|_{\frac{2^*}{2^*-2}}\Big) - \frac{1}{4\alpha 2^*_\mu}\big(\frac{2\alpha}{S_\mu}\big)^{2^*_\mu}\|(w,z)\|^{22^*_\mu}.
\end{aligned}$$

Now for $\gamma < \big(a - \frac{4a\eta}{(N-2)^2}\big)S\|g\|^{-1}_{\frac{2^*}{2^*-2}}$, we can choose $\|(w,z)\| = \epsilon << 1$ such that $\mathcal{I}_\gamma(w,z) \geq \rho > 0$. So Theorem 4.3(1) is proved.

Step 2. This follows from Lemma 3.9. So Theorem 4.3(2) is proved.

Step 3. We will show that $\mathcal{I}_\gamma(w,z)$ satisfies hypothesis Theorem 4.3(3) any finite-dimensional subspace M of $\mathcal{D}$ there exists $R_0 = R_0(M)$ such that $\mathcal{I}_\gamma(w,z) < 0$ for all $(w,z) \in \mathcal{D}\setminus B_{R_0}(M)$, where $B_{R_0}(M) = \{(w,z)\in\mathcal{D} : \|(w,z)\| \leq R_0\}$. Fix $(\psi,\phi)\in\mathcal{D}$, $\|(\psi,\phi)\| = 1$. For $t > 1$, using Lemma 2.2(3), (8), we obtain

$$\begin{aligned}
\mathcal{I}_\gamma(t\psi,t\phi) &\leq \frac{a}{2}t^2\|(\psi,\phi)\|^2 + \frac{b}{2\theta}t^{2\theta}\|(\psi,\phi)\|^{2\theta}\\
&\quad - \frac{(f(1))^{4\alpha 2^*_\mu}}{4\alpha 2^*_\mu}t^{22^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|\psi(x)|^{2^*_\mu}|\psi(y)|^{2^*_\mu}+|\phi(x)|^{2^*_\mu}|\phi(y)|^{2^*_\mu})}{|x-y|^\mu}\,dx\,dy\\
&\leq \frac{a}{2\theta}\|(\psi,\phi)\|^{2\theta}t^2 + \frac{b}{2\theta}\|(\psi,\phi)\|^{2\theta}t^{2\theta} - \frac{(f(t))^{4\alpha 2^*_\mu}}{4\alpha 2^*_\mu}t^{22^*_\mu}\|(\psi,\phi)\|_\mu^{22^*_\mu}.
\end{aligned} \tag{4.5}$$

Since M is finite dimensional all norms are equivalent on M, which yields that there exists some constant $K(Y) > 0$ such that $K(Y)\|(\psi,\phi)\| \leq \|(\psi,\phi)\|_\mu$. Therefore from (4.5), we obtain

$$\begin{aligned}
\mathcal{I}_\theta(t\psi,t\phi) &\leq \frac{a}{2\theta}\|(\psi,\phi)\|^{2\theta}t^2 + \frac{b}{2\theta}\|(\psi,\phi)\|^{2\theta}t^{2\theta} - \frac{(f(t))^{4\alpha 2^*_\mu}}{4\alpha 2^*_\mu}(K(Y))^{22^*_\mu}t^{22^*_\mu}\|(\psi,\phi)\|^{22^*_\mu}\\
&= K_1t^2 + K_2t^{2\theta} - K_3(K(Y))^{22^*_\mu}t^{22^*_\mu} \to -\infty
\end{aligned}$$

as $t\to\infty$. Hence, there exists $R_0 > 0$ large enough such that $\mathcal{I}_\gamma(w,z) < 0$ for all $(w,z)\in\mathcal{D}$ with $\|(w,z)\| = R$ and $R \geq R_0$. Therefore, $\mathcal{I}_\gamma$ satisfies Theorem 4.3(3). So, step 3 is proved.

Step 4. We will prove that there exists a sequence $\{s_n\}\subset(0,\infty)$ with $s_n < s_{n+1}$, such that

$$c_n^\gamma := \inf_{U\in\Upsilon_n}\max_{(w,z)\in U}\mathcal{I}_\gamma(w,z) < s_n,$$

where Υ_n is defined in (4.4). To complete this step, in view the definition of c_n^γ and using Lemma 2.2(3),(8), we obtain

$$\begin{aligned}
c_n^\gamma &:= \inf_{U\in\Upsilon_n}\max_{(w,z)\in U}\mathcal{I}_\gamma(w,z)\\
&= \inf_{U\in\Upsilon_n}\max_{(w,z)\in U}\Big\{\frac{a}{2}\int_{\mathbb{R}^N}(|\nabla w|^2+|\nabla z|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2dx\Big)^\theta\Big]\\
&\quad - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(w)|^{4\alpha}}{|x|^2}dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(z)|^{4\alpha}}{|x|^2}dx - \frac{\gamma}{r+s}\int_{\mathbb{R}^N}g(x)|f(w)|^r|f(z)|^s\,dx\\
&\quad - \frac{1}{4\alpha 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w(x))|^{2\alpha 2^*_\mu}|f(w(y))|^{2\alpha 2^*_\mu}+|f(z(x))|^{2\alpha 2^*_\mu}|f(z(y))|^{2\alpha 2^*_\mu})}{|x-y|^\mu}\,dx\,dy\Big\}\\
&\leq \inf_{U\in\Upsilon_n}\max_{(w,z)\in U}\Big\{\frac{a}{2}\|(w,z)\|^2 + \frac{b}{2\theta}\|(w,z)\|^{2\theta} - \frac{(f(1))^{2\alpha 2^*_\mu}}{4\alpha 2^*_\mu}\|(w,z)\|_\mu^{22^*_\mu}\Big\}.
\end{aligned}$$

We set

$$s_n = \inf_{U\in\Upsilon_n} \max_{(w,z)\in U} \Big\{ \frac{a}{2}\|(w,z)\|^2 + \frac{b}{2\theta}\|(w,z)\|^{2\theta} - \frac{(f(1))^{2\alpha 2_\mu^*}}{4\alpha 2_\mu^*}\|(w,z)\|_\mu^{22_\mu^*} \Big\}.$$

Then clearly from the definition of Υ_n, it follows that $s_n < \infty$ and $s_n \le s_{n+1}$.

Step 5. We prove that problem (1.1) has at least n pairs of weak solutions. Now, we argue similarly as in [31]. We can choose $\hat{a} > 0$ sufficiently large such that for any $a > \hat{a}$,

$$\sup_n s_n < \frac{1}{8\alpha}\Big(1 - \frac{4\eta}{(N-2)^2}\Big)\big(aS_\mu\big)^{\frac{2_\mu^*}{2_\mu^*-1}} := c^*,$$

that is,

$$c_n^\gamma < s_n < \frac{1}{8\alpha}\Big(1 - \frac{4\eta}{(N-2)^2}\Big)\big(aS_\mu\big)^{\frac{2_\mu^*}{2_\mu^*-1}}.$$

Hence, for all $\gamma \in \big(0,\, a(1 - \frac{4\eta}{(N-2)^2})S\|g\|^{-1}_{\frac{2^*}{2^*-2}}\big)$ and $a > \hat{a}$, we have

$$0 < c_1^\gamma \le c_2^\gamma \le \dots \le c_n^\gamma < s_n < c^*.$$

Now by Theorem 4.3, we infer that levels $c_1^\gamma \le c_2^\gamma \le \dots \le c_n^\gamma$ are critical values of $\mathcal{I}_\gamma$. Therefore, if $c_1^\gamma < c_2^\gamma < \dots < c_n^\gamma$, then $\mathcal{I}_\gamma$ has at least n number of critical points. Furthermore, if $c_j^\gamma = c_{j+1}^\gamma$ for some $j = 1, 2, \dots, k-1$, then again Theorem 4.3 yields that $K_{c_j^\gamma}$ is an infinite set and hence, problem (2.11) has infinitely many solutions. Therefore, we can conclude that problem (2.11) has at least n pairs of weak solutions. □

To prove Theorem 1.4 using Theorem 4.3. We show that $\mathcal{I}_\gamma$ verifies all the hypotheses of Theorem 4.3 when $4\alpha < r + s < 2\alpha 2^*$.

Lemma 4.4. *Let* (1.9) *hold and* $4\alpha < r + s < 2\alpha 2^*$. *Then* $\mathcal{I}_\gamma$ *satisfies conditions (1)–(3) of Theorem 4.3 for all* $\gamma \ge 0$.

Proof. Let $(w,z) \in \mathcal{D}$ be with $\|(w,z)\| < 1$. Using (2.9), (2.10), $\eta = max(\eta_1, \eta_2)$ and (2.3), we obtain

$$\begin{aligned}
\mathcal{I}_\gamma(w,z) &= \frac{a}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + |\nabla z|^2)\,dx + \frac{b}{2\theta}\Big[\Big(\int_{\mathbb{R}^N}|\nabla f(w)|^2 dx\Big)^\theta + \Big(\int_{\mathbb{R}^N}|\nabla f(z)|^2 dx\Big)^\theta\Big] \\
&\quad - \frac{a\eta_1}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(w)|^{4\alpha}}{|x|^2}dx - \frac{a\eta_2}{4\alpha}\int_{\mathbb{R}^N}\frac{|f(z)|^{4\alpha}}{|x|^2}dx - \frac{\gamma}{r+s}\int_{\mathbb{R}^N} g(x)|f(w)|^r|f(z)|^s\,dx \\
&\quad - \frac{1}{4\alpha 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|f(w)|^{2\alpha 2_\mu^*}|f(w)|^{2\alpha 2_\mu^*} + |f(z)|^{2\alpha 2_\mu^*}|f(z)|^{2\alpha 2_\mu^*})}{|x-y|^\mu}\,dx\,dy \\
&\ge \frac{a}{2}\Big(1 - \frac{4\eta}{(N-2)^2}\Big)\|(w,z)\|^2 - \frac{\gamma}{r+s}S^{-\frac{r+s}{4\alpha}}\|g\|_{\frac{2\alpha 2^*}{2\alpha 2^* - (r+s)}}\|(w,z)\|^{\frac{r+s}{2\alpha}} \\
&\quad - \frac{1}{4\alpha 2_\mu^*}\big(\frac{2\alpha}{S_\mu}\big)^{2_\mu^*}\|(w,z)\|^{22_\mu^*}.
\end{aligned}$$

Since $4\alpha < (r+s)$ and $2 < 22_\mu^*$, we can choose $0 < \epsilon < 1$ sufficiently small so that we obtain for all $(w,z) \in \mathcal{D}$ with $\|(w,z)\| = \epsilon$, $\mathcal{I}_\gamma(w,z) \ge \rho > 0$ for some $\rho > 0$ depending on ϵ. So condition (1) of Theorem 4.3 is proved.

It follows from Lemma 3.10, since $c^{**} > 0$. So, condition (2) of Theorem 4.3 is proved.

The argument replaces similarly as in the condition (3) of Theorem 4.3. □

Proof of Theorem 1.4. Using Lemma 4.4 and arguing in a similar way to the Step 4, 5 and as in Theorem 4.3, we can conclude that (2.11) has at least n pairs of solutions for all $\gamma > 0$ and $(u,v) = (f(w), f(z))$ must solve problem (1.1). □

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