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REARRANGEMENTS OF FUNCTIONS AND OPTIMIZATION PROBLEMS IN ORLICZ SPACES

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Abstract. We develop the rearrangement optimization theory in the setting of Orlicz spaces. By using this theory, we can investigate the rearrangement minimization and maximization problems related to the boundary value problems containing the Φ-Laplacian and the Φ-Kirchhoff operator separately. We show the existence of solutions and derive the corresponding optimality conditions for both the minimization and maximization problems. For the maximization problems, we also prove the uniqueness of solutions.

1. Introduction

In this article we consider the boundary value problem

$$\begin{gathered} -\nabla \cdot (a(|\nabla u|)\nabla u) = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary, $a : (0,\infty) \to (0,\infty)$ is a continuous function such that the map

$$\varphi(t) = \begin{cases} a(|t|)t & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

is an odd strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, and f is in the Orlicz space $L^{\Phi^*}(\Omega)$ with $\Phi(t) := \int_0^t \varphi(s)ds$ $(t \in [0,\infty))$, which will be defined later in Section 2.1. In Section 3.1, we will show the problem (1.1) has a unique solution, denoted u_f (to emphasize its dependence on f), in the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$. Associated with (1.1), our objective functional is defined as

$$J_1(f) := \int_\Omega \Phi(|\nabla u_f|)\,\mathrm{d}x - \int_\Omega f u_f \,\mathrm{d}x.$$

Now, we are interested in the following rearrangement optimization problems (ROPs):

$$\inf_{f\in\mathcal{R}(f_0)} J_1(f) \quad \text{and} \quad \sup_{f\in\mathcal{R}(f_0)} J_1(f) \tag{1.2}$$

where f_0 is a fixed nontrivial function in $L_+^{\Phi^*}(\Omega)$, and the admissible set $\mathcal{R}(f_0)$ contains all the (measurable) functions on Ω having the same distribution function with f_0, see Section 2.1 for the precise definition. In order to investigate the problems in (1.2), we need to build the framework of rearrangement optimization theory in the setting of Orlicz spaces. To the best of our knowledge, such a theory does not exist in the current literature. So, our main target in this paper is to investigate the rearrangement optimization theory in the setting of Orlicz spaces.

Before presenting the main results, let us briefly introduce the history of the problems in (1.2). If $a \equiv 1$ (and then $\Phi(t) = \frac{t^2}{2}$), the left hand side operator in (1.1) becomes (negative) Laplacian. In this case, the maximization problem in (1.2) has been investigated by Burton and Mcleod in [8], while Burton has considered the minimization problem of (1.2) in [6, 7]. Moreover, in

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[6, 7], Burton has developed a series of tools for analyzing rearrangement optimization problems in the setting of L^p spaces. If $a(t) = t^{p-2}$ with $p \in (1, \infty)$, the operator in (1.1) becomes the famous $p-$Laplacian. When $p \neq 2$, the corresponding problems in (1.2) have been researched by the authors in [14, 17, 26]. On the other hand, many other types of rearrangement optimization problems in the setting of L^p spaces have been investigated in the literature, see [3] and the related extensive literature in [30].

Now, we present two main results which are of fundamental importance in the analysis of rearrangement optimization problems, see (1.2), in the setting of Orlicz spaces. Before that, we need some preparations. If ψ satisfies the settings described in Section 2.2 and $\Psi(t) = \int_0^t \psi(s)\,\mathrm{d}s$, then the space $L^{\Psi}(\Omega)$ is reflexive and its dual space is $L^{\Psi^*}(\Omega)$, see Remark 2.4 below for details. Our first main result of the rearrangement optimization theory in the setting of Orlicz spaces is

Theorem 1.1. *Let $f_0 \in L^{\Psi}(\Omega)$, $g \in L^{\Psi^*}(\Omega)$, and $\mathcal{R} := \mathcal{R}(f_0)$. Suppose there is a increasing function η such that $\hat{f} = \eta(g) \in \mathcal{R}$. Then, $\hat{f}$ is the unique maximizer of the linear functional $L(f) = \int_\Omega fg\,\mathrm{d}x$ relative to $f \in \overline{\mathcal{R}}$. Here, $\overline{\mathcal{R}}$ denotes the weak closure of $\mathcal{R}$.*

Observing that in the conclusion of Theorem 1.1, $\hat{f}$ is not only the unique maximizer with respect to $\mathcal{R}$, but also with respect to a larger set $\overline{\mathcal{R}}$. Theorem 1.1 will play an important role in the analysis of the maximization problem in (1.2).

To resolve the minimization problem in (1.2), we introduce our second main result of the rearrangement optimization theory in the setting of Orlicz spaces as follows

Theorem 1.2. *Let $f_0 \in L^{\Psi}(\Omega)$ and $\mathcal{R} := \mathcal{R}(f_0)$. Suppose $J : \overline{\mathcal{R}} \to \mathbb{R}$ is convex, weakly sequentially continuous, and satisfies*

$$\lim_{t\to 0^+} \frac{J(f + t(g - f)) - J(f)}{t} = \int_\Omega G(f)(g - f)\,\mathrm{d}x, \quad \forall f, g \in \overline{\mathcal{R}}, \tag{1.3}$$

where $G : \overline{\mathcal{R}} \to L^{\Psi^*}(\Omega)$ is an operator. Then, we have

(i) *J attains a maximum value relative to $\mathcal{R}$.*

(ii) *Let $\hat{f}$ be a maximizer of J relative to $\mathcal{R}$, and suppose J is locally strictly convex at $\hat{f}$, i.e.*

$$\exists \varepsilon > 0 : \quad J \text{ is strictly convex on } \overline{\mathcal{R}} \cap B(\hat{f}, \varepsilon)$$

where $B(\hat{f}, \varepsilon)$ denotes the open ball centered at $\hat{f}$ with radius ε in $L^{\Psi}(\Omega)$. Then, $\hat{f} = \eta \circ G(\hat{f})$ a.e. in Ω, for some increasing function η.

By applying Theorems 1.1 and 1.2, we can show the existence of solutions for the problems in (1.2). Moreover, we can derive an optimality condition with a simple formula. For the maximization problem in (1.2), the uniqueness of the solution is also guaranteed. The precise statements of the results are presented in Theorems 3.5 and 3.6 below.

Next, we will use Theorems 1.1 and 1.2 to investigate the rearrangement optimization problems related to the nonlocal boundary value problem

$$\begin{gathered} -M\Big(\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\Big)\nabla \cdot (a(|\nabla v|)\nabla v) = f(x) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $M : [0, \infty) \to [0, \infty)$ is a continuous increasing function satisfying $M(t) > 0$ for all $t > 0$, and the conditions for a, Φ, Ω are the same as those for the problem (1.1). In Section 3.2, we will show that problem (1.4) has a unique solution, denoted v_f, in the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$. Regarding (1.4), our objective functional is defined as

$$J_2(f) = \hat{M}\Big(\int_\Omega \Phi(|\nabla v_f|)\,\mathrm{d}x\Big) - \int_\Omega f v_f\,\mathrm{d}x,$$

where $\hat{M}(t) = \int_0^t M(s)\,\mathrm{d}s$ $(t \geq 0)$.

Similarly as in (1.2), we are interested in the following problems:

$$\inf_{f\in\mathcal{R}(f_0)} J_2(f) \quad \text{and} \quad \sup_{f\in\mathcal{R}(f_0)} J_2(f), \tag{1.5}$$

with a prescribed function f_0 defined as above. In parallel with the results obtained for (1.2), we can use Theorems 1.1 and 1.2 to obtain similar results for the problems (1.5) involving a nonlocal operator here. Note that if $a \equiv 1$ and $M(s) = \alpha s + \beta$ with $\alpha, \beta > 0$, then the Φ-Kirchhoff problem (1.4) becomes the famous Kirchhoff problem, see [2, 23]. For $a(t) = t^{p-2}$ with $p \in (1, \infty)$, the corresponding problems in (1.5) have been investigated in [16].

Finally, we would like to mention other problems related to (1.1) and (1.2). For the boundary value problem (1.1), the authors in [12, 28] have investigated related problems. In recent years, rearrangement optimization theory has be used to analyze some optimization or control problems appearing from biology and control theory, see for example [4, 15, 18]. Two other interesting optimization problems in the setting of Orlicz spaces are discussed in [21, 27]. In addition, it is interesting to notice that if $a(t) = t^{p-2} + t^{q-2}$ for some different $p, q \in (1, \infty)$, the operator in (1.1) which we will call it Φ-Laplacian becomes the (p, q)-Laplacian, see Remark 3.7 for details.

This article is structured as follows: In Section 2.1, we introduce some backgrounds for Orlicz and Sobolev-Orlicz spaces. The rearrangement optimization theory in the setting of Orlicz spaces will be developed in Section 2.2. More precisely, we will investigate the properties of the rearrangement class $\mathcal{R}(f_0)$ and its weak closure, and then provide the proofs of the two main results Theorems 1.1 and 1.2. Section 3 is devoted to the analysis of the ROPs (1.2) and (1.5). In the final appendix, we will discuss the relations between two structural conditions for the function $a(\cdot)$ in the definition of the operator.

2. Rearrangement optimization theory in Orlicz spaces

We will introduce some backgrounds of Orlicz and Orlicz-Sobolev spaces in the first subsection. Based on this, we will continue to develop the theory for solving rearrangement optimization problems in the setting of Orlicz space in the second subsection.

2.1. **Orlicz spaces.**

Suppose $\varphi : [0, \infty) \to [0, \infty)$ is a function satisfying

$$\varphi \text{ is continuous, strictly increasing, } \varphi(0) = 0, \text{ and } \lim_{t \to \infty} \varphi(t) = \infty. \tag{2.1}$$

Thus, the function $\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s$, $t \in [0, \infty)$, is an N-function, see Chapter 8 in [1] for the definition. The conjugate of Φ, denoted Φ^*, is defined by $\Phi^*(t) = \int_0^t \varphi^{-1}(s)\,\mathrm{d}s$, for all $t \in [0, \infty)$. It is known that Φ^* is also an N-function, and can be reformulated as:

$$\Phi^*(t) = \sup_{s \geq 0}(st - \Phi(s)).$$

Let Ω be a bounded domain in $\mathbb{R}^N$. The set

$$K_\Phi(\Omega) = \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x < \infty\big\},$$

is called the *Orlicz class* while the *Orlicz space* is defined by

$$L^\Phi(\Omega) = \big\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \lim_{\tau \to 0^+} \int_\Omega \Phi(\tau|u(x)|)\,\mathrm{d}x = 0\big\}.$$

$L^\Phi(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$\|u\|_\Phi = \inf\big\{\tau > 0 : \int_\Omega \Phi\big(\frac{|u(x)|}{\tau}\big)\,\mathrm{d}x \leq 1\big\}.$$

The following Hölder type inequality holds, see [1, Section 8.11],

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \leq 2\|u\|_\Phi\|v\|_{\Phi^*}, \quad \forall u \in L^\Phi(\Omega),\ v \in L^{\Phi^*}(\Omega). \tag{2.2}$$

Now, we assume that there exist $t_0 > 0$ and two positive constants r and p such that

$$1 < r \leq \frac{t\varphi(t)}{\Phi(t)} \leq p < \infty, \quad \forall t \geq t_0. \tag{2.3}$$

Lemma 2.1. *The following embeddings are continuous*

$$L^p(\Omega) \hookrightarrow L^\Phi(\Omega) \hookrightarrow L^r(\Omega). \tag{2.4}$$

Proof. Since $\Phi(t)=\int_0^t \varphi(s)\,\mathrm{d}s$, we have

$$\frac{r}{t}\le \frac{\Phi'(t)}{\Phi(t)}\le \frac{p}{t}.$$

For $t>\tau\ge t_0$, we integrate the above inequality from τ to t and obtain

$$r\log\frac{t}{\tau}\le \log\frac{\Phi(t)}{\Phi(\tau)}\le p\log\frac{t}{\tau}.$$

Consequently, we have

$$\Phi(t)\ge \frac{\Phi(\tau)}{\tau^r}t^r \quad \text{and} \quad \Phi(t)\le \frac{\Phi(\tau)}{\tau^p}t^p. \tag{2.5}$$

Let $\tau=t_0$, and for $t\ge t_0$, the following inequalities hold

$$\Phi(t)\ge \Big(\frac{\Phi(t_0)^{\frac{1}{r}}}{t_0}t\Big)^r \quad \text{and} \quad \Phi(t)\le \Big(\frac{\Phi(t_0)^{\frac{1}{p}}}{t_0}t\Big)^p.$$

Finally, we can apply [1, Theorem 8.12 (b)] to conclude. □

Remark 2.2. By setting $t=2\tau$ in the second inequality of (2.5), we infer that Φ satisfies the Δ_2-condition near infinity, see [1, Section 8.6] for the definition, with

$$\Phi(2\tau)\le 2^p\Phi(\tau),\quad \forall \tau\ge t_0.$$

In addition, for nontrivial $u\in L^\Phi(\Omega)$, we can derive

$$\int_\Omega \Phi\Big(\frac{|u(x)|}{\|u\|_\Phi}\Big)\,\mathrm{d}x=1, \tag{2.6}$$

see [21, equation (2.2)].

Lemma 2.3. *For all $s\ge \varphi(t_0)$, it holds*

$$1<p'\le \frac{s\varphi^{-1}(s)}{\Phi^*(s)}\le r'<\infty,$$

where r',p' denote the conjugate exponent of r,p respectively.

Proof. By setting $s=\varphi(t)$, we know from a similar graph in Page 264 of [1] that

$$t\varphi(t)=s\varphi^{-1}(s)=\Phi^*(s)+\Phi(t).$$

Then, the result easily follows from this relation. □

Remark 2.4. From Lemma 2.1 and Lemma 2.3, we deduce the following embeddings are continuous

$$L^{r'}(\Omega)\hookrightarrow L^{\Phi^*}(\Omega)\hookrightarrow L^{p'}(\Omega).$$

On the other hand, the Δ_2-condition implies that $L^\Phi(\Omega)$ and $K_\Phi(\Omega)$ are identical, and the dual of $L^\Phi(\Omega)$ coincides with $L^{\Phi^*}(\Omega)$, see Section 8.14 and Theorem 8.19 in [1]. From Remark 2.2, we can deduce both Φ and Φ^* satisfy the Δ_2-condition near infinity. Consequently, we can apply [1, Theorem 8.20] to infer $L^\Phi(\Omega)$ is reflexive. Moreover, the following equivalence of convergence is true in $L^\Phi(\Omega)$,

$$\|u_n-u\|_\Phi\to 0 \iff \int_\Omega \Phi(|u_n(x)-u(x)|)\,\mathrm{d}x\to 0, \tag{2.7}$$

see [1, Section 8.13] and [21, Lemma 2.1]. Observe that (2.7) is valid if we replace Φ by any $N-$function satisfying the Δ_2-condition near infinity when Ω is bounded.

The *Orlicz-Sobolev space* is defined as

$$W^{1,\Phi}(\Omega)=\Big\{u\in L^\Phi(\Omega):\frac{\partial u}{\partial x_i}\in L^\Phi(\Omega), i=1,\dots,N\Big\}.$$

It is well known that $W^{1,\Phi}(\Omega)$ endowed with the norm $\|u\|_{1,\Phi}=\|\,|\nabla u|\,\|_\Phi+\|u\|_\Phi$ is a reflexive Banach space, see Remark 2.4 above or [1, Theorem 8.31(b)]. The space $W_0^{1,\Phi}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to $\|u\|_{1,\Phi}$-norm. Using the Poincaré inequality in Orlicz-Sobolev spaces (see [21, Lemma 2.4]), it follows that $\|u\| := \| \, |\nabla u| \, \|_\Phi$ is equivalent to $\|u\|_{1,\Phi}$.

The following useful result is in the spirit of [24, Lemma 2.2, p. 72]. For a Lebesgue measurable set $E \subseteq \mathbb{R}^N$, we will use $|E|$ to denote its Lebesgue measure.

Lemma 2.5. *Suppose $c, \alpha > 0$, $\|u_k\|_{\Phi,\Omega} \le c$ and*
$$\|f_k\|_{\Phi^*,\Omega'} \le \omega(|\Omega'|),$$
holds for all $k \in \mathbb{N}$ and $\Omega' \subseteq \Omega$ with $|\Omega'| \le \alpha$, where the function $\omega : [0,\alpha] \to [0,\infty)$ satisfies $\omega(0) = 0$ and ω is continuous at 0. If $u_k \to u$ a.e. on Ω and $f_k \rightharpoonup f$ in $L^{\Phi^*}(\Omega)$, then $\int_\Omega u_k f_k \,\mathrm{d}x \to \int_\Omega uf \,\mathrm{d}x$ as $k \to \infty$.

Proof. As $u_k \to u$ a.e. on Ω, we infer from Egorov's Theorem to obtain that u_n converge to u almost uniformly on Ω. So, for $\delta \in (0,\alpha)$, we infer the existence of $\Omega^\delta \subseteq \Omega$ with $|\Omega^\delta| \le \delta$ such that u_k converge to u uniformly on $\Omega \setminus \Omega^\delta$. Observe that
$$\begin{aligned}
&\Big| \int_\Omega u_k f_k \,\mathrm{d}x - \int_\Omega uf \,\mathrm{d}x \Big| \\
&= \Big| \int_\Omega (u_k - u) f_k \,\mathrm{d}x + \int_\Omega u(f_k - f) \,\mathrm{d}x \Big| \\
&\le \Big| \int_{\Omega^\delta} (u_k - u) f_k \,\mathrm{d}x \Big| + \Big| \int_{\Omega\setminus\Omega^\delta} (u_k - u) f_k \,\mathrm{d}x \Big| + \Big| \int_\Omega u(f_k - f) \,\mathrm{d}x \Big|.
\end{aligned}$$
On the other hand, by using Fatou's lemma, we know $u \in L^\Phi(\Omega)$ and $\|u\|_{\Phi,\Omega} \le c$. So, the above inequality leads to
$$\begin{aligned}
&\Big| \int_\Omega u_k f_k \,\mathrm{d}x - \int_\Omega uf \,\mathrm{d}x \Big| \\
&\le 2(\|u_k\|_{\Phi,\Omega} + \|u\|_{\Phi,\Omega}) \|f_k\|_{\Phi^*,\Omega^\delta} + \|u_k - u\|_{\infty,\Omega\setminus\Omega^\delta} \|f_k\|_{1,\Omega} + \Big| \int_\Omega u(f_k - f) \,\mathrm{d}x \Big| \\
&\le 4c\omega(|\Omega^\delta|) + c_1 \|u_k - u\|_{\infty,\Omega\setminus\Omega^\delta} \|f_k\|_{\Phi^*,\Omega} + \Big| \int_\Omega u(f_k - f) \,\mathrm{d}x \Big|,
\end{aligned}$$
for some $c_1 > 0$, where we have used the generalized Hölder's inequality (2.2) in the first inequality, and Lemmas 2.1 and 2.3 in the second inequality. Fix $\epsilon > 0$. As $\lim_{t\to 0} \omega(t) = 0$, we may find a sufficiently small $\delta \in (0,\alpha)$ such that
$$4c\omega(|\Omega^\delta|) < \epsilon \tag{2.8}$$
and u_k converge to u uniformly on $\Omega \setminus \Omega^\delta$. Remembering that $f_k \rightharpoonup f$ in $L^{\Phi^*}(\Omega)$, we deduce $\|f_k\|_{\Phi^*,\Omega}$ is bounded. So, for sufficiently large k, we have
$$c_1 \|u_k - u\|_{\infty,\Omega\setminus\Omega^\delta} \|f_k\|_{\Phi^*,\Omega} < \epsilon \quad \text{and} \quad \Big| \int_\Omega u(f_k - f) \,\mathrm{d}x \Big| < \epsilon. \tag{2.9}$$
Combining (2.8) and (2.9), we have $\int_\Omega u_k f_k \,\mathrm{d}x \to \int_\Omega uf \,\mathrm{d}x$ as $k \to \infty$, which is the desired result. □

Lemma 2.6. *If $f \in L^\Phi(\Omega)$, then $\varphi(|f|) \in L^{\Phi^*}(\Omega)$.*

Proof. It suffices to show that
$$\int_\Omega \Phi^*(\varphi(|f(x)|)) \,\mathrm{d}x < \infty.$$
Since φ, φ^{-1} are increasing, we infer that
$$\begin{aligned}
\int_\Omega \Phi^*(\varphi(|f(x)|)) \,\mathrm{d}x &= \int_\Omega \int_0^{\varphi(|f(x)|)} \varphi^{-1}(s) \,\mathrm{d}s \,\mathrm{d}x \\
&\le \int_{\{|f(x)|<t_0\}} \varphi(t_0) t_0 \,\mathrm{d}x + \int_{\{|f(x)|\ge t_0\}} \varphi(|f(x)|)|f(x)| \,\mathrm{d}x \\
&\le \varphi(t_0) t_0 |\Omega| + p \int_\Omega \Phi(|f(x)|) \,\mathrm{d}x < \infty,
\end{aligned}$$

where we have used (2.3) in the second inequality, and the fact $f \in L^{\Phi}(\Omega)$ in the last inequality. Therefore, $\varphi(|f(x)|) \in L^{\Phi^*}(\Omega)$ as desired. $\square$

2.2. **Rearrangements of functions and optimization principles.** In this section, we suppose (ψ, Ψ) is a pair of functions from $[0,\infty)$ to $[0,\infty)$ satisfying (2.1), $\Psi(t) = \int_0^t \psi(s)\,\mathrm{d}s$, and (2.3) for some valid constants t_0, r, p with $\psi = \varphi, \Psi = \Phi$. Let (X, Σ, μ), (X', Σ', μ') be two measure spaces satisfying $\mu(X) = \mu'(X') < \infty$. For two measurable functions $f : X \to \mathbb{R}$ and $g : X' \to \mathbb{R}$, we say f is a rearrangement of g, provided that for all $\beta \in \mathbb{R}$,

$$\lambda_{f,\mu}(\beta) := \mu(\{x \in X : f(x) \geq \beta\}) = \mu'(\{x \in X' : g(x) \geq \beta\}) =: \lambda_{g,\mu'}(\beta).$$

Given a measurable function $f_0 : X' \to \mathbb{R}$, the rearrangement class $\mathcal{R}_{\mu,X}(f_0)$ generated by f_0 on X is defined as

$$\mathcal{R}_{\mu,X}(f_0) := \{f : X \to \mathbb{R} \text{ measurable: } f \text{ is a rearrangement of } f_0\}.$$

When the measure μ and the space X are clear from the context, we write $\mathcal{R}(f_0)$ instead of $\mathcal{R}_{\mu,X}(f_0)$, for simplicity. Let us start by defining one important rearrangement.

Definition 2.7. Let $f : X \to \mathbb{R}$ be measurable. The decreasing rearrangement of f on $(0, \mu(X))$ is defined by $f^{\Delta}(s) := \max\{\alpha : \lambda_f(\alpha) \geq s\}$. Clearly, f^{Δ} is a rearrangement of f (see Lemma 1 in [6]).

Henceforth, in this section, we focus on the case $X = \Omega$ and $X' = \Omega'$, where Ω, Ω' are two Lebesgue measure spaces satisfying $|\Omega| = |\Omega'|$. For a set $E \subseteq L^{\Psi}(\Omega)$, we denote by $\overline{E}$ the closure of E in the weak topology $\sigma(L^{\Psi}, L^{\Psi^*})$ (this is well-defined, see Remark 2.4). First, we present a well-known result whose proof can be found in [7, Lemma 2.1].

Lemma 2.8. *Let $f : \Omega \to \mathbb{R}$ and $g : \Omega' \to \mathbb{R}$ be two measurable functions. Suppose f is a rearrangement of g and $\eta : \mathbb{R} \to \mathbb{R}$ is Borel measurable. Then, $\eta(f)$ is a rearrangement of $\eta(g)$. Moreover, if $\eta(f) \in L^1(\Omega)$, then $\eta(g) \in L^1(\Omega')$ and*

$$\int_{\Omega} \eta(f)\,\mathrm{d}x = \int_{\Omega'} \eta(g)\,\mathrm{d}x.$$

Now, we investigate the properties of a rearrangement class in the setting of Orlicz spaces.

Lemma 2.9. *Let $f_0 \in L^{\Psi}(\Omega)$, and let $\mathcal{R} := \mathcal{R}(f_0)$. Then*

(i) $\mathcal{R} \subseteq L^{\Psi}(\Omega)$, $\int_{\Omega} f_0\,\mathrm{d}x = \int_{\Omega} f\,\mathrm{d}x$ *and* $\|f_0\|_{\Psi} = \|f\|_{\Psi}$ *for all* $f \in \mathcal{R}$.

(ii) $\overline{\mathcal{R}}$ *is convex and weakly sequentially compact.*

(iii) $\overline{\mathcal{R}}$ *is metrizable with respect to the weak topology.*

(iv) $\overline{\mathcal{R}} = \{g \in L^1(\Omega) : \int_{\Omega} f_0\,\mathrm{d}x = \int_{\Omega} g\,\mathrm{d}x,\ \int_0^s g^{\Delta}\,\mathrm{d}t \leq \int_0^s f_0^{\Delta}\,\mathrm{d}t\ \forall s \in (0, |\Omega|)\}$.

(v) $\mathcal{R}$ *is the set of extreme points of* $\overline{\mathcal{R}}$.

Proof. (i) follows from Lemma 2.8. For (iii), as $L^{\Psi^*}(\Omega)$ is separable (see [1, Section 8.14 and Theorem 8.21]), from a classical result in functional analysis, see Theorem 3.29 in [5], it suffices to show $\overline{\mathcal{R}}$ is bounded in $L^{\Psi}(\Omega)$. This is clearly true as $\overline{\mathcal{R}} \subseteq \{f \in L^{\Psi}(\Omega) : \|f\|_{\Psi} \leq \|f_0\|_{\Psi}\}$ which is weakly closed.

For (iv), we set

$$A := \left\{ g \in L^1(\Omega) : \int_{\Omega} f_0\,\mathrm{d}x = \int_{\Omega} g\,\mathrm{d}x,\ \int_0^s g^{\Delta}\,\mathrm{d}t \leq \int_0^s f_0^{\Delta}\,\mathrm{d}t \ \ \forall s \in (0, |\Omega|) \right\}.$$

From [7, Lemma 2.3], we know that $A = \overline{\mathcal{R}}^{\sigma(L^1, L^{\infty})}$, which is the weak closure of $\mathcal{R}$ in $L^1(\Omega)$. Clearly, we have $\overline{\mathcal{R}} \subseteq \overline{\mathcal{R}}^{\sigma(L^1, L^{\infty})} = A$.

Now, we show that $A \subseteq \overline{\mathcal{R}}$. We argue by contradiction and suppose there exists a sequence $f_n \rightharpoonup f$ in $L^1(\Omega)$ with $\{f_n\} \subseteq \mathcal{R}$ and $f \notin \overline{\mathcal{R}}$. As $\{f_n\}$ is bounded in $L^{\Psi}(\Omega)$ and $L^{\Psi}(\Omega)$ is reflexive (see Remark 2.4), from a classical result in functional analysis, see for example [5, Theorem 3.18], we infer the existence of a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightharpoonup \tilde{f}$ in $L^{\Psi}(\Omega)$, and $\tilde{f} \in \overline{\mathcal{R}}$. Clearly, $\tilde{f} = f$ a.e. in Ω, and we must have $f = \tilde{f} \in \overline{\mathcal{R}}$ which is a contradiction. Therefore, $A \subseteq \overline{\mathcal{R}}$ and (iv) is proved.

For (ii), the convexity of $\overline{\mathcal{R}}$ follows from the fact $\overline{\mathcal{R}} = \overline{\mathcal{R}}^{\sigma(L^1,L^\infty)}$ and [7, Lemma 2.2]. Moreover, the weakly sequentially compactness of $\overline{\mathcal{R}}$ easily follows from (iii), the reflexivity of $L^\Psi(\Omega)$ (see Remark 2.4), and a classical result in functional analysis, see [5, Theorem 3.16]. The last assertion also follows from the fact $\overline{\mathcal{R}} = \overline{\mathcal{R}}^{\sigma(L^1,L^\infty)}$ and [7, Lemma 2.3]. □

Corollary 2.10. *$\overline{\mathcal{R}}$ is the closed convex hull of $\mathcal{R}$.*

Proof. We set $\mathcal{R}_1$ to be the closed convex hull of $\mathcal{R}$. From [5, Theorem 3.7] and [11, Theorem 2.2], we know $\mathcal{R}_1$ is weakly closed in $L^\Psi(\Omega)$. So, $\overline{\mathcal{R}} \subseteq \mathcal{R}_1$. By using Lemma 2.9 and [5, Theorem 3.7] again, we infer that $\overline{\mathcal{R}}$ is convex and closed. So, we must have $\mathcal{R}_1 \subseteq \overline{\mathcal{R}}$. Therefore, $\mathcal{R}_1 = \overline{\mathcal{R}}$. □

The rearrangement operation in Definition 2.7 is non-expansive in Orlicz spaces.

Lemma 2.11. *For all $f, g \in L^\Psi(\Omega)$, we have $\|f^\Delta - g^\Delta\|_\Psi \le \|f - g\|_\Psi$.*

Proof. Using the Theorem in [9], or [13, Corollary 1] (see also the Remark after Lemma 2.7 in [7]), we can show that

$$\int_0^{|\Omega|} \Psi\Big(\frac{|f^\Delta(t) - g^\Delta(t)|}{\tau}\Big)\,\mathrm{d}t \le \int_\Omega \Psi\Big(\frac{|f(x) - g(x)|}{\tau}\Big)\,\mathrm{d}x$$

for all $\tau > 0$. The assertion follows from this. □

As we are going to apply some approximation arguments, let us set the truncation functions

$$T_k(t) = \begin{cases} k, & \text{if } t > k, \\ t, & \text{if } |t| \le k, \\ -k, & \text{if } t < -k, \end{cases}$$

for $k > 0$.

Lemma 2.12. *Let $f_0 \in L^\Psi(\Omega)$, $g \in L^{\Psi^*}(\Omega)$, and $\mathcal{R} := \mathcal{R}(f_0)$. Then, the linear functional $L(f) = \int_\Omega fg\,\mathrm{d}x$ has a maximizer $\hat{f}$ relative to $f \in \mathcal{R}$. Moreover, we have*

$$\int_\Omega fg\,\mathrm{d}x \le \int_\Omega \hat{f}g\,\mathrm{d}x = \int_0^{|\Omega|} f_0^\Delta g^\Delta\,\mathrm{d}t, \quad \forall f \in \mathcal{R}. \tag{2.10}$$

Proof. If $f_0 \in L^\infty(\Omega)$, then the result follows from [6, Theorems 1 and 4]. For $f_0 \in L^\Psi(\Omega)$, we intend to apply an approximation argument. Indeed, set $f_k = T_k(f_0)$. For each $k \in \mathbb{N}$, we know there exists $\hat{f}_k \in \mathcal{R}(f_k)$ such that

$$\int_\Omega fg\,\mathrm{d}x \le \int_\Omega \hat{f}_k g\,\mathrm{d}x = \int_0^{|\Omega|} f_k^\Delta g^\Delta\,\mathrm{d}t, \quad \forall f \in \mathcal{R}(f_k). \tag{2.11}$$

Let us construct our maximizer $\hat{f}$ through an inductive argument. To be specific, let $k = 1$, and we obtain a maximizer $\hat{f}_1 \in \mathcal{R}(f_1)$ such that the inequality (2.11) holds. Let $i \in \mathbb{N}$, and suppose we can get a maximizer $\hat{f}_i$ such that the inequality (2.11) holds. In $(i+1)$th step, setting $E_i = \{x \in \Omega : \hat{f}_i(x) \ge i\} = \{x \in \Omega : \hat{f}_i(x) = i\}$ and $F_i = \{x \in \Omega : \hat{f}_i(x) \le -i\} = \{x \in \Omega : \hat{f}_i(x) = -i\}$, we know that f_0 has a rearrangement on Ω, denoted w_i, such that $w_i = \hat{f}_i$ a.e. in $\Omega \setminus (E_i \cup F_i)$. Let us consider the following linear maximization problem over the set $E_i \cup F_i$,

$$\sup_{f \in \mathcal{R}_{E_i \cup F_i}(T_{i+1}(w_i)|_{E_i \cup F_i})} \int_{E_i \cup F_i} fg\,\mathrm{d}x,$$

where $T_{i+1}(w_i)|_{E_i \cup F_i}$ denotes the restriction of $T_{i+1}(w_i)$ on $E_i \cup F_i$, and we know it has a maximizer $\tilde{f}_i$ (which is only defined over $E_i \cup F_i$) satisfying

$$\int_{E_i \cup F_i} \tilde{f}_i g\,\mathrm{d}x = \int_0^{|E_i \cup F_i|} \tilde{f}_i^{\,\Delta} g|_{E_i \cup F_i}^\Delta\,\mathrm{d}x. \tag{2.12}$$

We define

$$\eta_i(x) = \begin{cases} \tilde{f}_i(x) - i, & \text{if } x \in E_i,\\ \tilde{f}_i(x) + i, & \text{if } x \in F_i,\\ 0, & \text{if } x \in \Omega \setminus (E_i \cup F_i). \end{cases}$$

Clearly $\hat{f}_{i+1} = \hat{f}_i + \eta_i \in \mathcal{R}(f_{i+1})$. We claim that $\hat{f}_{i+1} = \hat{f}_i + \eta_i$ is a maximizer described in (2.11) with $k = i+1$. Indeed, it suffices to show that

$$\operatorname{ess\,inf}_{E_i} g \geq \operatorname{ess\,sup}_{\Omega\setminus(E_i\cup F_i)} g \geq \operatorname{ess\,inf}_{\Omega\setminus(E_i\cup F_i)} g \geq \operatorname{ess\,sup}_{F_i} g\,. \tag{2.13}$$

If (2.13) is true, then it is readily seen that

$$\begin{aligned} \int_\Omega \hat{f}_{i+1} g \,\mathrm{d}x &= \int_\Omega \hat{f}_i g \,\mathrm{d}x + \int_\Omega \eta_i g \,\mathrm{d}x \\ &= \int_0^{|\Omega|} \hat{f}_i^\Delta g^\Delta \,\mathrm{d}t + \int_{E_i} (\tilde{f}_i - i) g \,\mathrm{d}x + \int_{F_i} (\tilde{f}_i + i) g \,\mathrm{d}x \\ &= \int_0^{|\Omega|} \hat{f}_i^\Delta g^\Delta \,\mathrm{d}t + \int_0^{|E_i|} (\tilde{f}_i^\Delta - i) g^\Delta \,\mathrm{d}t + \int_{|\Omega|-|F_i|}^{|\Omega|} (\tilde{f}_i^\Delta + i) g^\Delta \,\mathrm{d}t \\ &= \int_0^{|\Omega|} f_{i+1}^\Delta g^\Delta \,\mathrm{d}t, \end{aligned}$$

where we have used (2.11) with $k = i$ in the second equality, and (2.12), (2.13) in conjunction with the definitions of E_i, F_i in the last two equalities. So, $\hat{f}_{i+1}$ is a maximizer described in (2.11). To complete the proof at this step, we only need to show (2.13). We only prove that the left-hand side inequality in (2.13) as the middle one is trivial and the right-hand side one is similar. To derive a contradiction, we suppose there exist $A \subseteq E_i$, $B \subseteq \Omega \setminus (E_i \cup F_i)$ and $\alpha > \beta$ such that $g \geq \alpha$ in B and $g \leq \beta$ in A. Without loss of generality, we may assume $|A| = |B|$. Let $\rho : A \to B$ be a measure preserving bijection, and such a map exists (see [29]). Now, we define $h : \Omega \to \mathbb{R}$ by

$$h(x) = \begin{cases} \hat{f}_i(x), & \text{if } x \in \Omega \setminus (A \cup B),\\ \hat{f}_i(\rho(x)), & \text{if } x \in A,\\ \hat{f}_i(\rho^{-1}(x)), & \text{if } x \in B. \end{cases}$$

Observing that $h \in \mathcal{R}(f_i)$, we compute

$$\begin{aligned} \int_\Omega hg \,\mathrm{d}x - \int_\Omega \hat{f}_i g \,\mathrm{d}x &= \int_{A\cup B} hg \,\mathrm{d}x - \int_{A\cup B} \hat{f}_i g \,\mathrm{d}x \\ &= \int_A (\hat{f}_i \circ \rho) g \,\mathrm{d}x + \int_B (\hat{f}_i \circ \rho^{-1}) g \,\mathrm{d}x - \int_A \hat{f}_i g \,\mathrm{d}x - \int_B \hat{f}_i g \,\mathrm{d}x \\ &= \int_A (\hat{f}_i \circ \rho) g \,\mathrm{d}x + \int_A \hat{f}_i (g\circ\rho) \,\mathrm{d}x - \int_A \hat{f}_i g \,\mathrm{d}x - \int_A (\hat{f}_i \circ \rho)(g\circ\rho) \,\mathrm{d}x \\ &= -\int_A \hat{f}_i \circ \rho [g\circ\rho - g] \,\mathrm{d}x + i \int_A [g\circ\rho - g] \,\mathrm{d}x \\ &= \int_A [i - \hat{f}_i \circ \rho] \cdot [g\circ\rho - g] \,\mathrm{d}x \geq (\alpha - \beta) \int_A [i - \hat{f}_i \circ \rho] \,\mathrm{d}x > 0, \end{aligned}$$

which contradicts the maximality of $\hat{f}_i$ in (2.11), where we have also used the facts that $\hat{f}_i = i$ in $A \subseteq E_i$ and $\hat{f}_i < i$ in $B \subseteq \Omega \setminus (E_i \cup F_i)$.

Finally, setting $\hat{f}(x) = \lim_{k\to\infty} \hat{f}_k(x)$ a.e. in Ω, it is obvious that $\hat{f} \in \mathcal{R}$. By letting k go to infinity, the equality in (2.10) follows from the one in (2.11) by using dominated convergence theorem, (2.7) and Lemma 2.11. For the inequality in (2.10), we argue by contradiction and suppose there exists $f \in \mathcal{R}$ such that $\int_\Omega fg \,\mathrm{d}x > \int_\Omega \hat{f} g \,\mathrm{d}x$. Then, we can find a sufficiently large k such that

$$\int_\Omega T_k(f) g \,\mathrm{d}x > \int_\Omega \hat{f}_k g \,\mathrm{d}x$$

which contradicts (2.11). ∎

Remark 2.13. From Lemma 2.12, we can deduce that

$$\int_\Omega fg\,\mathrm{d}x \le \int_0^{|\Omega|} f^\Delta g^\Delta\,\mathrm{d}t \quad \forall f\in L^\Psi(\Omega),\ g\in L^{\Psi^*}(\Omega),$$

which is the Hardy-Littlewood inequality in Orlicz spaces.

Lemma 2.14. *Let $f_0\in L^\Psi(\Omega)$, $g\in L^{\Psi^*}(\Omega)$, and $\mathcal{R}:=\mathcal{R}(f_0)$. Suppose there is a increasing function η such that $\hat f=\eta(g)\in\mathcal{R}$. Then, $\hat f$ is a maximizer of the linear functional $L(f)=\int_\Omega fg\,\mathrm{d}x$ relative to $f\in\mathcal{R}$.*

Proof. Setting $\zeta(t)=t\eta(t)$, we know from Lemma 2.8 that $\zeta(g)$ is a rearrangement of $\zeta(g^\Delta)$. Using Lemma 2.8 again, we derive

$$\begin{aligned}\int_\Omega g\hat f\,\mathrm{d}x &= \int_\Omega g\eta(g)\,\mathrm{d}x=\int_\Omega \zeta(g)\,\mathrm{d}x\\ &=\int_0^{|\Omega|}\zeta(g^\Delta)\,\mathrm{d}t=\int_0^{|\Omega|} g^\Delta\eta(g^\Delta)\,\mathrm{d}t\\ &=\int_0^{|\Omega|} g^\Delta\hat f^\Delta\,\mathrm{d}t,\end{aligned}$$

where we have used that $\hat f^\Delta=[\eta(g)]^\Delta=\eta(g^\Delta)$ in the last equality. We can conclude by applying Lemma 2.12. □

Now, we are ready to prove Theorem 1.1 which is an improvement of Lemma 2.14.

Proof of Theorem 1.1. From Lemma 2.14, we know that $\hat f$ is a maximizer of $L(f)$ relative to $f\in\mathcal{R}$. It is not hard to see that $\hat f$ is a maximizer of $L(f)$ relative to $f\in\overline{\mathcal{R}}$. Indeed, we fix any $\bar f\in\overline{\mathcal{R}}$. Then, there exists $\{f_n\}\subseteq\mathcal{R}$ such that $f_n\rightharpoonup\bar f$ in $L^\Psi(\Omega)$, and it follows that

$$L(\hat f)\ge L(f_n)\to L(\bar f)$$

as desired.

Next, we claim that if $\{f_n\}$ is a sequence in $\mathcal{R}$ such that $L(f_n)\to L(\hat f)$, then $f_n\to\hat f$ in $L^\Psi(\Omega)$. We leave the proof of this claim to the end, and suppose it is true at this moment. Let $\tilde f\in\overline{\mathcal{R}}$ be a maximizer of $L(f)$ relative to $f\in\overline{\mathcal{R}}$. Then, there exists $\{f_n\}\subseteq\mathcal{R}$ such that $f_n\rightharpoonup\tilde f$ in $L^\Psi(\Omega)$, consequently $L(f_n)\to L(\tilde f)=L(\hat f)$. By using the claim, we derive $f_n\to\hat f$ in $L^\Psi(\Omega)$. Recalling that $f_n\rightharpoonup\tilde f$ in $L^\Psi(\Omega)$, we must have $\tilde f=\hat f$ and this will complete the proof of the theorem. So, we only need to prove the claim. Indeed, fix $k>0$, and let us define

$$\begin{gathered}\gamma_1(s)=\min\{0,s+k\},\\ \gamma_2(s)=\max\{0,s-k\},\\ \gamma_3(s)=s-\gamma_1(s)-\gamma_2(s).\end{gathered}$$

Observe that $\gamma_3(s)=T_k(s)$, and γ_i are increasing for $i=1,2,3$. So, for $i=1,2,3$, the functions $\gamma_i\circ\eta$ are increasing, and we have $\gamma_i\circ f_n,\gamma_i\circ\hat f\in\mathcal{R}(\gamma_i\circ f_0)$. As

$$\int_\Omega(\hat f-f_n)g\,\mathrm{d}x=\sum_{i=1}^3\int_\Omega(\gamma_i(\hat f)-\gamma_i(f_n))g\,\mathrm{d}x$$

and $\gamma_i(\hat f)=(\gamma_i\circ\eta)(g)$, we may use Lemma 2.14 to obtain

$$\int_\Omega(\gamma_i(\hat f)-\gamma_i(f_n))g\,\mathrm{d}x\ge 0$$

for $i=1,2,3$. Recalling that $L(f_n)\to L(\hat f)$, we must have

$$\int_\Omega(\gamma_i(\hat f)-\gamma_i(f_n))g\,\mathrm{d}x\to 0 \tag{2.14}$$

for $i = 1, 2, 3$. Observe that

$$\begin{aligned} \|\hat{f} - f_n\|_\Psi &= \Big\| \sum_{i=1}^{3} (\gamma_i(\hat{f}) - \gamma_i(f_n)) \Big\|_\Psi \\ &\le \|\gamma_1(\hat{f})\|_\Psi + \|\gamma_1(f_n)\|_\Psi + \|\gamma_2(\hat{f})\|_\Psi + \|\gamma_2(f_n)\|_\Psi + \|\gamma_3(\hat{f}) - \gamma_3(f_n)\|_\Psi . \end{aligned} \tag{2.15}$$

Now, let us fix $\epsilon > 0$. After an application of dominated convergence theorem and Lemma 2.9 (i), we can find a sufficiently large $k > 0$ such that

$$\|\gamma_1(\hat{f})\|_\Psi = \|\gamma_1(f_n)\|_\Psi < \epsilon, \quad \text{and} \quad \|\gamma_2(\hat{f})\|_\Psi = \|\gamma_2(f_n)\|_\Psi < \epsilon.$$

On the other hand, as $L(\gamma_3(f_n)) \to L(\gamma_3(\hat{f}))$ (see (2.14)), we may use [6, Theorem 2] and (2.4) to imply that

$$\|\gamma_3(f_n) - \gamma_3(\hat{f})\|_\Psi \to 0.$$

So, for all sufficiently large n, it follows from (2.15) that

$$\|\hat{f} - f_n\|_\Psi < 5\epsilon,$$

which proves the claim. This completes the proof of the theorem. □

The following two lemmas will help us utilize Theorem 1.1 in applications. We need some preparations before stating the results. Let $f : \Omega \to \mathbb{R}$ be measurable, and S be a measurable subset of Ω. We say the graph of f has no significant flat sections on S if

$$|\{x \in S : f(x) = \alpha\}| = 0 \quad \forall \alpha \in \mathbb{R}.$$

In addition, for $g \in L^1_+(\Omega)$, we denote the support of g on Ω by $S(g)$, i.e. $S(g) \equiv \{x \in \Omega : g(x) > 0\}$.

Lemma 2.15. *Let $f, g : \Omega \to \mathbb{R}$ be measurable functions. Suppose the graph of g has no significant flat sections on Ω. Then, there exists a increasing (or decreasing) function η such that $\eta(g)$ is a rearrangement of f.*

For a proof of the above lemma, see [7, Lemma 2.9], or [17, Lemma 2.2].

Lemma 2.16. *Let $f_0 \in L^\Psi_+(\Omega)$, $g \in L^{\Psi^*}(\Omega)$, and $\mathcal{R} := \mathcal{R}(f_0)$. Let $\hat{f}$ be a minimizer of the linear functional $L(f) = \int_\Omega fg \, dx$ relative to $f \in \overline{\mathcal{R}}$. Suppose the graph of g has no significant flat sections on $S(\hat{f})$. Then, there exists a decreasing function η such that $\eta(g)$ is a rearrangement of f_0.*

Proof. The proof is a minor variant of the proof of [25, Lemma 2.12]. For the convenience of readers, we sketch the proof here. Firstly, we observe that $|S(f_0)| \le |S(\hat{f})|$ (see [25, Lemma 2.11]). Consequently, there exists $f_1 \in \mathcal{R}$ such that $S(f_1) \subseteq S(\hat{f})$. Since the graph of g has no significant flat sections on $S(\hat{f})$, by using Lemma 2.15, we infer the existence of a decreasing function η_S such that $\eta_S(g|_{S(\hat{f})})$ is a rearrangement of $f_1|_{S(\hat{f})}$. On the other hand, by using a similar proof of (2.13) (or see the proof of [25, Lemma 2.12] for details), we can show

$$\operatorname{ess\,inf}_{S(\hat{f})^c} g \ge \operatorname{ess\,sup}_{S(\hat{f})} g \equiv \gamma.$$

Now, we construct η as follows,

$$\eta(t) = \begin{cases} \eta_S(t), & \text{if } t < \gamma, \\ 0, & \text{if } t \ge \gamma. \end{cases}$$

Clearly, η is decreasing and $\eta(g) \in \mathcal{R}(f_1) = \mathcal{R}$ as desired. □

Lemma 2.17. *Let $f_0 \in L^\Psi(\Omega)$, $g \in L^{\Psi^*}(\Omega)$, and $\mathcal{R} := \mathcal{R}(f_0)$. If the linear functional $L(f) = \int_\Omega fg \, dx$ has a unique maximizer $\hat{f}$ relative to $f \in \mathcal{R}$, then there is an increasing function η such that $\hat{f} = \eta(g)$ a.e. in Ω.*

Proof. The proof is a minor variant of the proof of [6, Theorem 5]. More precisely, we define a function
$$\eta(\alpha) = \operatorname{ess\,inf} \hat{f}(\{x \in \Omega : g(x) \geq \alpha\})$$
and try to show that $\hat{f} = \eta(g)$ a.e. in Ω. This is the desired result as η is increasing.

For convenience of the readers, we provide another proof by an approximation argument in conjunction with [6, Theorem 5] here. Firstly, if $g \in L^\infty(\Omega)$, then the result follows from [6, Theorem 5]. Suppose $g \in L^{\Psi^*}(\Omega)$. We claim that for all measurable $F \subseteq \Omega$ with $|F| > 0$:
$$\hat{f}|_F \text{ is the unique maximizer of } L_F(f) = \int_F fg\,\mathrm{d}x \text{ relative to } f \in \mathcal{R}_F(\hat{f}). \tag{2.16}$$
Indeed, to derive a contradiction, we suppose $h \in \mathcal{R}_F(\hat{f})$, $h \neq \hat{f}|_F$ on some measurable subset of F with positive measure, and $\int_F hg\,\mathrm{d}x \geq \int_F \hat{f}g\,\mathrm{d}x$. Now, we define
$$\tilde{f}(x) = \begin{cases} \hat{f}(x), & \text{if } x \in \Omega \setminus F, \\ h(x), & \text{if } x \in F. \end{cases}$$
It follows that
$$\int_\Omega \tilde{f}g\,\mathrm{d}x = \int_{\Omega\setminus F} \hat{f}g\,\mathrm{d}x + \int_F hg\,\mathrm{d}x \geq \int_{\Omega\setminus F} \hat{f}g\,\mathrm{d}x + \int_F \hat{f}g\,\mathrm{d}x = \int_\Omega \hat{f}g\,\mathrm{d}x,$$
and this implies $\tilde{f}$ is a maximizer over $\mathcal{R}$. Since $\tilde{f}$ is different from $\hat{f}$, we reach a contradiction. Next, we observe that there are at most countably many $\alpha \in \mathbb{R}$ such that $|\{x \in \Omega : |g(x)| = \alpha\}| > 0$ since $|\Omega| < \infty$. So, we can find a increasing positive sequence $\{k_n\}$ such that $k_n \to \infty$ and
$$|\{x \in \Omega : g(x) = k_n\} \cup \{x \in \Omega : g(x) = -k_n\}| = 0 \quad \forall n \in \mathbb{N}. \tag{2.17}$$
We set $E_n = \{x \in \Omega : |g(x)| < k_n\}$. Observing that g is bounded in E_n, we can construct a desired increasing function by using (2.16) and an inductive argument. To be specific, let $i = 1$, and we obtain a increasing function $\eta_1 : (-k_1, k_1) \to \mathbb{R}$ such that $\hat{f} = \eta_1(g)$ a.e. in E_1 by using (2.16) with $F = E_1$ and [6, Theorem 5]. Let $i \in \mathbb{N}$, and suppose we can find a increasing function $\eta_i : (-k_i, k_i) \to \mathbb{R}$ such that $\hat{f} = \eta_i(g)$ a.e. in E_i. We intend to extend η_i to a desired increasing function on $(-k_{i+1}, k_{i+1})$. Denoting $F_1 = \{x \in \Omega : k_i \leq g(x) < k_{i+1}\}$ and $F_2 = \{x \in \Omega : -k_{i+1} < g(x) \leq -k_i\}$, we have $E_{i+1} = E_i \cup F_1 \cup F_2$. Now, we apply (2.16) with $F = F_1, F_2$ and [6, Theorem 5] to obtain two increasing functions $\zeta_1 : [k_i, k_{i+1}) \to \mathbb{R}$, $\zeta_2 : (-k_{i+1}, -k_i] \to \mathbb{R}$ such that $\hat{f} = \zeta_1(g)$ a.e. in F_1 and $\hat{f} = \zeta_2(g)$ a.e. in F_2. On the other hand, by using a similar proof of (2.13) (in the proof of Lemma 2.12), we can show that
$$\operatorname{ess\,inf}_{F_1} \hat{f} \geq \operatorname{ess\,sup}_{E_i} \hat{f} \geq \operatorname{ess\,inf}_{E_i} \hat{f} \geq \operatorname{ess\,sup}_{F_2} \hat{f}. \tag{2.18}$$
Defining $\eta_{i+1} : (-k_{i+1}, k_{i+1}) \to \mathbb{R}$ by
$$\eta_{i+1}(s) = \begin{cases} \zeta_1(s), & \text{if } s \in [k_i, k_{i+1}), \\ \eta_i(s), & \text{if } s \in (-k_i, k_i), \\ \zeta_2(s), & \text{if } s \in (-k_{i+1}, -k_i], \end{cases}$$
we infer from (2.17), (2.18) that $\hat{f} = \eta_{i+1}(g)$ a.e. in E_{i+1} and η_{i+1} is increasing. (Note that if $|F_1| = 0$, then we can set $\eta_{i+1}(s) = \sup\{\eta_i(s) : s \in (-k_i, k_i)\}$ for all $s \in [k_i, k_{i+1})$. The case $|F_2| = 0$ can be treated similarly.) Finally, we do this process inductively, and get a increasing function $\eta : \mathbb{R} \to \mathbb{R}$ satisfying $\eta(s) = \eta_i(s)$ if $|s| < k_i$ and $\hat{f} = \eta(g)$ a.e. in Ω as desired. □

Proof of Theorem 1.2. We follow the ideas of the proofs of [6, Theorem 7], and [15, Lemma 3].
(i) We begin by considering the maximization problem
$$\sup_{f \in \overline{\mathcal{R}}} J(f). \tag{2.19}$$
Since J is weakly sequentially continuous, and $\overline{\mathcal{R}}$ is weakly sequentially compact (see Lemma 2.9 (ii)), (2.19) is solvable and let $\tilde{f} \in \overline{\mathcal{R}}$ be a solution. By applying the formula (1.3), we infer that $\tilde{f}$ maximizes the linear functional $L(h) = \int_\Omega G(\tilde{f})h\,\mathrm{d}x$ relative to $h \in \overline{\mathcal{R}}$. From Lemma 2.12, we

know that the functional L has a maximizer $\hat{f}$ relative to $\mathcal{R}$. It follows that $\hat{f}$ maximizes L relative to $\overline{\mathcal{R}}$ as well, as L is weakly sequentially continuous. Whence, in particular, $L(\hat{f}) = L(\tilde{f})$. By using the convexity of J and the formula (1.3), we have

$$J(\hat{f}) - J(\tilde{f}) \geq \lim_{t\to 0^+} \frac{J(\tilde{f} + t(\hat{f} - \tilde{f})) - J(\tilde{f})}{t} = \int_\Omega G(\tilde{f})(\hat{f} - \tilde{f})\,\mathrm{d}x = L(\hat{f} - \tilde{f}) = 0.$$

Thus, $J(\hat{f}) = J(\tilde{f})$, and $\hat{f} \in \mathcal{R}$ is a solution of (2.19), as desired.

(ii) Let us fix a maximizer $\hat{f}$ satisfying the local strict convexity. For any $g \in \mathcal{R}$, there exists a sufficiently small $t > 0$ such that

$$0 \geq J(g) - J(\hat{f}) > \frac{J(\hat{f} + t(g - \hat{f})) - J(\hat{f})}{t} \geq \int_\Omega G(\hat{f})(g - \hat{f})\,\mathrm{d}x,$$

where the second inequality follows from the local strict convexity of Ψ at $\hat{f}$, and the third inequality is derived from the convexity of J in combined with the formula (1.3). Thus, $\hat{f}$ is the unique maximizer of the linear functional $L(h) = \int_\Omega G(\hat{f})h\,\mathrm{d}x$ relative to $h \in \mathcal{R}$. By applying Lemma 2.17, we infer $\hat{f} = \eta \circ G(\hat{f})$ a.e. in Ω for some increasing function η as desired. □

Remark 2.18. In Theorem 1.2, if J is a Gâteaux differentiable function defined on $L^\Psi(\Omega)$ with derivative J', then $G = J'$ in (1.3).

Before proving the last lemma, we state a basic result, which can be found in [6, Lemma 2].

Lemma 2.19. *Let $f \in L^1(\Omega)$. Then for any measurable $\Omega' \subseteq \Omega$, $\int_{\Omega'} f\,\mathrm{d}x \leq \int_0^{|\Omega'|} f^\Delta\,\mathrm{d}t$.*

Finally, we present a lemma which will help us apply the convergence result in Lemma 2.5.

Lemma 2.20. *Let $f_0 \in L^\Psi(\Omega)$ and $\mathcal{R} := \mathcal{R}(f_0)$. Then, there exists a increasing function $\omega : [0, |\Omega|] \to [0, \infty)$ such that $\omega(0) = 0$, $\lim_{t\to 0} \omega(t) = 0$ and the following inequality*

$$\|f\|_{\Psi,\Omega'} \leq \omega(|\Omega'|)$$

holds for all $f \in \overline{\mathcal{R}}$ and $\Omega' \subseteq \Omega$. In fact, we can choose $\omega(t) = \||f_0|^\Delta\|_{\Psi,(0,t)}$.

Proof. Firstly, we prove the case $f_0 \geq 0$. Let us fix $f \in \overline{\mathcal{R}}$ and $\Omega' \subseteq \Omega$ with $|\Omega'| = \alpha$. It follows from Lemma 2.9 (iv) that

$$\int_0^\beta f^\Delta\,\mathrm{d}s \leq \int_0^\beta f_0^\Delta\,\mathrm{d}s$$

holds for all $\beta \in [0, |\Omega|]$. By using an approximation argument with simple functions (see [19, proof of Lemma 2.4]), we can derive

$$\int_0^\alpha f^\Delta g^\Delta\,\mathrm{d}s \leq \int_0^\alpha f_0^\Delta g^\Delta\,\mathrm{d}s, \quad \forall g \in L_+^{\Psi^*}(\Omega). \tag{2.20}$$

On the other hand, as $\Psi_\lambda(t) := \Psi(t/\lambda)$ for $\lambda > 0$ is convex with $\Psi'_\lambda(t) = \frac{\psi(t/\lambda)}{\lambda}$, we obtain

$$\int_0^\alpha \Psi_\lambda(f_0^\Delta(s))\,\mathrm{d}s - \int_0^\alpha \Psi_\lambda(f^\Delta(s))\,\mathrm{d}s \geq \frac{1}{\lambda}\int_0^\alpha \psi(f^\Delta(s)/\lambda)(f_0^\Delta(s) - f^\Delta(s))\,\mathrm{d}s.$$

From Lemma 2.6, we know that $\psi(f/\lambda) \in L_+^{\Psi^*}(\Omega)$. Observing that $\psi(f/\lambda)^\Delta = \psi(f^\Delta/\lambda)$, by using (2.20), we deduce that

$$\int_0^\alpha \Psi_\lambda(f_0^\Delta(s))\,\mathrm{d}s \geq \int_0^\alpha \Psi_\lambda(f^\Delta(s))\,\mathrm{d}s. \tag{2.21}$$

On the other hand, by Lemma 2.19 and the fact $\Psi_\lambda(f)^\Delta = \Psi_\lambda(f^\Delta)$, we obtain

$$\int_{\Omega'} \Psi_\lambda(f(x))\,\mathrm{d}x \leq \int_0^\alpha \Psi_\lambda(f^\Delta(s))\,\mathrm{d}s. \tag{2.22}$$

Combining (2.21) and (2.22), we infer $\|f\|_{\Psi,\Omega'} \leq \|f_0^\Delta\|_{\Psi,(0,\alpha)}$. We claim that the function $\omega(\alpha) := \|f_0^\Delta\|_{\Psi,(0,\alpha)}$ satisfies the required property. Clearly, ω is increasing and $\omega(0) = 0$. We only need to show $\lim_{t\to 0}\omega(t)=0$. To derive a contradiction, suppose there exists a sequence $\{t_n\}$ decreasing to zero and it satisfies $\omega(t_n)>\epsilon>0$ for all n. This implies

$$\int_0^{t_n}\Psi\Big(\frac{f_0^{\Delta}(x)}{\epsilon}\Big)\,\mathrm{d}x>1$$

holds for all n which is an obvious contradiction, as the left hand side integral will decrease to zero as $n\to\infty$. Therefore, we have proved the case $f_0\geq 0$.

Secondly, we consider the general case $f_0\in L^{\Psi}(\Omega)$. We can repeat the above arguments if we prove that

$$\int_0^{\beta}|f|^{\Delta}\,\mathrm{d}s\leq\int_0^{\beta}|f_0|^{\Delta}\,\mathrm{d}s \tag{2.23}$$

holds for all $\beta\in[0,|\Omega|]$ and $f\in\overline{\mathcal{R}}$. Obviously, the above inequality holds for all $f\in\mathcal{R}$. So, let us fix any $f\in\overline{\mathcal{R}}\setminus\mathcal{R}$ and $\beta\in(0,|\Omega|]$. We set the sign function

$$\operatorname{sign}(t)=\begin{cases}1, & \text{if } t>0,\\ 0, & \text{if } t=0,\\ -1, & \text{if } t<0,\end{cases}$$

and we know from a fundamental result that there exists a Borel set $F\subseteq\Omega$ with $|F|=\beta$ and

$$\int_0^{\beta}|f|^{\Delta}(s)\,\mathrm{d}s=\int_F|f(x)|\,\mathrm{d}x=\int_{\Omega}f(x)\cdot(\operatorname{sign}(f(x))\chi_F(x))\,\mathrm{d}x, \tag{2.24}$$

see for example [3, relation (12)]. As $f\in\overline{\mathcal{R}}\setminus\mathcal{R}$, there exists a sequence $\{f_n\}\subseteq\mathcal{R}$ such that $f_n\rightharpoonup f$ in $L^{\Psi}(\Omega)$. This implies

$$\int_{\Omega}f_n(x)\cdot(\operatorname{sign}(f(x))\chi_F(x))\,\mathrm{d}x\to\int_{\Omega}f(x)\cdot(\operatorname{sign}(f(x))\chi_F(x))\,\mathrm{d}x. \tag{2.25}$$

On the other hand, we infer from Lemma 2.19 that

$$\int_{\Omega}f_n(x)\cdot(\operatorname{sign}(f(x))\chi_F(x))\,\mathrm{d}x\leq\int_F|f_n(x)|\,\mathrm{d}x\leq\int_0^{\beta}|f_n|^{\Delta}\,\mathrm{d}s=\int_0^{\beta}|f_0|^{\Delta}\,\mathrm{d}s. \tag{2.26}$$

Therefore, the inequality (2.23) follows from (2.24), (2.25) and (2.26). Following the arguments in the first step, we can show that the function $\omega(\alpha):=\||f_0|^{\Delta}\|_{\Psi,(0,\alpha)}$ is the desired one. This completes the proof of the lemma. □

3. Optimization problems

Let $a:(0,\infty)\to(0,\infty)$ be a continuous function such that the map

$$\varphi(t)=\begin{cases}a(|t|)t & \text{if } t\neq 0,\\ 0 & \text{if } t=0,\end{cases}$$

is an odd strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Then, the function $\Phi(t)=\int_0^t\varphi(s)\,\mathrm{d}s$, $t\in[0,\infty)$, becomes an N-function. Instead of (2.3), we suppose there are two positive constants r and p such that

$$1<r\leq\frac{t\varphi(t)}{\Phi(t)}\leq p<\infty,\quad\forall t>0. \tag{3.1}$$

in this section. With this stronger condition, we can derive a useful result.

Lemma 3.1. *If $\|u\|_{\Phi}>1$, then*

$$\|u\|_{\Phi}^{r}\leq\int_{\Omega}\Phi(|u(x)|)\,\mathrm{d}x\leq\|u\|_{\Phi}^{p}. \tag{3.2}$$

Proof. The proof is of minor variant of the proof of [28, Prop. 2.1]. For convenience of the readers, let us prove the left-hand side inequality in (3.2). For $\sigma > 1$ and $t > 0$, integrating the second inequality in (3.1) from t to σt yields

$$\log(\sigma^r) = \int_t^{\sigma t} \frac{r}{s}\, \mathrm{d}s \le \int_t^{\sigma t} \frac{\varphi(s)}{\Phi(s)}\, \mathrm{d}s = \log \frac{\Phi(\sigma t)}{\Phi(t)},$$

and this implies

$$\sigma^r \Phi(t) \le \Phi(\sigma t).$$

Clearly, the above inequality is also true for $t = 0$. Since $\|u\|_\Phi > 1$, we deduce that

$$\int_\Omega \Phi(|u(x)|)\, \mathrm{d}x = \int_\Omega \Phi\Big(\|u\|_\Phi \cdot \frac{|u(x)|}{\|u\|_\Phi}\Big)\, \mathrm{d}x \ge \|u\|_\Phi^r \int_\Omega \Phi\Big(\frac{|u(x)|}{\|u\|_\Phi}\Big)\, \mathrm{d}x = \|u\|_\Phi^r,$$

where we have used (2.6) in the last equality. □

Let us present some applications of the results in Section 2. Henceforth, we use the symbol C to indicate a generic positive constant that may vary from one place to another.

3.1. **Φ-Laplacian case.** Consider the boundary value problem

$$\begin{gathered} -\nabla \cdot (a(|\nabla u|)\nabla u) = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \ge 2)$ with Lipschitz boundary, and $f \in L^{\Phi^*}(\Omega)$. For a solution $u \in W_0^{1,\Phi}(\Omega)$ of (3.3), we mean

$$\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla w\, \mathrm{d}x = \int_\Omega f w\, \mathrm{d}x, \quad \forall w \in W_0^{1,\Phi}(\Omega). \tag{3.4}$$

Under this definition, we have the following result.

Theorem 3.2. *Problem* (3.3) *has a unique solution $u_f \in W_0^{1,\Phi}(\Omega)$. Moreover, the solution u_f is the unique minimizer of the energy functional $E_f : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by*

$$E_f(u) = \int_\Omega \Phi(|\nabla u|)\, \mathrm{d}x - \int_\Omega f u\, \mathrm{d}x,$$

relative to $u \in W_0^{1,\Phi}(\Omega)$. In addition, if $f \in L^{\Phi^*}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega)$, $a \in C^1(0,\infty)$ and it satisfies

$$r - 2 \le \frac{t a'(t)}{a(t)} \le p - 2 \quad \forall t > 0. \tag{3.5}$$

then $a(|\nabla u_f|)\nabla u_f \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

Proof. It is clear that $E_f \in C^1(W_0^{1,\Phi}(\Omega))$, see [28, Prop. 4.1]. For $\|u\| = \|\, |\nabla u|\, \|_\Phi > 1$, it follows from Lemma 3.1 that

$$E_f(u) \ge \|u\|^r - 2\|f\|_{\Phi^*}\|u\|_\Phi \ge \|u\|^r - C\|f\|_{\Phi^*}\|u\|,$$

where we have also used Hölder's inequality (2.2) in the first inequality, and the Poincaré inequality (see Lemma 2.4 in [21]) in the second inequality. So, E_f is coercive as $r > 1$. Since Φ is strictly convex and strictly increasing, we infer E_f is strictly convex. To show E_f is weakly lower semicontinuous, it suffices to show the functional $\Gamma : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by

$$\Gamma(u) = \int_\Omega \Phi(|\nabla u|)\, \mathrm{d}x \tag{3.6}$$

is weakly lower semicontinuous. By using (2.7) with $u = 0$, we can show Γ is continuous at $u = 0$. As Γ is convex, from a classical result in convex analysis, see for example [11, Theorem 2.34], Γ is continuous (even locally Lipschitz) over $W_0^{1,\Phi}(\Omega)$. Thus, by using [20, Prop. 4.26], we can derive Γ is weakly lower semicontinuous, which implies E_f is also weakly lower semicontinuous. Now, we can apply the direct method to the minimization problem

$$\inf_{u \in W_0^{1,\Phi}(\Omega)} E_f(u), \tag{3.7}$$

to deduce that it has a unique solution. It is straightforward to show that (3.7) and (3.3) have the same solutions. The last result is a direct consequence of [10, Theorem 2.1]. $\square$

Remark 3.3. If $a \in C^1(0,\infty)$, then the condition (3.5) is 'strictly' stronger than that of (3.1). A concrete discussion of the relations between (3.1) and (3.5) is included in Appendix, and see also Remark 3.7 below.

For any $f \in L^{\Phi^*}(\Omega)$, we use $u_f \in W_0^{1,\Phi}(\Omega)$ to denote the unique solution of (3.3) (see Theorem 3.2). Now, we are ready to recall the rearrangement optimization problems we are interested in. Let f_0 be a nonnegative and nontrivial function in $L^{\Phi^*}(\Omega)$ (or $f_0 \in L_+^{\Phi^*}(\Omega)$). We consider the following optimization problems:

$$\inf_{f\in\mathcal{R}(f_0)} J_1(f) := E_f(u_f) = \int_\Omega \Phi(|\nabla u_f|)\,\mathrm{d}x - \int_\Omega f u_f\,\mathrm{d}x, \tag{3.8}$$

and

$$\sup_{f\in\mathcal{R}(f_0)} J_1(f). \tag{3.9}$$

To solve the above optimization problems, we first investigate the properties of the energy functional J_1. For simplicity, we set $\mathcal{R} = \mathcal{R}(f_0)$.

Lemma 3.4. *The following statements are valid:*

(i) $J_1 : \overline{\mathcal{R}} \to \mathbb{R}$ *is weakly sequentially continuous in* $L^{\Phi^*}(\Omega)$.
(ii) J_1 *is strictly concave.*
(iii) *Given* $f_1, f_2 \in \overline{\mathcal{R}}$, *it holds*

$$\lim_{t\to 0^+} \frac{J_1(f_1 + t(f_2 - f_1)) - J_1(f_1)}{t} = -\int_\Omega (f_2 - f_1)u_{f_1}\,\mathrm{d}x. \tag{3.10}$$

Proof. (i) Let $\{f_n\} \subseteq \overline{\mathcal{R}}$ satisfying $f_n \rightharpoonup f$ in $L^{\Phi^*}(\Omega)$ and set $u_n \equiv u_{f_n}$ for simplicity. By using (3.4) and the fact $\Phi(t) = \int_0^t \varphi(s)ds \le t\varphi(t)$ (as φ is increasing), we deduce

$$\begin{aligned}
\int_\Omega \Phi(|\nabla u_n|)\,\mathrm{d}x &\le \int_\Omega \varphi(|\nabla u_n|)|\nabla u_n|\,\mathrm{d}x = \int_\Omega a(|\nabla u_n|)|\nabla u_n|^2\,\mathrm{d}x \\
&= \int_\Omega f_n u_n\,\mathrm{d}x \\
&\le 2\|f_n\|_{\Phi^*}\|u_n\|_\Phi \le C\|u_n\|,
\end{aligned}$$

where we used Hölder's inequality in the second inequality, and the Poincaré inequality in the last one. Applying this and Lemma 3.1, we can find $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. Thus, we infer the existence of a subsequence, still denoted $\{u_n\}$, and $u \in W_0^{1,\Phi}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$, $u_n \to u$ in $L^1(\Omega)$, and $u_n \to u$ a.e. in Ω, where the second convergence follows from the embedding $W_0^{1,\Phi}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ and the compact Sobolev embedding. Since the functional Γ defined in (3.6) is weakly lower semicontinuous (see the proof Theorem 3.2), we infer that

$$\begin{aligned}
E_f(u) &= \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x - \int_\Omega f u\,\mathrm{d}x \\
&\le \liminf_{n\to\infty} \Big\{ \int_\Omega \Phi(|\nabla u_n|)\,\mathrm{d}x - \int_\Omega f_n u_n\,\mathrm{d}x \Big\} \\
&\le \limsup_{n\to\infty} \Big\{ \int_\Omega \Phi(|\nabla u_n|)\,\mathrm{d}x - \int_\Omega f_n u_n\,\mathrm{d}x \Big\} \\
&\le \limsup_{n\to\infty} \Big\{ \int_\Omega \Phi(|\nabla u_f|)\,\mathrm{d}x - \int_\Omega f_n u_f\,\mathrm{d}x \Big\} \\
&= \int_\Omega \Phi(|\nabla u_f|)\,\mathrm{d}x - \int_\Omega f u_f\,\mathrm{d}x = E_f(u_f),
\end{aligned} \tag{3.11}$$

where we used Lemma 2.5 in conjunction with Lemma 2.20 in the first inequality, and the minimality of u_n in the third inequality (see Theorem 3.2). Using Theorem 3.2 again, we obtain

$u = u_f$. Furthermore, all the inequalities in (3.11) become equalities, and we have

$$\lim_{n\to\infty} J_1(f_n) = \lim_{n\to\infty} E_{f_n}(u_n) = E_f(u_f) = J_1(f)$$

as desired.

(ii) We first show that J_1 is concave. Observe that

$$J_1(f) = E_f(u_f) = \inf_{u\in W_0^{1,\Phi}(\Omega)} \Big\{ \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x \Big\}.$$

So, J_1, as the infimum over a collection of affine functions of f, is concave (see for example Proposition 2.20 in [11]).

Now, we show that J_1 is in fact strictly concave. Let us fix two functions $f_1 \neq f_2$ in $\overline{\mathcal{R}}$ and let $f_t = tf_1 + (1-t)f_2$, $t \in (0,1)$. It is clear that $f_t \in \overline{\mathcal{R}}$ as $\overline{\mathcal{R}}$ is convex (see Lemma 2.9 (ii)). For simplicity, let us set $u_1 = u_{f_1}$, $u_2 = u_{f_2}$ and $u_t = u_{f_t}$. We need to show that

$$J_1(f_t) > tJ_1(f_1) + (1-t)J_1(f_2) \quad \forall t \in (0,1). \tag{3.12}$$

Let us suppose (3.12) is false and seek a contradiction. So we assume there exists $t \in (0,1)$ such that

$$J_1(f_t) = tJ_1(f_1) + (1-t)J_1(f_2). \tag{3.13}$$

Note that in (3.13) we have used the concavity of J_1. Writing out the left-hand side of (3.13), we have

$$\int_\Omega \Phi(|\nabla u_t|)\,\mathrm{d}x - \int_\Omega f_t u_t\,\mathrm{d}x = tJ_1(f_1) + (1-t)J_1(f_2).$$

Hence,

$$\begin{aligned} &t\overbrace{\Big(\int_\Omega \Phi(|\nabla u_t|)\,\mathrm{d}x - \int_\Omega f_1 u_t\,\mathrm{d}x - J_1(f_1)\Big)}^{A} \\ &+ (1-t)\overbrace{\Big(\int_\Omega \Phi(|\nabla u_t|)\,\mathrm{d}x - \int_\Omega f_2 u_t\,\mathrm{d}x - J_1(f_2)\Big)}^{B} = 0. \end{aligned} \tag{3.14}$$

Clearly, (3.14) implies $A = B = 0$. So, recalling the minimality of u_1 and u_2, we infer $u_1 = u_t = u_2$. Furthermore, it follows from (3.4) that $\int_\Omega (f_1 - f_2)w\,\mathrm{d}x = 0$ for all $w \in W_0^{1,\Phi}(\Omega)$. Thus, $f_1 = f_2$ a.e. in Ω, which is a contradiction.

(iii) To prove (3.10), it suffices to show that for a numerical sequence $\{t_n\}$ such that $t_n \downarrow 0$,

$$\lim_{n\to\infty} \frac{J_1(f_1 + t_n(f_2 - f_1)) - J_1(f_1)}{t_n} = -\int_\Omega (f_2 - f_1)u_{f_1}\,\mathrm{d}x. \tag{3.15}$$

Let us set $u_1 = u_{f_1}$, $u_2 = u_{f_2}$, and $w_n = u_{h_n}$, where $h_n = f_1 + t_n(f_2 - f_1)$. We have

$$\begin{aligned} J_1(h_n) &= \int_\Omega \Phi(|\nabla w_n|)\,\mathrm{d}x - \int_\Omega h_n w_n\,\mathrm{d}x \\ &\le \int_\Omega \Phi(|\nabla u_1|)\,\mathrm{d}x - \int_\Omega h_n u_1\,\mathrm{d}x \\ &= \int_\Omega \Phi(|\nabla u_1|)\,\mathrm{d}x - \int_\Omega f_1 u_1\,\mathrm{d}x + \int_\Omega (f_1 - h_n)u_1\,\mathrm{d}x \\ &= J_1(f_1) - t_n \int_\Omega (f_2 - f_1)u_1\,\mathrm{d}x, \end{aligned} \tag{3.16}$$

where we used the minimality of w_n to derive the inequality. Similarly, we can find that

$$J_1(f_1) \le J_1(h_n) + t_n \int_\Omega (f_2 - f_1)w_n\,\mathrm{d}x. \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$-\int_\Omega (f_2 - f_1)w_n\,\mathrm{d}x \le \frac{J_1(h_n) - J_1(f_1)}{t_n} \le -\int_\Omega (f_2 - f_1)u_1\,\mathrm{d}x. \tag{3.18}$$

Using the results in part (i), we know $w_n \rightharpoonup u_1$ in $W_0^{1,\Phi}(\Omega)$. Therefore, (3.15) follows from (3.18). $\square$

Now, we have enough tools to solve the optimization problems (3.8) and (3.9).

Theorem 3.5. *The minimization problem* (3.8) *is solvable. Moreover, if $\check{f} \in \mathcal{R}$ is a minimizer, then $\check{f} = \zeta(u_{\check{f}})$ a.e. in Ω for some increasing function ζ.*

Proof. The assertions follow directly from Lemma 3.4 and Theorem 1.2 with $J(f) = -J_1(f)$ and $G(f) = u_f$. $\square$

Theorem 3.6. *Suppose $f_0 \in L_+^{\Phi^*}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega)$, $a \in C^1(0,\infty)$ and it also satisfies* (3.5). *The maximization problem* (3.9) *has a unique solution $\hat{f} \in \mathcal{R}$. Moreover, we have $\hat{f} = \eta(u_{\hat{f}})$ a.e. in Ω for some decreasing function η.*

Proof. First, we consider the relaxed problem

$$\sup_{f \in \overline{\mathcal{R}}} J_1(f). \tag{3.19}$$

Since $\overline{\mathcal{R}}$ is weakly sequentially compact (Lemma 2.9 (ii)) and J_1 is weakly sequentially continuous (Lemma 3.4 (i)), the problem (3.19) is solvable. Because of the convexity of $\overline{\mathcal{R}}$ (Lemma 2.9 (ii)) and strict concavity of J_1 (Lemma 3.4 (ii)), the solution to (3.19) is unique and we denote it by $\hat{f}$. For any $g \in \overline{\mathcal{R}}$, we have $\hat{f} + t(g - \hat{f}) \in \overline{\mathcal{R}}$ for all $t \in (0,1)$. So, by applying Lemma 3.4 (iii), it follows that

$$0 \geq \lim_{t \to 0^+} \frac{J_1(\hat{f} + t(g - \hat{f})) - J_1(\hat{f})}{t} = -\int_\Omega (g - \hat{f}) u_{\hat{f}} \,\mathrm{d}x.$$

This means

$$\int_\Omega g u_{\hat{f}} \,\mathrm{d}x \geq \int_\Omega \hat{f} u_{\hat{f}} \,\mathrm{d}x \quad \forall g \in \overline{\mathcal{R}}.$$

Hence, $\hat{f}$ minimizes the linear functional $L(h) = \int_\Omega h u_{\hat{f}} \,\mathrm{d}x$ relative to $g \in \overline{\mathcal{R}}$. On the other hand, from the differential equation in (3.3) and Theorem 3.2 (in particular, the last assertion), in conjunction with [22, Lemma 7.7], we deduce that the graph of $u_{\hat{f}}$ has no significant flat sections on $S(\hat{f})$. By using Lemma 2.16, we infer the existence of a decreasing function η such that $\eta(u_{\hat{f}}) \in \mathcal{R}$. Applying Theorem 1.1, we must have $\hat{f} = \eta(u_{\hat{f}}) \in \mathcal{R}$ as desired. $\square$

Remark 3.7. The Φ-Laplacian operator includes the famous $p-$Laplacian operator if we set $a(t) = t^{p-2}$. Clearly, the structural conditions (3.1) and (3.5) are satisfied with $r = p$. Moreover, if we set $a(t) = t^{p-2} + t^{q-2}$ for some $p, q \in (1,\infty)$ satisfying $p < q$, then the Φ-Laplacian operator becomes the $(p,q)-$Laplacian one. The structural conditions (3.1) and (3.5) are also satisfied in this case. Indeed, it is elementary to show the function g defined by $g(t) = \frac{t^p + t^q}{\frac{t^p}{p} + \frac{t^q}{q}}$ is increasing on $(0,\infty)$. Consequently, for all $t > 0$, we have

$$p \leq \lim_{t \to 0^+} g(t) \leq \frac{t\varphi(t)}{\Phi(t)} = \frac{t^p + t^q}{\frac{t^p}{p} + \frac{t^q}{q}} = g(t) \leq \lim_{t \to \infty} g(t) = q,$$

and this proves (3.1). Condition (3.5) can be shown in a similar way. On the other hand, considering the delicate example constructed in Appendix, see (4.1), it satisfies the structural condition (3.1), but not for (3.5).

Remark 3.8. Our main results in this subsection, i.e. Theorems 3.5 and 3.6, extend the corresponding results in [6, 7, 8] for the classical Laplacian operator, and that in [14, 17, 26] for the p-Laplacian operator.

3.2. **Φ-Kirchhoff case.** Consider the boundary value problem

$$\begin{gathered} -M\Big(\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\Big)\nabla\cdot(a(|\nabla v|)\nabla v) = f(x) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.20}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary, $f \in L^{\Phi^*}(\Omega)$, and $M : [0,\infty) \to [0,\infty)$ is a continuous increasing function satisfying $M(t) > 0$ for all $t > 0$. For a solution $v \in W_0^{1,\Phi}(\Omega)$ of (3.20), we mean

$$M\Big(\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\Big)\int_\Omega a(|\nabla v|)\nabla v\cdot\nabla w\,\mathrm{d}x = \int_\Omega fw\,\mathrm{d}x, \quad \forall w \in W_0^{1,\Phi}(\Omega). \tag{3.21}$$

Using this definition we obtain the following result.

Theorem 3.9. *Problem* (3.20) *has a unique solution* $v_f \in W_0^{1,\Phi}(\Omega)$. *Moreover, the solution* v_f *is the unique minimizer of the energy functional* $I_f : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ *defined by*

$$I_f(v) = \hat{M}\Big(\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\Big) - \int_\Omega fv\,\mathrm{d}x,$$

relative to $v \in W_0^{1,\Phi}(\Omega)$, *where* $\hat{M}(t) = \int_0^t M(s)\,\mathrm{d}s$ $(t \geq 0)$. *In addition, if* $f \in L^{\Phi^*}(\Omega) \cap L^2_{\rm loc}(\Omega)$, $a \in C^1(0,\infty)$ *and it satisfies* (3.5), *then* $a(|\nabla v|)\nabla v \in W^{1,2}_{\rm loc}(\Omega)$.

Proof. Using that M is continuous, we can show $I_f \in C^1(W_0^{1,\Phi}(\Omega))$ by a variant of the proof of [28, Prop. 4.1]. Observing that $\hat{M}$ is increasing, for $\|v\| > 1$, it follows from Lemma 3.1 that

$$I_f(v) \geq \hat{M}\Big(\|v\|^r\Big) - 2\|f\|_{\Phi^*}\|v\|_\Phi \geq (\|v\|^r - 1)M(1) - C\|f\|_{\Phi^*}\|v\|,$$

where we used Hölder's inequality (2.2) in the first inequality, and the Poincaré inequality in the last one. Since $M(1) > 0$ and $r > 1$, I_f is coercive. Observing that the functional $\Gamma(v) = \int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x$ is strictly convex (as Φ is strictly convex and strictly increasing), it follows from the fact $\hat{M}$ is strictly increasing (as $M(t) > 0$ for all $t > 0$) and convex that I_f is strictly convex. On the other hand, remembering that Γ is weakly lower semicontinuous (see the proof of Theorem 3.2), we can show the functional $\hat{\Gamma} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\Gamma}(v) = \hat{M}\Big(\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\Big) \tag{3.22}$$

is weakly lower semicontinuous as $\hat{M}$ is continuous and increasing. This implies I_f is weakly lower semicontinuous. Now, we can apply the direct method to the minimization problem

$$\inf_{v \in W_0^{1,\Phi}(\Omega)} I_f(v), \tag{3.23}$$

to obtain a unique solution. It is straightforward to show that (3.23) and (3.20) have the same solutions. For the last assertion, if $v_f = 0$, then it is trivial. Otherwise, $M(\int_\Omega \Phi(|\nabla v_f|)\,\mathrm{d}x) > 0$ and it follows from [10, Theorem 2.1]. □

For each $f \in L^{\Phi^*}(\Omega)$, we use $v_f \in W_0^{1,\Phi}(\Omega)$ to denote the unique solution of (3.20) (see Theorem 3.9). Now, we can recall the rearrangement optimization problems in the Φ-Kirchhoff setting. Let $f_0 \in L_+^{\Phi^*}(\Omega)$ and we consider the following optimization problems

$$\inf_{f \in \mathcal{R}(f_0)} J_2(f) := I_f(v_f) = \hat{M}\Big(\int_\Omega \Phi(|\nabla v_f|)\,\mathrm{d}x\Big) - \int_\Omega fv_f\,\mathrm{d}x, \tag{3.24}$$

and

$$\sup_{f \in \mathcal{R}(f_0)} J_2(f). \tag{3.25}$$

Again, we firstly investigate the properties of the energy functional J_2. For simplicity, we set $\mathcal{R} = \mathcal{R}(f_0)$.

Lemma 3.10. *The following statements are valid:*

(i) *$J_2 : \overline{\mathcal{R}} \to \mathbb{R}$ is weakly sequentially continuous in $L^{\Phi^*}(\Omega)$.*
(ii) *J_2 is strictly concave.*
(iii) *Given $f_1, f_2 \in \overline{\mathcal{R}}$, it holds*

$$\lim_{t\to 0^+} \frac{J_2(f_1 + t(f_2 - f_1)) - J_2(f_1)}{t} = -\int_\Omega (f_2 - f_1) v_{f_1}\,\mathrm{d}x. \tag{3.26}$$

Proof. (i) Let $\{f_n\} \subseteq \overline{\mathcal{R}}$ satisfying $f_n \rightharpoonup f$ in $L^{\Phi^*}(\Omega)$ and set $v_n \equiv v_{f_n}$ for simplicity. Similarly as the proof of Lemma 3.4 (i), we can derive

$$\begin{aligned}
&M\Big(\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x\Big)\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x \\
&\le M\Big(\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x\Big)\int_\Omega \varphi(|\nabla v_n|)|\nabla v_n|\,\mathrm{d}x \\
&= M\Big(\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x\Big)\int_\Omega a(|\nabla v_n|)|\nabla v_n|^2\,\mathrm{d}x \\
&= \int_\Omega f_n v_n\,\mathrm{d}x \\
&\le 2\|f_n\|_{\Phi^*}\|v_n\|_\Phi \\
&\le C\|v_n\|.
\end{aligned}$$

As M is increasing, for $\|v_n\| > 1$, it follows from Lemma 3.1 that

$$M(\|v_n\|^r)\|v_n\|^r \le C\|v_n\|.$$

Thus, $\{v_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. Consequently, we infer the existence of a subsequence, still denoted $\{v_n\}$, and $v \in W_0^{1,\Phi}(\Omega)$ such that $v_n \rightharpoonup v$ in $W_0^{1,\Phi}(\Omega)$, $v_n \to v$ in $L^1(\Omega)$, and $v_n \to v$ a.e. in Ω. Since the functional $\hat{\Gamma}$ defined in (3.22) is weakly lower semicontinuous (see the proof Theorem 3.9), we infer that

$$\begin{aligned}
I_f(v) &= \hat{M}\Big(\int_\Omega \Phi(|\nabla v|)\,\mathrm{d}x\Big) - \int_\Omega f v\,\mathrm{d}x \\
&\le \liminf_{n\to\infty}\Big\{\hat{M}\Big(\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x\Big) - \int_\Omega f_n v_n\,\mathrm{d}x\Big\} \\
&\le \limsup_{n\to\infty}\Big\{\hat{M}\Big(\int_\Omega \Phi(|\nabla v_n|)\,\mathrm{d}x\Big) - \int_\Omega f_n v_n\,\mathrm{d}x\Big\} \\
&\le \limsup_{n\to\infty}\Big\{\hat{M}\Big(\int_\Omega \Phi(|\nabla v_f|)\,\mathrm{d}x\Big) - \int_\Omega f_n v_f\,\mathrm{d}x\Big\} \\
&= \hat{M}\Big(\int_\Omega \Phi(|\nabla v_f|)\,\mathrm{d}x\Big) - \int_\Omega f v_f\,\mathrm{d}x \\
&= I_f(v_f)
\end{aligned} \tag{3.27}$$

where we used Lemma 2.5 in conjunction with Lemma 2.20 in the first inequality, and the minimality of v_n in the third inequality (see Theorem 3.9). Using Theorem 3.9 again, we obtain $v = v_f$. Furthermore, all the inequalities in (3.27) become equalities, and we have

$$\lim_{n\to\infty} J_2(f_n) = \lim_{n\to\infty} I_{f_n}(v_n) = I_f(v_f) = J_2(f)$$

as desired.

The proofs of (ii) and (iii) are essentially along the same lines of that of Lemma 3.4 (ii) and (iii), so omitted. $\square$

Theorem 3.11. *The minimization problem* (3.24) *is solvable. Moreover, if $\check{f} \in \mathcal{R}$ is a minimizer, then $\check{f} = \zeta(v_{\check{f}})$ a.e. in Ω for some increasing function ζ.*

Proof. The assertions directly follow from Lemma 3.10 and Theorem 1.2 with $J(f) = -J_2(f)$ and $G(f) = v_f$. $\square$

Theorem 3.12. *Suppose $f_0 \in L_+^{\Phi^*}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega)$, $a \in C^1(0,\infty)$ and it also satisfies* (3.5)*. The maximization problem* (3.25) *has a unique solution $\hat{f} \in \mathcal{R}$. Moreover, we have $\hat{f} = \eta(v_{\hat{f}})$ a.e. in Ω for some decreasing function η.*

Proof. By using Lemma 3.10 and Theorem 3.9, the proof is essentially along the same lines of that of Theorem 3.6, so omitted. □

Remark 3.13. When $a(t) = 1$ and $M(s) = \alpha s + \beta$ with $\alpha, \beta > 0$, the Φ-Kirchhoff problem (3.20) becomes the famous Kirchhoff problem. The problem (3.20) is nonlocal and it has a connection with the Φ-Laplacian problem (3.3). Specifically, you can find a description of this in [2, Section 2] for $a(t) = 1$, and [16, Lemma 2.2] for $a(t) = t^{p-2}$. The homogeneity of the function $a(t)$ plays a role in these connections. Moreover, if $a(t) = t^{p-2} + t^{q-2}$ with $1 < p < q < \infty$, then the problem (3.20) is a nonlocal problem of $(p,q)-$Kirchhoff type, see also Remark 3.7. In addition, our main results in the subsection, i.e. Theorem 3.11 and 3.12, extend the corresponding results in [16] for the $p-$Kirchhoff operator.

4. Appendix: Relations between the conditions (3.1) and (3.5)

Suppose $a \in C^1(0,\infty)$ satisfies the conditions described in the first paragraph of Section 3 without (3.1). Firstly, we intend to show the condition (3.5) implies that of (3.1). Let us fix any $t \in (0,\infty)$. Recalling that $\varphi(t) = ta(t)$ and $\Phi(t) = \int_0^t sa(s)\,\mathrm{d}s$, by Cauchy's mean value theorem, there exists $c \in (0,t)$ such that

$$\frac{t\varphi(t)}{\Phi(t)} = \frac{t\varphi(t) - 0}{\Phi(t) - 0} = \frac{(t\varphi)'(c)}{\Phi'(c)} = \frac{c^2 a'(c) + 2ca(c)}{ca(c)} = \frac{ca'(c)}{a(c)} + 2.$$

This, in combined with (3.5), implies (3.1) as desired.

Secondly, we want to construct an intriguing example of $a(t)$ such that it conforms (3.1) for certain $p, r \in (1,\infty)$ while not satisfying (3.5) for any $p, r \in (1,\infty)$. This example shows the converse implication from (3.5) to (3.1) is not true in general. We start with the construction of the function $\varphi : [0,\infty) \to [0,\infty)$. Let $\eta \in C^1([0,1])$ be any strictly increasing function satisfying $\eta(0) = 0$, $\eta(1) = 1$, $\eta'(0) = 1$, and $\eta'(1) = 0$ (such a function clearly exists). Let $N \in \mathbb{Z}^+$ be large, to be determined later. The function φ is defined as follows

$$\varphi(t) = \begin{cases} t, & \text{if } t \in [0, 2N-1], \\ \eta(t - 2N + 1) + 2N - 1, & \text{if } t \in (2N-1, 2N], \\ (t-2k)^2 + 2k, & \text{if } t \in (2k, 2k+1],\ \forall k \in \{N, N+1, \dots\}, \\ (t-2k-2)^2 + 2k + 2, & \text{if } t \in (2k+1, 2k+2],\ \forall k \in \{N, N+1, \dots\}, \end{cases} \tag{4.1}$$

see Figure 1 for part of the graph.

Consequently, we find

$$a(t) = \begin{cases} 1, & \text{if } t \in (0, 2N-1], \\ \frac{\eta(t-2N+1)+2N-1}{t}, & \text{if } t \in (2N-1, 2N], \\ \frac{(t-2k)^2+2k}{t}, & \text{if } t \in (2k, 2k+1],\ \forall k \in \{N, N+1, \dots\}, \\ \frac{-(t-2k-2)^2+2k+2}{t}, & \text{if } t \in (2k+1, 2k+2],\ \forall k \in \{N, N+1, \dots\}. \end{cases}$$

It is easy to show that $a \in C^1(0,\infty)$. At the points $t = 2k$ ($k \in \{N, N+1, \dots\}$), we have

$$\frac{ta'(t)}{a(t)}\Big|_{t=2k} = \frac{2k \cdot \left(-\frac{1}{2k}\right)}{1} = -1.$$

Clearly, this contradicts (3.5) (and [10, condition (2.2)]). On the other hand, let us estimate the condition (3.1). We fix any $t \in [i, i+1]$ with $i \in \{2N-1, 2N, \dots\}$. Remembering that $\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s$, we infer from Figure 1 that

$$i^2 \le t\varphi(t) \le (i+1)^2,$$

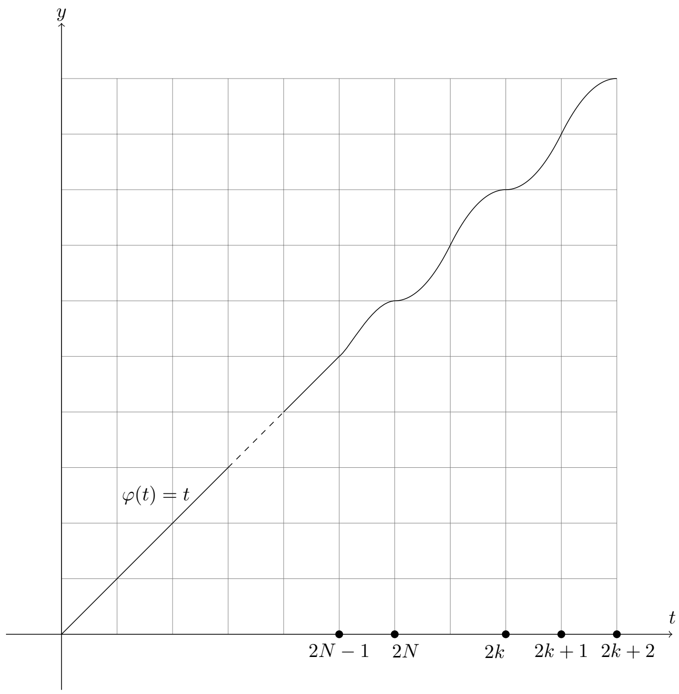

Figure 1. Graph of $\varphi(t)$

$$\frac{i^2}{2} - \frac{i-2N+1}{2} \le \Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s \le \frac{(i+1)^2}{2} + \frac{i-2N+2}{2}.$$

It yields

$$\begin{aligned}
2\frac{i^2}{(i+1)^2+i-2N+2} &= \frac{i^2}{\frac{(i+1)^2}{2} + \frac{i-2N+2}{2}} \\
&\le \frac{t\varphi(t)}{\Phi(t)} \\
&\le \frac{(i+1)^2}{\frac{i^2}{2} - \frac{i-2N+1}{2}} \\
&= 2\frac{(i+1)^2}{i^2-i+2N-1}.
\end{aligned} \tag{4.2}$$

We define $f, g : [2N-1, \infty) \to \mathbb{R}$ by $f(t) = \frac{(t+1)^2}{t^2-t+2N-1}$ and $g(t) = \frac{t^2}{(t+1)^2+t-2N+2}$. Direct calculations show that $f'(t) \le 0$ and $g'(t) \ge 0$ for all $t \in [2N-1, \infty)$. Therefore, it follows from (4.2) that

$$2\frac{(2N-1)^2}{(2N)^2+1} = 2g(2N-1) \le \frac{t\varphi(t)}{\Phi(t)} \le 2f(2N-1) = 2\frac{(2N)^2}{(2N-1)^2},$$

for all $t \in [2N-1, \infty)$. It is obvious that $\frac{t\varphi(t)}{\Phi(t)} = 2$ for all $t \in [0, 2N-1]$. Whence, for any $\epsilon > 0$, we can find a sufficiently large $N \in \mathbb{Z}^+$ such that

$$2 - \epsilon \le \frac{t\varphi(t)}{\Phi(t)} \le 2 + \epsilon, \quad \forall t > 0,$$

which shows the function $\varphi(t)$ will satisfy the condition (3.1) for a large N when $r \in (1, 2)$ and $p \in (2, \infty)$.

Remark 4.1. We can replace the exponent 2 in the definition of φ, see (4.1), by any $r \in (1, \infty)$ to get other interesting examples. More precisely, we only need to replace $(t-2k)^2$ and $(t-2k-2)^2$ in (4.1) by $(t-2k)^r$ and $(2k+2-t)^r$ respectively. The details are left to the readers.

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