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INTERFACE CONDITION FOR THE POISSON EQUATION IN A PARTIALLY POROUS MEDIUM

TOMISLAV FRATROVIĆ, EDUARD MARUŠIĆ-PALOKA

Abstract. In this article, we derive a new effective interface condition between two parts of a partially porous medium. We study the steady-state Poisson equation in a two-dimensional domain that combines an upper porous part with a periodic microstructure and a lower free region. Using the homogenization approach and a formal asymptotic expansion, we establish a second-order correction to the classical interface condition that accounts for microscale effects, including those arising from both the local flux and the source term redistribution across the interface. This refinement is essential for high-precision applications in which the pore size is small but not negligible compared to the system scale.

1. Introduction

A partially porous medium is a high-contrast composite system in which two distinct regions are in contact: a free-flow region (or a non-perforated solid) and a porous matrix (see e.g. [21, 22, 23]). The interface between these two domains is not merely a mathematical boundary - it is a physical transition zone where the microstructure of the porous medium interacts with the macroscopic behavior of the free domain. Understanding and accurately modeling this interface are crucial for predicting the overall response of the system to external stimuli, such as heat flux or mechanical loading.

Modeling fluid flow through a partially porous medium is a classical problem in fluid mechanics and has been studied extensively in the literature (see e.g. [5, 10, 18, 19, 24]). The interface condition, which describes the behavior of the solution across the boundary separating the porous and the free region, is a key component of these models. A closely related but slightly different problem appears in the design of modern thermal engineering devices such as micro-lattice and metal-foam heat exchangers. Applications range from the cooling of electronic components and porous burners to the management of geothermal reservoirs. These devices rely on the efficient transfer of heat across interfaces between free-flowing fluids and porous structures [20]. The classical interface condition, which assumes continuity of temperature and flux, is often insufficient for capturing the complex physics involved. For high-precision applications, this "perfect contact" assumption can lead to non-negligible errors in the temperature field. This is particularly true when the characteristic pore size ε is not negligible compared to the system size L. In such cases, higher-order corrections to the interface condition become necessary to account for the microscale effects that influence macroscopic behavior. Near the boundary, the flow and heat paths are distorted by the solid obstacles of the porous matrix, creating a thermal boundary layer.
In this type of systems, the Poisson equation governs the steady-state temperature distribution:

$$-\Delta u = f$$

where the source term f represents internal heat generation. While the Beavers-Joseph approach ([1, 3]) is more phenomenological (experimental), the Bakhvalov-Panasenko approach or the homogenization method in general [2, 4, 8, 7, 9, 17, 18], provides a rigorous mathematical framework

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for deriving effective interface conditions. The Beavers-Joseph-Saffman condition [25] introduces a first-order correction to the classical interface condition, accounting for the slip velocity at the interface in that setting. The second-order correction, which we derive here using a formal asymptotic expansion, incorporates an additional term. This leads to a more accurate representation of the temperature jump across the interface, especially when ε is not negligible. To match the microscale (pore size ε) and the macroscale (system size L), the Bakhvalov-Panasenko method uses a formal asymptotic expansion. By introducing a fast variable $y = x/\varepsilon$, the interface is no longer a sharp line but a transition zone. While first-order corrections ($\mathcal{O}(\varepsilon)$) introduce a jump in temperature proportional to the local flux, the second-order correction ($\mathcal{O}(\varepsilon^2)$) provides a deeper refinement.

This second-order term incorporates the effects of a source-term correction, or how the heat generation f in the immediate vicinity of the pores contributes to the macroscopic jump. The physical significance for heat conduction is that the newly proposed model, with interface condition (2.7), accounts for the fact that heat does not simply "jump" across the interface - it is rather redistributed tangentially as a thermal lag at the ε^2 level.

For our study, we use interior-layer expansions and the matching procedure developed first for the effective condition on a porous wall in [13, 14, 15] and then [18] (see also [6] for the nonlinear case).

2. Setting of the problem and the main result

We assume that the $2D$ porous medium Ω_ε^+ is obtained by the periodic repetition of a canonical cell. More precisely, let $Y = \langle -\frac{1}{2}, \frac{1}{2}\rangle \times \langle 0, 1\rangle$ and $A \subset\subset Y$ be a smooth, solid, and strict inclusion in the center of the cell, such that its translationally equivalent prototype A_0, centered at the origin, satisfies $-A_0 = A_0$. The boundary of the pore is denoted by $\Gamma = \partial A$, and $Y^* = Y \setminus \overline{A}$ (see Figure 1).

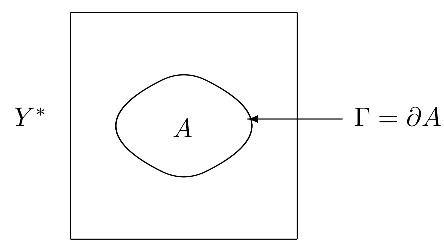

Figure 1. Canonical cell.

Let $\varepsilon \ll 1$ be a small parameter representing the size of a typical pore and the period of our porous medium. Let $m, \ell \in \mathbf{N}$ and let $L = m\,\varepsilon\,,\ K = \ell\,\varepsilon$. Now,

$$\Omega_\varepsilon^+ = \cup_{j=0}^{\ell-1} \cup_{i=0}^{m-1} \varepsilon\big[\big(i+\frac{1}{2}, j\big) + Y^*\big]\,.$$

Let $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega^-$ be the combined domain as seen in Figure 2, consisting of the upper porous part $\Omega_\varepsilon^+ \subset \Omega^+ = \langle 0, L\rangle \times \langle 0, K\rangle$ and the lower part $\Omega^- = \langle 0, L\rangle \times \langle -M, 0\rangle$ (free, no perforations). The interface between these two parts is denoted by Σ and the rest of the boundaries by $\Gamma^\pm$ and Γ_ε.

We study a simple steady-state model problem consisting of the Poisson equation in Ω_ε, endowed with the appropriate boundary conditions as follows:

$$-\Delta u^\varepsilon = f \quad \text{in } \Omega_\varepsilon\,, \tag{2.1}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon\,, \tag{2.2}$$

$$\frac{\partial u^\varepsilon}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma^\pm\,, \tag{2.3}$$

$$u^\varepsilon \text{ is } L\text{-periodic with respect to } x_1\,. \tag{2.4}$$

We assume for now that f is smooth (its regularity will be specified later) and L-periodic in x_1. In the sequel, all functions depending on x_1 are assumed to be L-periodic.

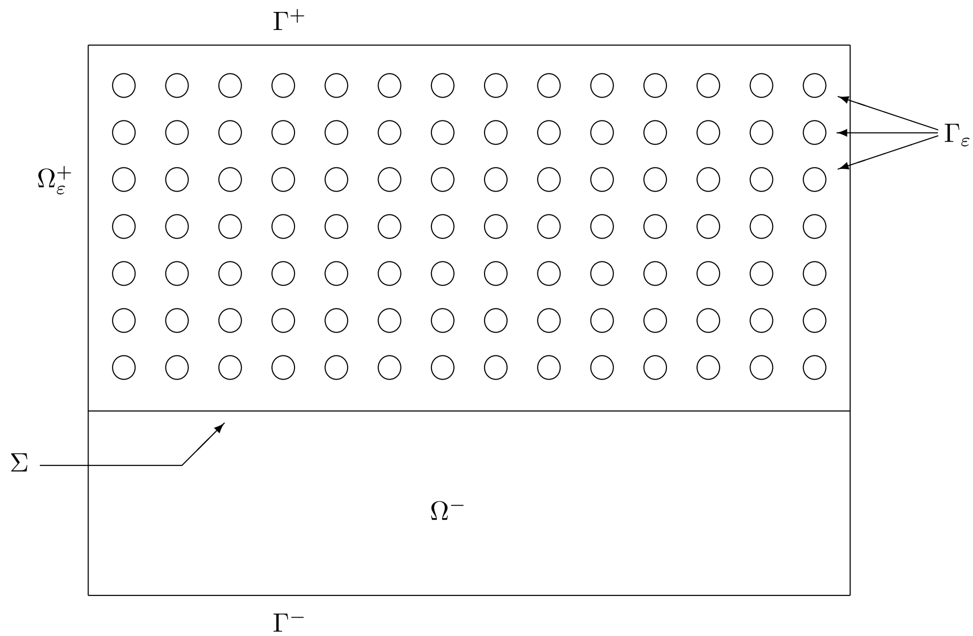

Figure 2. Partially porous domain.

The solution represents a state of dynamic equilibrium where a continuous, non-zero flux streams across the domains. The source term constantly injects energy, which moves across the interface, making the accurate modeling of the boundary behavior crucial for capturing this ongoing energy transport. For example, the Poisson equation solves a heat problem, f being a heat source.

The goal of this paper is to derive analytically the effective interface condition on Σ, the interface separating the porous from the non-perforated (free) part of the domain. The matching conditions required are continuity of the solution and its normal derivative on this interface.

Using higher-order asymptotics, here we propose a different boundary condition that improves the Beavers-Joseph-Saffman (BJS) condition, obtained if only scale of order ε is considered. Employing multiscale asymptotic expansion and the boundary-layer analysis, we find that the effective interface condition is in fact of the Robin type, with parameters calculated from the auxiliary problems. More precisely, in the free part Ω^-, the effective problem reads

$$-\Delta v = f \quad \text{in } \Omega^- , \tag{2.5}$$

$$\frac{\partial v}{\partial x_2} = 0 \quad \text{on } \Gamma^- , \tag{2.6}$$

$$v(x_1, 0-) + \varepsilon d \frac{\partial v}{\partial x_2}(x_1, 0-) = \varepsilon^2 \lambda f(x_1, 0) , \tag{2.7}$$

whereas in the porous medium Ω^+ (including the perforations) the flux satisfies a simple algebraic equation (4.4) that can be seen as a scalar version of the Darcy law. The effective interface condition (2.7) resembles a second order correction for (BJS), if we look at the normal derivative (outward for Ω^+!) of v on Σ:

$$\frac{\partial v}{\partial \mathbf{n}} = -\frac{\partial v}{\partial x_2} .$$

The parameters d and λ are obtained from the auxiliary problems defined in the infinite strip $\mathcal{T}$ (see Figure 3) and examined in more details in Appendices B and C.

3. Asymptotic expansion

We seek for the solution in the form of an asymptotic expansion. We do the construction step by step and explain the matching procedure that leads to the final form given by (3.19).

In the two-scale expansion we use the notation $y = \frac{x}{\varepsilon}$ for the fast variable, together with the solutions of the so-called auxiliary cell problems. We also use several interior-layer correctors and also correctors of the exterior boundary layer. All the auxiliary problems defining the solutions and correctors are explained in details in the Appendices.

3.1. **Interior expansion.** We start by classical two-scale expansion (see e.g. [2, 12, 16]) for describing the behavior inside the porous part, away from its boundary. Thus, for the moment, we neglect the exterior boundaries of the porous medium Ω_ε^+. Inside Ω_ε^+ we just need to correct the solution on the boundary of pores (Γ_ε). So, we look for an asymptotic expansion of the form

$$U_I^\varepsilon(x) = \begin{cases} \varepsilon^2 w(y) f(x) + \varepsilon^3 \sum_{k=1}^2 w^k(y) \frac{\partial f}{\partial x_k}(x) + \varepsilon^4 \sum_{k,j=1}^2 w^{kj}(y) \frac{\partial^2 f}{\partial x_k \partial x_j}(x) & \text{in } \Omega_\varepsilon^+\,, \\ v(x) & \text{in } \Omega^-\,. \end{cases} \tag{3.1}$$

The auxiliary functions w, w^k and w^{kj} are defined in the cell Y^* and their basic properties can be found in Appendix A.

Taking the negative Laplacian of (3.1) and noting that $\Delta = \Delta_x + \frac{2}{\varepsilon}\Delta_{xy} + \frac{1}{\varepsilon^2}\Delta_y$ we obtain

$$\begin{aligned} -\Delta U_I^\varepsilon = -\Delta_y w f - \varepsilon \sum_{k=1}^2 \big[\Delta_y w^k + 2\frac{\partial w}{\partial y_k}\big] \frac{\partial f}{\partial x_k} - \varepsilon^2 \sum_{k,j=1}^2 \big[\Delta_y w^{kj} + 2\frac{\partial w^k}{\partial y_j} + \delta_{jk} w\big] \frac{\partial^2 f}{\partial x_k \partial x_j} \\ + \mathcal{O}(\varepsilon^3)\,. \end{aligned} \tag{3.2}$$

Now we see the reason for introducing the auxiliary problems (6.1)-(6.3), (6.5)-(6.7) (6.9)-(6.11) in the cell Y^*, whose solutions are designed to influence the first three terms in (3.2). We see that, by construction, the interior expansion in Ω_ε^+ satisfies (2.1) up to order ε^3.

In the free domain Ω^- we clearly have a simple Poisson problem (2.5)-(2.6) for v,

$$-\Delta v = f \quad \text{in } \Omega^-\,, \quad \frac{\partial v}{\partial x_2} = 0 \quad \text{on } \Gamma^-\,, \tag{3.3}$$

but we still need to provide an appropriate boundary condition on the interface Σ between the porous medium and the non-perforated free domain.

3.2. **Interface correctors.** Our main interest is the behavior on the interface $\Sigma = \langle 0, L\rangle \times \{0\}$ separating the upper (porous) domain Ω_ε^+ and the lower (free) domain Ω^-. We use the classical matching procedure (see e.g. the monograph [2] by Bakhvalov and Panasenko).

For now, in Ω^- we have $v(x)$ and in Ω_ε^+ there is

$$U_I^\varepsilon = \varepsilon^2 w\big(\frac{x}{\varepsilon}\big) f(x) + \varepsilon^3 \sum_{k=1}^2 w^k\big(\frac{x}{\varepsilon}\big) \frac{\partial f}{\partial x_k}(x) + \varepsilon^4 \sum_{k,j=1}^2 w^{kj}\big(\frac{x}{\varepsilon}\big) \frac{\partial^2 f}{\partial x_k \partial x_j}(x)\,.$$

To match those two, we introduce the interior-layer correctors and this correction has the form

$$U_L^\varepsilon = \varepsilon^2 P\big(\frac{x}{\varepsilon}\big) f(x_1, 0) + \varepsilon^3 \sum_{k=1}^2 P^k\big(\frac{x}{\varepsilon}\big) \frac{\partial f}{\partial x_k}(x_1, 0) + \varepsilon^4 \sum_{k,j=1}^2 P^{kj}\big(\frac{x}{\varepsilon}\big) \frac{\partial^2 f}{\partial x_k \partial x_j}(x_1, 0)\,. \tag{3.4}$$

This is designed to correct the jump of the solution and the normal derivative over Σ. The correctors P, P^k and P^{kj} are defined in a semi-perforated, infinite strip $\mathcal{T}$ obtained by the periodic repetition of the canonical cell Y^* in the vertical direction for the upper part (see Figure 3):

$$\mathcal{T} = \langle -\frac{1}{2}, \frac{1}{2}\rangle \times \mathbf{R} \setminus \Big(\cup_{i=0}^{+\infty} A_i\Big), \quad A_i = A + i\mathrm{e}_2\,.$$

P, P^k and P^{kj} are 1-periodic in y_1 and vanish on

$$\cup_i \Gamma_i = \cup_i \partial A_i$$

(see more details in Appendix B).

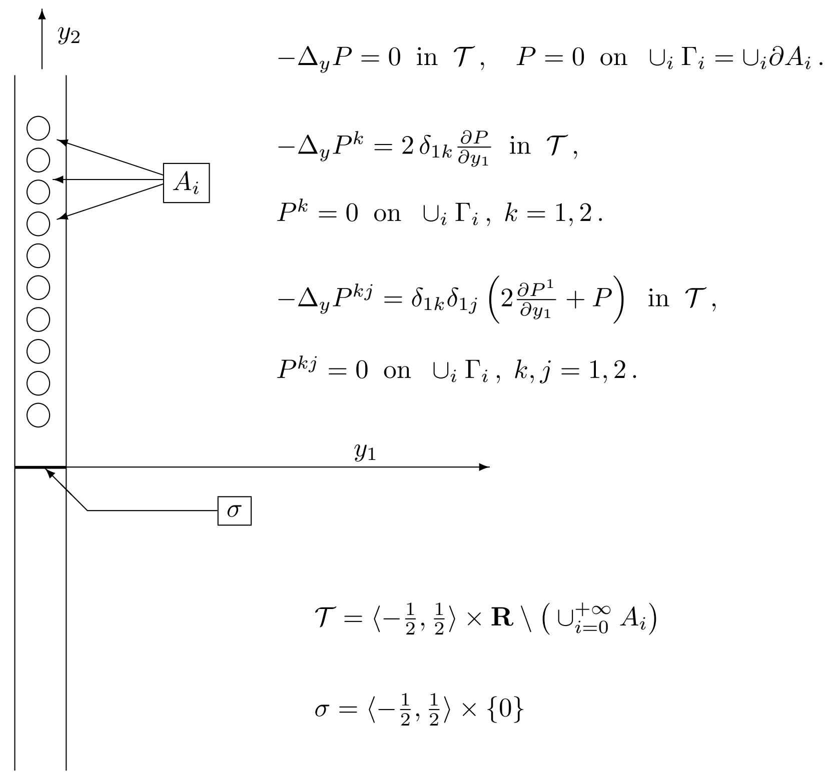

FIGURE 3. Infinite strip $\mathcal{T}$ and interior-layer correctors.

Adding the interior-layer correctors (3.4) to the interior approximation (3.1), in the free region Ω^- below Σ, we now have $U^\varepsilon = U_I^\varepsilon + U_L^\varepsilon$ equal to

$$v(x) + \varepsilon^2 P(y) f(x_1, 0) + \varepsilon^3 \sum_{k=1}^{2} P^k(y) \frac{\partial f}{\partial x_k}(x_1, 0) + \varepsilon^4 \sum_{k,j=1}^{2} P^{kj}(y) \frac{\partial^2 f}{\partial x_j \partial x_k}(x_1, 0) \tag{3.5}$$

with the boundary-layer functions P, P^k and P^{kj} such that

$$\lim_{y_2 \to \pm\infty} P = 0\,, \quad \lim_{y_2 \to \pm\infty} P^k = 0\,, \quad \lim_{y_2 \to \pm\infty} P^{kj} = 0\,, \quad \text{exponentially for } k, j = 1, 2\,. \tag{3.6}$$

Thus, outside of the small interior layer near Σ, in Ω^- the expansion is approximately equal to v. In the porous domain above Σ, our asymptotic expansion U^ε gets the form

$$\begin{aligned} U^\varepsilon &= U_I^\varepsilon + U_L^\varepsilon \\ &= \varepsilon^2 \big[w(y) f(x) + P(y) f(x_1, 0) \big] \\ &\quad + \varepsilon^3 \sum_{k=1}^{2} \Big[w^k(y) \frac{\partial f}{\partial x_k}(x) + P^k(y) \frac{\partial f}{\partial x_k}(x_1, 0) \Big] \\ &\quad + \varepsilon^4 \sum_{k,j=1}^{2} \Big[w^{kj}(y) \frac{\partial^2 f}{\partial x_j \partial x_k}(x) + P^{kj}(y) \frac{\partial^2 f}{\partial x_j \partial x_k}(x_1, 0) \Big]\,. \end{aligned} \tag{3.7}$$

We would like to impose the continuity of our approximation and of its normal derivative on the interface Σ up to the order $\mathcal{O}(\varepsilon^3)$, which is the best we can do with this kind of approximation.

The continuity of the approximation yields

$$
\begin{aligned}
&- v(x_1,0-) + \varepsilon^2 \,\{[P]_\sigma\,(y_1) + w(y_1,0)\} f(x_1,0)\\
&+ \varepsilon^3 \sum_{k=1}^{2} \{\,[P^k]_\sigma\,(y_1) + w^k(y_1,0)\} \frac{\partial f}{\partial x_k}(x_1,0)\\
&+ \varepsilon^4 \sum_{k,j=1}^{2} \{[P^{kj}]_\sigma(y_1) + w^{kj}(y_1,0)\} \frac{\partial^2 f}{\partial x_j \partial x_k}(x_1,0) = 0\,.
\end{aligned} \tag{3.8}
$$

Here, and in the sequel, the brackets $[\cdot]_\sigma$ will denote the jump across σ, i.e.

$$
[F]_\sigma(y_1) = F(y_1,0+) - F(y_1,0-)\,.
$$

We would like to set

$$
\begin{gathered}
[P]_\sigma(y_1) + w(y_1,0) = 0\,,\\
[P^k]_\sigma(y_1) + w^k(y_1,0) = 0\,,\\
[P^{kj}]_\sigma(y_1) + w^{kj}(y_1,0) = 0\,,
\end{gathered}
$$

but, since we expect P, P^k and P^{kj} to decay to zero as $y_2 \to \pm\infty$, that might not be possible. So, at the moment, we only impose

$$
\begin{gathered}
[P]_\sigma(y_1) + w(y_1,0) = \lambda\,,\\
[P^k]_\sigma(y_1) + w^k(y_1,0) = \lambda^k\,,\\
[P^{kj}]_\sigma(y_1) + w^{kj}(y_1,0) = \lambda^{kj}\,,
\end{gathered} \tag{3.9}
$$

for some constants λ, λ^k and λ^{kj} (independent on y_1) and we address them later in Appendix B. These constants are a part of the auxiliary problems and serve to get the decay of the interior-layer correctors (3.6). It is important to notice that λ is not given in advance and will appear in the macroscopic interface condition.

Clearly, due to (3.8) and (3.9), to get the continuity of the matched asymptotic approximation (jump of order $\mathcal{O}(\varepsilon^3)$), we *temporarily* impose

$$
v(x_1,0-) = \varepsilon^2\,\lambda\, f(x_1,0)\,, \tag{3.10}
$$

which is to be corrected, taking into account the normal derivative and its jump across the interface Σ:

$$
\begin{aligned}
\Big[\frac{\partial U^\varepsilon}{\partial x_2}\Big]_\Sigma &= \frac{\partial U^\varepsilon}{\partial x_2}(x_1,0+) - \frac{\partial U^\varepsilon}{\partial x_2}(x_1,0-)\\
&= \varepsilon\Big\{\Big[\frac{\partial P}{\partial y_2}\Big]_\sigma + \frac{\partial w}{\partial y_2}(y_1,0)\Big\} f(x_1,0)\\
&\quad - \frac{\partial v}{\partial x_2}(x_1,0-) + \varepsilon^2 \sum_{k=1}^{2} \Big\{\Big[\frac{\partial P^k}{\partial y_2}\Big]_\sigma + \frac{\partial w^k}{\partial y_2}(y_1,0)\Big\} \frac{\partial f}{\partial x_k}(x_1,0) + \mathcal{O}(\varepsilon^3)\,.
\end{aligned} \tag{3.11}
$$

We can impose (see Appendix B)

$$
\Big[\frac{\partial P}{\partial y_2}\Big]_\sigma + \frac{\partial w}{\partial y_2}(y_1,0) = 0\,,\quad \Big[\frac{\partial P^k}{\partial y_2}\Big]_\sigma + \frac{\partial w^k}{\partial y_2}(y_1,0) = 0\,.
$$

Keeping the normal derivative jump of order $\mathcal{O}(\varepsilon^3)$ would lead to imposing also

$$
\frac{\partial v}{\partial x_2}(x_1,0-) = 0\,. \tag{3.12}
$$

Clearly, we cannot at the same time ask for (3.10) and (3.12) to hold, as that would lead to an overposed problem for our Poisson equation (2.5). Thus, to eliminate $\frac{\partial v}{\partial x_2}(x_1,0-)$ we need another boundary-layer corrector of the form

$$
\varepsilon B(y) \frac{\partial v}{\partial x_2}(x_1,0-) \tag{3.13}
$$

to be added to (3.5) and (3.7), such that

$$\big[\varepsilon\frac{\partial B}{\partial x_2}\big]_\Sigma\frac{\partial v}{\partial x_2}(x_1,0-)=\big[\frac{\partial B}{\partial y_2}\big]_\sigma\frac{\partial v}{\partial x_2}(x_1,0-)=\frac{\partial v}{\partial x_2}(x_1,0-)$$

The function B has been studied in details in Appendix C. The idea is that (3.13) corrects the term $\frac{\partial v}{\partial x_2}(x_1,0-)$ in (3.11) and does not spoil the continuity of the approximation. Clearly, it needs to satisfy the Laplace equation

$$-\Delta B=0\quad\text{in } \mathcal{T} \tag{3.14}$$

and we also require the following jump conditions on σ,

$$\big[\frac{\partial B}{\partial y_2}\big]_\sigma=\frac{\partial B}{\partial y_2}(y_1,0+)-\frac{\partial B}{\partial y_2}(y_1,0-)=1\,, \tag{3.15}$$

$$\big[B\big]_\sigma=B(y_1,0+)-B(y_1,0-)=0\,. \tag{3.16}$$

That, however, is not possible if we want B to decay to zero as $y_2\to\pm\infty$ (see Appendix C). We must keep (3.15) as that was the whole purpose of adding the new corrector, which means we have to sacrify (3.16) and replace it with

$$[B]_\sigma=B(y_1,0+)-B(y_1,0-)=-d\,, \tag{3.17}$$

where d is some constant. It turns out (from Appendix C) that

$$d=\int_{\mathcal{T}}|\nabla B|^2>0\,. \tag{3.18}$$

That is important, as this coefficient will also appear in our effective (macroscopic) interface condition for v.

After correcting the normal derivative, we just need to correct the boundary layer on the exterior boundary Γ^+ (see Appendix D) and can construct the final form of the approximation. With all the auxiliary functions defined in the appendices, we choose the final form of the asymptotic expansion $U^\varepsilon(x)$ equal to

$$\begin{aligned}
&\varepsilon^2\big\{w\big(\frac{x}{\varepsilon}\big)f(x)+T\Big(\frac{x-K\mathbf{e}_2}{\varepsilon}\Big)f(x_1,K)+P\big(\frac{x}{\varepsilon}\big)f(x_1,0)\big\}\\
&+\varepsilon^3\sum_{k=1}^{2}\big\{w^k\big(\frac{x}{\varepsilon}\big)\frac{\partial f}{\partial x_k}(x)+T^k\big(\frac{x-K\mathbf{e}_2}{\varepsilon}\big)\frac{\partial f}{\partial x_k}(x_1,K)+P^k\big(\frac{x}{\varepsilon}\big)\frac{\partial f}{\partial x_k}(x_1,0)\,\big\}\\
&+\varepsilon^4\sum_{k,j=1}^{2}\big\{w^{kj}\big(\frac{x}{\varepsilon}\big)\frac{\partial^2 f}{\partial x_j\partial x_k}(x)+P^{kj}\big(\frac{x}{\varepsilon}\big)\frac{\partial^2 f}{\partial x_j\partial x_k}(x_1,0)\,\big\}+\varepsilon B\big(\frac{x}{\varepsilon}\big)\frac{\partial v}{\partial x_2}(x_1,0-)\\
&+\varepsilon^2B^1\big(\frac{x}{\varepsilon}\big)\frac{\partial^2 v}{\partial x_1\partial x_2}(x_1,0-)+\varepsilon^3B^{11}\big(\frac{x}{\varepsilon}\big)\frac{\partial^3 v}{\partial x_1^2\partial x_2}(x_1,0-)\quad\text{in } \Omega_\varepsilon^+\,, \qquad (3.19)\\
&v(x)+\varepsilon^2P\big(\frac{x}{\varepsilon}\big)f(x_1,0)+\varepsilon^3\sum_{k=1}^{2}P^k\big(\frac{x}{\varepsilon}\big)\frac{\partial f}{\partial x_k}(x_1,0)\\
&+\varepsilon^4\sum_{k,j=1}^{2}P^{kj}\big(\frac{x}{\varepsilon}\big)\frac{\partial^2 f}{\partial x_j\partial x_k}(x_1,0)+\varepsilon B\big(\frac{x}{\varepsilon}\big)\frac{\partial v}{\partial x_2}(x_1,0-)\\
&+\varepsilon^2B^1\big(\frac{x}{\varepsilon}\big)\frac{\partial^2 v}{\partial x_1\partial x_2}(x_1,0-)+\varepsilon^3B^{11}\big(\frac{x}{\varepsilon}\big)\frac{\partial^3 v}{\partial x_1^2\partial x_2}(x_1,0-)\quad\text{in } \Omega^-
\end{aligned}$$

Functions B^1 and B^{11} are the higher order correctors needed only to prove error estimates and are defined in Appendix C. Functions T and T^k are the exterior boundary layer correctors and also present an unessential technicality. The details of their construction can be found in Appendix D. The normal derivative jump is taken care of by the construction and it now has the order $\mathcal{O}(\varepsilon^3)$. As done before, we require the jump of the final form of the approximation U^ε to be zero across

Σ. That leads to the final version of the effective interface condition causing the jump to be of order $\mathcal{O}(\varepsilon^3)$. From (3.9), (3.17) and (8.20), and because of the construction, we obtain

$$\begin{aligned}
0 &= [U^\varepsilon]_\Sigma = U^\varepsilon(x_1,0+) - U^\varepsilon(x_1,0-)\\
&= -v(x_1,0-) + \varepsilon^2\{[P]_\sigma + w(y_1,0)\}f(x_1,0)\\
&\quad + \varepsilon^3\sum_{k=1}^{2}\{\left[P^k\right]_\sigma + w^k(y_1,0)\}\frac{\partial f}{\partial x_k}(x_1,0)\\
&\quad + \varepsilon^4\sum_{k,j=1}^{2}\{[P^{kj}]_\sigma + w^{kj}(y_1,0)\}\frac{\partial^2 f}{\partial x_j\partial x_k}(x_1,0)\\
&\quad + \varepsilon\,[B]_\sigma\frac{\partial v}{\partial x_2}(x_1,0-) + \varepsilon^2[B^1]_\sigma\frac{\partial^2 v}{\partial x_1\partial x_2}(x_1,0-) + \varepsilon^3[B^{11}]_\sigma\frac{\partial^3 v}{\partial x_1^2\partial x_2}(x_1,0-)\\
&= -v(x_1,0-) + \varepsilon^2\lambda\, f(x_1,0) - \varepsilon\, d\,\frac{\partial v}{\partial x_2}(x_1,0-) + \mathcal{O}(\varepsilon^3)\,,
\end{aligned} \tag{3.20}$$

leading to the final version of the effective interface condition for v,

$$v(x_1,0-) + \varepsilon\, d\,\frac{\partial v}{\partial x_2}(x_1,0-) = \varepsilon^2\lambda f(x_1,0)\,. \tag{3.21}$$

4. Macroscopic problem

Summing up the results from Section 3, we recover the obtained macroscopic problem for v.

In the free part Ω^-, it reads

$$-\Delta v = f \quad \text{in } \Omega^-\,, \tag{4.1}$$

$$\frac{\partial v}{\partial x_2} = 0 \quad \text{on } \Gamma^-\,, \tag{4.2}$$

$$v(x_1,0-) + \varepsilon\, d\,\frac{\partial v}{\partial x_2}(x_1,0-) = \varepsilon^2\lambda f(x_1,0)\,. \tag{4.3}$$

In the porous medium Ω^+, we have the scalar Darcy equation for $\overline{v}$, obtained as the weak limit of $\frac{u^\varepsilon}{\varepsilon^2}$ as $\varepsilon \to 0$ in $L^2(\Omega_\varepsilon^+)$:

$$\overline{v} = \mathrm{K}f \quad \text{in } \Omega^+\,. \tag{4.4}$$

This scalar approximation accurately represents the bulk behavior of the homogeneous porous matrix: $\overline{v}$ is in fact the localized flux, K represents the porous medium's transport coefficient, and f is the driving gradient.

So, on the interface Σ, the effective boundary condition (4.3) for v is of the Robin type. Since $d > 0$ (see Appendix C), the problem is well-posed. Assuming f is, at least an $L^2(\Omega^-)$ function. Indeed, its weak formulation reads:

find $v \in H^1_{\rm per}(\Omega^-)$ such that

$$\int_{\Omega^-}\nabla v\,\nabla w + \frac{1}{\varepsilon\, d}\int_\Sigma v\, w = \int_{\Omega^-} f\, w + \frac{\varepsilon\lambda}{d}\int_\Sigma f w\,, \quad w \in H^1_{\rm per}(\Omega^-)\,. \tag{4.5}$$

The form on the left-hand side is obviously positive implying that, due to the Lax-Milgram theorem, the problem has a unique solution $v \in H^1_{\rm per}(\Omega^-)$. The standard elliptic regularity theory applies so that $f \in H^m_{\rm per}(\Omega^-)$ implies $v \in H^{m+2}_{\rm per}(\Omega^-)$.

4.1. **Asymptotic behavior of solution to the macroscopic problem.** The proposed system, given by (4.1)-(4.4), still depends on the small parameter ε. We recall that

$$v(x_1,0-) + \varepsilon\, d\,\frac{\partial v}{\partial x_2}(x_1,0-) = \varepsilon^2\,\lambda\, f(x_1,0)\,. \tag{4.6}$$

Thus, we can look for an asymptotic expansion of the solution in the form

$$v \approx v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots\,. \tag{4.7}$$

Now, we plug the expansion above in the system (4.1)-(4.3) and collect the terms with equal powers of ε. The zero-order term is given by

$$\begin{aligned} -\Delta v^0 &= f \quad \text{in } \Omega^- \,, \\ \frac{\partial v^0}{\partial x_2} &= 0 \quad \text{on } \Gamma^- \,, \\ v^0(x_1, 0-) &= 0 \,. \end{aligned} \tag{4.8}$$

The first-order term satisfies

$$\begin{aligned} -\Delta v^1 &= 0 \quad \text{in } \Omega^- \,, \\ \frac{\partial v^1}{\partial x_2} &= 0 \quad \text{on } \Gamma^- \,, \\ v^1(x_1, 0-) + d \, \frac{\partial v^0}{\partial x_2}(x_1, 0-) &= 0 \,. \end{aligned} \tag{4.9}$$

The second-order term satisfies

$$\begin{aligned} -\Delta v^2 &= 0 \quad \text{in } \Omega^- \,, \\ \frac{\partial v^2}{\partial x_2} &= 0 \quad \text{on } \Gamma^- \,, \\ v^2(x_1, 0-) + d \frac{\partial v^1}{\partial x_2}(x_1, 0-) &= \lambda f(x_1, 0) \,. \end{aligned} \tag{4.10}$$

We conclude that the zero-order term in such expansion satisfies the no-slip condition, while the sum of the zero and the first-order term satisfy the Beavers-Joseph condition. If one more term is added, we recover the Darcy coupling condition that we propose here. The approximation can be justified, assuming sufficient regularity for f.

Theorem 4.1. *Let $f \in H^2_{\rm per}(\Omega^-)$. Then there exists a constant $C > 0$ such that*

$$\|v - (v^0 + \varepsilon v^1 + \varepsilon^2 v^2)\|_{H^1(\Omega^-)} \le C \varepsilon^{5/2} \,, \tag{4.11}$$

$$\|v - (v^0 + \varepsilon v^1 + \varepsilon^2 v^2)\|_{L^2(\Sigma)} \le C \varepsilon^3 \,. \tag{4.12}$$

Proof. The regularity of f implies $v^0 \in H^4_{\rm per}(\Omega^-)$, $v^1 \in H^3_{\rm per}(\Omega^-)$ and $v^2 \in H^2_{\rm per}(\Omega^-)$, all independent on ε. Denoting

$$R^\varepsilon = v - (v^0 + \varepsilon v^1 + \varepsilon^2 v^2) \,,$$

by direct computation we obtain

$$\begin{aligned} -\Delta R^\varepsilon &= 0 \quad \text{in } \Omega^- \,, \\ \frac{\partial R^\varepsilon}{\partial x_2} &= 0 \quad \text{on } \Gamma^- \,, \\ R^\varepsilon(x_1, 0-) &= v(x_1, 0-) - (v^0(x_1, 0-) + \varepsilon v^1(x_1, 0-) + \varepsilon^2 v^2(x_1, 0-)) \\ &= v(x_1, 0-) + \varepsilon d \, \frac{\partial v^0}{\partial x_2}(x_1, 0-) + \varepsilon^2 d \, \frac{\partial v^1}{\partial x_2}(x_1, 0-) - \varepsilon^2 \lambda \, f(x_1, 0) \,, \\ \frac{\partial R^\varepsilon}{\partial x_2}(x_1, 0-) &= \frac{\partial v}{\partial x_2}(x_1, 0-) - \frac{\partial v^0}{\partial x_2}(x_1, 0-) - \varepsilon \frac{\partial v^1}{\partial x_2}(x_1, 0-) - \varepsilon^2 \frac{\partial v^2}{\partial x_2}(x_1, 0-) \,. \end{aligned}$$

From here, we can write a Robin-type boundary condition for R^ε on Σ as

$$R^\varepsilon(x_1, 0-) + \varepsilon d \frac{\partial R^\varepsilon}{\partial x_2}(x_1, 0-) = -\varepsilon^3 d \frac{\partial v^2}{\partial x_2}(x_1, 0-) \,.$$

The proof now follows from the Robin boundary substitution into the energy identity,

$$\begin{aligned} \|\nabla R^\varepsilon\|^2_{L^2(\Omega^-)} &= \int_\Sigma R^\varepsilon(x_1, 0-) \frac{\partial R^\varepsilon}{\partial x_2}(x_1, 0-) \\ &= -\frac{1}{\varepsilon d} \|R^\varepsilon\|^2_{L^2(\Sigma)} - \varepsilon^2 \int_\Sigma R^\varepsilon(x_1, 0-) \frac{\partial v^2}{\partial x_2}(x_1, 0-) \end{aligned}$$

$$\|\nabla R^\varepsilon\|^2_{L^2(\Omega^-)} + \frac{1}{\varepsilon d}\|R^\varepsilon\|^2_{L^2(\Sigma)} \le \big|\varepsilon^2\int_\Sigma R^\varepsilon \frac{\partial v^2}{\partial x_2}\big| \le \int_\Sigma |R^\varepsilon|\big|\varepsilon^2\frac{\partial v^2}{\partial x_2}\big|\,.$$

We estimate the right-hand side by weighted Young's inequality, $|ab| \le \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$, with parameter $\delta = \frac{1}{\varepsilon d}$,

$$\int_\Sigma |R^\varepsilon|\big|\varepsilon^2\frac{\partial v^2}{\partial x_2}\big| \le \frac{\delta}{2}\int_\Sigma |R^\varepsilon|^2 + \frac{\varepsilon^4}{2\delta}\int_\Sigma \big|\frac{\partial v^2}{\partial x_2}\big|^2 = \frac{1}{2\varepsilon d}\|R^\varepsilon\|^2_{L^2(\Sigma)} + \frac{\varepsilon^5 d}{2}\|\frac{\partial v^2}{\partial x_2}\|^2_{L^2(\Sigma)}\,, \tag{4.13}$$

which gives us

$$\|\nabla R^\varepsilon\|^2_{L^2(\Omega^-)} + \frac{1}{\varepsilon d}\|R^\varepsilon\|^2_{L^2(\Sigma)} \le \frac{1}{2\varepsilon d}\|R^\varepsilon\|^2_{L^2(\Sigma)} + \frac{\varepsilon^5 d}{2}\|\frac{\partial v^2}{\partial x_2}\|^2_{L^2(\Sigma)}. \tag{4.14}$$

From here, we obtain the upper bounds for ∇R^ε and R^ε

$$\|\nabla R^\varepsilon\|^2_{L^2(\Omega^-)} + \frac{1}{2\varepsilon d}\|R^\varepsilon\|^2_{L^2(\Sigma)} \le C\varepsilon^5 \tag{4.15}$$

and the desired estimates now follow with the use of Poincaré inequality. □

5. Justification of the effective model by error estimate

Theorem 5.1. *Let $f \in C^4(\overline{\Omega_\varepsilon})$ be L-periodic in x_1. Let u^ε be the solution to the boundary value problem (2.1)-(2.4) and let U^ε be defined by (3.19). There exists a constant $C > 0$ such that*

$$\|u^\varepsilon - U^\varepsilon\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{5/2}\,, \tag{5.1}$$

$$\|u^\varepsilon - U^\varepsilon\|_{L^2(\Omega_\varepsilon^+)} \le C\varepsilon^{7/2}\,. \tag{5.2}$$

Proof. We subtract the solution u^ε and its approximation U^ε and the difference

$$E^\varepsilon = u^\varepsilon - U^\varepsilon$$

satisfies (for some functions Z_ε, ζ_ε and η_ε),

$$\begin{gathered} -\Delta E^\varepsilon = Z_\varepsilon \quad \text{in } \Omega_\varepsilon\,, \\ \frac{\partial E^\varepsilon}{\partial x_2} = 0 \quad \text{on } \Gamma^-\,, \\ [E^\varepsilon]_\Sigma = 0\,, \quad \big[\frac{\partial E^\varepsilon}{\partial x_2}\big]_\Sigma = \zeta_\varepsilon\,, \\ \frac{\partial E^\varepsilon}{\partial x_2} = \eta_\varepsilon \quad \text{on } \Gamma^+\,. \end{gathered} \tag{5.3}$$

The next step is to show that

$$\|Z_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} \le C\varepsilon^{5/2}\,, \tag{5.4}$$

$$\|Z_\varepsilon\|_{L^2(\Omega^-)} \le C\varepsilon^{5/2}\,, \tag{5.5}$$

$$\|\zeta_\varepsilon\|_{L^2(\Sigma)} \le C\varepsilon^3\,, \tag{5.6}$$

$$\|\eta_\varepsilon\|_{L^2(\Gamma^+)} \le C\varepsilon^3\,. \tag{5.7}$$

We start by analyzing Z_ε in both parts of the domain. In Ω_ε^+ we have

$$\begin{aligned} Z_\varepsilon = {}& \varepsilon^2\big\{B^1 + 2\frac{\partial B^{11}}{\partial y_1}\big\}\big(\frac{x}{\varepsilon}\big)\frac{\partial^4 v}{\partial x_1^3\partial x_2}(x_1,0-) + \varepsilon^3 B^{11}\big(\frac{x}{\varepsilon}\big)\frac{\partial^5 v}{\partial x_1^4\partial x_2}(x_1,0-) \\ &+ \varepsilon^2\sum_{j=1}^2\big\{T\delta_{j1} + 2\frac{\partial T^j}{\partial y_1}\big\}\Big(\frac{x-K\mathbf{e}_2}{\varepsilon}\Big)\frac{\partial^2 f}{\partial x_1\partial x_j}(x_1,K) \\ &+ \varepsilon^3\sum_{j=1}^2 T^j\Big(\frac{x-K\mathbf{e}_2}{\varepsilon}\Big)\frac{\partial^3 f}{\partial x_1^2\partial x_j}(x_1,K) \\ &+ \varepsilon^3\sum_{i,j,k=1}^2\big\{w^k\delta_{ji} + 2\frac{\partial w^{kj}}{\partial y_i}\big\}\big(\frac{x}{\varepsilon}\big)\frac{\partial^3 f}{\partial x_i\partial x_j\partial x_k}(x) \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^4 \sum_{j,k=1}^{2} w^{kj}\big(\frac{x}{\varepsilon}\big)\Delta_x\big(\frac{\partial^2 f}{\partial x_j \partial x_k}\big)(x)\\
&+\varepsilon^3 \sum_{j,k=1}^{2} \big\{P^k\delta_{j1} + 2\frac{\partial P^{kj}}{\partial y_1}\big\}\big(\frac{x}{\varepsilon}\big)\frac{\partial^3 f}{\partial x_1 \partial x_j \partial x_k}(x_1,0)\\
&+\varepsilon^4 \sum_{j,k=1}^{2} P^{kj}\big(\frac{x}{\varepsilon}\big)\frac{\partial^4 f}{\partial x_1^2 \partial x_j \partial x_k}(x_1,0)\,.
\end{aligned}
$$

Consequently, we can estimate the L^2 norm in Ω_ε^+ as follows

$$
\begin{aligned}
\|Z_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} &\le \|\varepsilon^2\Big(B^1 + 2\,\frac{\partial B^{11}}{\partial y_1}\Big)\big(\frac{x}{\varepsilon}\big)\frac{\partial^4 v}{\partial x_1^3 \partial x_2}(x_1,0-)\|_{L^2(\Omega_\varepsilon^+)}\\
&\quad + \varepsilon^2\sum_{j=1}^{2}\|\Big(T\delta_{j1} + 2\,\frac{\partial T^j}{\partial y_1}\Big)\Big(\frac{x-K\mathbf{e}_2}{\varepsilon}\Big)\frac{\partial^2 f}{\partial x_1 \partial x_j}(x_1,K)\|_{L^2(\Omega_\varepsilon^+)} + \mathcal{O}(\varepsilon^3)\\
&\le C\varepsilon^2\|B^1\big(\frac{x}{\varepsilon}\big)\|_{L^2(\Omega_\varepsilon^+)} + C\varepsilon^2\|\frac{\partial B^{11}}{\partial y_1}\big(\frac{x}{\varepsilon}\big)\|_{L^2(\Omega_\varepsilon^+)}\\
&\quad + C\varepsilon^2\|T\Big(\frac{x-K\mathbf{e}_2}{\varepsilon}\Big)\|_{L^2(\Omega_\varepsilon^+)} + C\varepsilon^2\sum_{j=1}^{2}\|\frac{\partial T^j}{\partial y_1}\Big(\frac{x-K\mathbf{e}_2}{\varepsilon}\Big)\|_{L^2(\Omega_\varepsilon^+)} + C\varepsilon^3\\
&\le C\varepsilon^2\sqrt{\varepsilon}\,.
\end{aligned}
$$

In Ω^-,

$$
\begin{aligned}
Z_\varepsilon &= \varepsilon^2\Big(B^1 + 2\frac{\partial B^{11}}{\partial y_1}\Big)\big(\frac{x}{\varepsilon}\big)\frac{\partial^4 v}{\partial x_1^3 \partial x_2}(x_1,0-) + \varepsilon^3 B^{11}\big(\frac{x}{\varepsilon}\big)\frac{\partial^5 v}{\partial x_1^4 \partial x_2}(x_1,0-)\\
&\quad + \varepsilon^3 \sum_{j,k=1}^{2} \big\{P^k\delta_{j1} + 2\frac{\partial P^{kj}}{\partial y_1}\big\}\big(\frac{x}{\varepsilon}\big)\frac{\partial^3 f}{\partial x_1 \partial x_j \partial x_k}(x_1,0)\\
&\quad + \varepsilon^4 \sum_{j,k=1}^{2} P^{kj}\big(\frac{x}{\varepsilon}\big)\frac{\partial^4 f}{\partial x_1^2 \partial x_j \partial x_k}(x_1,0)\,.
\end{aligned}
$$

And analogously,

$$
\begin{aligned}
\|Z_\varepsilon\|_{L^2(\Omega^-)} &\le \|\varepsilon^2\Big(B^1 + 2\frac{\partial B^{11}}{\partial y_1}\Big)\big(\frac{x}{\varepsilon}\big)\frac{\partial^4 v}{\partial x_1^3 \partial x_2}(x_1,0-)\|_{L^2(\Omega^-)} + \mathcal{O}(\varepsilon^3)\\
&\le C\varepsilon^2\|B^1\big(\frac{x}{\varepsilon}\big)\|_{L^2(\Omega^-)} + C\varepsilon^2\|\frac{\partial B^{11}}{\partial y_1}\big(\frac{x}{\varepsilon}\big)\|_{L^2(\Omega^-)} + C\varepsilon^3\\
&\le C\varepsilon^2\sqrt{\varepsilon}\,,
\end{aligned}
$$

which proves (5.4) and (5.5).

From (3.20) and (3.21) we obtain (5.6). It remains to prove (5.7). We first notice that the interior-layer correctors P, P^k, P^{kj}, B, B^1 and B^{11} decay exponentially far from Σ. Thus, their contribution on Γ^+ is negligible. We focus only on significant terms and using the properties of the exterior boundary layer correctors T and T^k we obtain

$$
\begin{aligned}
\frac{\partial U^\varepsilon}{\partial x_2}(x_1,K) &= \varepsilon\big\{\frac{\partial w}{\partial y_2}\Big(\frac{x_1}{\varepsilon},0\Big) + \frac{\partial T}{\partial y_2}\Big(\frac{x_1}{\varepsilon},0\Big)\big\}f(x_1,K)\\
&\quad + \varepsilon^2\sum_{j=1}^{2}\big\{w\Big(\frac{x_1}{\varepsilon},0\Big)\delta_{2j} + \frac{\partial w^j}{\partial y_2}\Big(\frac{x_1}{\varepsilon},0\Big) + \frac{\partial T^j}{\partial y_2}\Big(\frac{x_1}{\varepsilon},0\Big)\big\}\frac{\partial f}{\partial x_j}(x_1,K)\\
&\quad + \varepsilon^3\sum_{i,j=1}^{2}\big\{w^i\Big(\frac{x_1}{\varepsilon},0\Big)\delta_{2j} + \frac{\partial w^{ij}}{\partial y_2}\Big(\frac{x_1}{\varepsilon},0\Big) + T^i\Big(\frac{x_1}{\varepsilon},0\Big)\delta_{2j}\big\}\frac{\partial f}{\partial x_j \partial x_i}(x_1,K)
\end{aligned}
$$

$$= \varepsilon^3 \sum_{i,j=1}^{2} \big\{ w^i\Big(\frac{x_1}{\varepsilon},0\Big)\delta_{2j} + \frac{\partial w^{ij}}{\partial y_2}\Big(\frac{x_1}{\varepsilon},0\Big) + T^i\Big(\frac{x_1}{\varepsilon},0\Big)\delta_{2j} \big\} \frac{\partial f}{\partial x_j \partial x_i}(x_1,K)\,,$$

leading to (5.7). We can now test (5.3) by E^ε and use (5.4)-(5.7) to estimate term by term

$$\begin{aligned} \|\nabla E^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} &= \int_{\Omega_\varepsilon} Z^\varepsilon E^\varepsilon + \int_{\Gamma^+} \eta_\varepsilon E^\varepsilon - \int_\Sigma \zeta_\varepsilon E^\varepsilon \le \\ &\le C\varepsilon^{5/2}\|E^\varepsilon\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^3\|E^\varepsilon\|_{L^2(\Gamma^+)} + C\varepsilon^3\|E^\varepsilon\|_{L^2(\Sigma)}\,. \end{aligned}$$

At this point we use technical results (10.1), (10.2) and (10.3) from Appendix E (the Poincaré inequality in the porous part), to obtain (5.1) and (5.2). □

To prove the above error estimate, we needed the complex approximation (3.19). However, since the obtained error estimate is of order $\varepsilon^{5/2}$ in $H^1(\Omega_\varepsilon)$ and ε^3 in $L^2(\Sigma)$, we can now erase the terms of order superior to that. Those were useful for the proof, but do not actually contribute to the approximation, as far as we know. The simpler approximation reads

$$V^\varepsilon = \varepsilon B\Big(\frac{x}{\varepsilon}\Big)\frac{\partial v}{\partial x_2}(x_1,0) + \begin{cases} \varepsilon^2 w\big(\frac{x}{\varepsilon}\big) f(x) & \text{in } \Omega^+_\varepsilon \\ v(x) & \text{in } \Omega^-\,. \end{cases} \tag{5.8}$$

Corollary 5.2. *Let f and u^ε be like in Theorem 5.1. Let v be the solution to the effective problem* (4.1)-(4.4) *and let V^ε be defined by* (5.8)*. Then, there exists $C>0$ such that*

$$\|u^\varepsilon - V^\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{5/2}\,. \tag{5.9}$$

In fact, if we cut out the boundary layers

$$G^+_\delta = \Omega^+_\varepsilon \cap \{0 < x_2 < \delta\}\,, \quad G^-_\delta = \Omega^- \cap \{-\delta < x_2 < 0\}$$

we do not need any correctors at all.

Corollary 5.3. *Let f and u^ε be like in Theorem 5.1. Let v be the solution to the effective problem* (4.1)-(4.4)*. Then, for any $\delta>0$ there exists $C>0$ such that*

$$\|u^\varepsilon - v\|_{L^2(\Omega^-\setminus G^-_\delta)} \le C\varepsilon^{5/2}\,, \tag{5.10}$$

$$\|u^\varepsilon - \varepsilon^2 w\Big(\frac{x}{\varepsilon}\Big) f\|_{L^2(\Omega^+_\varepsilon\setminus G^+_\delta)} \le C\varepsilon^{5/2}\,. \tag{5.11}$$

6. Appendix A. Auxiliary cell problems

For our interior approximation we use two types of auxiliary problems and the properties of their solutions. For reader's convenience, here we list their main properties. We pick w as the solution of the auxiliary cell problem

$$-\Delta_y w = 1 \quad \text{in } Y^*\,, \tag{6.1}$$

$$w = 0 \quad \text{on } \Gamma\,, \tag{6.2}$$

$$w \text{ is 1-periodic in both } y_1 \text{ and } y_2\,. \tag{6.3}$$

We use it to define the number

$$K = \int_{Y^*} w = \int_{Y^*} |\nabla_y w|^2 > 0\,. \tag{6.4}$$

representing the permeability of the porous medium Ω^+_ε.

The second type auxiliary problems reads (for $k=1,2$)

$$-\Delta_y w^k = 2\frac{\partial w}{\partial y_k} \quad \text{in } Y^*\,, \tag{6.5}$$

$$w^k = 0 \quad \text{on } \Gamma\,, \tag{6.6}$$

$$w^k \text{is 1-periodic in } y_1, y_2\,. \tag{6.7}$$

It is easy to see (testing (6.5) by w and (6.1) by w^k) that

$$\int_{Y^*} w^k = \int_{Y^*} \nabla_y w \, \nabla_y w^k = 0 \,. \tag{6.8}$$

The third type auxiliary problems reads (for $j,k \in \{1,2\}$)

$$-\Delta_y w^{kj} = 2\frac{\partial w^k}{\partial y_j} + \delta_{jk} w \quad \text{in } Y^* \,, \tag{6.9}$$

$$w^{kj} = 0 \quad \text{on } \Gamma \,, \tag{6.10}$$

$$w^{kj} \text{ is 1-periodic in } y_1, y_2 \,. \tag{6.11}$$

We test (6.9) by w, (6.1) by w^{kj} and (6.5) by w^j and combine that to obtain

$$\int_{Y^*} w^{kj} = \delta_{jk} \int_{Y^*} w^2 - \int_{Y^*} \nabla w^i \nabla w^k \,. \tag{6.12}$$

7. Appendix B. Interior-layer problem

To correct the jumps of the approximation and its normal derivative on the interface Σ separating the porous and the free part of the domain, we need the interior-layer correctors. The fundamental results about them can be found in papers by Mikelić and collaborators [7, 8, 9, 11]. For reader's convenience, we formulate those problems and recall their basic properties. The first problem that we have introduced, to keep the approximation continuous and minimize the derivative jump, has the form

$$\Delta_y P = 0 \quad \text{in } \mathcal{T} \,, \tag{7.1}$$

$$P = 0 \quad \text{on } \cup_i \Gamma_i \,, \tag{7.2}$$

$$P(y_1, 0+) + w(y_1, 0) = P(y_1, 0-) + \lambda \,, \tag{7.3}$$

$$\frac{\partial P}{\partial y_2}(y_1, 0+) + \frac{\partial w}{\partial y_2}(y_1, 0) = \frac{\partial P}{\partial y_2}(y_1, 0-) \,, \tag{7.4}$$

$$\nabla_y P \in L^2(\mathcal{T}) \,, \tag{7.5}$$

$$P \text{ is 1-periodic in } y_1 \,. \tag{7.6}$$

The constant λ is not given in advance. It is a part of the problem and is chosen to assure that P vanishes at infinity

$$\lim_{y_2 \to -\infty} P = 0 \,. \tag{7.7}$$

The problem similar to (7.1)-(7.6) has been studied in [7]. Adapted to our notation and geometry, they have studied the problem:

$$\Delta \omega = 0 \quad \text{in } \mathcal{T} \,, \tag{7.8}$$

$$\omega = 0 \ \text{ on } \ \cup_i \Gamma_i \,, \tag{7.9}$$

$$[\omega]_\sigma = -w(y_1, 0) \,, \tag{7.10}$$

$$[\frac{\partial \omega}{\partial y_2}]_\sigma = -\frac{\partial w}{\partial y_2}(y_1, 0) \,, \tag{7.11}$$

$$\nabla \omega \in L^2(\mathcal{T}^+) \,, \quad \nabla \omega \in L^2(\mathcal{T}^-) \,, \tag{7.12}$$

$$\omega \text{ is 1-periodic in } y_1 \,. \tag{7.13}$$

The only difference in [7] is that the considered geometry was upside down compared to ours, i.e. the porous part was down. So, from [7, (1.10)-(1.13)], the problem (7.8)-(7.13) has a unique weak solution such that, for some $\gamma \in \langle 0, 1 \rangle$,

$$\omega \in L^2_{loc}(\mathcal{T}) \,, \quad e^{\gamma |y_2|} \omega \in L^2(\mathcal{T}^+) \,, e^{\gamma |y_2|} \nabla \omega \in L^2(\mathcal{T}^+) \,, \quad e^{\gamma |y_2|} \nabla \omega \in L^2(\mathcal{T}^-) \,.$$

Furthermore, there exists a constant, lets call it λ, such that

$$e^{\gamma |y_2|} (\omega - \lambda) \in L^2(\mathcal{T}^-) \,. \tag{7.14}$$

In addition, ω is smooth (except across Σ) and the exponential decay (7.14) is pointwise in the sense

$$|\omega(y_1, y_2) - \lambda| \le Ce^{-\gamma |y_2|}, \quad y_2 < 0. \tag{7.15}$$

Our boundary-layer function is now constructed as

$$P = \begin{cases} \omega(y_1, y_2) & \text{for } y_2 > 0 \\ \omega(y_1, y_2) - \lambda & \text{for } y_2 < 0 \end{cases} \tag{7.16}$$

to obtain (7.3).

Integrating (7.1) with respect to y_1 from $-1/2$ to $1/2$, and $y_2 \le 0$ and taking into account that $\nabla_y P$ decays to zero as $y_2 \to -\infty$, gives

$$\frac{d^2}{dy_2^2} \int_{-1/2}^{1/2} P(y_1, y_2) dy_1 = 0 \implies \int_{-1/2}^{1/2} P(y_1, y_2) dy_1 = \text{const}.$$

This now ensures that P decays to zero for $y_2 \to -\infty$. We owe the positivity of λ to the maximum principle. The couple of second order boundary layer correctors are

$$\Delta_y P^k + 2\,\delta_{1k} \frac{\partial P}{\partial y_1} = 0 \quad \text{in } \mathcal{T}, \tag{7.17}$$

$$P^k = 0 \quad \text{on } \cup_i \Gamma_i, \tag{7.18}$$

$$P^k(y_1, 0+) + w^k(y_1, 0) = P^k(y_1, 0-) + \lambda^k, \tag{7.19}$$

$$\frac{\partial P^k}{\partial y_2}(y_1, 0+) + \frac{\partial w^k}{\partial y_2}(y_1, 0) = \frac{\partial P^k}{\partial y_2}(y_1, 0-) \tag{7.20}$$

$$\nabla_y P^k \in L^2(\mathcal{T}) \tag{7.21}$$

$$P^k \text{ is 1-periodic in } y_1. \tag{7.22}$$

The constants λ^k are chosen to assure that$\lim_{y_2 \to -\infty} P^k = 0$. Clearly

$$\lambda^k = \int_{-1/2}^{1/2} \big(P^k(y_1, 0+) - P^k(y_1, 0-)\big)\, dy_1 + \int_{-1/2}^{1/2} w^k(y_1, 0)\, dy_1\,,\ k = 1, 2.$$

From the definition of w^k $(k = 1, 2)$ we have

$$-\frac{d^2}{dy_2^2} \int_{-1/2}^{1/2} w^k(y_1, y_2)\, dy_1 = 2 \int_{-1/2}^{1/2} \frac{\partial w}{\partial y_k} dy_1\,.$$

In both cases, the right-hand side is zero, so together with the fact that the only periodic polynomial in y_2 is a constant, we obtain that for any y_2,

$$\int_{-1/2}^{1/2} w^k(y_1, y_2)\, dy_1 = \mathit{const}\,.$$

If the inclusions A are symmetric in the y_1 direction, then w^k and P^k are odd with respect to y_1 and consequently $\lambda^k = 0$. In the end, we need the couple of third order boundary layer correctors P^{kj}, $k, j = 1, 2$ that satisfy

$$\Delta_y P^{kj} + \delta_{1k}\delta_{1j}\Big(2\frac{\partial P^1}{\partial y_1} + P\Big) = 0 \quad \text{in } \mathcal{T}, \tag{7.23}$$

$$P^{kj} = 0 \quad \text{on } \cup_i \Gamma_i, \tag{7.24}$$

$$[P^{kj}]_\sigma + w^{kj}(y_1, 0) = \lambda^{kj}, \tag{7.25}$$

$$[\frac{\partial P^{kj}}{\partial y_2}]_\sigma + \frac{\partial w^{kj}}{\partial y_2}(y_1, 0) = 0, \tag{7.26}$$

$$\nabla_y P^{kj} \in L^2(\mathcal{T}), \tag{7.27}$$

$$P^{kj} \text{ is 1-periodic in } y_1. \tag{7.28}$$

The constants λ^{kj} are chosen to assure that $\lim_{y_2 \to -\infty} P^{kj} = 0$.

8. Appendix C. Interior layer for the normal derivative correction

To correct the normal derivative jump on the interface Σ we introduce the particular form of the interior-layer corrector that will, eventually, produce the effective interface condition on Σ. It reads:

$$-\Delta_y B = 0; \quad \text{in } \mathcal{T}, \tag{8.1}$$

$$B = 0 \quad \text{on } \cup_i \Gamma_i, \tag{8.2}$$

$$B(y_1, 0+) - B(y_1, 0-) = -d, \tag{8.3}$$

$$\frac{\partial B}{\partial y_2}(y_1, 0+) - \frac{\partial B}{\partial y_2}(y_1, 0-) = 1, \tag{8.4}$$

$$\nabla_y B \in L^2(\mathcal{T}), \tag{8.5}$$

$$B \text{ is 1-periodic in } y_1. \tag{8.6}$$

Again, the constant d is chosen such that

$$\lim_{y_2 \to -\infty} B = 0. \tag{8.7}$$

To prove that such B exists we use again the result from [7]. Adapted to our notation and geometry (from [7, (1.17)-(1.20)]), we first obtain the solution M of the similar problem, where constant d is taken to be zero and decay at $-\infty$ is imposed only on $\nabla_y M$:

$$-\Delta_y M = 0 \quad \text{in } \mathcal{T}, \tag{8.8}$$

$$M = 0 \quad \text{on } \cup_i \Gamma_i, \tag{8.9}$$

$$M(y_1, 0+) - M(y_1, 0-) = 0, \tag{8.10}$$

$$\frac{\partial M}{\partial y_2}(y_1, 0+) - \frac{\partial M}{\partial y_2}(y_1, 0-) = -1, \tag{8.11}$$

$$\nabla_y M \in L^2(\mathcal{T}), \tag{8.12}$$

$$M \text{ is 1-periodic in } y_1. \tag{8.13}$$

Such M exists, it is unique and in fact smooth (except across σ). Moreover, there exists $\gamma_0 \in \langle 0, 1\rangle$ such that

$$M \in H^1_{loc}(\mathcal{T}), \quad e^{\gamma_0 |y_2|} M \in L^2(\mathcal{T}^+), \quad e^{\gamma_0 |y_2|} \nabla_y M \in L^2(\mathcal{T}) \tag{8.14}$$

and there exists a constant, lets call it d, such that

$$e^{\gamma_0 |y_2|}\,(M - d) \in L^2(\mathcal{T}^-). \tag{8.15}$$

Recalling, from [7], the weak formulation: find $M \in V$

$$V = \{\phi \in L^2_{loc}(\mathcal{T})\,; \quad \nabla\phi \in L^2(\mathcal{T}), \quad \phi = 0 \quad \text{on } \cup_i \Gamma_i, \quad \phi \text{ is 1-periodic in } y_1\}$$

such that, for any $\phi \in V$,

$$\int_{\mathcal{T}} \nabla_y M \, \nabla_y \phi = \int_\sigma \phi(y_1, 0)\, dy_1.$$

Due to the Lax-Milgram theorem such M exists. Taking M for the test function ϕ gives

$$\int_\sigma M(y_1, y_2)\, dy_1 = \int_{-1/2}^{1/2} M(y_1, 0)\, dy_1 = \int_{\mathcal{T}} |\nabla_y M|^2 > 0.$$

As with the corrector P, here we integrate (8.8) with respect to y_1 from $-1/2$ to $1/2$ and $y_2 \le 0$. Taking into account that $\nabla_y M$ decays to zero as $y_2 \to -\infty$, we obtain

$$\frac{d^2}{dy_2^2} \int_{-1/2}^{1/2} M(y_1, y_2) dy_1 = 0 \implies \int_{-1/2}^{1/2} M(y_1, y_2) dy_1 = const.$$

The constant has to be equal to d to ensure the exponential decay of M for $y_2 \to -\infty$ and it satisfies

$$d = \int_{\mathcal{T}} |\nabla_y M|^2 > 0. \tag{8.16}$$

Since for the corrector B we want to assure that it decays to zero when $y_2 \to -\infty$ we define B as

$$B = \begin{cases} -M(y_1, y_2) & \text{for } y_2 > 0 \\ -M(y_1, y_2) + d & \text{for } y_2 < 0\,. \end{cases} \tag{8.17}$$

Now, because $[M]_\sigma = 0$, consequently $[B]_\sigma = B(y_1, 0+) - B(y_1, 0-) = -d$. Also,

$$\big[\frac{\partial B}{\partial y_2}\big]_\sigma = -\big[\frac{\partial M}{\partial y_2}\big]_\sigma = 1\,.$$

To get the error estimate, we also need the corrector B^1 such that

$$-\Delta_y B^1 - 2\frac{\partial B}{\partial y_1} = 0 \quad \text{in } \mathcal{T}\,, \tag{8.18}$$

$$B^1 = 0 \quad \text{on } \cup_i \Gamma_i\,, \tag{8.19}$$

$$B^1(y_1, 0+) - B^1(y_1, 0-) = 0\,, \tag{8.20}$$

$$\frac{\partial B^1}{\partial y_2}(y_1, 0+) - \frac{\partial B^1}{\partial y_2}(y_1, 0-) = 0\,, \tag{8.21}$$

$$\nabla_y B^1 \in L^2(\mathcal{T})\,, \tag{8.22}$$

$$B^1 \text{ is 1-periodic in } y_1\,. \tag{8.23}$$

Again, such B^1 exponentially decays when $y_2 \to \pm\infty$ towards some constants. More precisely, there exist constants $d^1 \in \mathbf{R}$ and $\gamma_1 > 0$, such that

$$B^1 \in H^1_{loc}(\mathcal{T}), \quad e^{\gamma_1 |y_2|} B^1 \in L^2(\mathcal{T}^+), \quad e^{\gamma_1 |y_2|}(B^1 - d^1) \in L^2(\mathcal{T}^+)\,, \quad e^{\gamma_1 |y_2|}\, \nabla_y B^1 \in L^2(\mathcal{T})\,. \tag{8.24}$$

Clearly, the constant at $y_2 \to +\infty$ is zero, due to the Dirichlet condition on solid inclusions. For d^1 (the one at $y_2 \to -\infty$), simple calculations show that it is also zero. Indeed, testing (8.18) by B and (8.1) by B^1 gives

$$\int_\Sigma B^1(y_1, 0)\, dy_1 = \int_{\mathcal{T}} \nabla B \nabla B^1 = 0\,.$$

On the other hand, integrating (8.18) implies that for $y_2 < 0$

$$\frac{d^2}{dy_2^2} \int_{-1/2}^{1/2} B^1(y_1, y_2)\, dy_1 \implies \int_{-1/2}^{1/2} B^1(y_1, y_2)\, dy_1 = const. = 0\,.$$

Thus $d^1 = 0$ and $B^1 \in H^1(\mathcal{T})$. We need the next order corrector B^{11} as well, defined by the boundary value problem

$$-\Delta_y B^{11} - 2\frac{\partial B^1}{\partial y_1} - B = 0 \quad \text{in } \mathcal{T}\,, \tag{8.25}$$

$$B^{11} = 0 \quad \text{on } \cup_i \Gamma_i\,, \tag{8.26}$$

$$B^{11}(y_1, 0+) - B^{11}(y_1, 0-) = 0\,, \tag{8.27}$$

$$\frac{\partial B^{11}}{\partial y_2}(y_1, 0+) - \frac{\partial B^{11}}{\partial y_2}(y_1, 0-) = 0\,, \tag{8.28}$$

$$\nabla_y B^{11} \in L^2(\mathcal{T})\,, \tag{8.29}$$

$$B^{11} \text{ is 1- periodic in } y_1\,. \tag{8.30}$$

As for the previous two problems, there exist constants $d^{11} \in \mathbf{R}$ and $\gamma_{11} > 0$, such that

$$B^{11} \in H^1_{loc}(\mathcal{T}), \quad e^{\gamma_{11} |y_2|} B^{11} \in L^2(\mathcal{T}^+), \quad e^{\gamma_{11} |y_2|}(B^{11} - d^{11}) \in L^2(\mathcal{T}^+)\,, \quad e^{\gamma_{11} |y_2|}\, \nabla_y B^{11} \in L^2(\mathcal{T}) \tag{8.31}$$

We will not study d^{11} here as it influences our approximation only in the higher order terms and is irrelevant for our result.

9. Appendix D. Exterior boundary layer

The approximation in the porous part Ω_ε^+ near Γ^+ satisfies

$$u^\varepsilon \approx U_I^\varepsilon = \varepsilon^2 w(y) f(x) + \varepsilon^3 \sum_{k=1}^{2} w^k(y) \frac{\partial f}{\partial x_k}(x) + \varepsilon^4 \sum_{k,j=1}^{2} w^{kj}(y) \frac{\partial^2 f}{\partial x_j \partial x_k}(x)\,,$$

but does not satisfy the upper boundary condition on Γ^+, i.e. for $x_2 = K$. The normal derivative on Γ^+ is of order ε which is not enough for our purpose. We need, at least ε^2. Thus we correct it by adding the boundary layer correctors T and T^k, $k = 1, 2$, and get the approximation of the form

$$u^\varepsilon \approx \varepsilon^2 \Big[w(y) + T\Big(\frac{x - K\mathbf{e}_2}{\varepsilon}\Big)\Big] f(x_1, K) + \varepsilon^3 \sum_{k=1}^{2} \Big[w^k(y) + T^k\Big(\frac{x - K\mathbf{e}_2}{\varepsilon}\Big)\Big] \frac{\partial f}{\partial x_k}(x_1, K)\,.$$

We do not correct the ε^4 term (even though it would be easy) since we do not need to, at this level of approximation.

Functions T and T^k are defined in a semi-infinite perforated strip

$$\mathcal{G} = \langle -1/2, 1/2\rangle \times \langle -\infty, 0\rangle \setminus \Big(\cup_{k=1}^{+\infty} (A - k\,\mathbf{e}_2)\Big)\,.$$

We choose them such that they vanish far from the upper boundary and that

$$\frac{\partial}{\partial y_2}\left[w(y) + T(y)\right] = 0 \quad \text{for } y_2 = 0\,, \tag{9.1}$$

$$\frac{\partial}{\partial y_2}\left[w^k(y) + T^k(y)\right] = -w(y_1, 0)\delta_{2k} \quad \text{for } y_2 = 0\,. \tag{9.2}$$

They must satisfy the Laplace/Poisson equations and decay at infinity. Thus

$$-\Delta_y T = 0 \quad \text{in } \mathcal{G} \tag{9.3}$$

$$T = 0 \quad \text{on } \cup_k \Gamma_k = \cup_k \partial(A - k\,\mathbf{e}_2) \tag{9.4}$$

$$T \text{ is 1-periodic in } y_1 \tag{9.5}$$

$$\frac{\partial T}{\partial y_2}(y_1, 0) = -\frac{\partial w}{\partial y_2}(y_1, 0) \tag{9.6}$$

$$\nabla_y T \in L^2(\mathcal{G})\,. \tag{9.7}$$

Analogously,

$$-\Delta_y T^k = 2\delta_{1k}\frac{\partial T}{\partial y_k} \ \text{ in } \ \mathcal{G} T^k = 0 \ \text{ on } \ \cup_k \Gamma_k \frac{\partial T^k}{\partial y_2}(y_1, 0) = -\frac{\partial w^k}{\partial y_2}(y_1, 0) - w(y_1, 0)\delta_{2k} T^k \ \text{ is 1-periodic in } y_1 \nabla_y T^k \in L^2(\mathcal{G})\,. \tag{9.8}$$

It is known that, because (9.4) and (9.8), T and T^k decay to zero as $y_2 \to -\infty$ and they do so exponentially (see e.g. [7]) in the sense that for some $\beta > 0$,

$$e^{\beta|y_2|} T\,, \quad e^{\beta|y_2|} T^k \in H^1(\mathcal{G})\,. \tag{9.9}$$

10. Appendix E. Technical lemma

For reader's convenience, we recall the technical result that is used in the error estimate. It is a version of the Poincaré inequality for perforated domains.

Lemma 10.1. *For any $\phi \in H^1(\Omega_\varepsilon^+)$ such that $\phi = 0$ on Γ_ε there exists a constant $C > 0$, independent on ε, such that*

$$|\phi|_{L^2(\Omega_\varepsilon^+)} \le C\varepsilon |\nabla\phi|_{L^2(\Omega_\varepsilon^+)} \tag{10.1}$$

$$|\phi|_{L^2(\Sigma)} \le C\sqrt{\varepsilon} |\nabla\phi|_{L^2(\Omega_\varepsilon^+)} \tag{10.2}$$

$$|\phi|_{L^2(\Gamma^+)} \le C\sqrt{\varepsilon} |\nabla\phi|_{L^2(\Omega_\varepsilon^+)}\,. \tag{10.3}$$

The proof can be found in [7, Proposition 1.1] or in [8].

11. Conclusion

We derived an effective interface condition for the Poisson problem in a partially porous medium by combining homogenization with asymptotic matching. The resulting Robin-type condition captures the influence of the porous microstructure on the macroscopic solution through a correction involving the normal derivative and a source-term contribution of order ε^2. The error estimates are of the order higher than that, so the result is justified. This improves the classical interface description and provides a more accurate model whenever the pore size is not negligible compared to the macroscopic scale.

From the physical point of view, this effective interface condition includes the additional source term which accounts for the flux contribution from the porous region. This provides a deeper understanding of the physical phenomena involved, justifying e.g. the thermal lag appearing in the transition zone of a partially porous medium.

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Tomislav Fratrović
Faculty of Transport and Traffic Sciences, University of Zagreb, Croatia
Email address: tfratrovic@fpz.unizg.hr

Eduard Marušić-Paloka
Department of Mathematics, Faculty of Science, University of Zagreb, Croatia
Email address: emarusic@math.hr