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EXISTENCE OF GROUND STATE SOLUTIONS FOR FRACTIONAL CHOQUARD SYSTEMS

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Abstract. This article studies the existence of ground state solutions for nonlinear fractional Choquard systems. Under Berestycki-Lions type assumptions on the nonlinearity, we develop a variational framework in fractional Sobolev spaces to tackle the system. The main difficulties arising from the nonlocality of the fractional Laplacian and the associated loss of compactness are overcome by combining the concentration-compactness principle, the Pohožaev identity, and s-harmonic extension techniques. We establish the existence of ground state solutions of the fractional Choquard systems, and a numerical example is provided to verify the main result obtained in this paper.

1. Introduction and Main Results

This article investigates the existence of ground state solutions for the fractional nonlinear Choquard system

$$\begin{aligned} &(-\Delta)^s u_1 + u_1 = (I_\mu * F(u_2)) f(u_1), \quad x \in \mathbb{R}^N, \\ &(-\Delta)^s u_2 + u_2 = (I_\mu * F(u_1)) f(u_2), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $N \geq 3, s \in (0,1), F$ is the primitive of f, and the function $I_\mu : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential of order $\mu \in (0,N)$, i.e.,

$$I_\mu(x) = \frac{\Gamma\big(\frac{N-\mu}{2}\big)}{\Gamma\big(\frac{\mu}{2}\big)\pi^{N/2}2^\mu |x|^{N-\mu}}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Nonlinear Choquard-type equations are a class of nonlocal elliptic equations featuring convolution terms. Such equations arise in various contexts in mathematical physics and nonlinear analysis. The classical form of the Choquard equation was introduced by Pekar in 1954 in his study of the quantum theory of the polaron [33], and takes the form

$$-\Delta u + u = \left(I_2 * |u|^2\right) u, \quad x \in \mathbb{R}^3. \tag{1.2}$$

where $I_2(x) = \frac{1}{4\pi|x|}$ denotes the Newtonian potential in $\mathbb{R}^3$, equivalently, the Riesz potential of order 2. In 1976, Choquard employed this equation within the Hartree-Fock theory to model a one-component plasma. The existence and uniqueness of a positive ground state solution for equation (1.2) were subsequently established by Lieb [27] via symmetric decreasing rearrangement. Using variational methods, Lions [29] later proved the existence of infinitely many radial solutions. Later, Penrose [34, 35] revisited the equation in the context of self-gravitating quantum systems, interpreting it as a model for the wave function of a particle subject to its own gravitational field.

The generalized nonlinear Choquard equation

$$-\Delta u + u = (I_\mu * F(u)) f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

has been extensively studied in the framework of nonlocal variational problems. The foundational work of Berestycki and Lions [6] established sharp conditions on the nonlinearity f for the existence

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of ground states in the local semilinear elliptic equation $-\Delta u + u = f(u)$. These celebrated Berestycki-Lions conditions have since been adapted to the nonlocal setting of (1.3) by Moroz and Van Schaftingen [32], who developed a comprehensive existence theory for Choquard-type equations under optimal growth assumptions on F and f. Choquard equations with power-type nonlinearities of the form $|u|^{p-2}u$ have been extensively investigated. Ma and Zhao [30] established the radial symmetry and monotonicity of positive solutions via the method of moving planes. Further contributions address the existence, uniqueness, and qualitative properties of solutions under various ranges of the exponent p; see for instance [1, 5, 8, 9, 10, 24, 31, 41] and the references therein for an overview of the related topic. Additional developments concerning normalized solutions and concentration phenomena can be found in [3, 12, 19, 25, 26, 40] and the references therein.

In recent years, the fractional Laplacian has received considerable attention due to its ability to effectively model physical phenomena involving long-range interactions. It has found applications in diverse areas such as quantum mechanics, anomalous diffusion, and mathematical finance; see for instance [4, 16, 21, 23]. The fractional Laplacian, a nonlocal operator that arises naturally in the study of anomalous diffusion and fractional quantum mechanics, was introduced in the physical context by Laskin [22]. For $s \in (0,1)$, it is defined via the singular integral

$$(-\Delta)^s u(x) = C(N,s)\ \mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(z)}{|x-z|^{N+2s}}dz,$$

where P.V. denotes the Cauchy principal value and the normalization constant is

$$C(N,s) = \frac{4^s \Gamma(\frac{N+2s}{2})}{\pi^{N/2}|\Gamma(-s)|},$$

(see, e.g., [14]). The corresponding fractional Choquard equation

$$(-\Delta)^s u + u = (I_\mu * F(u))f(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

has been extensively investigated. In the power-type case $f(u) = |u|^{p-2}u$, D'Avenia, Siciliano, and Squassina [13] established comprehensive results on existence, nonexistence, regularity, and symmetry of solutions. For general nonlinearities, Shen and Gao [36] proved the existence of ground state solutions for equation (1.4) under Berestycki-Lions type conditions suitably adapted to the fractional Sobolev framework.

The analysis of Choquard systems entails additional analytical difficulties compared to scalar equations, primarily due to the presence of nonlocal coupling terms that intertwine the components. In the local case $s = 1$, variational systems of gradient type, where the nonlinearities derive from a common potential, have been investigated by several authors. Alves and Soares [2] established existence results via variational methods based on the Mountain Pass Theorem. While Chen and Liu [11] and Yang [39] addressed multiplicity and concentration phenomena under various assumptions on the coupling. More recently, Sun and Chang [38] considered a class of gradient systems with a Choquard term, and proved the existence of ground state solutions under Berestycki-Lions type conditions adapted to the system setting. Obviously, the system under consideration features a cross-coupled nonlinearity that does not arise from a single variational potential; consequently, it lies outside the class of gradient-type systems and differs fundamentally from pure power-coupled models. However, research on fractional Choquard systems has predominantly addressed equations with specific structural assumptions, such as pure powers or Hamiltonian coupling; see for instance [17, 18]. The existence theory for fractional Choquard systems with general nonlinear couplings remains largely undeveloped and constitutes a natural direction for future investigation.

Inspired by [32, 36, 38], this paper aims to investigate the existence of ground state solutions for the fractional Choquard system (1.1). Assuming that the nonlinear term f satisfies the following Berestycki-Lions type conditions in the fractional Sobolev space:

(A1) There exists a constant $C > 0$ such that $|uf(u)| \le C(|u|^{\frac{N+\mu}{N}} + |u|^{\frac{N+\mu}{N-2s}})$ for all $u \in \mathbb{R}$;

(A2) Let $F : u \in \mathbb{R} \mapsto \int_0^u f(\tau)d\tau$ and satisfying
$$\lim_{|u|\to 0} \frac{F(u)}{|u|^{\frac{N+\mu}{N}}} = 0, \quad \lim_{|u|\to \infty} \frac{F(u)}{|u|^{\frac{N+\mu}{N-2s}}} = 0;$$

(A3) There exists a $u_0 \in \mathbb{R} \setminus \{0\}$ such that $F(u_0) \neq 0$.

Let $E = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$. The norm of the space E can be expressed as
$$\|(u_1, u_2)\|_E^2 = \|u_1\|_{H^s(\mathbb{R}^N)}^2 + \|u_2\|_{H^s(\mathbb{R}^N)}^2, \quad \forall (u_1, u_2) \in E,$$
where $H^s(\mathbb{R}^N)$ denotes the fractional Sobolev space, defined as
$$H^s(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dxdy < +\infty\big\}$$
equipped with the norm
$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\, dx + \int_{\mathbb{R}^N} |u|^2\, dx.$$
We define the functional $J : E \to \mathbb{R}$ by
$$J(u_1, u_2) := \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2}u_1|^2 + |u_1|^2 + |(-\Delta)^{s/2}u_2|^2 + |u_2|^2\Big)dx - \int_{\mathbb{R}^N} (I_\mu * F(u_1))F(u_2)dx.$$
According to the discussion, we have $J \in C^1(E, \mathbb{R})$. We define the set
$$K = \{(u_1, u_2) \in E \setminus \{(0, 0)\} \mid J'(u_1, u_2) = 0\},$$
A pair $(u_1, u_2) \in E$ is called a ground state solution of the system (1.1) if it belongs to the set of nontrivial solutions K and minimizes the energy functional J over K, that is,
$$J(u_1, u_2) = \inf_{(v_1,v_2)\in K} J(v_1, v_2) =: c^*.$$
Our main result states as follows.

Theorem 1.1. *Assume that $N \geq 3$, $s \in (0, 1)$ and $\mu \in (0, 2s)$. If $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies conditions* (A1)–(A3)*, then system* (1.1) *admits at least one ground state solution.*

Remark 1.2. Extending the classical Choquard systems to the fractional case faces many difficulties: first, the embedding properties of the fractional Sobolev space are essentially different from those of the classical one; second, the combined nonlocality of the fractional operator and the convolution term makes the estimation of energy and the study of compactness more complex; finally, the proof method of the classical case cannot be directly extended to derive the Pohožaev identity corresponding to the fractional system, and the special properties of the fractional operator need to be considered. To solve these difficulties, we first establish a variational framework in the fractional product space E and prove that the corresponding energy functional satisfies the mountain pass geometry; then we construct the Pohožaev-Palais-Smale sequence by referring to the method of Jeanjean [20], and analyze the compactness of the sequence by using the properties of fractional spaces and the concentration-compactness principle; finally, we convert the nonlocal problem into a local one by the s-harmonic extension technique of the fractional Laplacian [7], derive the corresponding Pohožaev identity, and thus obtain the existence of ground state solutions for the system (1.1).

Remark 1.3. There exist many nonlinearities $f(u)$ satisfying conditions (A1)–(A3). For example, take $N = 3$, $s = \frac{3}{5}$, and $\mu = 1 \in (0, 2s)$. For all $u \in \mathbb{R}$, define
$$f(u) = \begin{cases} |u|^{1/3}ue^{-|u|} = |u|^{4/3}\frac{u}{|u|} \cdot e^{-|u|}, & u \neq 0,\\ 0, & u = 0. \end{cases}$$
It is clear that $f \in C^1(\mathbb{R}, \mathbb{R})$. Since
$$|uf(u)| = |u|^{7/3}e^{-|u|} \leq C\left(|u|^{4/3} + |u|^{20/9}\right)$$

for some $C > 0$, condition (A1) holds. Since $F(u) = \int_0^u f(\tau)d\tau = \int_0^u |\tau|^{4/3} \cdot \frac{\tau}{|\tau|} e^{-|\tau|} d\tau$, we have $e^{-|\tau|} \approx 1$ as $|u| \to 0$, hence

$$F(u) \approx \int_0^u |\tau|^{4/3}\, d\tau = \frac{3}{7}|u|^{7/3}.$$

Consequently,

$$\lim_{|u|\to 0} \frac{F(u)}{|u|^{\frac{N+\mu}{N}}} = \lim_{|u|\to 0} \frac{\frac{3}{7}|u|^{7/3}}{|u|^{4/3}} = 0.$$

As $|u| \to \infty$, $F(u)$ converges to some constant C_1, and therefore

$$\lim_{|u|\to \infty} \frac{F(u)}{|u|^{\frac{N+\mu}{N-2s}}} = \lim_{|u|\to \infty} \frac{C_1}{|u|^{20/9}} = 0.$$

Thus condition (A2) holds. Taking $u_0 = 1$, we obtain $F(1) = \int_0^1 \tau^{4/3} e^{-\tau}\, d\tau > 0$, which clearly gives $F(u_0) \neq 0$. Hence (f_3) also holds. In conclusion, this nonlinearity $f(u)$ satisfies all the assumptions of Theorem 1.1.

This paper is organized as follows. In Section 2, we collect the necessary functional analytic preliminaries and establish auxiliary lemmas. We then verify that the associated energy functional satisfies the mountain pass geometry and prove the existence of nontrivial critical points. Section 3 is devoted to proving the regularity of weak solutions and establishing the corresponding Pohožaev identity. In Section 4, we complete the proof of Theorem 1.1.

2. Preliminaries

To prove Theorem 1.1, we need some auxiliary lemmas, which will be crucial in dealing with ground state solutions of problem (1.1). First, we present the following fractional Sobolev embedding inequality and Hardy-Littlewood-Sobolev inequality.

Lemma 2.1 ([15]). *The fractional Sobolev space $H^s(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2, 2_s^*]$, and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2_s^*)$, that is , for any bounded domain $\Omega \subset \mathbb{R}^N$, every bounded sequence in $H^s(\mathbb{R}^N)$ has a convergent subsequence in $L^p(\Omega)$. Here $2_s^* = \frac{2N}{N-2s}$.*

Lemma 2.2 ([15]). *Let $2 \leq q \leq 2_s^*$, then for all $u \in H^s(\mathbb{R}^N)$, we have $\|u\|_{L^q} \leq C_q \|u\|_{H^s}$. Furthermore, if $2 \leq q < 2_s^*$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then every bounded sequence $\{u_k\}$ in $H^s(\mathbb{R}^N)$ has a convergent subsequence in $L^q(\Omega)$.*

Lemma 2.3 (Hardy-Littlewood-Sobolev inequality [28]). *Let $t, r > 1$ and $0 < \mu < N$ satisfy*

$$\frac{1}{t} + \frac{\mu}{N} + \frac{1}{r} = 2.$$

If $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a best constant $C(t, N, \mu, r)$, independent of f and h, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\, dx\, dy \leq C(t, N, \mu, r)\|f\|_{L^t}\|h\|_{L^r},$$

where $\|f\|_{L^t} = \left(\int_{\mathbb{R}^N} |f(x)|^t dx\right)^{\frac{1}{t}}$ denotes the L^t-norm of f.

Lemma 2.4. *Assume that* (A1)–(A3) *hold, then the functional J satisfies the mountain pass geometry, i.e.,*

- (i) *There exist constants $\rho, \delta > 0$ such that $J(u_1, u_2) \geq \delta > 0$ whenever $\|(u_1, u_2)\|_E = \rho$ (with ρ sufficiently small);*
- (ii) *There exists a pair $(e_1, e_2) \in E$ with $\|(e_1, e_2)\|_E > \rho$ such that $J(e_1, e_2) < 0$.*

Proof. (i) From (A1), we know that there exists a constant $C > 0$ such that $|F(u)| \leq C(|u|^{\frac{N+\mu}{N}} + |u|^{\frac{N+\mu}{N-2s}})$. By Lemmas 2.1 and 2.3, it holds

$$\int_{\mathbb{R}^N} (I_\mu * F(u_1))F(u_2)\, dx \leq C\|F(u_1)\|_{L^{\frac{2N}{N+\mu}}} \cdot \|F(u_2)\|_{L^{\frac{2N}{N+\mu}}}$$

$$
\begin{aligned}
&= C\Big(\int_{\mathbb{R}^N}|F(u_1)|^{\frac{2N}{N+\mu}}dx\Big)^{\frac{N+\mu}{2N}}\Big(\int_{\mathbb{R}^N}|F(u_2)|^{\frac{2N}{N+\mu}}dx\Big)^{\frac{N+\mu}{2N}}\\
&\le C\Big[\Big(\int_{\mathbb{R}^N}|F(u_1)|^{\frac{2N}{N+\mu}}dx\Big)^{\frac{N+\mu}{N}}+\Big(\int_{\mathbb{R}^N}|F(u_2)|^{\frac{2N}{N+\mu}}dx\Big)^{\frac{N+\mu}{N}}\Big]\\
&\le C\Big[\Big(\int_{\mathbb{R}^N}|u_1|^2+|u_1|^{2_s^*}dx\Big)^{\frac{N+\mu}{N}}+\Big(\int_{\mathbb{R}^N}|u_2|^2+|u_2|^{2_s^*}dx\Big)^{\frac{N+\mu}{N}}\Big]\\
&\le C\left(\|u_1\|_{H^s}^2+\|u_1\|_{H^s}^{2_s^*}\right)^{\frac{N+\mu}{N}}+C\left(\|u_2\|_{H^s}^2+\|u_2\|_{H^s}^{2_s^*}\right)^{\frac{N+\mu}{N}}\\
&\le C\|(u_1,u_2)\|_E^{\frac{2(N+\mu)}{N}}+C\|(u_1,u_2)\|_E^{\frac{2(N+\mu)}{N-2s}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
J(u_1,u_2)&=\frac{1}{2}\|(u_1,u_2)\|_E^2-\int_{\mathbb{R}^N}(I_\mu*F(u_1))F(u_2)\,dx\\
&\ge\frac{1}{2}\|(u_1,u_2)\|_E^2-C\|(u_1,u_2)\|_E^{\frac{2(N+\mu)}{N}}-C\|(u_1,u_2)\|_E^{\frac{2(N+\mu)}{N-2s}}.
\end{aligned}
$$

Consequently, there exists $\rho>0$ sufficiently small such that for all $(u_1,u_2)\in E$ with $\|(u_1,u_2)\|_E=\rho$, the energy functional J satisfies $J(u_1,u_2)\ge\frac{1}{4}\rho^2=\delta>0$.

(ii) From condition (f_3), there exists $u_0\in\mathbb{R}\setminus\{0\}$ such that $F(u_0)\neq0$. Take the test function $w_1=w_2=u_0\chi_{B_1}$, where χ_{B_1} is the characteristic function of the unit ball $B_1(0)$ in $\mathbb{R}^N$, i.e.,

$$
\chi_{B_1}(x)=\begin{cases}1, & \text{if } x\in B_1,\\ 0, & \text{otherwise.}\end{cases}
$$

Hence

$$
\int_{\mathbb{R}^N}(I_\mu*F(w_1))F(w_2)\,dx=F^2(u_0)\int_{B_1}\int_{B_1}I_\mu(x-y)\,dx\,dy>0.
$$

Since $H^s(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)\cap L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}(I_\mu*F(w_1))F(w_2)dx$ is continuous in $L^2(\mathbb{R}^N)\cap L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, there exist $v_1,v_2\in H^s(\mathbb{R}^N)$ such that

$$
\int_{\mathbb{R}^N}(I_\mu*F(v_1))F(v_2)\,dx>0.
$$

For $\tau>0$, define the scaling transformations $u_{1,\tau}(x)=v_1(\frac{x}{\tau}$, $u_{2,\tau}(x)=v_2(\frac{x}{\tau}$, then

$$
\begin{aligned}
J(u_{1,\tau},u_{2,\tau})&=\frac{\tau^{N-2s}}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{s/2}v_1|^2+|(-\Delta)^{s/2}v_2|^2\right)\\
&\quad+\frac{\tau^N}{2}\int_{\mathbb{R}^N}\left(|v_1|^2+|v_2|^2\right)-\tau^{N+\mu}\int_{\mathbb{R}^N}(I_\mu*F(v_1))F(v_2).
\end{aligned}
$$

For $\tau_0>0$ sufficiently large, the pair $(u_{1,\tau_0},u_{2,\tau_0})$ satisfies

$$
J(u_{1,\tau_0},u_{2,\tau_0})<0,\ \textit{and}\ \|(u_{1,\tau_0},u_{2,\tau_0})\|_E>\rho.
$$

Setting $(e_1,e_2)=(u_{1,\tau_0},u_{2,\tau_0})$, we obtain the required test function for the mountain pass geometry. Hence, Lemma 2.4 is proved. □

We define the mountain pass level as follows.

$$
c=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}J(\gamma(t)),
$$

where

$$
\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=(0,0),\,J(\gamma(1))<0\}.
$$

We introduce the associated Pohožaev functional $P:E\to\mathbb{R}$ by

$$
\begin{aligned}
P(u_1,u_2):=&\frac{N-2s}{2}\int_{\mathbb{R}^N}\left(|(-\Delta)^{s/2}u_1|^2+|(-\Delta)^{s/2}u_2|^2\right)dx+\frac{N}{2}\int_{\mathbb{R}^N}\left(|u_1|^2+|u_2|^2\right)dx\\
&-(N+\mu)\int_{\mathbb{R}^N}(I_\mu*F(u_1))F(u_2)dx,\quad\forall(u_1,u_2)\in E.
\end{aligned}
$$

WE set

$$\mathcal{R} = \{(u_1, u_2) \in E \setminus \{(0,0)\} : P(u_1, u_2) = 0\}.$$

For $\sigma \in \mathbb{R}$, $(v_1, v_2) \in E$ and $x \in \mathbb{R}^N$, we define the map $\Phi : \mathbb{R} \times E \to E$ by

$$\Phi(\sigma, v_1, v_2)(x) = (v_1(e^{-\sigma} x), v_2(e^{-\sigma} x)).$$

Then the composite functional $J \circ \Phi$ is

$$\begin{aligned}(J \circ \Phi)(\sigma, v_1, v_2) = \frac{e^{(N-2s)\sigma}}{2} \int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2} v_1|^2 + |(-\Delta)^{s/2} v_2|^2\Big) dx + \frac{e^{N\sigma}}{2} \int_{\mathbb{R}^N} \big(|v_1|^2 + |v_2|^2\big)\, dx \\ - e^{(N+\mu)\sigma} \int_{\mathbb{R}^N} (I_\mu * F(v_1)) F(v_2) dx.\end{aligned}$$

WE set

$$\tilde{\Gamma} = \{\tilde{\gamma} \in C([0,1], \mathbb{R} \times E) : \tilde{\gamma}(0) = (0,0,0),\, (J \circ \Phi)(\tilde{\gamma}(1)) < 0\}.$$

In fact, the admissible class Γ can be written as $\Gamma = \{\Phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}\}$, and therefore the mountain pass levels of the functionals J and $J \circ \Phi$ coincide. Consequently, the minimax value is

$$c = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{t \in [0,1]} (J \circ \Phi)(\tilde{\gamma}(t)) > 0.$$

Obviously, it follows from the condition (A1) that the functional $J \circ \Phi$ is continuously Fréchet differentiable on $\mathbb{R} \times E$. By the minimax principle, there exists a sequence $(\sigma_n, v_{1,n}, v_{2,n}) \subset \mathbb{R} \times E$ such that

$$(J \circ \Phi)(\sigma_n, v_{1,n}, v_{2,n}) \to c \quad \textit{and} \quad (J \circ \Phi)'(\sigma_n, v_{1,n}, v_{2,n}) \to 0,$$

as $n \to \infty$. For any $(h, \omega, z) \in \mathbb{R} \times E$, it holds

$$(J \circ \Phi)'(\sigma_n, v_{1,n}, v_{2,n})(h, \omega, z) = J'(\Phi(\sigma_n, v_{1,n}, v_{2,n}))\Phi(\sigma_n, \omega, z) + P(\Phi(\sigma_n, v_{1,n}, v_{2,n}))h.$$

Take $(u_{1,n}, u_{2,n}) = \Phi(\sigma_n, v_{1,n}, v_{2,n})$, we have

$$J(u_{1,n}, u_{2,n}) \to c, \quad J'(u_{1,n}, u_{2,n}) \to 0 \quad \text{and} \quad P(u_{1,n}, u_{2,n}) \to 0, \quad \text{as} \quad n \to \infty.$$

Lemma 2.5 ([13]). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^s(\mathbb{R}^N)$, if there exist $\sigma > 0$ and $2 \le q < 2_s^*$, such that*

$$\sup_{x \in \mathbb{R}^N} \int_{B_\sigma(x)} |u_n|^q\, dx \to 0 \quad (n \to \infty),$$

then $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2_s^*$.

Lemma 2.6. *Assume that conditions* (A1)–(A3) *hold, then the system* (1.1) *admits at least one nontrivial weak solution.*

Proof. Let $\{(u_{1,n}, u_{2,n})\} \subset E$ satisfy

$$J(u_{1,n}, u_{2,n}) \to c > 0, \quad J'(u_{1,n}, u_{2,n}) \to 0 \quad \text{and} \quad P(u_{1,n}, u_{2,n}) \to 0, \quad \text{as } n \to \infty.$$

For each positive integer n, we consider a suitable linear combination of the energy functional $J(u_{1,n}, u_{2,n})$ and Pohožaev functional $P(u_{1,n}, u_{2,n})$ in order to cancel the nonlocal nonlinear terms. This yields an identity involving only the linear part of the functional,

$$\begin{aligned}&J(u_{1,n}, u_{2,n}) - \frac{1}{N+\mu} P(u_{1,n}, u_{2,n}) \\ &= \frac{\mu + 2s}{2(N+\mu)} \int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2} u_{1,n}|^2 + |(-\Delta)^{s/2} u_{2,n}|^2\big) dx + \frac{\mu}{2(N+\mu)} \int_{\mathbb{R}^N} \big(|u_{1,n}|^2 + |u_{2,n}|^2\big) dx.\end{aligned}$$

As $n \to \infty$, we have $J(u_{1,n}, u_{2,n}) \to c > 0$ and $P(u_{1,n}, u_{2,n}) \to 0$. Consequently, the left-hand side of the identity remains bounded. Since the coefficients on the right-hand side of the identity are all nonnegative, and the integral terms

$$\int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2} u_{1,n}|^2 + |(-\Delta)^{s/2} u_{2,n}|^2\Big) dx, \quad \text{and} \int_{\mathbb{R}^N} \Big(|u_{1,n}|^2 + |u_{2,n}|^2\Big) dx$$

are also nonnegative, we can show that the sequence $\{(u_{1,n}, u_{2,n})\} \subset E$ is bounded.

Taking a subsequence (still denoted $(u_{1,n}, u_{2,n})$), then we have $(u_{1,n}, u_{2,n})$ converges weakly to $(u_{1,0}, u_{2,0})$ in E. We claim that both $\{u_{1,n}\}$ and $\{u_{2,n}\}$ are nonvanishing sequences. Let

$$\delta_1 = \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_{1,n}|^2\,dx.$$

If $\delta_1 = 0$, by Lemma 2.5, we must have $\|u_{1,n}\|_{L^p} \to 0$ for $p \in (2, 2_s^*)$. Fix $q \in \big(\frac{N+\mu}{N}, \frac{N+\mu}{N-2s}\big)$, from assumption (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(u)|^{\frac{2N}{N+\mu}} \le \varepsilon\big(|u|^2 + |u|^{\frac{2N}{N-2s}}\big) + C_\varepsilon |u|^{\frac{2Nq}{N+\mu}}, \quad \forall u \ge 0.$$

This implies $\lim_{n\to\infty} \int_{\mathbb{R}^N} |F(u_{1,n})|^{\frac{2N}{N+\mu}}\,dx = 0$. Since the sequence $\{u_{2,n}\}$ is bounded in $H^s(\mathbb{R}^N)$, according to Lemma 2.3, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\mu * F(u_{1,n}))F(u_{2,n})\,dx \le \lim_{n\to\infty} C\|F(u_{1,n})\|_{L^{\frac{2N}{N+\mu}}} \|F(u_{2,n})\|_{L^{\frac{2N}{N+\mu}}} = 0.$$

Therefore

$$\int_{\mathbb{R}^N} (I_\mu * F(u_{1,n}))F(u_{2,n})\,dx \to 0,\ n \to \infty.$$

Substituting into Pohožaev functional $P(u_{1,n}, u_{2,n})$ yields $\|(u_{1,n}, u_{2,n})\|_E \to 0$, as $n \to \infty$. This contradicts the fact that $J(u_{1,n}, u_{2,n}) \to c > 0$, and therefore $\delta_1 \ne 0$. Similarly, $\{u_{2,n}\}$ is also nonvanishing. Thus the claim holds.

From the nonvanishing of $\{u_{1,n}\}$ and $\{u_{2,n}\}$, then there exist sequences $\{y_{1,n}\}, \{y_{2,n}\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_{i,n})} |u_{i,n}|^2\,dx \ge \frac{\delta}{2} > 0, \quad i = 1, 2.$$

We define the translated sequences by $\omega_{1,n}(x) = u_{1,n}(x+y_{1,n})$ and $\omega_{2,n}(x) = u_{2,n}(x+y_{2,n})$, since translation is an isometry in $H^s(\mathbb{R}^N)$, the sequence $(\omega_{1,n}, \omega_{2,n})$ remains bounded in E and satisfies $\int_{B_1(0)} |\omega_{i,n}|^2\,dx \ge \frac{\delta}{2}$. Then there exists a subsequence (still denoted $(\omega_{1,n}, \omega_{2,n})$), and $(\omega_{1,0}, \omega_{2,0}) \in E$ such that $(\omega_{1,n}, \omega_{2,n})$ converges weakly to $(\omega_{1,0}, \omega_{2,0})$ in E. By the local compactness of the fractional Sobolev embedding, we have $\omega_{i,n} \to \omega_{i,0}$ in $L^q_{\rm loc}(\mathbb{R}^N)$ for any $2 \le q < 2_s^*$ and almost everywhere in $\mathbb{R}^N$. Together with Fatou's lemma, it holds

$$\int_{B_1(0)} \liminf_{n\to\infty} |\omega_{1,n}|^2\,dx \le \liminf_{n\to\infty} \int_{B_1(0)} |\omega_{1,n}|^2\,dx.$$

Thus we obtain $\omega_{1,0} \ne 0$. Similarly, we have $\omega_{2,0} \ne 0$, i.e., $(\omega_{1,0}, \omega_{2,0}) \ne (0, 0)$.

Since $\{\omega_{i,n}\}$ is bounded in $H^s(\mathbb{R}^N)$, and by the Sobolev embedding theorem, it is also bounded in $L^2(\mathbb{R}^N) \cap L^{2_s^*}(\mathbb{R}^N)$. Combined with condition (f_1), the sequence $\{F(\omega_{i,n})\}$ is bounded in $L^{\frac{2N}{N+\mu}}(\mathbb{R}^N)$. Moreover, since $\omega_{i,n}$ converges almost everywhere to $\omega_{i,0}$ in $\mathbb{R}^N$, by the continuity of F, we have $F(\omega_{i,n})$ converges almost everywhere to $F(\omega_{i,0})$ in $\mathbb{R}^N$. Hence, $F(\omega_{i,n})$ converges weakly to $F(\omega_{i,0})$ in $L^{\frac{2N}{N+\mu}}(\mathbb{R}^N)$. Since the Riesz potential I_μ is a linear continuous map from $L^{\frac{2N}{N+\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$, then the sequence $I_\mu * F(\omega_{i,n})$ converges weakly to $I_\mu * F(\omega_{i,0})$ in $L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$.

On the other hand, it follows from (f_1) and the fractional Rellich's theorem that the sequence $\{f(\omega_{i,n})\}$ converges strongly to $f(\omega_{i,0})$ in $L^p_{\rm loc}(\mathbb{R}^N)$ for all $p \in \big[1, \frac{2N}{\mu+2s}\big)$, $i = 1, 2$. Therefore, $(I_\mu * F(\omega_{j,n}))f(\omega_{i,n})$ converges weakly to $(I_\mu * F(\omega_{j,0}))f(\omega_{i,0})$ in $L^p(\mathbb{R}^N)$ for all $p \in \big[1, \frac{2N}{N+2s}\big)$, $i, j = 1, 2, i \ne j$.

For any test function $(\phi, \psi) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} J'(\omega_{1,0}, \omega_{2,0})(\phi, \psi) &= \int_{\mathbb{R}^N} (-\Delta)^{s/2}\omega_{1,0}(-\Delta)^{s/2}\phi\,dx + \int_{\mathbb{R}^N} \omega_{1,0}\phi\,dx \\ &\quad + \int_{\mathbb{R}^N} (-\Delta)^{s/2}\omega_{2,0}(-\Delta)^{s/2}\psi\,dx + \int_{\mathbb{R}^N} \omega_{2,0}\psi\,dx \\ &\quad - \int_{\mathbb{R}^N} (I_\mu * F(\omega_{2,0}))f(\omega_{1,0})\phi\,dx - \int_{\mathbb{R}^N} (I_\mu * F(\omega_{1,0}))f(\omega_{2,0})\psi\,dx \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\Big[\int_{\mathbb{R}^N}(-\Delta)^{s/2}\omega_{1,n}(-\Delta)^{s/2}\phi\,dx + \int_{\mathbb{R}^N}\omega_{1,n}\phi\,dx \\
&\quad + \int_{\mathbb{R}^N}(-\Delta)^{s/2}\omega_{2,n}(-\Delta)^{s/2}\psi\,dx + \int_{\mathbb{R}^N}\omega_{2,n}\psi\,dx \\
&\quad - \int_{\mathbb{R}^N}(I_\mu * F(\omega_{2,n}))f(\omega_{1,n})\phi\,dx - \int_{\mathbb{R}^N}(I_\mu * F(\omega_{1,n}))f(\omega_{2,n})\psi\,dx\Big] \\
&= 0,
\end{aligned}
$$

which implies that $(\omega_{1,0},\omega_{2,0})$ is a nontrivial weak solution of the system (1.1). The proof is complete. □

3. Regularity

We define $H_1,H_2:\mathbb{R}^N\to\mathbb{R}$ and $K_1,K_2:\mathbb{R}^N\to\mathbb{R}$ by

$$
\begin{gathered}
H_1(x)=\frac{F(u_1(x))}{u_1(x)},\quad H_2(x)=\frac{F(u_2(x))}{u_2(x)},\\
K_1(x)=f(u_1(x)),\quad K_2(x)=f(u_2(x)).
\end{gathered}
$$

Then system (1.1) can be rewritten as

$$
\begin{gathered}
(-\Delta)^s u_1+u_1=(I_\mu*(H_2u_2))K_1,\quad x\in\mathbb{R}^N,\\
(-\Delta)^s u_2+u_2=(I_\mu*(H_1u_1))K_2,\quad x\in\mathbb{R}^N.
\end{gathered}
$$

By (A1), it holds

$$
\begin{gathered}
|H_i(x)|\le C\Big(\frac{N}{N+\mu}|u_i(x)|^{\mu/N}+\frac{N-2s}{N+\mu}|u_i(x)|^{\frac{\mu+2s}{N-2s}}\Big),\quad i=1,2,\\
|K_i(x)|\le C\Big(|u_i(x)|^{\mu/N}+|u_i(x)|^{\frac{\mu+2s}{N-2s}}\Big),\quad i=1,2.
\end{gathered}
$$

Therefore, $H_i,K_i\in L^{\frac{2N}{\mu}}(\mathbb{R}^N)+L^{\frac{2N}{\mu+2s}}(\mathbb{R}^N)$, $i=1,2$. By an argument analogous to that in [36, Lemmas 3.1-3.3], we obtain $u_1,u_2\in L^p\left(\mathbb{R}^N\right)$ for every $p\in[2,\frac{N}{\mu}\ \frac{2\,N}{N-2\,s})$.

Lemma 3.1. *Let $N\ge 3$, $s\in(0,1)$ and $\mu\in(0,2s)$. Suppose that $f\in C(\mathbb{R},\mathbb{R})$ satisfies conditions (A1)–(A3) and $(u_1,u_2)\in E$ is a weak solution of system (1.1), then $u_1,u_2\in L^p\left(\mathbb{R}^N\right)$ for every $p\in[2,+\infty)$, i.e., $u_1,u_2\in L^\infty\left(\mathbb{R}^N\right)$.*

Proof. We already know $u_1,u_2\in L^p(\mathbb{R}^N)$ for any $p\in[2,\frac{2N^2}{\mu(N-2s)})$. Moreover, by (A1), we have $F(u_i)\in L^q(\mathbb{R}^N)$, for any $q\in[\frac{2N}{N+\mu},\frac{N}{\mu}\ \frac{2N}{N+\mu})$. Since

$$
\frac{2N}{N+\mu}<\frac{N}{\mu}<\frac{N}{\mu}\cdot\frac{2N}{N+\mu},
$$

by Lemma 2.3, it holds

$$
M_i(x)=I_\mu*F(u_i)\in L^\infty(\mathbb{R}^N).
$$

We denote $E_s:H^s(\mathbb{R}^N)\to X^s(\mathbb{R}^{N+1}_+)$ be the s-harmonic extension operator, where the weighted Sobolev space $X^s(\mathbb{R}^{N+1}_+)$ is defined as

$$
X^s(\mathbb{R}^{N+1}_+)=\big\{\omega:\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla\omega|^2\,dx\,dy<+\infty\big\}.
$$

Let $\omega_i=E_s(u_i)$ be the s-harmonic extension of u_i, which satisfies

$$
\begin{gathered}
\operatorname{div}(y^{1-2s}\nabla\omega_i)=0\quad\text{in }\mathbb{R}^{N+1}_+,\\
\omega_i(x,0)=u_i(x)\quad\text{on }\mathbb{R}^N,
\end{gathered}
$$

and

$$
(-\Delta)^s u_i(x)=-\frac{1}{\kappa_s}\lim_{y\to0^+}y^{1-2s}\frac{\partial\omega_i}{\partial y}(x,y),
$$

where κ_s is a normalization constant. Then system (1.1) is equivalent to

$$\begin{gathered}-\operatorname{div}(y^{1-2s}\nabla\omega_i)=0 \quad \text{in } \mathbb{R}_+^{N+1},\\ \partial_\nu^s\omega_i=-u_i+M_j(x)f(u_i) \quad \text{on } \mathbb{R}^N,\end{gathered}$$

for $i,j=1,2,i\neq j$, where

$$\partial_\nu^s\omega_i(x,0)=-\frac{1}{\kappa_s}\lim_{y\to 0^+}y^{1-2s}\frac{\partial\omega_i}{\partial y}(x,y),\quad \forall x\in\mathbb{R}^N.$$

Since $\omega_i\in X^s(\mathbb{R}_+^{N+1})$ is a critical point of the associated energy functional, it satisfies

$$\kappa_s\int_{\mathbb{R}_+^{N+1}}y^{1-2s}\nabla\omega_i\nabla\varphi_i=\int_{\mathbb{R}^N}(-u_i+M_j(x)f(u_i))\varphi_i,\qquad i\neq j,$$

for every $\varphi_i\in X^s(\mathbb{R}_+^{N+1})$. For $T>0$, define the truncated functions

$$\omega_{i,T}=\min\left\{(\omega_i)_+,T\right\},\quad u_{i,T}=\omega_{i,T}(\cdot,0),$$

where $(\omega_i)_+=\max\{0,\omega_i\}$.

Taking the test function $\varphi_i=|\omega_{i,T}|^{2\beta}\omega_i\in X^s(\mathbb{R}_+^{N+1})$ with $\beta>0$, it holds

$$\begin{aligned}&\kappa_s\int_{\mathbb{R}_+^{N+1}}y^{1-2s}\nabla\omega_i\nabla(|\omega_{i,T}|^{2\beta}\omega_i)\\ &=\kappa_s\int_{\mathbb{R}_+^{N+1}}y^{1-2s}\left\langle\nabla\omega_i,\nabla(|\omega_{i,T}|^{2\beta}\omega_i)\right\rangle\\ &=\kappa_s\int_{\mathbb{R}_+^{N+1}}y^{1-2s}|\omega_{i,T}|^{2\beta}|\nabla\omega_i|^2+(2\beta)\kappa_s\int_{\{\omega_i\le T\}}y^{1-2s}\omega_i^{2\beta}|\nabla\omega_i|^2\\ &=\int_{\mathbb{R}^N}(-u_i+M_j(x)f(u_i))|u_{i,T}|^{2\beta}u_i, i\neq j.\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}_+^{N+1}}\kappa_s y^{1-2s}|\nabla(|\omega_{i,T}|^{\beta}\omega_i)|^2\\ &=\int_{\mathbb{R}_+^{N+1}}\kappa_s y^{1-2s}|\omega_{i,T}|^{2\beta}|\nabla\omega_i|^2+(2\beta+\beta^2)\int_{\{\omega_i\le T\}}\kappa_s y^{1-2s}\omega_i^{2\beta}|\nabla\omega_i|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(|u_{i,T}|^{\beta}u_i)|^2+||u_{i,T}|^{\beta}u_i|^2\\ &=\int_{\mathbb{R}_+^{N+1}}\kappa_s y^{1-2s}|\nabla(|\omega_{i,T}|^{\beta}\omega_i)|^2+\int_{\mathbb{R}^N}||u_{i,T}|^{\beta}u_i|^2\\ &=\int_{\mathbb{R}_+^{N+1}}\kappa_s y^{1-2s}|\omega_{i,T}|^{2\beta}|\nabla\omega_i|^2+\int_{\mathbb{R}^N}||u_{i,T}|^{\beta}u_i|^2+(2\beta+\beta^2)\int_{\{\omega_i\le T\}}\kappa_s y^{1-2s}\omega_i^{2\beta}|\nabla\omega_i|^2\\ &\le C_\beta\int_{\mathbb{R}^N}M_j(x)f(u_i)|u_{i,T}|^{2\beta}u_i,\end{aligned}$$

where $C_\beta=1+\frac{\beta}{2}$. Since $M_j(x)\in L^\infty(\mathbb{R}^N)$, and by (A1), we obtain

$$f(u_i)\le C(|u_i|^{\mu/N}+|u_i|^{\frac{\mu+2s}{N-2s}}).$$

Since $u_i\in L^p(\mathbb{R}^N)$ for all $p\in[2,\frac{2N^2}{\mu(N-2s)})$, there exist a constant C_1 and a function $g\in L^{\frac{N}{2s}}(\mathbb{R}^N)$ with $g\ge 0$, independent of T and β, such that

$$M_j(x)f(u_i)|u_{i,T}|^{2\beta}u_i\le(C_1+g)|u_{i,T}|^{2\beta}u_i^2. \tag{3.1}$$

Consequently,

$$\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(|u_{i,T}|^{\beta}u_i)|^2+||u_{i,T}|^{\beta}u_i|^2\le C_1C_\beta\int_{\mathbb{R}^N}|u_{i,T}|^{2\beta}u_i^2+C_\beta\int_{\mathbb{R}^N}g|u_{i,T}|^{2\beta}u_i^2.$$

By Fatou's lemma and the monotone convergence theorem, we obtain as $T \to \infty$,
$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(u_{i+})^{\beta+1}|^2 + |(u_{i+})^{\beta+1}|^2 \le C_1 C_\beta \int_{\mathbb{R}^N} |(u_i)_+|^{2(\beta+1)} + C_\beta \int_{\mathbb{R}^N} g|(u_i)_+|^{2(\beta+1)}.$$
For any $M > 0$, let $A_1 = \{g \le M\}$ and $A_2 = \{g > M\}$. Since
$$\begin{aligned}\int_{\mathbb{R}^N} g|(u_i)_+|^{2(\beta+1)} &= \int_{A_1} g|(u_i)_+|^{2(\beta+1)} + \int_{A_2} g|(u_i)_+|^{2(\beta+1)} \\ &\le M \int_{A_1} |(u_i)_+|^{2(\beta+1)} + \big(\int_{A_2} g^{\frac{N}{2s}}\big)^{\frac{2s}{N}} \big(\int_{A_2} |(u_i)_+|^{(\beta+1)\frac{2N}{N-2s}}\big)^{\frac{N-2s}{N}} \\ &\le M\|(u_i)_+^{\beta+1}\|_{L^2}^2 + \varepsilon(M)\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2.\end{aligned}$$
Therefore,
$$\|(u_i)_+^{\beta+1}\|_{H^s}^2 \le C_\beta(C_1 + M)\|(u_i)_+^{\beta+1}\|_{L^2}^2 + C_\beta\varepsilon(M)\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2. \tag{3.2}$$
By Lemma 2.2, we have $\|u_i\|_{L^{2_s^*}} \le C_{2_s^*}\|u_i\|_{H^s}$ for $u_i \in H^s(\mathbb{R}^N)$. Thus, there exists $C_{2_s^*} > 0$ such that
$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}} \le C_{2_s^*}\|(u_i)_+^{\beta+1}\|_{H^s}.$$
Squaring both sides yields
$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 \le C_{2_s^*}^2\|(u_i)_+^{\beta+1}\|_{H^s}^2.$$
Substituting this estimate into inequality (3.2), we obtain
$$\begin{aligned}\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 &\le C_{2_s^*}^2\big[C_\beta(C_1 + M)\|(u_i)_+^{\beta+1}\|_{L^2}^2 + C_\beta\varepsilon(M)\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2\big] \\ &= C_\beta C_{2_s^*}^2(C_1 + M)\|(u_i)_+^{\beta+1}\|_{L^2}^2 + C_\beta C_{2_s^*}^2\varepsilon(M)\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2,\end{aligned}$$
which implies
$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 - C_\beta C_{2_s^*}^2\varepsilon(M)\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 \le C_\beta C_{2_s^*}^2(C_1 + M)\|(u_i)_+^{\beta+1}\|_{L^2}^2,$$
and equivalently,
$$\big[1 - C_\beta C_{2_s^*}^2\varepsilon(M)\big]\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 \le C_\beta C_{2_s^*}^2(C_1 + M)\|(u_i)_+^{\beta+1}\|_{L^2}^2.$$
Choosing M sufficiently large such that
$$C_\beta C_{2_s^*}^2\varepsilon(M) \le \frac{1}{2},$$
then
$$1 - C_\beta C_{2_s^*}^2\varepsilon(M) \ge \frac{1}{2}.$$
Therefore
$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 \le 2C_\beta C_{2_s^*}^2(C_1 + M)\|(u_i)_+^{\beta+1}\|_{L^2}^2.$$
Now a bootstrap argument start with $\beta+1 = \frac{N}{N-2s}$ shows that $(u_i)_+ \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty)$. Similarly, we know $(u_i)_- \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty)$, and hence $u_i \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty)$.

Next we prove that u_i is bounded in $\mathbb{R}^N$. Since $(u_i)_+ \in L^p(\mathbb{R}^N)$ for all $p \in [2, +\infty)$, we repeat the arguments in (3.1) to conclude that there exist the constant C_1 and function $g \in L^{\frac{N}{s}}(\mathbb{R}^N)$ with $g \ge 0$, independent of T and β such that
$$\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2}(|u_{i,T}|^\beta u_i)\big|^2 + \big||u_{i,T}|^\beta u_i\big|^2 \le C_1 C_\beta \int_{\mathbb{R}^N} |u_{i,T}|^{2\beta}u_i^2 + C_\beta \int_{\mathbb{R}^N} g|u_{i,T}|^{2\beta}u_i^2.$$
Applying Fatou's lemma and the monotone convergence theorem, we can take the limit as $T \to \infty$ to obtain
$$\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2}(u_i)_+^{\beta+1}\big|^2 + |(u_i)_+^{\beta+1}|^2 \le C_1 C_\beta \int_{\mathbb{R}^N} |(u_i)_+|^{2(\beta+1)} + C_\beta \int_{\mathbb{R}^N} g|(u_i)_+|^{2(\beta+1)}.$$
Applying Young's inequality, it holds
$$\int_{\mathbb{R}^N} g|(u_i)_+|^{2(\beta+1)} \le \|g\|_{L^{\frac{N}{s}}}\|(u_i)_+^{\beta+1}\|_{L^2}\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}} \le \|g\|_{L^{\frac{N}{s}}}\big(\lambda\|(u_i)_+^{\beta+1}\|_{L^2}^2 + \frac{1}{\lambda}\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2\big),$$

where $\lambda > 0$ is a constant to be chosen later. Therefore,

$$\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2}(u_i)_+^{\beta+1}\big|^2 + |(u_i)_+^{\beta+1}|^2 \le C_\beta(C_1 + \|g\|_{L^{\frac{N}{s}}}\lambda)\|(u_i)_+^{\beta+1}\|_{L^2}^2 + \frac{C_\beta\|g\|_{L^{\frac{N}{s}}}}{\lambda}\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2.$$

Applying the Lemma 2.2 and choosing λ sufficiently large such that $C_\beta\|g\|_{L^{\frac{N}{s}}} C_{2_s^*}^2 = \frac{\lambda}{2}$, we obtain

$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}}^2 \le 2C_{2_s^*}^2 C_\beta(C_1 + \|g\|_{L^{\frac{N}{s}}}\lambda)\|(u_i)_+^{\beta+1}\|_{L^2}^2 = M_\beta\|(u_i)_+^{\beta+1}\|_{L^2}^2,$$

where $M_\beta = 2C_{2_s^*}^2 C_\beta(C_1 + \|g\|_{L^{\frac{N}{s}}}\lambda)$. Taking the square root, we have

$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}} \le \sqrt{M_\beta}\|(u_i)_+^{\beta+1}\|_{L^2}.$$

Since

$$\|(u_i)_+^{\beta+1}\|_{L^{2_s^*}} = \Big(\int_{\mathbb{R}^N} |(u_i)_+|^{(\beta+1)2_s^*}\Big)^{1/2_s^*} = \|(u_i)_+\|_{L^{2_s^*(\beta+1)}}^{\beta+1},$$

and

$$\|(u_i)_+^{\beta+1}\|_{L^2} = \Big(\int_{\mathbb{R}^N} |(u_i)_+|^{2(\beta+1)}\Big)^{1/2} = \|(u_i)_+\|_{L^{2(\beta+1)}}^{\beta+1},$$

substituting these, we obtain $\|(u_i)_+\|_{L^{2_s^*(\beta+1)}}^{\beta+1} \le \sqrt{M_\beta}\|(u_i)_+\|_{L^{2(\beta+1)}}^{\beta+1}$, which implies

$$\|(u_i)_+\|_{L^{2_s^*(\beta+1)}} \le M_\beta^{\frac{1}{2(\beta+1)}}\|(u_i)_+\|_{L^{2(\beta+1)}}.$$

Since $C_\beta \le C(1+\beta)$ and $C_1, \|g\|_{L^{\frac{N}{s}}}, \lambda$ are independent of β, it follows that $M_\beta \le C(1+\beta)^2$. Moreover, since $(1+\beta)^2 \le e^{2\sqrt{1+\beta}}$ for sufficiently large β, we have $M_\beta \le M_0 e^{2\sqrt{1+\beta}}$, where M_0 is a constant independent of β. Therefore, we have

$$M_\beta^{\frac{1}{2(\beta+1)}} \le M_0^{\frac{1}{2(\beta+1)}} e^{\frac{1}{\sqrt{1+\beta}}}.$$

Then

$$\|(u_i)_+\|_{L^{2_s^*(\beta+1)}} \le M_0^{\frac{1}{2(\beta+1)}} e^{\frac{1}{\sqrt{1+\beta}}}\|(u_i)_+\|_{L^{2(\beta+1)}}.$$

We define the iteration $2(\beta_{n+1}+1) = 2_s^*(\beta_n+1), \beta_0 = 0$, solving this recurrence relation, we obtain

$$\beta_n + 1 = \big(\frac{2_s^*}{2}\big)^n = \big(\frac{N}{N-2s}\big)^n.$$

For each β_n, it holds

$$\|(u_i)_+\|_{L^{2_s^*(\beta_n+1)}} \le M_0^{\frac{1}{2(\beta_n+1)}} e^{\frac{1}{\sqrt{1+\beta_n}}}\|(u_i)_+\|_{L^{2(\beta_n+1)}}.$$

Since $2(\beta_{n+1}+1) = 2_s^*(\beta_n+1)$, it follows that $\|(u_i)_+\|_{L^{2(\beta_{n+1}+1)}} = \|(u_i)_+\|_{L^{2_s^*(\beta_n+1)}}$, hence,

$$\|(u_i)_+\|_{L^{2(\beta_{n+1}+1)}} \le M_0^{\frac{1}{2(\beta_n+1)}} e^{\frac{1}{\sqrt{1+\beta_n}}}\|(u_i)_+\|_{L^{2(\beta_n+1)}}.$$

Iterating from $n = 0$ to n, we obtain

$$\begin{aligned}
\|(u_i)_+\|_{L^{2(\beta_{n+1}+1)}} &\le M_0^{\frac{1}{2(\beta_n+1)}} e^{\frac{1}{\sqrt{1+\beta_n}}}\|(u_i)_+\|_{L^{2(\beta_n+1)}} \\
&\le M_0^{\frac{1}{2(\beta_n+1)}+\frac{1}{2(\beta_{n-1}+1)}} e^{\frac{1}{\sqrt{1+\beta_n}}+\frac{1}{\sqrt{1+\beta_{n-1}}}}\|(u_i)_+\|_{L^{2(\beta_{n-1}+1)}} \\
&\le \dots \\
&\le M_0^{\frac{1}{2}\sum_{k=0}^n \frac{1}{\beta_k+1}} e^{\sum_{k=0}^n \frac{1}{\sqrt{1+\beta_k}}}\|(u_i)_+\|_{L^{2(\beta_0+1)}}.
\end{aligned}$$

Since $\beta_0 = 0$ and $2(\beta_0+1) = 2$, we have

$$\|(u_i)_+\|_{L^{2(\beta_{n+1}+1)}} \le M_0^{\frac{1}{2}\sum_{k=0}^n \frac{1}{\beta_k+1}} e^{\sum_{k=0}^n \frac{1}{\sqrt{1+\beta_k}}}\|(u_i)_+\|_{L^2}.$$

Because $\beta_n + 1 = \big(\frac{N}{N-2s}\big)^n$, letting $r = \frac{N}{N-2s} > 1$, it holds

$$\sum_{k=0}^\infty \frac{1}{\beta_k+1} = \sum_{k=0}^\infty \frac{1}{r^k} = \frac{1}{1-1/r} < \infty, \quad \sum_{k=0}^\infty \frac{1}{\sqrt{1+\beta_k}} = \sum_{k=0}^\infty \frac{1}{r^{k/2}} = \frac{1}{1-1/\sqrt{r}} < \infty.$$

Thus,

$$\limsup_{n\to\infty}\|(u_i)_+\|_{L^{2(\beta_n+1)}} \le M_0^{\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{\beta_k+1}} e^{\sum_{k=0}^{\infty}\frac{1}{\sqrt{1+\beta_k}}}\|(u_i)_+\|_{L^2} < \infty.$$

By the monotonicity of the L^p-norm, we have

$$\|(u_i)_+\|_{L^\infty} = \lim_{p\to\infty}\|(u_i)_+\|_{L^p} \le \limsup_{n\to\infty}\|(u_i)_+\|_{L^{2(\beta_n+1)}} < \infty.$$

Therefore $(u_i)_+ \in L^\infty(\mathbb{R}^N)$. Similarly, we have $(u_i)_- \in L^\infty(\mathbb{R}^N)$. Thus we have $u_i \in L^\infty(\mathbb{R}^N)$ for $i = 1, 2$. The proof is complete. $\square$

Proposition 3.2 ([37]). *Let $\omega = (-\Delta)^s u$. If $\omega \in C^{0,\alpha}(\mathbb{R}^N)$ and $u \in L^\infty(\mathbb{R}^N)$, for $\alpha \in (0, 1]$, then*

(i) *If $\alpha + 2s \le 1$, then $u \in C^{0,\alpha+2s}(\mathbb{R}^N)$, and there exists a constant $C > 0$, depending only on N, α, s, such that*

$$\|u\|_{C^{0,\alpha+2s}(\mathbb{R}^N)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^N)} + \|\omega\|_{C^{0,\alpha}(\mathbb{R}^N)}\big);$$

(ii) *If $\alpha + 2s > 1$, then $u \in C^{1,\alpha+2s-1}(\mathbb{R}^N)$, and there exists a constant $C > 0$, depending only on N, α, s, such that*

$$\|u\|_{C^{1,\alpha+2s-1}(\mathbb{R}^N)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^N)} + \|\omega\|_{C^{0,\alpha}(\mathbb{R}^N)}\big).$$

Proposition 3.3 ([37]). *Let $\omega = (-\Delta)^s u$. If $\omega \in L^\infty(\mathbb{R}^N)$ and $u \in L^\infty(\mathbb{R}^N)$ for $s > 0$, then*

(i) *If $2s \le 1$, then $u \in C^{0,\alpha}(\mathbb{R}^N)$ for any $\alpha < 2s$, moreover*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^N)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^N)} + \|\omega\|_{L^\infty(\mathbb{R}^N)}\big),$$

where the constant C depends only on N, α, s;

(ii) *If $2s > 1$, then $u \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha < 2s - 1$, moreover,*

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^N)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^N)} + \|\omega\|_{L^\infty(\mathbb{R}^N)}\big),$$

where the constant C depends only on N, α, s.

From Lemma 3.1 and Propositions 3.2 and 3.3, the weak solution (u_1, u_2) is in fact a classical solution of system (1.1); that is, it satisfies the equations pointwise in E and enjoys higher regularity. From Lemma 3.1, we know $u_1, u_2 \in L^\infty(\mathbb{R}^N)$. Combined with Proposition 3.3, there exists $\alpha \in (0, 1)$ depending only on s, such that $u_1, u_2 \in C^{0,\alpha}(\mathbb{R}^N)$. Furthermore, we obtain

$$(I_\mu * F(u_2))f(u_1) - u_1 \in C^{0,\alpha}(\mathbb{R}^N) \quad \text{and} \quad (I_\mu * F(u_1))f(u_2) - u_2 \in C^{0,\alpha}(\mathbb{R}^N).$$

By Proposition 3.2, if $\alpha + 2s \le 1$, then $u_1, u_2 \in C^{0,\alpha+2s}(\mathbb{R}^N)$; if $\alpha + 2s > 1$, then we can further conclude $u_1, u_2 \in C^{1,\alpha+2s-1}(\mathbb{R}^N)$.

Lemma 3.4. *Assume $f \in C^1(\mathbb{R}, \mathbb{R})$, then there exists a constant α depending on s, such that $u_1, u_2 \in C^{2,\alpha}(\mathbb{R}^N)$.*

Proof. Let $W_i(u_1, u_2) = (I_\mu * F(u_j))f(u_i)$, $i \ne j$, $i, j = 1, 2$. Since $f \in C^1(\mathbb{R}, \mathbb{R})$ and $u_i \in L^\infty(\mathbb{R}^N)$, we have $F(u_i) \in C^1(\mathbb{R}^N)$. Combining with the regularity of the convolution of I_μ, we obtain $W_1, W_2 \in C^1(\mathbb{R}^N)$. We now prove the result by considering the following three cases.

Case I: $s \in (\frac{1}{2}, 1)$. By Lemma 3.1, we obtain $u_1, u_2 \in L^\infty(\mathbb{R}^N)$. Since $\omega_i = (-\Delta)^s u_i = W_i - u_i \in L^\infty(\mathbb{R}^N)$ and $2s > 1$. By Proposition 3.3 (ii), we deduce that $u_1, u_2 \in C^{1,\alpha_1}(\mathbb{R}^N)$ for some $\alpha_1 < 2s - 1$. Taking the partial derivative of u_i, then $\partial_{x_k} u_1$ satisfies

$$(-\Delta)^s \partial_{x_k} u_1 = \partial_{x_k} W_1 - \partial_{x_k} u_1 = (I_\mu * F'(u_2)\partial_{x_k} u_2) f(u_1) + (I_\mu * F(u_2)) f'(u_1)\partial_{x_k} u_1 - \partial_{x_k} u_1.$$

Similarly, we obtain the equation for $\partial_{x_k} u_2$. Since $W_1 \in C^1(\mathbb{R}^N)$, it follows that $\partial_{x_k} W_1 \in C^{0,\alpha_1}(\mathbb{R}^N)$ for some $\alpha_1 \in (0, 2s-1)$, and hence $\partial_{x_k} W_1 \in L^\infty(\mathbb{R}^N)$. Moreover, as $\partial_{x_k} u_1 \in L^\infty(\mathbb{R}^N)$, the fractional Laplacian satisfies $(-\Delta)^s \partial_{x_k} u_1 \in L^\infty(\mathbb{R}^N)$. By Proposition 3.3 (ii), we deduce that $\partial_{x_k} u_1 \in C^{1,\alpha_2}(\mathbb{R}^N)$ for some $\alpha_2 < 2s-1$. Consequently, $u_1 \in C^{2,\alpha}(\mathbb{R}^N)$ with $\alpha = \alpha_2$. An identical argument yields $u_2 \in C^{2,\alpha}(\mathbb{R}^N)$.

Case II: $s = \frac{1}{2}$. Since $W_i \in C^1$, $u_i \in L^\infty$ and $2s = 2 \cdot \frac{1}{2} = 1$. By Proposition 3.3 (i), we know that $u_i \in C^{0,\alpha_1}(\mathbb{R}^N)$ for some $\alpha_1 < 1$. Thus, $\omega_i = W_i - u_i \in C^{0,\alpha_1}(\mathbb{R}^N)$. It follows from $u_i \in L^\infty(\mathbb{R}^N)$, Proposition 3.2 (ii) and $\alpha_1 + 2s > 1$ that $u_i \in C^{1,\alpha_2}(\mathbb{R}^N)$ with $\alpha_2 = \alpha_1 + 1 - 1 = \alpha_1 \in (0, 1)$.

Following the argument employed in Case I and applying the corresponding regularity results to $\partial_{x_k} u_i$, we conclude that $u_i \in C^{2,\alpha}(\mathbb{R}^N)$ for $i = 1, 2$.

Case III: $s \in (0, \frac{1}{2})$. Since $\omega_i = (-\Delta)^s u_i = W_i - u_i \in L^\infty(\mathbb{R}^N)$, $u_i \in L^\infty(\mathbb{R}^N)$ and $2s < 1$. By Proposition 3.3 (i), we deduce that $u_i \in C^{0,\alpha_1}(\mathbb{R}^N)$ for any $\alpha_1 < 2s$, we choose $\alpha_1 \in (0, 2s)$. Since $\omega_i = W_i - u_i \in C^{0,\alpha_1}(\mathbb{R}^N)$, $u_i \in L^\infty(\mathbb{R}^N)$ and $\alpha_1 + 2s < 2s + 2s = 4s$.

If $s \in (\frac{1}{4}, \frac{1}{2})$, then $4s > 1$. Hence, by Proposition refprop3.2 (ii) we obtain that $u_i \in C^{1,\alpha_2}(\mathbb{R}^N)$ with $\alpha_2 = \alpha_1 + 2s - 1 \in (0, 2s + 2s - 1) = (0, 4s - 1) \subset (0, 1)$. Since $u_i \in C^{1,\alpha_2}(\mathbb{R}^N)$, it follows that $\partial_{x_k} u_i \in C^{0,\alpha_2}(\mathbb{R}^N)$. Moreover, as $\partial_{x_k} W_i \in C^{0,\alpha_2}(\mathbb{R}^N)$, the fractional Laplacian satisfies $(-\Delta)^s(\partial_{x_k} u_i) = \partial_{x_k} W_i - \partial_{x_k} u_i \in C^{0,\alpha_2}(\mathbb{R}^N)$. Combining with $\partial_{x_k} u_i \in L^\infty(\mathbb{R}^N)$, and Proposition 3.2 (ii), we obtain that $\partial_{x_k} u_i \in C^{1,\alpha_3}(\mathbb{R}^N)$ with $\alpha_3 = \alpha_2 + 2s - 1$, i.e., $u_i \in C^{2,\alpha}(\mathbb{R}^N)$ for $i = 1, 2$.

If $s \in (0, \frac{1}{4}]$, obviously, $4s \le 1$. Then we need to iterate Proposition 3.2 (i) again to obtain $u_i \in C^{0,\alpha_2}(\mathbb{R}^N)$ with $\alpha_2 = \alpha_1 + 2s \le 1$. Since $\alpha_2 + 2s = \alpha_1 + 4s$, when $s \in (\frac{1}{6}, \frac{1}{4}]$, we have $\alpha_1 + 4s > 1$, applying Proposition 3.2 (ii), we obtain that $u_i \in C^{1,\alpha_3}(\mathbb{R}^N)$ with $\alpha_3 = \alpha_2 + 2s - 1$. Iterating Proposition 3.2 (ii) for $\partial_{x_k} u_i$, we can conclude that $u_i \in C^{2,\alpha}(\mathbb{R}^N)$ by choosing $\alpha = \alpha_3$.

In general, for $s \in (\frac{1}{2k+2}, \frac{1}{2k}]$ with ($k \in \mathbb{N}$), a finite number of bootstrap iterations based on the regularity improvement scheme yields that $u_i \in C^{2,\alpha}(\mathbb{R}^N)$, where $\alpha = \alpha_1 + 2ks - 1$ and $\alpha_1 < 2s$. The proof is complete. □

Lemma 3.5. *If $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies* (A1)–(A3) *and $(u_1, u_2) \in E$ is a solution of system* (1.1). *Then the following Pohožaev identity holds:*

$$\begin{aligned} &\frac{N-2s}{2} \int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2} u_1|^2 + |(-\Delta)^{s/2} u_2|^2 \Big) dx + \frac{N}{2} \int_{\mathbb{R}^N} \big(|u_1|^2 + |u_2|^2 \big) dx \\ &= (N + \mu) \int_{\mathbb{R}^N} (I_\mu * F(u_1)) F(u_2) dx. \end{aligned} \tag{3.3}$$

Proof. Since $u_1, u_2 \in C^{2,\alpha}(\mathbb{R}^N)$, their s-harmonic extensions $\omega_i = E_s(u_i)$ satisfy $-\operatorname{div}(y^{1-2s} \nabla \omega_i) = 0$, and

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_i|^2 dx = \kappa_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla \omega_i|^2 dx\, dy, \quad \kappa_s > 0,$$

which implies $\omega_i \in C^{2,\alpha}(\mathbb{R}^{N+1}_+)$.

Let $D := \{z = (x, y) \in \mathbb{R}^N \times [0, \infty) : |z| \le 1\}$. Choose a cutoff function $\varphi \in C^1_c(\mathbb{R}^{N+1}_+)$ such that $\varphi = 1$ on D. For $\lambda \in (0, \infty)$, define the rescaled cutoff $\varphi_\lambda := \varphi(\lambda x, \lambda y)$, and introduce the test functions for $z \in \mathbb{R}^{N+1}_+$

$$\omega_{i,\lambda}(z) = \varphi_\lambda(z) \cdot z \cdot \nabla \omega_i(z) = \varphi(\lambda x, \lambda y) \big(x \cdot \nabla_x \omega_i(x, y) + y \cdot \nabla_y \omega_i(x, y) \big), \ i = 1, 2.$$

Then the weak form of the extension equation is

$$\kappa_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \omega_i \cdot \nabla \omega_{i,\lambda} dx\, dy = \int_{\mathbb{R}^N} \big(-u_i + (I_\mu * F(u_j)) f(u_i) \big) \cdot \omega_{i,\lambda}(x, 0) dx.$$

For $i = 1$, $j = 2$, we have

$$\kappa_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \omega_1 \cdot \nabla \omega_{1,\lambda} dx\, dy = \int_{\mathbb{R}^N} \big(-u_1 + (I_\mu * F(u_2)) f(u_1) \big) \cdot \omega_{1,\lambda}(x, 0) dx.$$

Let $L_1(\lambda)$ and $R_1(\lambda)$ denote the left- and right-hand sides of the above equation, respectively.

We now estimate the limits of $L_1(\lambda)$ and $R_1(\lambda)$ separately. On the one hand, since

$$\nabla \omega_{1,\lambda} = \lambda (\nabla \varphi)(\lambda x, \lambda y) [x \cdot \nabla_x \omega_1 + y \cdot \nabla_y \omega_1] + \varphi(\lambda x, \lambda y) \nabla [x \cdot \nabla_x \omega_1 + y \cdot \nabla_y \omega_1],$$

we have

$$\begin{aligned} L_1(\lambda) &= \kappa_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \omega_1 \cdot \big[\lambda (\nabla \varphi)(\lambda x, \lambda y)(x \cdot \nabla_x \omega_1 + y \cdot \nabla_y \omega_1) \big] dx\, dy \\ &\quad + \kappa_s \int_{\mathbb{R}^{N+1}_+} y^{1-2s} \nabla \omega_1 \cdot \big[\varphi(\lambda x, \lambda y) \nabla (x \cdot \nabla_x \omega_1 + y \cdot \nabla_y \omega_1) \big] dx\, dy \\ &=: L_1^{(1)}(\lambda) + L_1^{(2)}(\lambda). \end{aligned}$$

Since $\nabla\varphi$ is bounded, we have $\lambda(\nabla\varphi)(\lambda x, \lambda y) \to 0$ as $\lambda \to 0^+$. Furthermore, since the integrand is dominated by $C|\nabla\omega_1|^2$ for some constant C, the dominated convergence theorem implies $\lim\limits_{\lambda\to 0^+} L_1^{(1)}(\lambda) = 0$.

For

$$L_1^{(2)}(\lambda) = \kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}\varphi(\lambda x, \lambda y)\nabla\omega_1 \cdot \nabla(x \cdot \nabla_x\omega_1 + y \cdot \nabla_y\omega_1)dx\,dy.$$

Because,

$$\begin{aligned}\nabla(x \cdot \nabla_x\omega_1 + y \cdot \nabla_y\omega_1) &= (\nabla_x\omega_1 + x \cdot \nabla_x^2\omega_1) + (\nabla_y\omega_1 + y \cdot \nabla_y^2\omega_1)\\ &= \nabla\omega_1 + (x \cdot \nabla_x + y \cdot \nabla_y)\nabla\omega_1.\end{aligned}$$

Then it holds

$$\nabla\omega_1 \cdot \nabla(x \cdot \nabla_x\omega_1 + y \cdot \nabla_y\omega_1) = |\nabla\omega_1|^2 + \nabla\omega_1 \cdot \big[(x \cdot \nabla_x + y \cdot \nabla_y)\nabla\omega_1\big].$$

Moreover, it holds

$$(x \cdot \nabla_x + y \cdot \nabla_y)|\nabla\omega_1|^2 = \sum_{k=1}^{N+1} x_k\Big(\sum_{j=1}^{N+1} 2(\partial_{x_j}\omega_1)\partial_{x_jx_k}\omega_1\Big) = 2\nabla\omega_1 \cdot \big[(x \cdot \nabla_x + y \cdot \nabla_y)\nabla\omega_1\big],$$

which implies that

$$\nabla\omega_1 \cdot \big[(x \cdot \nabla_x + y \cdot \nabla_y)\nabla\omega_1\big] = \frac{1}{2}(x \cdot \nabla_x + y \cdot \nabla_y)|\nabla\omega_1|^2.$$

Therefore,

$$\nabla\omega_1 \cdot \nabla(x \cdot \nabla_x\omega_1 + y \cdot \nabla_y\omega_1) = |\nabla\omega_1|^2 + \frac{1}{2}(x \cdot \nabla_x + y \cdot \nabla_y)|\nabla\omega_1|^2.$$

Let $\lambda \to 0^+$, $\varphi(\lambda x, \lambda y) \to 1$, the dominated convergence theorem yields

$$\lim_{\lambda\to 0^+} L_1^{(2)}(\lambda) = \kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}\big[|\nabla\omega_1|^2 + \frac{1}{2}(x \cdot \nabla_x + y \cdot \nabla_y)|\nabla\omega_1|^2\big]dx\,dy.$$

By the divergence theorem, we have

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s}(x \cdot \nabla_x + y \cdot \nabla_y)|\nabla\omega_1|^2 dx\,dy = -(N + 2 - 2s)\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\omega_1|^2 dx\,dy.$$

Then

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s}\big[|\nabla\omega_1|^2 + \frac{1}{2}(x \cdot \nabla_x + y \cdot \nabla_y)|\nabla\omega_1|^2\big]dx\,dy = -\frac{N-2s}{2}\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\omega_1|^2 dx\,dy.$$

Thus,

$$\lim_{\lambda\to 0^+} L_1^{(2)}(\lambda) = \kappa_s\Big(-\frac{N-2s}{2}\Big)\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\omega_1|^2 dx\,dy = -\frac{N-2s}{2}\kappa_s\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\omega_1|^2 dx\,dy.$$

By the norm equivalence in the Caffarelli-Silvestre extension theorem (see, e.g., [7]), it holds

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_1|^2 dx = \kappa_s\int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla\omega_1|^2 dx\,dy.$$

Substituting this identity into the previous expression yields

$$\lim_{\lambda\to 0^+} L_1(\lambda) = -\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_1|^2 dx.$$

Similarly, for $i = 2$, $j = 1$, we have

$$\lim_{\lambda\to 0^+} L_2(\lambda) = -\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_2|^2 dx.$$

On the other hand, we have

$$R_1(\lambda) = \int_{\mathbb{R}^N} \big(-u_1 + (I_\mu * F(u_2))f(u_1)\big)\omega_{1,\lambda}(x,0)dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \big(-u_1 + (I_\mu * F(u_2))f(u_1)\big) \cdot \varphi(\lambda x)x \cdot \nabla u_1 dx \\
&= -\int_{\mathbb{R}^N} u_1 \cdot \varphi(\lambda x)x \cdot \nabla u_1 dx + \int_{\mathbb{R}^N} (I_\mu * F(u_2))f(u_1) \cdot \varphi(\lambda x)x \cdot \nabla u_1 dx \\
&=: R_{11}(\lambda) + R_{12}(\lambda),
\end{aligned}
$$

where

$$
R_{11}(\lambda) = -\int_{\mathbb{R}^N} u_1 \cdot \varphi(\lambda x)x \cdot \nabla u_1 dx,\ R_{12}(\lambda) = \int_{\mathbb{R}^N} (I_\mu * F(u_2))f(u_1) \cdot \varphi(\lambda x)x \cdot \nabla u_1 dx.
$$

Then, we obtain

$$
\begin{aligned}
R_{11}(\lambda) &= -\frac{1}{2}\int_{\mathbb{R}^N} \varphi(\lambda x) \cdot x \cdot \nabla({u_1}^2) dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} {u_1}^2 \big[\varphi(\lambda x)\nabla \cdot x + \lambda x \cdot \nabla\varphi(\lambda x)\big] dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} {u_1}^2 \big[\varphi(\lambda x) \cdot N + \lambda x \cdot \nabla\varphi(\lambda x)\big] dx \\
&= \int_{\mathbb{R}^N} \frac{N}{2}{u_1}^2 \varphi(\lambda x) dx + \int_{\mathbb{R}^N} \frac{{u_1}^2}{2} \cdot \lambda x \cdot \nabla\varphi(\lambda x) dx,
\end{aligned}
$$

which implies $\lim_{\lambda\to0^+} R_{11}(\lambda) = \frac{N}{2}\int_{\mathbb{R}^N} |u_1|^2 dx$. Similarly, we have $\lim_{\lambda\to0^+} R_{21}(\lambda) = \frac{N}{2}\int_{\mathbb{R}^N} |u_2|^2 dx$.

Furthermore, we have

$$
\begin{aligned}
R_{12}(\lambda) &= \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{F(u_2(y))}{|x-y|^{N-\mu}} dy\Big) f(u_1(x))\varphi(\lambda x)x \cdot \nabla u_1(x) dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(u_2(y))\Big(\frac{1}{|x-y|^{N-\mu}}\varphi(\lambda x)x \cdot \nabla F(u_1(x))\Big) dx\,dy \\
&= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(u_1(x))F(u_2(y))\Big[\frac{\varphi(\lambda x)}{|x-y|^{N-\mu}} \cdot N \\
&\quad + \frac{x\lambda\nabla\varphi(\lambda x)}{|x-y|^{N-\mu}} + x\varphi(\lambda x) \cdot \nabla\Big(\frac{1}{|x-y|^{N-\mu}}\Big)\Big] dx\,dy \\
&= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_1(x))F(u_2(y))}{|x-y|^{N-\mu}}\big(N\varphi(\lambda x) + \lambda x\nabla\varphi(\lambda x)\big) dx\,dy \\
&\quad + (N-\mu)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_1(x))F(u_2(y))}{|x-y|^{N-\mu}} \frac{(x-y)x \cdot \varphi(\lambda x)}{|x-y|^2} dx\,dy,
\end{aligned}
$$

and for $i=2$, $j=1$,

$$
\begin{aligned}
R_{22}(\lambda) &= \int_{\mathbb{R}^N} (I_\mu * F(u_1))f(u_2) \cdot \varphi(\lambda x)x \cdot \nabla u_2 dx \\
&= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_2(x))F(u_1(y))}{|x-y|^{N-\mu}}\big(N\varphi(\lambda x) + \lambda x\nabla\varphi(\lambda x)\big) dx\,dy \\
&\quad + (N-\mu)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F(u_2(x))F(u_1(y))}{|x-y|^{N-\mu}} \frac{(y-x)y \cdot \varphi(\lambda x)}{|x-y|^2} dx\,dy.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{\lambda\to0^+}\big[R_{12}(\lambda) + R_{22}(\lambda)\big] &= \lim_{\lambda\to0^+}\Big[\int_{\mathbb{R}^N} (I_\mu * F(u_2))f(u_1) \cdot \varphi(\lambda x)x \cdot \nabla u_1 dx \\
&\quad + \int_{\mathbb{R}^N} (I_\mu * F(u_1))f(u_2) \cdot \varphi(\lambda x)x \cdot \nabla u_2 dx\Big] \\
&= -(N+\mu)\int_{\mathbb{R}^N} (I_\mu * F(u_2))F(u_1) dx.
\end{aligned}
$$

Consequently, the Pohožaev identity (3.3) holds. The proof is now complete. $\square$

4. Proof of the main results

Proof of Theorem 1.1. Since $(\omega_{1,0},\omega_{2,0})$ is a nontrivial solution of system (1.1), by the definition of ground state energy, we have $J(\omega_{1,0},\omega_{2,0})\geq c^*$.

We define the path $\widetilde{\gamma}:[0,\infty)\to E$ by

$$\widetilde{\gamma}(\tau)(x)=\begin{cases}\big(u_1(x/\tau),u_2(x/\tau)\big), & \tau>0,\\ (0,0), & \tau=0.\end{cases}$$

Since $\widetilde{\gamma}$ is continuous on $(0,\infty)$, it holds for any $\tau>0$ that

$$\begin{aligned}\|\widetilde{\gamma}(\tau)\|_E^2&=\sum_{i=1}^2\Big[\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big(\tfrac{x}{\tau}\big)\big|^2dx+\int_{\mathbb{R}^N}\big|u_i\big(\tfrac{x}{\tau}\big)\big|^2dx\Big]\\ &=\tau^{N-2s}\sum_{i=1}^2\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big|^2dx+\tau^N\sum_{i=1}^2\int_{\mathbb{R}^N}\big|u_i\big|^2dx.\end{aligned}$$

This implies

$$\lim_{\tau\to0^+}\|\widetilde{\gamma}(\tau)\|_E^2=0=\|\widetilde{\gamma}(0)\|_E^2.$$

By the norm convergence, we have $\widetilde{\gamma}(\tau)\to\widetilde{\gamma}(0)=(0,0)$ in E as $\tau\to0^+$. Thus $\widetilde{\gamma}$ is continuous at $\tau=0$. This leads to $\widetilde{\gamma}\in C([0,\infty);E)$. For $\tau>0$, it holds

$$\begin{aligned}J(\widetilde{\gamma}(\tau))&=\frac12\|\widetilde{\gamma}(\tau)\|_E^2-\int_{\mathbb{R}^N}\big(I_\mu*F(\widetilde{\gamma}_1(\tau))\big)F(\widetilde{\gamma}_2(\tau))dx\\ &=\frac{\tau^{N-2s}}{2}\sum_{i=1}^2\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big|^2dx+\frac{\tau^N}{2}\sum_{i=1}^2\int_{\mathbb{R}^N}\big|u_i\big|^2dx-\tau^{N+\mu}\int_{\mathbb{R}^N}\big(I_\mu*F(u_1)\big)F(u_2)dx.\end{aligned}$$

This together with the Pohožaev identity (3.3), we have

$$J(\widetilde{\gamma}(\tau))=\Big(\frac{\tau^{N-2s}}{2}-\frac{(N-2s)\tau^{N+\mu}}{2(N+\mu)}\Big)\sum_{i=1}^2\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big|^2dx+\Big(\frac{\tau^N}{2}-\frac{N\tau^{N+\mu}}{2(N+\mu)}\Big)\sum_{i=1}^2\int_{\mathbb{R}^N}|u_i|^2dx.$$

Let $\phi(\tau)=J(\widetilde{\gamma}(\tau))$ for $\tau\in[0,\infty)$. Then we have $\phi(0)=0$, $\phi(1)=J(u_1,u_2)>0$, and

$$\begin{aligned}\phi'(\tau)&=\Big(\frac{(N-2s)\tau^{N-2s-1}}{2}-\frac{(N-2s)(N+\mu)\tau^{N+\mu-1}}{2(N+\mu)}\Big)\sum_{i=1}^2\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big|^2dx\\ &\quad+\Big(\frac{N\tau^{N-1}}{2}-\frac{N(N+\mu)\tau^{N+\mu-1}}{2(N+\mu)}\Big)\sum_{i=1}^2\int_{\mathbb{R}^N}|u_i|^2dx\\ &=\Big(\frac{(N-2s)\tau^{N-2s-1}}{2}-\frac{(N-2s)\tau^{N+\mu-1}}{2}\Big)\sum_{i=1}^2\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big|^2dx\\ &\quad+\Big(\frac{N\tau^{N-1}}{2}-\frac{N\tau^{N+\mu-1}}{2}\Big)\sum_{i=1}^2\int_{\mathbb{R}^N}|u_i|^2dx\\ &=\frac{(N-2s)\tau^{N-2s-1}}{2}\big(1-\tau^{\mu+2s}\big)\sum_{i=1}^2\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u_i\big|^2dx+\frac{N\tau^{N-1}}{2}\big(1-\tau^{\mu}\big)\sum_{i=1}^2\int_{\mathbb{R}^N}|u_i|^2dx.\end{aligned}$$

Then, it follows that $\phi'(\tau)>0$ for $0<\tau<1$; $\phi'(1)=0$ for $\tau=1$; and $\phi'(\tau)<0$ for $\tau>1$. Thus, $\phi(\tau)$ attains its strict global maximum at $\tau=1$, with $\phi(1)=J(u_1,u_2)$. That is, for any $\tau\in[0,1)\cup(1,\infty)$, we have $J(\widetilde{\gamma}(\tau))<J(u_1,u_2)$ and $\lim_{\tau\to+\infty}\phi(\tau)=-\infty$.

By a suitable change of variable, define $\gamma(t)=\tilde{\gamma}(\tau(t))$. Because $\tilde{\gamma}\in C([0,\infty);E)$ and $\tau\in C([0,1);[0,\infty))$ with $\lim_{t\to1^-}\tau(t)=\infty$, we have $\gamma\in C([0,1];E)$ satisfying

$$\gamma(0)=\widetilde{\gamma}(0)=(0,0),\quad\text{and}\quad\gamma(\tfrac12)=\widetilde{\gamma}(1)=(u_1,u_2).$$

For any $t \in [0,1] \setminus \{\frac{1}{2}\}$, we have $\tau(t) \neq 1$, so

$$J(\gamma(t)) = \phi(\tau(t)) < \phi(1) = J(u_1, u_2).$$

Then as $t \to 1^-$, $\tau(t) \to \infty$, it holds

$$J(\gamma(1)) = \lim_{t\to 1^-} \phi(\tau(t)) = -\infty < 0.$$

Thus, we may define the following set of mountain pass path Γ:

$$\Gamma = \{\gamma \in C([0,1]; E) : \gamma(0) = (0,0),\ J(\gamma(1)) < 0\},$$

Obviously, $\gamma \in \Gamma$, then Γ is nonempty, and we can define the mountain pass level c as follows

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)).$$

By the Pohožaev identity (3.3), we have $P(\omega_{1,0}, \omega_{2,0}) = 0$ for the solution $(\omega_{1,0}, \omega_{2,0})$. Thus,

$$J(\omega_{1,0}, \omega_{2,0}) = J(\omega_{1,0}, \omega_{2,0}) - \frac{1}{N+\mu} P(\omega_{1,0}, \omega_{2,0}).$$

For the sequence $(\omega_{1,n}, \omega_{2,n})$, because $P(\omega_{1,n}, \omega_{2,n}) \to 0$, it holds

$$J(\omega_{1,n}, \omega_{2,n}) - \frac{1}{N+\mu} P(\omega_{1,n}, \omega_{2,n}) \to c.$$

By lower semicontinuity, we obtain

$$\begin{aligned}
J(\omega_{1,0}, \omega_{2,0}) &= \frac{2s+\mu}{2(N+\mu)} \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\omega_{i,0}\right|^2 dx + \frac{\mu}{2(N+\mu)} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\omega_{i,0}|^2 dx \\
&\le \liminf_{n\to\infty} \Big[\frac{2s+\mu}{2(N+\mu)} \sum_{i=1}^{2} \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\omega_{i,n}\right|^2 dx + \frac{\mu}{2(N+\mu)} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\omega_{i,n}|^2 dx \Big] \\
&= \liminf_{n\to\infty} \Big[J(\omega_{1,n}, \omega_{2,n}) - \frac{1}{N+\mu} P(\omega_{1,n}, \omega_{2,n}) \Big] \\
&= \liminf_{n\to\infty} J(\omega_{1,n}, \omega_{2,n}) = c.
\end{aligned}$$

Let (v_1, v_2) be any nontrivial solution of system (1.1), then there exists a path $\gamma \in \Gamma$ such that

$$\gamma(0) = (0,0), \quad \gamma(\tfrac{1}{2}) = (v_1, v_2), \quad \sup_{t\in[0,1]} J(\gamma(t)) = J(v_1, v_2), \quad J(\gamma(1)) < 0.$$

Thus, for the path γ corresponding to the solution (v_1, v_2), we have

$$c \le \sup_{t\in[0,1]} J(\gamma(t)) = J(v_1, v_2),$$

for any (v_1, v_2). This implies

$$c \le \inf\{J(v_1, v_2) : (v_1, v_2) \text{ is a solution}\} = c^*.$$

Combining $c^* \le J(\omega_{1,0}, \omega_{2,0})$, $J(\omega_{1,0}, \omega_{2,0}) \le c$ and $c \le c^*$, we have

$$c^* \le J(\omega_{1,0}, \omega_{2,0}) \le c \le c^*,$$

which implies

$$J(\omega_{1,0}, \omega_{2,0}) = c = c^*.$$

Therefore, $(\omega_{1,0}, \omega_{2,0})$ is a ground state solution of system (1.1). The proof is completed. □

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