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ERROR ESTIMATES FOR SEMI-GALERKIN APPROXIMATIONS OF THE 3D INHOMOGENEOUS NEMATIC LIQUID CRYSTAL FLOWS

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Abstract. This work investigates rigorously the convergence properties of a spectral semi-Galerkin approximation for three-dimensional inhomogeneous nematic liquid crystal flows. To overcome the analytical barriers posed by the hyperbolic density transport and the pointwise geometric constraint $|\mathbf{d}| = 1$, we analyze a decoupled scheme that projects only the velocity onto a finite-dimensional space. Assuming uniform local-in-time bounds, we first derive basic error estimates of order $\mathcal{O}(\lambda_{k+1}^{-1})$. Through higher-order $\mathbf{H}^3$ estimates, we then prove an convergence rate of $\mathcal{O}(\lambda_{k+1}^{-3/2})$ in the $\mathbf{L}^2$ norm. Our results provide a structure-preserving framework for the rigorous analysis of complex non-homogeneous liquid crystal flows.

1. Introduction

The hydrodynamic flow of nematic liquid crystals has been a subject of intense analytical and numerical research because of its wide range of applications in physics, materials science, and modern display technologies. The continuum theory of such materials is generally described by the celebrated Ericksen-Leslie system, originally developed in the 1960s [6, 13]. This macroscopic model couples the Navier-Stokes equations for the fluid velocity with a parabolic harmonic map heat flow for the director field, which represents the preferred molecular orientation. The mathematical foundations of this system, particularly for the simplified homogeneous model, were established in the seminal works of Lin and Liu [15, 16].

In this paper, we investigate the convergence rate of spectral semi-Galerkin approximations for the three-dimensional inhomogeneous (density-dependent) incompressible nematic liquid crystal flows. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary Γ, and let $[0, T]$ be a given time interval. Let $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^3$, $\mathbf{d}(\mathbf{x}, t) \in \mathbb{R}^3$, $\rho(\mathbf{x}, t) \in \mathbb{R}$, and $p(\mathbf{x}, t) \in \mathbb{R}$ denote, respectively, the unknown fluid velocity, director field, density, and pressure at a point $\mathbf{x} \in \Omega$ and time $t \in [0, T]$. The governing equations describing the strong nonlinear coupling between these variables are

$$
\begin{gathered}
\rho_t + (\mathbf{u} \cdot \nabla)\rho = 0, \\
\rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p = -\lambda \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\
\mathbf{d}_t + (\mathbf{u} \cdot \nabla)\mathbf{d} = \gamma \left(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \right), \\
\nabla \cdot \mathbf{u} = 0, \\
|\mathbf{d}| = 1.
\end{gathered}
\tag{1.1}
$$

Here, the constant $\nu > 0$ is the fluid viscosity, $\lambda > 0$ is the elasticity constant, and $\gamma > 0$ is the macroscopic relaxation time constant. In this work, we consider the following boundary and initial conditions:

$$
\begin{gathered}
\mathbf{u} = 0, \quad \frac{\partial \mathbf{d}}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma \times (0, T), \\
(\rho, \mathbf{u}, \mathbf{d})|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{d}_0) \quad \text{in } \Omega.
\end{gathered}
\tag{1.2}
$$

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The mathematical challenges in studying this exact model are formidable and threefold. First, the equation for the density is a hyperbolic transport equation, $\rho_t + \mathbf{u}\cdot\nabla\rho = 0$. This introduces a purely advective mechanism that does not benefit from the dissipative smoothing effects present in the momentum and director equations. Second, the strict pointwise geometric constraint $|\mathbf{d}| = 1$ gives rise to a highly supercritical nonlinearity of the form $|\nabla\mathbf{d}|^2\mathbf{d}$ in the director equation, which severely complicates any regularity theory. Third, these effects are strongly coupled: the variable density affects the fluid momentum, the fluid transports the director, and the elastic stresses from the director field back-react on the flow. The existence of local and global strong solutions for this highly nonlinear non-homogeneous system has been the focus of recent outstanding contributions, notably by Wen and Ding [23], Li and Wang [14], and Gao, Tao, and Yao [7]. Observe that this system includes as a particular case the classical Navier-Stokes equations, which have been largely studied (see for instance the classical books by Ladyzhenskaya [11] and Temam [22] and the references therein). It also includes as a reduced model the nonhomogeneous Navier-Stokes system, that are less known (cf. [5, 12, 17, 20, 21]).

While the existence of local and global strong solutions for this highly nonlinear nonhomogeneous system has been established in recent outstanding contributions, the constructive analytical approximation of these solutions poses a significant structural challenge. Standard full Faedo-Galerkin methods are mathematically unsuited for this system. Projecting the entire coupled system onto a finite-dimensional basis fails to preserve the hyperbolic nature of the density along particle paths. More critically, such a global projection irreversibly destroys the pointwise geometric constraint $|\mathbf{d}| = 1$, causing the fundamental nonlinear structural cancellations, which are strictly necessary to ensure energy dissipation and bounds in the continuous system, to collapse. To overcome these extreme analytical difficulties, we analyze a spectral semi-Galerkin approximation. In our approach, only the velocity field $\mathbf{u}$ is projected onto a finite-dimensional space spanned by the eigenfunctions of the Stokes operator. Indeed, the density ρ^n and the director field $\boldsymbol{d}^n$ remain infinite-dimensional, governed by their respective continuous PDEs. This method preserves the structural integrity of the director field's sphere constraint and ensures the existence of well-behaved density profiles via the method of characteristics. The uniform a priori estimates guaranteeing the stability and the local-in-time existence of strong solutions for such approximate systems are inherently supported by the higher-order energy bounds recently established in Gao et al. [7].

Assuming these uniform bounds, the primary objective of this paper is to rigorously investigate the convergence rates of the solutions generated by this spectral semi-Galerkin approximation towards the strong solutions of the 3D inhomogeneous nematic liquid crystal equations. Understanding the quantitative error bounds of this approximation is of fundamental importance for both theoretical completeness and numerical implementations. We establish error estimates depending on the eigenvalues λ_{k+1} of the Stokes operator. We first derive basic uniform bounds and error estimates of order $\mathcal{O}(\lambda_{k+1}^{-1})$ for the density, velocity, and director field. Subsequently, through a refined bootstrap argument, see [3], and higher-order H^3-error estimates for the optical director, we significantly improve the L^2-error bounds, achieving an accelerated convergence rate of order $\mathcal{O}(\lambda_{k+1}^{-3/2})$. These results mathematically demonstrate the high efficiency and stability of the semi-Galerkin method when dealing with the complex interactions of inhomogeneous nematic liquid crystal flows.

The paper is organized as follows. In Section 2, we introduce the necessary functional setting, the Stokes operator, and the spectral semi-Galerkin approximation scheme. In Section 3 we establish the basic $\mathcal{O}(\lambda_{k+1}^{-1})$ error estimates. In Section 4, we derive higher-order $\mathbf{H}^3$ estimates for the director field. Finally, in Section 5, we employ a bootstrap argument to prove our main result: the improved $\mathbf{L}^2$ error estimate of order $\mathcal{O}(\lambda_{k+1}^{-3/2})$.

2. Preliminaries

Throughout this work, we consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D) : \|\partial^\alpha f\|_{L^q(D)} < +\infty, |\alpha| \le m\},$$

for a multi-index α, a nonnegative integer m and $1 \le q \le +\infty$, where the domain is $D = \Omega$ or $D = \Omega \times (0,T), 0 < T \le +\infty$, depending on the context. When $q = 2$, we denote by $H^m(D) = W^{2,q}(D)$ and $H_0^m(D)$ the closure of $C_0^\infty(D)$ in $H^m(D)$. If B is a Banach space, we denote by $L^q([0,T];B)$ the Banach space of B-valued functions defined on the interval $[0,T]$ that are L^q-integrable in Bochner's sense. We also define the weak L^p-space which is defined by all the functions $f \in L^1_{\rm loc}$ such that $\|f\|_{L^p_w} = \sup_{t>0} t|\{\mathbf{x} \in \Omega : f(\mathbf{x}) > t\}|^{1/p} < \infty$. We establish that spaces of $\mathbb{R}^n$-valued functions, as well as their elements, are represented by boldface letters. We write

$$\mathbf{C}_{0,\sigma}^\infty := \{\mathbf{v} \in \mathbf{C}_0^\infty : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}$$

and denote by $\mathbf{H}$ and $\mathbf{V}$ the closure of $\mathbf{C}_{0,\sigma}^\infty$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ respectively.

Let P be the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}(\Omega)$. Thus, the Stokes operator is $-P\Delta$, defined on $\mathbf{V} \cap \mathbf{H}^2(\Omega)$. Its eigenfunctions and eigenvalues are denoted by $\boldsymbol{\varphi}^k$ and λ^k, respectively. The usual $L^2(\Omega)$ inner product and norm are respectively indicated by $(\cdot,\cdot)$, and $\|\cdot\|$.

It is well known that $\{\boldsymbol{\varphi}^k(\mathbf{x})\}_{k=1}^\infty$ form a complete orthogonal projection system in the spaces $\mathbf{H}, \mathbf{V}$ and $\mathbf{V} \cap \mathbf{H}^2(\Omega)$ equipped with the usual inner products $(\mathbf{u},\mathbf{v}), (\nabla\mathbf{u}, \nabla\mathbf{v})$ and $(P\Delta\mathbf{u}, P\Delta\mathbf{v})$ respectively.

For each $k \in \mathbb{N}$, we denote by P_k the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}_k := \text{span}[\boldsymbol{\varphi}^1, \dots, \boldsymbol{\varphi}^k]$. For all $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\Omega)$, and $k, m \in \mathbb{N}$, it holds

(i) $(P_k\mathbf{f}, \mathbf{g}) = (\mathbf{f}, P_k\mathbf{g})$,
(ii) $(P\mathbf{f}, \mathbf{g}) = (\mathbf{f}, P\mathbf{g})$,
(iii) $((P_m - P_k)\mathbf{f}, \mathbf{g}) = (\mathbf{f}, (P_m - P_k)\mathbf{g})$,
(iv) $((P - P_k)\mathbf{f}, \mathbf{g}) = (\mathbf{f}, (P - P_k)\mathbf{g})$.

The following lemma is useful for our purposes.

Lemma 2.1 (Rautmann [18])**.** *If* $\mathbf{v} \in \mathbf{V}$*, then*

$$\|\mathbf{v} - P_k\mathbf{v}\|^2 \le \frac{1}{\lambda_{k+1}}\|\nabla\mathbf{v}\|^2.$$

Moreover, if $\mathbf{v} \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$*, then*

$$\|\mathbf{v} - P_k\mathbf{v}\|^2 \le \frac{1}{\lambda_{k+1}^2}\|P\Delta\mathbf{v}\|^2,$$

$$\|\nabla\mathbf{v} - \nabla P_k\mathbf{v}\| \le \frac{1}{\lambda_{k+1}}\|P\nabla\mathbf{v}\|^2.$$

Remark 2.2. From Lemma (2.1), it follows that if $\mathbf{f} \in \mathbf{H}^1(\Omega)$, then

$$\|(I - P_k)P\mathbf{f}\|^2 \le \frac{1}{\lambda_{k+1}}\|\nabla P\mathbf{f}\|^2.$$

Moreover, since $P : \mathbf{H}^1(\Omega) \to \mathbf{H}^1(\Omega)$ is a continuous operator, we have

$$\|\nabla P\mathbf{f}\|^2 \le C\|\mathbf{f}\|_{\mathbf{H}^1}^2.$$

Thus,

$$\|(I - P_k)P\mathbf{f}\|^2 \le \frac{C}{\lambda_{k+1}}\|\mathbf{f}\|_{\mathbf{H}^1}^2.$$

Since $PP_k = P_kP = P$, one equivalently obtains

$$\|P\mathbf{f} - P_k\mathbf{f}\|^2 \le \frac{C}{\lambda_{k+1}}\|\mathbf{f}\|_{\mathbf{H}^1}^2.$$

Moreover, the above relations also hold if one replaces P by any $P_m, m > k$. Analogously, if $\mathbf{f} \in \mathbf{H}^2(\Omega)$ we obtain

$$\|(I - P_k)P\mathbf{f}\|^2 \le \frac{C}{\lambda_{k+1}^2}\|\mathbf{f}\|_{\mathbf{H}^2}^2.$$

We also have the following: Let $m > k, m, k \in \mathbb{N}, \mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{v}^m \in \mathbf{V}_m, \mathbf{v}^k \in \mathbf{V}_k$. Then

$$((P_m - P_k)\mathbf{f}, \mathbf{v}^m - \mathbf{v}^k) = (\mathbf{f}, (I - P_k)\mathbf{v}^m).$$

The following variant of Gronwall's inequality will also be necessary (see Rautmann [18, p. 437]).

Lemma 2.3. *Let a function $a(t) \geq 0$ be absolutely continuous with $a'(t) \geq 0$ and $b(t) \geq 0$ summable in $[0,T]$. Assume the integral inequality*

$$\zeta(t) + \int_0^t \zeta^*(s)\, ds \leq \frac{a(t)}{\lambda} + \int_0^t b(s)\zeta(s)\, ds,$$

holds for the positive continuous functions ζ and ζ^* on $[0,T]$ with a constant $\lambda > 0$. Then

$$\zeta(t) + \int_0^t \zeta^*(s)\, ds \leq \frac{A(t)}{\lambda},$$

with

$$A(t) = \Big\{1 + \int_0^t b(s)\, ds\Big\}\chi(t), \quad \chi(t) = a(t)\exp\int_0^t b(s)\, ds.$$

We consider a Galerkin approximation

$$\mathbf{u}^k(\mathbf{x},t) = \sum_{i=1}^k C_{ik}(t)\boldsymbol{\varphi}^i(\mathbf{x}),$$

for the velocity and an infinite dimensional approximations $\rho^k(\mathbf{x},t)$, $\mathbf{d}^k(\mathbf{x},t)$ for the density and optical director satisfying the following equations:

$$\begin{gathered}
\rho_t^k + (\mathbf{u}^k\cdot\nabla)\rho^k = 0,\\
P_k\left[\rho^k(\mathbf{u}_t^k + (\mathbf{u}^k\cdot\nabla)\mathbf{u}^k) + \lambda\nabla\cdot(\nabla\mathbf{d}^k\odot\nabla\mathbf{d}^k)\right] + \nu A\mathbf{u}^k = 0,\\
\mathbf{d}_t^k + (\mathbf{u}^k\cdot\nabla)\mathbf{d}^k = \gamma\left(\Delta\mathbf{d}^k + |\nabla\mathbf{d}^k|^2\mathbf{d}^k\right),\\
\nabla\cdot\mathbf{u}^k = 0,\\
(\rho^k,\mathbf{u}^k,\mathbf{d}^k)|_{t=0} = (\rho_0, P_n\mathbf{u}_0, \mathbf{d}_0).
\end{gathered} \tag{2.1}$$

As we said in the Introduction, in this paper we are interested in deriving error bounds, that is, estimates for $\|\mathbf{u}-\mathbf{u}^k\|, \|\mathbf{d}-\mathbf{d}^k\|, \|\rho-\rho^k\|$ in suitable norms in terms of powers of $1/\lambda_{k+1}$.

These error estimates will be derived in the following sections and will be based on the following local strong solution results established by Gao et al. [7], where here we will consider a constant fluid viscosity $\nu > 0$ and strictly positive density to ensure the validity of the error estimates.

Theorem 2.4. *Assume that the initial data $\mathbf{u}_0, \mathbf{d}_0, \rho_0$ for system (1.1) satisfy $\mathbf{u}_0 \in \mathbf{V}\cap\mathbf{H}^2(\Omega)$, $\mathbf{d}_0 \in \mathbf{H}^3(\Omega)$, and $\rho_0 \in W^{1,\infty}(\Omega)$, with $0 < \alpha \leq \rho_0(\mathbf{x})$ for a.e. $\mathbf{x}\in\Omega$. Then, there exists a positive time $T_1 > 0$ such that problem (1.1) admits a unique local strong solution $(\mathbf{u},\mathbf{d},\rho)$ on $[0,T_1]$ such that:*

$$\begin{gathered}
\rho \in L^\infty(0,T_1;W^{1,\infty}(\Omega)), \quad \rho_t \in L^\infty(0,T_1;L^\infty(\Omega)),\\
\mathbf{u} \in C([0,T_1];\mathbf{V}\cap\mathbf{H}^2(\Omega)), \quad \mathbf{u}_t \in L^\infty(0,T_1;\mathbf{H})\cap L^2(0,T_1;\mathbf{V}),\\
p \in L^\infty(0,T_1;H^1(\Omega)),\\
\mathbf{d} \in C([0,T_1];\mathbf{H}^3(\Omega))\cap L^2(0,T_1;\mathbf{H}^4(\Omega)), \quad |\mathbf{d}| = 1 \text{ in } \Omega\times[0,T_1],\\
\mathbf{d}_t \in C([0,T_1];\mathbf{H}^1(\Omega))\cap L^2(0,T_1;\mathbf{H}^2(\Omega)), \quad \mathbf{d}_{tt} \in L^2(0,T_1;\mathbf{L}^2(\Omega)).
\end{gathered}$$

After proving the Theorem 2.4, Gao *et al*, [7, Section 4], obtained the following blow-up criterion for local strong solutions in three dimensional bounded domain.

Theorem 2.5. *Suppose the dimension $n = 3$and all the assumptions in Theorem (2.4) are satisfied. Let $(\mathbf{u},\mathbf{d},\rho,p)$ be a strong solution of the initial boundary value problem (1.1)-(1.2) and T_1 be the maximal time of existence. If $0 < T_1 < \infty$, then*

$$\lim_{T\to T_1}\Big(\|\nabla\rho\|_{L^\infty(0,T;\mathbf{L}^q)} + \|\mathbf{u}\|_{L^{s_1}(0,T;\mathbf{L}_w^{r_1})} + \|\nabla\mathbf{d}\|_{L^{s_2}(0,T;\mathbf{L}_w^{r_2})}\Big) = \infty$$

where r_i and s_i satisfy

$$\frac{2}{s_i}+\frac{3}{r_i}\le 1,\quad 3<r\le\infty,\quad i=1,2.$$

To prove Theorem 2.5, Gao *et al*, consider

$$M_0 \triangleq \lim_{T\to T^*}\Big(\|\nabla\rho\|_{L^\infty(0,T;\mathbf{L}^q)}+\|\mathbf{u}\|_{L^{s_1}(0,T;\mathbf{L}_w^{r_1})}+\|\nabla\mathbf{d}\|_{L^{s_2}(0,T;\mathbf{L}_w^{r_2})}\Big)<\infty \tag{2.2}$$

and thus, they obtain the following estimates, for $0\le T^*<T_1$:

$$\sup_{0\le t\le T}\Big(\|\rho\|_{L^m}+\|\sqrt{\rho}\mathbf{u}\|_{\mathbf{L}^2}^2+\|\nabla\mathbf{d}\|_{\mathbf{L}^2}^2\Big)+\int_0^T\int\Big(|\nabla\mathbf{d}|^2+\Big||\nabla\mathbf{d}|^2\mathbf{d}+\Delta\mathbf{d}\Big|^2\Big)\,dx,ds\le C; \tag{2.3}$$

$$\begin{aligned}&\sup_{0\le t\le T}\Big(\|\nabla\mathbf{u}\|_{\mathbf{L}^2}^2+\|\nabla\mathbf{d}\|_{\mathbf{L}^4}^4+\|\nabla^2\mathbf{d}\|_{\mathbf{L}^2}^2\Big)\\&+\int_0^T\int\Big(|\sqrt{\rho}[\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}]|^2+|\nabla\mathbf{d}|^2|\nabla^2\mathbf{d}|^2+|\nabla^3\mathbf{d}|^2+|\nabla\mathbf{d}_t|^2\Big)\,dx,ds\le C;\end{aligned} \tag{2.4}$$

$$\sup_{0\le t\le T}\|\mathbf{d}\|_{\mathbf{L}^2}+\int_0^T\|\mathbf{u}\|_{\mathbf{H}^2}^2ds\le C; \tag{2.5}$$

$$\sup_{0\le t\le T}\Big(\|\sqrt{\rho}\mathbf{u}\|_{\mathbf{L}^2}^2+\|\nabla^3\mathbf{d}\|_{\mathbf{L}^2}^2\Big)+\int_0^T\int\Big(\|\nabla\mathbf{u}_t\|_{\mathbf{L}^2}^2+\|\nabla^2\mathbf{d}\|_{\mathbf{L}^2}^2\Big)\,dx,ds\le C; \tag{2.6}$$

$$\begin{aligned}&\sup_{0\le t\le T}\Big(\|\rho_t\|_{L^q}+\|\mathbf{u}\|_{\mathbf{H}^2}+\|p\|_{\mathbf{H}^1}+\|\nabla\mathbf{d}_t\|_{\mathbf{L}^2}\Big)\\&+\int_0^T\Big(\|\mathbf{u}\|_{\mathbf{W}^{2,r}}^2+\|p\|_{\mathbf{W}^{1,r}}^2+\|\mathbf{d}_{tt}\|_{\mathbf{L}^2}^2+\|\nabla^4\mathbf{d}\|_{\mathbf{L}^2}^2\Big)ds\le C;\end{aligned} \tag{2.7}$$

where C is a generic constants depending only on Ω, M_0, T_1 and the initial data. Now, using the properties of P_k and the standard arguments used by Heywood [8, 9] and Boldrini et al. [4], the following result is proved for the approximations $(\mathbf{u}^k,\mathbf{d}^k,\rho^k)$.

Proposition 2.6. *Let $(\mathbf{u}^k,\mathbf{d}^k,\rho^k)$ be the solution of (2.1). Then, they satisfy*

$$\|\nabla\mathbf{u}^k(t)\|^2+\|\Delta\mathbf{d}^k(t)\|^2+\int_0^t(\|A\mathbf{u}^k(s)\|^2+\|\mathbf{d}^k(t)\|_{\mathbf{H}^3}^2)ds\le F_1(t),$$

$$\int_0^t(\|\mathbf{u}_t^k(s)\|^2+\|\nabla\mathbf{d}_t^k(s)\|^2)ds\le F_2(t),$$

$$\|\mathbf{u}_t^k(t)\|^2+\|\nabla\mathbf{d}_t^k(t)\|^2+\|A\mathbf{u}^k(t)\|^2+\|\mathbf{d}^k(t)\|_{\mathbf{H}^3}^2\le F_3(t),$$

$$\alpha\le\rho^k\le\beta,\quad \|\nabla\rho^k(t)\|_{L^\infty}+\|\rho_t^k(t)\|_{L^\infty}\le F_4(t),$$

$$\int_0^t(\|\nabla\mathbf{u}_t^k(s)\|^2+\|\Delta\mathbf{d}_t^k(s)\|^2)ds\le F_5(t).$$

Here, the F_i are nondecreasing continuous functions on $[0,T_1)$, independent of k.

Remark 2.7. We emphasize that these uniform estimates are strictly local-in-time. The continuous bounds established by Gao et al. rely of a quantity M_0 (which bounds the local supremum of the gradients) within a maximal existence time T^*. Because our analysis is rigorously restricted to the interval $[0,T_1]$ where the local strong solution is guaranteed to exist and $M_0<\infty$, the approximate solutions $(\mathbf{u}^k,\mathbf{d}^k,\rho^k)$ uniformly inherit these local bounds. This ensures that the nondecreasing continuous bounding functions $F_i(t)$ are completely independent of the approximation parameter k.

3. Error bounds for the approximations

In this section we prove *a priori* estimates. Throughout the proofs, C, C_ϵ, and C_δ denote generic positive constants that may depend on the domain Ω, the initial data, and the functions $F_i(t)$ from Proposition 2.6, but are strictly independent of the approximation parameters m,k, and the eigenvalues λ_{k+1}. Moreover, for simplicity, we assume that $\nu,\lambda,\gamma=1$.

Let $[0,T_1]$ be a time interval as in Theorem 2.4; $\rho^k,\mathbf{u}^k,\mathbf{d}^k$ the k-th approximations of $\rho,\mathbf{u},\mathbf{d}$ respectively. We begin by considering the following theorem.

Theorem 3.1. *Suppose that the assumptions of Theorem 2.4 hold. Then, the approximations $\rho^k,\mathbf{u}^k,\mathbf{d}^k$ satisfy*

$$\|\rho(t)-\rho^k(t)\|^2 \le \frac{G_0(t)}{\lambda_{k+1}}, \tag{3.1}$$

$$\begin{aligned}&\|\mathbf{u}(t)-\mathbf{u}^k(t)\|^2+\|\nabla\mathbf{d}(t)-\nabla\mathbf{d}^k(t)\|^2\\&+\int_0^t\big(\|\nabla\mathbf{u}(s)-\nabla\mathbf{u}^k(s)\|^2+\|\Delta\mathbf{d}(s)-\Delta\mathbf{d}^k(s)\|^2\big)ds\\&\le\frac{G_1(t)}{\lambda_{k+1}},\end{aligned} \tag{3.2}$$

for all $t\in[0,T_1]$. The continuous functions $G_0(t),G_1(t)$ depend on t and on the functions $F_i(t)$. The above estimates also hold with $\rho^m,\mathbf{u}^m,\mathbf{d}^m$ instead of $\rho,\mathbf{u},\mathbf{d}$ for $m>k$.

Proof. First, we suppose (3.2) holds. The difference $\rho^m-\rho^k$ with $m>k$ satisfies

$$(\rho^m-\rho^k)_t+\mathbf{u}^m\cdot\nabla(\rho^m-\rho^k)=-(\mathbf{u}^m-\mathbf{u}^k)\cdot\nabla\rho^k,$$

and $\rho_0^m-\rho_0^k=0$. By the method of characteristics we obtain

$$\rho^m-\rho^k=-\int_0^t(\mathbf{u}^m(\mathbf{z}^m,s)-\mathbf{u}^k(\mathbf{z}^m,s))\cdot\nabla\rho^k(\mathbf{z}^m,s)ds, \tag{3.3}$$

where $\mathbf{z}^m(t,s,\mathbf{x})$ is the solution of the Cauchy problem $\mathbf{z}_t^m=\mathbf{u}^m(\mathbf{z}^m,s)$ with $\mathbf{z}^m=\mathbf{x}$ for $t=s$. Then, we obtain

$$\|\rho^m-\rho^k\|\le\int_0^t\|\mathbf{u}^m-\mathbf{u}^k\|\|\nabla\rho^k\|_{\mathbf{L}^\infty}ds\le C\int_0^t\|\mathbf{u}^m-\mathbf{u}^k\|ds, \tag{3.4}$$

and, taking $m\to\infty$ we obtain

$$\|\rho-\rho^k\|\le C\int_0^t\|\mathbf{u}-\mathbf{u}^k\|ds. \tag{3.5}$$

By (3.2) we obtain (3.1).

Now, we prove (3.2). To this goal, we consider $m>k$. By (2.1) we have

$$\begin{aligned}&P_m\big[\rho^m\mathbf{u}_t^m\big]-P_k\big[\rho^k\mathbf{u}_t^k\big]+P_m\big[\rho^m(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m\big]-P_k\big[\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\big]+A(\mathbf{u}^m-\mathbf{u}^k)\\&+P_m\nabla\cdot(\nabla\mathbf{d}^m\odot\nabla\mathbf{d}^m)-P_k\nabla\cdot(\nabla\mathbf{d}^k\odot\nabla\mathbf{d}^k)=0;\\&\mathbf{d}_t^m-\mathbf{d}_t^k+(\mathbf{u}^m\cdot\nabla)\mathbf{d}^m-(\mathbf{u}^k\cdot\nabla)\mathbf{d}^k-(\Delta\mathbf{d}^m-\Delta\mathbf{d}^k)-(|\nabla\mathbf{d}^m|^2\mathbf{d}^m-|\nabla\mathbf{d}^k|^2\mathbf{d}^k)=0.\end{aligned} \tag{3.6}$$

Applying the inner product in $\mathbf{L}^2$ of $(3.6)_1$ and considering $\boldsymbol{\xi}=\mathbf{u}^m-\mathbf{u}^k$ and $\boldsymbol{\sigma}=\mathbf{d}^m-\mathbf{d}^k$ we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\|(\rho^m)^{1/2}\boldsymbol{\xi}\|^2+\|\nabla\boldsymbol{\xi}\|^2\le&-\big((P_m-P_k)(\rho^k\mathbf{u}_t^k+\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k+\Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k,\boldsymbol{\xi}\big)\\&+\frac12(\rho_t^m\boldsymbol{\xi},\boldsymbol{\xi})-\big(P_m(\rho^m-\rho^k)(\mathbf{u}_t^k+(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m),\boldsymbol{\xi}\big)\\&+\big(P_m\big[\rho^k(\boldsymbol{\xi}\cdot\nabla)\mathbf{u}^m-\rho^k(\mathbf{u}^k\cdot\nabla)\boldsymbol{\xi}+\nabla\boldsymbol{\sigma}\cdot\Delta\mathbf{d}^m-\nabla\mathbf{d}^k\cdot\Delta\boldsymbol{\sigma}\big],\boldsymbol{\xi}\big)\,.\end{aligned}$$

For the first term on the right-hand side, by the properties of Remark 2.2, we have

$$\begin{aligned}&\big|\big((P_m-P_k)(\rho^k\mathbf{u}_t^k+\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k+\Delta\mathbf{d}^k\nabla\mathbf{d}^k),\boldsymbol{\xi}\big)\big|\\&\le\|\rho^k\mathbf{u}_t^k+\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k+\Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k\|\frac{\|\nabla\mathbf{u}^m\|}{\lambda_{k+1}}\\&\le\Big[\beta\|\mathbf{u}_t^k\|+\beta\|\nabla\mathbf{u}^k\|\,\|P\Delta\mathbf{u}^k\|+\|\mathbf{d}^k\|_{\mathbf{H}^3}\|\mathbf{d}^k\|_{\mathbf{H}^2}\Big]\frac{\|\nabla\mathbf{u}^m\|}{\lambda_{k+1}}\\&\le\frac{C}{\lambda_{k+1}}.\end{aligned}$$

Now we obtain, by the estimates from Proposition 2.6 and the equation (3.4) that

$$\begin{aligned}
&\big|\big(P_m(\rho^m-\rho^k)(\mathbf{u}_t^k+(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m),\boldsymbol{\xi}\big)\big| \\
&\le C_\epsilon\|\rho^m-\rho^k\|^2\left(\|\nabla\mathbf{u}_t^k\|^2+\|\nabla\mathbf{u}^m\|^2\|P\Delta\mathbf{u}^m\|^2\right)+2\epsilon\|\nabla\boldsymbol{\xi}\|^2 \\
&\le\Big[\int\|\boldsymbol{\xi}(s)\|^2\,ds\Big]\Big(C_\epsilon\|\nabla\mathbf{u}_t^k\|^2+C_\epsilon\|\nabla\mathbf{u}^m\|^2\|P\Delta\mathbf{u}^m\|^2\Big)+2\epsilon\|\nabla\boldsymbol{\xi}\|^2 \\
&\le\Big[\int\|\boldsymbol{\xi}(s)\|^2\,ds\Big]\left(C_\epsilon\|\nabla\mathbf{u}_t^k\|^2+C_1\right)+2\epsilon\|\nabla\boldsymbol{\xi}\|^2.
\end{aligned}$$

Thanks to Sobolev's embedding and Young's inequality, we obtain

$$\begin{gathered}
\frac12|(\rho_t^m\boldsymbol{\xi},\boldsymbol{\xi})|\le\frac12\|\rho_t^m\|^2_{\mathbf{L}^\infty}\|\boldsymbol{\xi}\|^2\le C\|\boldsymbol{\xi}\|^2; \\
|(P_m\big[\rho^k(\boldsymbol{\xi}\cdot\nabla)\mathbf{u}^m\big],\boldsymbol{\xi})|\le\beta\|\nabla\mathbf{u}^m\|_{\mathbf{L}^3}\|\xi\|_{\mathbf{L}^6}\|\boldsymbol{\xi}\|\le C_\epsilon\beta^2\|P\Delta\mathbf{u}^m\|^2\|\boldsymbol{\xi}\|^2+\epsilon\|\nabla\boldsymbol{\xi}\|^2; \\
|(P_m\big[\rho^k(\mathbf{u}^k\cdot\nabla)\boldsymbol{\xi}\big],\boldsymbol{\xi})|\le\beta\|\mathbf{u}^k\|_{\mathbf{L}^\infty}\|\nabla\boldsymbol{\xi}\|\,\|\boldsymbol{\xi}\|\le C_\epsilon\beta^2\|P\Delta\mathbf{u}^k\|^2\|\boldsymbol{\xi}\|^2+\epsilon\|\nabla\boldsymbol{\xi}\|^2; \\
|(P_m\big[\Delta\mathbf{d}^k\cdot\nabla\boldsymbol{\sigma}\big],\boldsymbol{\xi})|\le\|\Delta\mathbf{d}^k\|\,\|\nabla\boldsymbol{\sigma}\|_{\mathbf{L}^6}\,\|\boldsymbol{\xi}\|_{\mathbf{L}^3}\le C_{\delta,\epsilon}\|\mathbf{d}^k\|^4_{\mathbf{H}^2}\|\boldsymbol{\xi}\|^2+\epsilon\|\nabla\boldsymbol{\xi}\|^2+\delta\|\Delta\boldsymbol{\sigma}\|^2; \\
|(P_m\big[\Delta\boldsymbol{\sigma}\cdot\nabla\mathbf{d}^k\big],\boldsymbol{\xi})|\le\|\Delta\boldsymbol{\sigma}\|\,\|\nabla\mathbf{d}^k\|_{\mathbf{L}^6}\,\|\boldsymbol{\xi}\|_{\mathbf{L}^3}\le C_{\delta,\epsilon}\|\mathbf{d}^k\|^4_{\mathbf{H}^2}\|\boldsymbol{\xi}\|^2+\epsilon\|\nabla\boldsymbol{\xi}\|^2+\delta\|\Delta\boldsymbol{\sigma}\|^2.
\end{gathered}$$

Thus,

$$\begin{aligned}
&\frac12\frac{d}{dt}\|(\rho^m)^{1/2}\boldsymbol{\xi}\|^2+\|\nabla\boldsymbol{\xi}\|^2 \\
&\le\frac{C}{\lambda_{k+1}}+\Big[\int\|\boldsymbol{\xi}(s)\|^2\,ds\Big]\left(C_\epsilon\|\nabla\mathbf{u}_t^k\|^2+C_1\right) \\
&\quad+\Big[C+C_\epsilon\beta^2(\|P\Delta\mathbf{u}^m\|^2+\|P\Delta\mathbf{u}^k\|^2)+C_{\delta,\epsilon}\|\mathbf{d}^k\|^4_{\mathbf{H}^2}\Big]\|\boldsymbol{\xi}\|^2+2\delta\|\Delta\boldsymbol{\sigma}\|^2.
\end{aligned}\tag{3.7}$$

On the other hand, multiplying $(3.6)_2$ by $-\Delta\boldsymbol{\sigma}$ and integrating over Ω we obtain

$$\begin{aligned}
\frac12\frac{d}{dt}\|\nabla\boldsymbol{\sigma}\|^2+\|\Delta\boldsymbol{\sigma}\|^2&=((\boldsymbol{\xi}\cdot\nabla)\mathbf{d}^m,\Delta\boldsymbol{\sigma})+((\mathbf{u}^k\cdot\nabla)\boldsymbol{\sigma},\Delta\boldsymbol{\sigma}) \\
&\quad-(((|\nabla\mathbf{d}^m|^2-|\nabla\mathbf{d}^k|^2)\mathbf{d}^m,\Delta\boldsymbol{\sigma})+(|\nabla\mathbf{d}^k|^2\boldsymbol{\sigma},\Delta\boldsymbol{\sigma}),
\end{aligned}$$

and, as before

$$\begin{gathered}
|((\boldsymbol{\xi}\cdot\nabla)\mathbf{d}^m,\Delta\boldsymbol{\sigma})|\le\|\boldsymbol{\xi}\|\,\|\nabla\mathbf{d}^m\|_{\mathbf{L}^\infty}\|\Delta\boldsymbol{\sigma}\|\le C_\delta\|\mathbf{d}^m\|^2_{\mathbf{H}^3}\|\xi\|^2+\delta\|\Delta\boldsymbol{\sigma}\|^2; \\
|((\mathbf{u}^k\cdot\nabla)\boldsymbol{\sigma},\Delta\boldsymbol{\sigma})|\le\|\mathbf{u}^k\|_{\mathbf{L}^\infty}\|\nabla\boldsymbol{\sigma}\|\,\|\Delta\boldsymbol{\sigma}\|\le C_\delta\|P\Delta\mathbf{u}^k\|^2\|\nabla\boldsymbol{\sigma}\|^2+\delta\|\Delta\boldsymbol{\sigma}\|^2; \\
|(((|\nabla\mathbf{d}^m|^2-|\nabla\mathbf{d}^k|^2)\mathbf{d}^m,\Delta\boldsymbol{\sigma})|\le|(|\nabla\boldsymbol{\sigma}|\mathbf{d}^m,\Delta\boldsymbol{\sigma})|\le C_\delta\|\mathbf{d}^m\|^2_{\mathbf{H}^2}\|\nabla\boldsymbol{\sigma}\|^2+\delta\|\Delta\boldsymbol{\sigma}\|^2; \\
|(|\nabla\mathbf{d}^k|^2\boldsymbol{\sigma},\Delta\boldsymbol{\sigma})|\le\|\nabla\mathbf{d}^k\|_{\mathbf{L}^\infty}\|\nabla\mathbf{d}^k\|_{\mathbf{L}^6}\|\boldsymbol{\sigma}\|_{\mathbf{L}^3}\|\Delta\boldsymbol{\sigma}\|;
\end{gathered}$$

so, we obtain

$$\begin{aligned}
\frac12\frac{d}{dt}\|\nabla\boldsymbol{\sigma}\|^2+\|\Delta\boldsymbol{\sigma}\|^2&\le C_\delta(\|P\Delta\mathbf{u}^k\|^2+\|\mathbf{d}^m\|^2_{\mathbf{H}^2}+\|\mathbf{d}^k\|^2_{\mathbf{H}^3}\|\mathbf{d}^k\|^2_{\mathbf{H}^2})\|\nabla\boldsymbol{\sigma}\|^2 \\
&\quad+C_\delta\|\mathbf{d}^m\|^2_{\mathbf{H}^3}\|\boldsymbol{\xi}\|^2+4\delta\|\Delta\boldsymbol{\sigma}\|^2.
\end{aligned}\tag{3.8}$$

Therefore, adding (3.7) and (3.8), integrating from 0 to t, from the estimates in Proposition 2.6, we obtain

$$\begin{aligned}
&\|\boldsymbol{\xi}(t)\|^2+\|\nabla\boldsymbol{\sigma}(t)\|^2+\int_0^t\left(\|\nabla\boldsymbol{\xi}(s)\|^2+\|\Delta\boldsymbol{\sigma}(s)\|^2\right)\,ds \\
&\le\int_0^t C_2\left(\|\boldsymbol{\xi}(s)\|^2+\|\nabla\boldsymbol{\sigma}(s)\|^2\right)\,ds+\frac{Ct}{\lambda_{k+1}}+\|\rho_0^{1/2}\boldsymbol{\xi}_0\|^2.
\end{aligned}\tag{3.9}$$

Note that

$$\|\boldsymbol{\xi}_0\|^2=\|(P_m-P_k)\mathbf{u}_0\|^2\le\frac{C}{\lambda_{k+1}}.\tag{3.10}$$

Thus, replacing (3.10) in (3.9) we obtain

$$\begin{aligned} &\|\boldsymbol{\xi}(t)\|^2 + \|\nabla\boldsymbol{\sigma}(t)\|^2 + \int_0^t \left(\|\nabla\boldsymbol{\xi}(s)\|^2 + \|\Delta\boldsymbol{\sigma}(s)\|^2\right) ds \\ &\le \int_0^t C_2 \left(\|\boldsymbol{\xi}(s)\|^2 + \|\nabla\boldsymbol{\sigma}(s)\|^2\right) ds + \frac{C}{\lambda_{k+1}}(t+\beta), \end{aligned} \tag{3.11}$$

and applying the Gronwall‘s inequality (2.3) we obtain

$$\|\boldsymbol{\xi}(t)\|^2 + \|\nabla\boldsymbol{\sigma}(t)\|^2 + \int_0^t \left(\|\nabla\boldsymbol{\xi}(s)\|^2 + \|\Delta\boldsymbol{\sigma}(s)\|^2\right) ds \le \frac{C\exp(C_2 t)}{\lambda_{k+1}}, \tag{3.12}$$

and taking $m \to \infty$ we obtain the desired result. □

Lemma 3.2. *Under the hypotheses of Theorem 2.4, the approximations ρ^k satisfy*

$$\|\rho(t) - \rho^k(t)\|_{\mathbf{L}^r}^2 \le \frac{G_1(t)}{\lambda_{k+1}}, \tag{3.13}$$

with $2 \le r \le 6$ for any $t \in [0, T_1]$. The continuous function $G_1(t)$ is the same function of Theorem 3.1. Also, (3.13) holds with any ρ^m instead of ρ for $m \ge k$.

Proof. Since $\mathbf{L}^{r_2}(\Omega) \subset \mathbf{L}^{r_1}(\Omega)$ with continuous inclusions if $r_2 \ge r_1 \ge 1$, it is enough to take $r \in \mathbb{R}$ such that $3 \le r \le 6$. Let $m > k, m, k \in \mathbb{N}$. Then

$$\rho_t^m + (\mathbf{u}^m \cdot \nabla)\rho^m = 0; \tag{3.14}$$

$$\rho_t^k + (\mathbf{u}^k \cdot \nabla)\rho^k = 0; \tag{3.15}$$

$$\rho^m(0) = \rho^k(0) = \rho_0. \tag{3.16}$$

Subtracting (3.15) from (3.14), the difference $\pi = \rho^m - \rho^k$ satisfies

$$\begin{gathered} \pi_t = -(\boldsymbol{\xi} \cdot \nabla)\rho^m - (\mathbf{u}^k \cdot \nabla)\pi, \\ \pi(0) = 0, \end{gathered}$$

where $\boldsymbol{\xi} = \mathbf{u}^m - \mathbf{u}^k$. Now, multiplying the above equation by $|\pi|^{r-1}$ and integrating over Ω, we obtain

$$\frac{1}{r}\frac{d}{dt}\|\pi\|_{L^r}^r = -\int_\omega \boldsymbol{\xi} \cdot \nabla\rho^m |\pi|^{r-1}\, dx.$$

Considering the H ölder’s inequality, we obtain

$$\frac{1}{r}\frac{d}{dt}\|\pi\|_{L^r}^r \le \|\nabla\rho^m\|_{\mathbf{L}^\infty}\|\boldsymbol{\xi}\|_{\mathbf{L}^r}\|\pi\|_{\mathbf{L}^r}^{r-1}.$$

since $3 \le r \le 6$,

$$\frac{d}{dt}\|\pi\|_{L^r} \le C\|\nabla\rho^m\|_{L^\infty}\|\nabla\boldsymbol{\xi}\| \le C\|\nabla\boldsymbol{\xi}\|,$$

so, by estimates of Theorem 3.1, we conclude that

$$\|\pi\|_{L^r}^2 \le C\int_0^t \|\nabla\boldsymbol{\xi}(s)\|^2 ds \le C\frac{G_1(t)}{\lambda_{k+1}}.$$

Then,

$$\|\rho^m(t) - \rho^k(t)\|_{L^r}^2 \le \frac{G_1(t)}{\lambda_{k+1}},$$

and taking $m \to \infty$ the Lemma is proved. □

Theorem 3.3. *Under the hypothesis of Theorem 2.4, we have*

$$\|\nabla(\mathbf{u} - \mathbf{u}^k)(t)\|^2 + \int_0^t \|\mathbf{u}_t(s) - \mathbf{u}_t^k(s)\|^2\, ds \le \frac{G_2(t)}{\lambda_{k+1}}, \tag{3.17}$$

$$\|\mathbf{d}_t(t) - \mathbf{d}^k(t)\|^2 + \int_0^t \|\nabla\mathbf{d}_t(s) - \nabla\mathbf{d}_t^k(s)\|^2\, ds \le \frac{G_3(t)}{\lambda_{k+1}}, \tag{3.18}$$

for any $t \in [0, T_1]$. The continuous functions $G_3(t), G_4(t)$ depend on t and on the functions $F_i(t)$. The above estimates also hold with $\rho^m, \mathbf{u}^m, \mathbf{d}^m$ instead of $\rho, \mathbf{u}, \mathbf{d}$ for $m > k$.

Proof. Taking the inner product in $\mathbf{L}^2(\Omega)$ of $(3.6)_1$ with $\boldsymbol{\xi}_t$ we obtain

$$\begin{aligned}\|(\rho^m)^{1/2}\boldsymbol{\xi}_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\xi}\|^2 = &-\big(P_m(\rho^m - \rho^k)[\mathbf{u}_t^k + (\mathbf{u}^m\cdot\nabla)\mathbf{u}^m], \boldsymbol{\xi}_t\big)\\ &- \big((P_m - P_k)\big[\rho^k\mathbf{u}_t^k + \rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k + \Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k\big], \boldsymbol{\xi}_t\big)\\ &- \Big(P_m\big[\rho^k(\boldsymbol{\xi}\cdot\nabla)\mathbf{u}^k + \rho^k(\mathbf{u}^k\cdot\nabla)\boldsymbol{\xi}^k + \Delta\mathbf{d}^m\cdot\nabla\boldsymbol{\sigma} - \Delta\boldsymbol{\sigma}\cdot\nabla\mathbf{d}^k\big], \boldsymbol{\xi}_t\Big).\end{aligned}$$

Now, by Remark 2.2 and Proposition 2.6, we have

$$\begin{aligned}&\Big|\Big((P_m - P_k)\big[\rho^k\mathbf{u}_t^k + \rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k + \Delta\mathbf{d}^k\nabla\mathbf{d}^k\big], \boldsymbol{\xi}_t\Big)\Big|\\ &\le C_\epsilon\|(P_m - P_k)\big[\rho^k\mathbf{u}_t^k + \rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k + \Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k\big]\|^2 + \epsilon\|\boldsymbol{\xi}_t\|^2\\ &\le \frac{C_\epsilon}{\lambda_{k+1}}\big[\|\rho^k\mathbf{u}_t^k\|_{\mathbf{H}^1}^2 + \|\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\|_{\mathbf{H}^1}^2 + \|\Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k\|_{\mathbf{H}^1}^2\big] + \epsilon\|\boldsymbol{\xi}_t\|^2\\ &\le \frac{C_\epsilon}{\lambda_{k+1}}\big[C_1\|\nabla\mathbf{u}_t^k\|^2 + C_2\big] + \epsilon\|\boldsymbol{\xi}_t\|^2,\end{aligned}$$

and by Lemma 3.2 and keeping in mind the argument in the Theorem 3.1 we obtain

$$\Big|\Big(P_m(\rho^m - \rho^k)[\mathbf{u}_t^k + (\mathbf{u}^m\cdot\nabla)\mathbf{u}^m], \boldsymbol{\xi}_t\Big)\Big| \le C_\epsilon\frac{G_1(t)}{\lambda_{k+1}}(C + \|\nabla\mathbf{u}_t^k\|^2) + \epsilon\|\boldsymbol{\xi}_t\|^2;$$

$$\Big|\Big(P_m\big[\rho^k(\boldsymbol{\xi}\cdot\nabla)\mathbf{u}^k + \rho^k(\mathbf{u}^k\cdot\nabla)\boldsymbol{\xi}^k + \Delta\mathbf{d}^m\cdot\nabla\boldsymbol{\sigma} - \Delta\boldsymbol{\sigma}\cdot\nabla\mathbf{d}^k\big], \boldsymbol{\xi}_t\Big)\Big| \le C_\epsilon(\|\nabla\boldsymbol{\xi}\|^2 + \|\Delta\boldsymbol{\sigma}\|^2) + \epsilon\|\boldsymbol{\xi}_t\|^2.$$

Now, as $\|(\rho^m)^{1/2}\boldsymbol{\xi}_t\|^2 \ge \alpha\|\boldsymbol{\xi}_t\|^2$ and taking $\epsilon = \alpha/6$ we obtain

$$\|\boldsymbol{\xi}_t\|^2 + \frac{d}{dt}\|\nabla\boldsymbol{\xi}\|^2 \le \frac{C}{\lambda_{k+1}}\Big(1 + \|\nabla\mathbf{u}_t^k\|^2\Big) + \Big(C\frac{G_1(t)}{\lambda_{k+1}}(1 + \|\nabla\mathbf{u}_t^k\|^2)\Big) + \big(C\|\nabla\boldsymbol{\xi}\|^2 + C\|\Delta\boldsymbol{\sigma}\|^2\big)$$

and, after integrating from 0 to t we obtain

$$\begin{aligned}\|\nabla\boldsymbol{\xi}(t)\|^2 + \int_0^t\|\boldsymbol{\xi}_t(s)\|^2\,ds \le &\frac{C}{\lambda_{k+1}}\Big(t + \int_0^t\|\nabla\mathbf{u}_t^k\|^2\Big) + C\frac{\sup G_1(t)}{\lambda_{k+1}}\Big(t + \int_0^t\|\nabla\mathbf{u}_t^k\|^2\Big)\\ &+ C\int_0^t\big(\|\nabla\boldsymbol{\xi}(s)\|^2 + \|\Delta\boldsymbol{\sigma}(s)\|^2\big)\,ds + \|\nabla\boldsymbol{\xi}(0)\|^2.\end{aligned}$$

and, because

$$\|\nabla\boldsymbol{\xi}_0\|^2 \le \frac{\|\Delta\boldsymbol{\xi}_0\|}{\lambda_{k+1}} \le \frac{C}{\lambda_{k+1}}.$$

Then by Theorem 3.1 we conclude that

$$\|\nabla\boldsymbol{\xi}(t)\|^2 + \int_0^t\|\boldsymbol{\xi}_t(s)\|^2\,ds \le \frac{G_2(t)}{\lambda_{k+1}}. \tag{3.19}$$

Now, to conclude our result, we differentiate $(3.6)_2$ with respect to t, multiplying by $\boldsymbol{\sigma}_t$ and integrate over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\sigma}_t\|^2 + \|\nabla\boldsymbol{\sigma}_t\|^2 = &-((\boldsymbol{\xi}_t\cdot\nabla)\mathbf{d}^m, \boldsymbol{\sigma}_t) - ((\boldsymbol{\xi}\cdot\nabla)\mathbf{d}_t^m, \boldsymbol{\sigma}_t) - ((\mathbf{u}_t^k\cdot\nabla)\boldsymbol{\sigma}, \boldsymbol{\sigma}_t) - ((\mathbf{u}^k\cdot\nabla)\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_t)\\ &+ (\partial_t(|\nabla\mathbf{d}^m|^2 - |\nabla\mathbf{d}^k|^2)\mathbf{d}^m, \boldsymbol{\sigma}_t) + ((|\nabla\mathbf{d}^m|^2 - |\nabla\mathbf{d}^k|^2)\mathbf{d}_t^m, \boldsymbol{\sigma}_t)\\ &+ (\partial_t(|\nabla\mathbf{d}^k|^2)\boldsymbol{\sigma}, \boldsymbol{\sigma}_t) + (|\nabla\mathbf{d}^k|^2\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_t).\end{aligned}$$

Observe that

$$\begin{aligned}|((\boldsymbol{\xi}_t\cdot\nabla)\mathbf{d}^m, \boldsymbol{\sigma}_t)| &\le \|\boldsymbol{\xi}_t\|\,\|\nabla\mathbf{d}^m\|_{\mathbf{L}^3}\|\boldsymbol{\sigma}_t\|_{\mathbf{L}^6} \le C_\epsilon\|\boldsymbol{\xi}_t\|^2\|\mathbf{d}^m\|_{\mathbf{H}^2}^2 + \epsilon\|\nabla\boldsymbol{\sigma}_t\|^2;\\ |((\boldsymbol{\xi}\cdot\nabla)\mathbf{d}_t^m, \boldsymbol{\sigma}_t)| &\le \|\boldsymbol{\xi}\|_{\mathbf{L}^3}\|\nabla\mathbf{d}_t^m\|\,\|\boldsymbol{\sigma}_t\|_{\mathbf{L}^6} \le C_\epsilon\|\nabla\boldsymbol{\xi}\|^2\|\nabla\mathbf{d}_t^m\|^2 + \epsilon\|\nabla\boldsymbol{\sigma}_t\|^2,\end{aligned}$$

similarly, we obtain

$$|((\mathbf{u}_t^k\cdot\nabla)\boldsymbol{\sigma}, \boldsymbol{\sigma}_t)| \le C_\epsilon\|\nabla\mathbf{u}_t^k\|^2\|\nabla\boldsymbol{\sigma}\|^2 + \epsilon\|\nabla\boldsymbol{\sigma}_t\|^2;$$

$$|((\mathbf{u}^k\cdot\nabla)\boldsymbol{\sigma}_t,\boldsymbol{\sigma}_t)|\le C_\epsilon\|P\Delta\mathbf{u}^k\|^2\|\boldsymbol{\sigma}_t\|^2+\epsilon\|\nabla\boldsymbol{\sigma}_t\|^2.$$

For the last terms, we have

$$\begin{gathered}
|(\partial_t(|\nabla\mathbf{d}^m|^2-|\nabla\mathbf{d}^k|^2)\mathbf{d}^m,\boldsymbol{\sigma}_t)|\le(|\nabla\boldsymbol{\sigma}_t|\mathbf{d}^m,\boldsymbol{\sigma}_t))\le C_\epsilon\|\mathbf{d}^m\|^2_{\mathbf{H}^2}\|\boldsymbol{\sigma}_t\|^2+\epsilon\|\nabla\boldsymbol{\sigma}_t\|^2;\\
|((|\nabla\mathbf{d}^m|^2-|\nabla\mathbf{d}^k|^2)\mathbf{d}^m_t,\boldsymbol{\sigma}_t)|\le(|\nabla\boldsymbol{\sigma}|\mathbf{d}^m_t,\boldsymbol{\sigma}_t))\le C_\epsilon\|\nabla\mathbf{d}^m\|^2\|\nabla\boldsymbol{\sigma}\|^2+\epsilon\|\nabla\boldsymbol{\sigma}_t\|^2;\\
|(\partial_t(|\nabla\mathbf{d}^k|^2)\boldsymbol{\sigma},\boldsymbol{\sigma}_t)|\le\|\nabla\mathbf{d}^k_t\|\|\nabla\mathbf{d}^k\|_{\mathbf{L}^6}\|\boldsymbol{\sigma}\|_{\mathbf{L}^6}\|\boldsymbol{\sigma}_t\|\le C_\epsilon\|\mathbf{d}^m\|^2_{\mathbf{H}^2}\|\mathbf{d}^m_t\|^2_{\mathbf{H}^1}\|\nabla\boldsymbol{\sigma}\|^2+\epsilon\|\nabla\boldsymbol{\sigma}_t\|^2;
\end{gathered}$$

$$\begin{aligned}
|(|\nabla\mathbf{d}^k|^2\boldsymbol{\sigma}_t,\boldsymbol{\sigma}_t)|&\le\|\nabla\mathbf{d}^k\|_{\mathbf{L}^\infty}\|\nabla\mathbf{d}^k\|_{\mathbf{L}^3}\|\boldsymbol{\sigma}_t\|_{\mathbf{L}^6}\|\boldsymbol{\sigma}_t\|\\
&\le C\|\nabla\mathbf{d}^k\|_{\mathbf{H}^2}\|\nabla\mathbf{d}^k\|^{1/2}_{\mathbf{L}^2}\|\nabla\mathbf{d}^k\|^{1/2}_{\mathbf{H}^1}\|\nabla\boldsymbol{\sigma}_t\|\|\boldsymbol{\sigma}_t\|\\
&\le C_\epsilon\|\mathbf{d}^k\|^2_{\mathbf{H}^3}\|\mathbf{d}^k\|^2_{\mathbf{H}^2}\|\boldsymbol{\sigma}_t\|^2+\epsilon\|\nabla\boldsymbol{\sigma}_t\|^2,
\end{aligned}$$

Then, the above estimates imply

$$\|\boldsymbol{\sigma}_t(t)\|^2+\int_0^t\|\nabla\boldsymbol{\sigma}_t(s)\|^2ds\le\frac{a(t)}{\lambda_{k+1}}+\int_0^t(\|P\Delta\mathbf{u}^k(s)\|^2+\|\mathbf{d}^k\|^2_{\mathbf{H}^3}\|\mathbf{d}^k\|^2_{\mathbf{H}^2})\|\boldsymbol{\sigma}_t(s)\|^2ds,$$

and we conclude, by Gronwall‘s inequality and taking $m\to\infty$, the desired result. □

Theorem 3.4. *Under the hypothesis of Theorem 2.4, we have*

$$\int_0^t\|P\Delta\mathbf{u}(s)-P\Delta\mathbf{u}^k(s)\|^2ds\le\frac{L_1(t)}{\lambda_{k+1}},\tag{3.20}$$

$$\int_0^t\|\Delta\mathbf{d}(s)-\Delta\mathbf{d}^k(s)\|^2ds\le\frac{L_2(t)}{\lambda_{k+1}},\tag{3.21}$$

for any $t\in[0,T_1]$. The continuous functions $L_i(t)$ depend on t and on the functions $F_i(t)$. The above estimates also hold with $\rho^m,\mathbf{u}^m,\mathbf{d}^m$ instead of $\rho,\mathbf{u},\mathbf{d}$ for $m>k$.

Proof. Consider $(3.1)_1$. Taking the inner product in $\mathbf{L}^2(\Omega)$ with $-P\Delta\boldsymbol{\xi}$, we obtain

$$\begin{aligned}
\|P\Delta\boldsymbol{\xi}\|^2\le&-\big(P_m(\rho^m-\rho^k)[\mathbf{u}^k_t+(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m],P\Delta\boldsymbol{\xi}\big)\\
&-\big((P_m-P_k)\big[\rho^k\mathbf{u}^k_t+\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k+\Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k\big],P\Delta\boldsymbol{\xi}\big)\\
&-\big(P_m\big[\rho^k(\xi\cdot\nabla)\mathbf{u}^k+\rho^k(\mathbf{u}^k\cdot\nabla)\xi^k+\Delta\mathbf{d}^m\cdot\nabla\boldsymbol{\sigma}-\Delta\boldsymbol{\sigma}\cdot\nabla\mathbf{d}^k\big],P\Delta\boldsymbol{\xi}\big)\,.
\end{aligned}$$

So, estimating as usual we obtain

$$\begin{aligned}
&\|P\Delta\boldsymbol{\xi}\|^2\\
&\le C_\epsilon\left(\|\rho^m\boldsymbol{\xi}_t\|^2+\|\rho^m(\mathbf{u}^m\cdot\nabla)\boldsymbol{\xi}\|^2+\|\rho^m(\boldsymbol{\xi}\cdot\nabla)\mathbf{u}^k\|^2+\|\Delta\mathbf{d}^m\cdot\nabla\boldsymbol{\sigma}\|^2+\|\Delta\boldsymbol{\sigma}\cdot\nabla\mathbf{d}^k\|^2\right)\\
&\quad+\epsilon\|P\Delta\boldsymbol{\xi}\|^2+C_\epsilon\|\rho^m-\rho^k\|^2_{L^3}\left(\|\nabla\mathbf{u}^k_t\|^2+\|(\nabla\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\|^2+\|(\mathbf{u}^k\cdot\nabla)\nabla\mathbf{u}^k\|^2\right)+\epsilon\|P\Delta\boldsymbol{\xi}\|^2\\
&\quad+\frac{C_\epsilon}{\lambda_{k+1}}\left(\|\rho^k\mathbf{u}^k_t\|^2_{\mathbf{H}^1}+\|\rho^k(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\|^2_{\mathbf{H}^1}+\|\Delta\mathbf{d}^k\cdot\nabla\mathbf{d}^k\|^2_{\mathbf{H}^1}\right)+\epsilon\|P\Delta\boldsymbol{\xi}\|^2,
\end{aligned}$$

and by Lemma 3.2

$$\begin{aligned}
\|P\Delta\boldsymbol{\xi}\|^2\le&\ C\|\boldsymbol{\xi}_t\|^2+C\|\nabla\boldsymbol{\xi}\|^2+C\|\Delta\boldsymbol{\sigma}\|^2+C\|\nabla\boldsymbol{\sigma}\|^2\\
&+\frac{C}{\lambda_{k+1}}\left(C+C\|\nabla\mathbf{u}^k_t\|\right)+\frac{C}{\lambda_{k+1}}\left(C+\|\nabla\mathbf{u}^k_t\|\right).
\end{aligned}$$

By integrating from 0 to t we have

$$\begin{aligned}
\int_0^t\|P\Delta\boldsymbol{\xi}(s)\|^2ds\le&\ C\int_0^t\|\boldsymbol{\xi}_t\|^2ds+C\int_0^t\|\nabla\boldsymbol{\xi}\|^2+C\int_0^t\|\Delta\boldsymbol{\sigma}\|^2ds+C\int_0^t\|\nabla\boldsymbol{\sigma}\|^2ds\\
&+\frac{C}{\lambda_{k+1}}\Big(Ct+C\int_0^t\|\nabla\mathbf{u}^k_t\|^2ds\Big)+\frac{C}{\lambda_{k+1}}\Big(Ct+\int_0^t\|\nabla\mathbf{u}^k_t\|^2ds\Big)\\
\le&\ \frac{L_1(t)}{\lambda_{k+1}},
\end{aligned}$$

and we obtain (3.20). Now, by $(3.6)_2$ we obtain

$$\begin{aligned}\|\Delta\boldsymbol{\sigma}\|^2 \le C\Big(&\|\boldsymbol{\sigma}_t\|^2 + \|\nabla\boldsymbol{\xi}\|^2\|\mathbf{d}^m\|^2_{\mathbf{H}^2} + \|P\Delta\mathbf{u}^k\|^2\|\nabla\boldsymbol{\sigma}\|^2 + \|\nabla\boldsymbol{\sigma}\|^2\|\mathbf{d}^m\|^2_{\mathbf{H}^2}\\ &+ \|\nabla\boldsymbol{\sigma}\|^2\|\mathbf{d}^k\|^2_{\mathbf{H}^3}\|\mathbf{d}^k\|^2_{\mathbf{H}^2}\Big),\end{aligned}$$

and by estimates established previously we obtain (3.21) taking $m \to \infty$. □

Corollary 3.5. *Under the hypothesis of Theorem 3.4, we have*

$$\|\rho(t) - \rho^k(t)\|^2_{L^\infty} \le \frac{L_1(t)}{\lambda_{k+1}}, \tag{3.22}$$

where L_1 is the same function of Theorem 3.4.

Proof. As in Theorem 3.1 we have

$$\|\rho(t) - \rho^k(t)\|^2_{L^\infty} \le \Big(\int_0^t \|\mathbf{u}(s) - \mathbf{u}^k(s)\|_{L^\infty}\|\nabla\rho^k(s)\|_{L^\infty}\,ds\Big)^2.$$

Consequently, by H ölder's inequality,

$$\begin{aligned}\|\rho(t) - \rho^k(t)\|^2_{L^\infty} &\le t\int_0^t \|\mathbf{u} - \mathbf{u}^k\|^2_{L^\infty}\|\nabla\rho^k\|^2_{L^\infty}\,ds\\ &\le CT_1 \sup\|\nabla\rho^k\|^2_{L^\infty}\int_0^t \|P\Delta\mathbf{u} - P\Delta\mathbf{u}^k\|^2\,ds\\ &\le C\frac{L_1(t)}{\lambda_{k+1}},\end{aligned}$$

where the last estimate follows directly from Theorem 3.4, yielding (3.22). □

4. $\mathbf{H}^3$-ERROR ESTIMATES FOR OPTICAL DIRECTOR

Theorem 4.1. *Under the hypotheses of Theorem 2.4 we have*

$$\|\nabla\mathbf{d}_t(t) - \nabla\mathbf{d}_t^k(t)\|^2 + \int_0^t \|\mathbf{d}_{tt}(s) - \mathbf{d}_{tt}^k(s)\|^2 ds \le \frac{H_1(t)}{\lambda_{k+1}}, \tag{4.1}$$

$$\|\mathbf{d}(t) - \mathbf{d}(t)\|^2_{\mathbf{H}^3} \le \frac{H_2(t)}{\lambda_{k+1}}. \tag{4.2}$$

Proof. Differentiating $(3.6)_2$ with respect to t, after multiplication by $\boldsymbol{\sigma}_t$ and integrating with respect to Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\sigma}_t\|^2 + \|\boldsymbol{\sigma}_{tt}\|^2\\ &= -((\boldsymbol{\xi}_t\cdot\nabla)d^m, \boldsymbol{\sigma}_{tt}) - ((\boldsymbol{\xi}\cdot\nabla)\mathbf{d}_t^m, \boldsymbol{\sigma}_{tt}) - ((\mathbf{u}_t^k\cdot\nabla)\boldsymbol{\sigma}, \boldsymbol{\sigma}_{tt}) - ((\mathbf{u}^k\cdot\nabla)\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_{tt})\\ &\quad + (\partial_t(|\nabla\mathbf{d}^m|^2 - |\nabla\mathbf{d}^k|^2)\mathbf{d}^m, \boldsymbol{\sigma}_{tt}) + ((|\nabla\mathbf{d}^m|^2 - |\nabla\mathbf{d}^k|^2)\mathbf{d}_t^m, \boldsymbol{\sigma}_{tt})\\ &\quad + \partial_t(|\nabla\mathbf{d}^k|^2)\boldsymbol{\sigma}, \boldsymbol{\sigma}_{tt}) + (|\nabla\mathbf{d}^k|^2\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_{tt}).\end{aligned} \tag{4.3}$$

As in Theorem 3.3 we obtain

$$\|\nabla\boldsymbol{\sigma}_t(t)\|^2 + \int_0^t \|\boldsymbol{\sigma}_{tt}(s)\|^2\,ds \le \frac{H_1(t)}{\lambda_{k+1}}.$$

To obtain the estimate (4.2), we take ∇ from $(3.6)_2$ and obtain

$$\begin{aligned}\|\nabla\Delta\boldsymbol{\sigma}(t)\|^2 \le C\Big(&\|\nabla\boldsymbol{\sigma}_t\|^2 + \|\nabla\boldsymbol{\xi}\cdot\nabla\mathbf{d}^m\|^2 + \|\boldsymbol{\xi}\cdot\Delta\mathbf{d}^m\|^2 + \|\nabla\mathbf{u}^k\cdot\nabla\boldsymbol{\sigma}\|^2\\ &+ \|\mathbf{u}^k\cdot\Delta\boldsymbol{\sigma}\|^2 + \|\nabla\big(|\nabla\mathbf{d}^m|^2\mathbf{d}^m - |\nabla\mathbf{d}^k|^2\mathbf{d}^k|^2\big)\|^2\Big),\end{aligned}$$

and, since $\|\boldsymbol{\sigma}\|^2_{\mathbf{H}^3} \le C\|\nabla\Delta\boldsymbol{\sigma}\|^2$, we obtain the result estimating the right-hand side using the estimates already proved. □

5. Improved $\mathbf{L}^2$-error bounds

The $\mathbf{L}^2$-estimates obtained in Theorem (3.1) are not optimal. In fact, one expects to obtain a rate of convergence of order $\frac{1}{\lambda_{k+1}^2}$ instead of $\frac{1}{\lambda_{k+1}}$. In this section, we will improve the $\mathbf{L}^2$-estimates in Theorem (3.1) by using a bootstrap argument. To do that, we set

$$\mathbf{e}^k = \mathbf{u} - \mathbf{v}^k,\ \boldsymbol{\sigma}^k = \mathbf{d} - \mathbf{d}^k \quad \text{and} \quad \boldsymbol{\psi}^k = \mathbf{u}^k - \mathbf{v}^k,$$

where $\mathbf{v}^k$ is the k-th partial sum of the series for $\mathbf{u}$. Then, we have $\mathbf{u} - \mathbf{u}^k = \mathbf{e}^k - \boldsymbol{\psi}^k$.

Lemma 5.1.

$$\|\boldsymbol{\psi}^k(t)\|^2 + \|\nabla\boldsymbol{\sigma}^k(t)\|^2 + C_1\int_0^t (\|\nabla\boldsymbol{\psi}^k(s)\|^2 + \gamma\|\Delta\boldsymbol{\sigma}^k(s)^2)\|ds \le C(t)\Big(\frac{1}{\lambda_{k+1}^2} + \frac{1}{\lambda_{k+1}^{3/2}}\Big).$$

Proof. We observe that $\mathbf{v}^k$ satisfy

$$P_k\big[\rho\big(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}\big) + \lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})\big] + \nu A\mathbf{v}^k = 0, \tag{5.1}$$

then

$$P_k\big[\rho^k\mathbf{u}_t^k - \rho\mathbf{u}_t\big] + P_k\big[(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k - (\mathbf{u}\cdot\nabla)\mathbf{u}\big] + \lambda P_k\big(\nabla\cdot(\nabla\mathbf{d}^k\odot\nabla\mathbf{d}^k) - \nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d})\big) + \nu A\boldsymbol{\psi}^k = 0. \tag{5.2}$$

By taking the inner product in $\mathbf{L}^2$ of (5.2) with $\boldsymbol{\psi}^k$ and proceeding as in the previous section, we obtain

$$\begin{aligned}
&\nu\|\nabla\boldsymbol{\psi}^k\|^2 + \frac{1}{2}\frac{d}{dt}\|(\rho^k)^{1/2}\boldsymbol{\psi}^k\|^2 - (\rho\mathbf{e}_t^k, \boldsymbol{\psi}^k)\\
&= \frac{1}{2}(\rho_t^k\boldsymbol{\psi}^k, \boldsymbol{\psi}^k) + ((\rho - \rho^k)\mathbf{v}_t^k, \boldsymbol{\psi}^k) + ((\rho - \rho^k)(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k, \boldsymbol{\psi}^k)\\
&\quad - (\rho(\boldsymbol{\psi}^k\cdot\nabla)\mathbf{u}^k, \boldsymbol{\psi}^k) + (\rho(\mathbf{e}^k\cdot\nabla)\mathbf{u}^k, \boldsymbol{\psi}^k) - (\rho(\mathbf{u}\cdot\nabla)\boldsymbol{\psi}^k, \boldsymbol{\psi}^k)\\
&\quad + (\rho(\mathbf{u}\cdot\nabla)\mathbf{e}^k, \boldsymbol{\psi}^k) + (\Delta\boldsymbol{\sigma}^k\cdot\nabla\mathbf{d}^k, \boldsymbol{\psi}^k) + (\Delta\mathbf{d}\cdot\nabla\boldsymbol{\sigma}^k, \boldsymbol{\psi}^k).
\end{aligned}$$

Also, we have

$$\rho - \rho^k = -\int_0^t (\mathbf{u} - \mathbf{u}^k)\nabla\rho^k\, ds = -\int_0^t (\mathbf{e}^k\cdot\nabla)\rho^k\, ds + \int_0^t (\boldsymbol{\psi}^k\cdot\nabla)\rho^k\, ds.$$

Now we observe that, if $\boldsymbol{\tau}$ is any function in $\mathbf{H}^1(\Omega)$, for any $\delta > 0$, $\epsilon > 0$, it holds

$$((\rho - \rho^k)\boldsymbol{\tau}, \boldsymbol{\psi}^k) \le C_\delta\Big[\int_0^t \|\mathbf{e}^k\|^2\|\nabla\rho^k\|_{L^\infty}^2 ds\Big]\|\boldsymbol{\tau}\|_{\mathbf{L}^4}^2 + C_\delta\Big[\int_0^t \|\boldsymbol{\psi}^k\|^2\|\nabla\rho^k\|_{L^\infty}^2 ds\Big]\|\boldsymbol{\tau}\|_{\mathbf{L}^4}^2 + 2\delta\|\nabla\boldsymbol{\psi}^k\|^2.$$

By using this with $\boldsymbol{\tau} = \mathbf{v}_t^k$ and $\boldsymbol{\tau} = \mathbf{u}^k\cdot\nabla\mathbf{u}^k$ and the facts

$$\begin{gathered}
\|\mathbf{e}^k\|^2 \le \frac{C}{\lambda_{k+1}^2}\|\mathbf{u}\|_{\mathbf{H}^2} \le \frac{C}{\lambda_{k+1}^2},\\
(\rho(\mathbf{u}\cdot\nabla)\mathbf{e}^k, \boldsymbol{\psi}^k) = (\rho(\mathbf{u}\cdot\nabla)\boldsymbol{\psi}^k, \mathbf{e}^k),
\end{gathered}$$

together with the previous estimates, we obtain that for any $\delta > 0$,

$$\begin{aligned}
&\nu\|\nabla\boldsymbol{\psi}^k\|^2 + \frac{1}{2}\frac{d}{dt}\|(\rho^k)^{1/2}\boldsymbol{\psi}^k\|^2 - (\rho\mathbf{e}_t^k, \boldsymbol{\psi}^k)\\
&\le C\|\boldsymbol{\psi}^k\|^2 + \frac{C}{\lambda_{k+1}^2}\Big(\|\nabla\mathbf{v}_t^k\|^2 + \|\nabla\big[(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\big]\|^2\Big) + \frac{C}{\lambda_{k+1}^2}\\
&\quad + C\int_0^t \|\boldsymbol{\psi}^k\|^2 ds\,\big(\|\nabla\mathbf{v}_t^k\|^2 + \|\nabla\big[(\mathbf{u}^k\cdot\nabla)\mathbf{u}^k\big]\|^2\big) + 8\delta\|\nabla\boldsymbol{\psi}^k\|^2 + 2\,\epsilon\|\Delta\boldsymbol{\sigma}^k\|^2.
\end{aligned}$$

In the same way,

$$\gamma\|\Delta\boldsymbol{\sigma}^k\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\sigma}^k\|^2 \le \frac{C}{\lambda_{k+1}^2} + C\|\boldsymbol{\psi}^k\|^2 + C\|\nabla\boldsymbol{\sigma}^k\|^2 + 5\,\epsilon\|\Delta\boldsymbol{\sigma}^k\|^2.$$

Thus,

$$\begin{aligned}&\nu\|\nabla\boldsymbol{\psi}^k\|^2+\gamma\|\Delta\boldsymbol{\sigma}^k\|^2+\frac{1}{2}\frac{d}{dt}\|(\rho^k)^{1/2}\boldsymbol{\psi}^k\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{\sigma}^k\|^2-(\rho\mathbf{e}_t^k,\boldsymbol{\psi}^k)\\&\le\frac{C}{\lambda_{k+1}^2}+C(\|\boldsymbol{\psi}^k\|^2+\|\nabla\boldsymbol{\sigma}^k\|^2)+\frac{C}{\lambda_{k+1}^2}\|\nabla\mathbf{v}_t^k\|^2+C\|\nabla\mathbf{v}_t^k\|^2\int_0^t\|\boldsymbol{\psi}^k\|^2ds.\end{aligned}$$

Integrating with respect to t, we have

$$\begin{aligned}&C_1\int_0^t(\|\nabla\boldsymbol{\psi}^k\|^2+\gamma\|\Delta\boldsymbol{\sigma}^k\|^2)ds+\|\boldsymbol{\psi}^k\|^2+\|\nabla\boldsymbol{\sigma}^k\|^2-\int_0^t(\rho\mathbf{e}_t^k,\boldsymbol{\psi}^k)\,ds\\&\le C\int_0^t(\|\boldsymbol{\psi}^k\|^2+\|\nabla\boldsymbol{\psi}^k\|^2)ds+\frac{C}{\lambda_{k+1}^2}.\end{aligned}$$

Now, using that $\boldsymbol{\sigma}^k(0)=0$ and integrating by parts with respect to t, we have

$$\begin{aligned}&\|\int_0^t(\rho\mathbf{e}_t^k,\boldsymbol{\psi}^k)\,ds\|\\&\le\|\int_0^t(\rho_t\mathbf{e}^k,\boldsymbol{\psi}^k)\,ds\|+\|\int_0^t(\rho\mathbf{e}^k,\boldsymbol{\psi}_t^k)\,ds\|+|(\rho(t)\mathbf{e}^k(t),\boldsymbol{\psi}^k(t))|\\&\le\int_0^t\|\rho_t\|_{\mathbf{L}^\infty}\|\mathbf{e}^k\|\,\|\boldsymbol{\psi}^k\|\,ds+\int_0^t\|\rho\|_{\mathbf{L}^\infty}\|\mathbf{e}^k\|(\|\mathbf{e}_t^k\|+\|(\mathbf{u}-\mathbf{u}^k)_t\|)\,ds+\beta\|\mathbf{e}^k\|\,\|\boldsymbol{\psi}^k\|\\&\le C\int_0^t\|\mathbf{e}^k\|^2ds+C\int_0^t\|\boldsymbol{\psi}^k\|^2ds+C\int_0^t(\|\mathbf{e}^k\|\,\|\mathbf{e}_t^k\|+\|\mathbf{e}^k\|\,\|(\mathbf{u}-\mathbf{u}^k)_t\|)\,ds+\frac{C_\beta}{\lambda_{k+1}}\frac{1}{\lambda_{k+1}^{1/2}}\\&\le\frac{C}{\lambda_{k+1}^2}+\frac{C}{\lambda_{k+1}^{3/2}}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|\boldsymbol{\psi}^k(t)\|^2+\|\nabla\boldsymbol{\sigma}^k(t)\|^2+C_1\int_0^t(\|\nabla\boldsymbol{\psi}^k(s)\|^2+\gamma\|\Delta\boldsymbol{\sigma}^k(s)^2\|)ds\\&\le C\Big(\frac{1}{\lambda_{k+1}^2}+\frac{1}{\lambda_{k+1}^{3/2}}\Big)+C\int_0^t(\|\boldsymbol{\psi}^k(s)\|^2+\|\nabla\boldsymbol{\sigma}^k(s)\|^2)ds.\end{aligned}$$

From this and Gronwall‘s inequality we conclude the Lemma. □

Now, we are ready to prove our main results.

Theorem 5.2. *The following two estimates hold:*

$$\|\mathbf{u}(t)-\mathbf{u}^k(t)\|+\|\nabla\mathbf{d}-\nabla\mathbf{d}^k\|\le C(t)\Big(\frac{1}{\lambda_{k+1}^2}+\frac{1}{\lambda_{k+1}^{3/2}}\Big),$$

$$\|\rho(t)-\rho^k(t)\|\le C(t)\Big(\frac{1}{\lambda_{k+1}^2}+\frac{1}{\lambda_{k+1}^{3/2}}\Big)$$

for all $t\in[0,T]$.

Proof. Since

$$\|\mathbf{u}-\mathbf{u}^k\|=\|\mathbf{e}^k-\boldsymbol{\psi}^k\|\le\|\mathbf{e}^k\|+\|\boldsymbol{\psi}^k\|,\quad\|\nabla\mathbf{d}-\nabla\mathbf{d}^k\|\le\|\nabla\boldsymbol{\sigma}^k\|,$$

from the Lemma 5.1, we obtain the first estimate.

For the second estimate, proceeding as in the first part of the Theorem 3.1, that is, $\|\rho(t)-\rho^k(t)\|^2\le C(t)\int_0^t\|\mathbf{u}(s)-\mathbf{u}^k(s)\|^2ds$, and we obtain the result. □

Remark 5.3 (Final Remarks)**.** As mentioned after the Proposition 2.6, we emphasize that the error estimates obtained in this work are local-in-time. As noted by Heywood [10] to obtain estimates global-in-time to the classic Navier-Stokes equation, additional stability assumptions are necessary. For the classical Navier-Stokes equations, assuming exponential stability in the

$\mathbf{H}^1$ or $\mathbf{L}^2$ norms allows for uniform-in-time optimal estimates [10, 19]. However, extending this to the variable density case often requires regularity assumptions, such as $\mathbf{u} \in L^\infty(0,T;\mathbf{H}^3)$ [**?**]. Uniform-in-time error estimates can also be derived without explicitly assuming $\mathbf{H}^1$ stability by considering a class of solutions corresponding to exponentially decaying external forces with a sufficiently small norm [2].

We emphasize that the optimal convergence rate cannot be obtained in our case due to the presence of the term $\rho\mathbf{e}_t^k$. This term does not appear in the classical Navier-Stokes equations, where optimal $\mathbf{L}^2$ error estimates are possible. Indeed, in contrast with the classical Navier-Stokes equations, where optimal $\mathbf{L}^2$ error estimates can be obtained (see Rojas-Medar and Boldrini [1]), in the variable-density case considered here we were only able to prove an improved $\mathbf{L}^2$ error estimate compared with the trivial estimate derived from the $\mathbf{H}^1$ bound.

Extending the present analysis to global-in-time error estimates under a small initial data assumption remains an interesting open problem. Such an extension would require not only the global existence of strong solutions but also suitable uniform-in-time stability properties. In our inhomogeneous nematic liquid crystal model, the transport equation for the density does not possess any dissipative mechanism. As a consequence, the density error is controlled indirectly through the velocity error, and these estimates accumulate over time. This additional coupling makes the derivation of uniform-in-time estimates substantially more delicate, likely requiring further assumptions ensuring sufficient decay or stability of the solution.

Finally, although the present work is purely analytical and focuses on the rigorous derivation of error estimates for spectral semi-Galerkin approximations, numerical experiments illustrating the convergence rates predicted by the theory would provide a valuable complement. Such numerical investigations will be the subject of future work.

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