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# GENERALIZATION OF THE CHORD SOBOLEV INEQUALITY FOR ORLICZ SPACES

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Abstract. The relationship between the classical chord power integrals and radial mean bodies, as well as the classical chord Sobolev inequalities, have already been established. This article extends the concepts of chord power integrals and chord Sobolev inequalities to the Orlicz space framework. We provide the definitions of the Orlicz chord power integrals and the Orlicz radial mean bodies, and we establish their equivalence. We prove the Orlicz chord Sobolev inequality for two distinct cases: $0 < \alpha < n$ and $\alpha > n$.

## 1. Introduction

In convex geometric analysis, the classical chord power integral is a fundamental geometric quantity characterizing the chord length distribution of convex bodies, systematically established by Lutwak et al. within the framework of chord Sobolev inequalities. This inequality sits at the intersection of analysis and geometry, bridging isoperimetric phenomena and functional inequalities in [8, 17, 18, 19, 22, 26, 27].

In recent years, the theory of chord power integrals has undergone substantial advancements, driven by its deep connections to emerging topics such as chord measures, Minkowski-type problems [21, 25], and affine functional inequalities [4]. The anisotropic fractional Sobolev inequalities has also attracted extensive in-depth investigations; see [12, 24]. Lutwak, Xi, Yang, and Zhang [25] pioneered the study of chord measures, a new family of translation-invariant geometric measures derived from the differentials of chord power integrals, solving the associated chord Minkowski problems [14] and logarithmic variants [13], which correspond to novel fully nonlinear partial differential equations involving dual quermassintegrals. Concurrently, Baêta and Cai [5, 6] established the complete chord Sobolev inequality theory, unifying the analytic inequalities for both $\alpha > 0$ (affine-type) and $\alpha \in (-1, 0)$ (fractional-type), and characterized the equality cases, significantly strengthening the classical Euclidean isoperimetric inequalities in [2, 3]. Beyond convex bodies, the functional extensions of chord power integrals have been systematically developed, with applications to log-concave and $s$-concave functions, as well as radial mean bodies in [2, 7]. Complementary studies include the $L_p$ and Orlicz extensions of chord Minkowski problems, variational formulas for chord power integrals, and integral representations for specific convex bodies like simplices in [16]. For other relevant references, see [10, 15, 20].

Relying on the theoretical framework of the chord Sobolev inequality, we introduce rigorous definitions of the *geometric chord power integral for convex bodies* and the *functional chord power integral for nonnegative measurable function*, and gives equivalent analytic forms and applicable conditions according to different ranges of the parameter $\alpha \geq -1$ in section 2.3, clarifying the structural differences and geometric meanings under different intervals.

The standard definitions of the chord power integral in [6] for convex bodies and for functions are as follows.

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Let $K \subset \mathbb{R}^n$ be a convex body a compact convex set with nonempty interior, $\omega_n$ be the volume of the unit ball in $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ be the $(n-1)$-dimensional Hausdorff measure. For a convex body $K \subset \mathbb{R}^n$ and a parameter $\alpha \geq -1$, the *chord power integral* $I_{\alpha+1}(K)$ is defined as the integral of the $(\alpha+1)$-th power of the length of intersection of a random line with the convex body

$$I_{\alpha+1}(K) = \int_{\mathrm{Aff}(n,1)} |K \cap l|^{\alpha+1} dl.$$

For any unit vector $u \in \mathbb{S}^{n-1}$, denote by $K|u^\perp$ the orthogonal projection of $K$ onto the orthogonal complement of $u$, and let

$$X_K(x,u) = |K \cap (x + \mathbb{R}u)|$$

be the *chord length* (X-ray function) of $K$ in direction $u$ through the point $x \in K|u^\perp$. Then the *classical chord power integral of order* $\alpha+1$ for $K$ is defined as

$$I_{\alpha+1}(K) = \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \int_{K|u^\perp} X_K(x,u)^{\alpha+1} d\mathcal{H}^{n-1}(x) du.$$

Three common properties of the chord power integral are as follows:

- $I_1(K) = |K|$, $I_2(K)$ is related to the surface area;
- Affine invariance and monotonicity: $K \subset L \Rightarrow I_{\alpha+1}(K) \leq I_{\alpha+1}(L)$;
- Homogeneity: $I_{\alpha+1}(tK) = t^{n+\alpha} I_{\alpha+1}(K)$.

For a nonnegative measurable function $f : \mathbb{R}^n \to \mathbb{R}_+$, whose level sets are $\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\}$, the *functional chord power integral* is defined via the layer cake representation of the chord power integral of its level sets

$$I_{\alpha+1}(f) = \int_0^\infty I_{\alpha+1}(\{f \geq t\}) dt = \frac{\alpha(\alpha+1)}{n\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^{n-\alpha}} \, dx \, dy,$$

where $\alpha > 0$.

The relationship between the classical chord power integral and the classical radial mean body is as follows

$$I_{\alpha+1}(K) = \frac{(\alpha+1)|K|}{n\omega_n} \int_{\mathbb{S}^{n-1}} \rho_{R_\alpha K}(u)^\alpha du.$$

where $R_\alpha K$ is the classical radial mean body in section 2.4, and

$$\rho_{R_\alpha K}(u)^\alpha = \frac{1}{|K|} \int_K \rho_{K-x}(u)^\alpha dx.$$

The corresponding relationship between the Orlicz chord power integral and the Orlicz radial mean body as follows.

**Theorem 1.1.** *For $\alpha > -1$ and a convex body $K \subset \mathbb{R}^n$, we have*

$$I_{G,\alpha+1}(K) = \frac{(\alpha+1)|K|}{n\omega_n} \int_{\mathbb{S}^{n-1}} G\left(\rho_{R_{G,\alpha} K}(u)\right)^\alpha du,$$

*where $R_{G,\alpha} K$ is the Orlicz radial mean body satisfying*

$$G\left(\rho_{R_{G,\alpha} K}(u)\right)^\alpha = \frac{1}{|K|} \int_K G\left(\rho_{K-x}(u)\right)^\alpha dx.$$

It perfectly replicates the structure of the classical framework and we present a series of lemmas along with detailed proofs in section 3.

In what follows, Section 4 establishes the Orlicz chord Sobolev inequality, which serves as the main theorem of this paper. We further present a comparison with the classical chord Sobolev inequality in [6].

**Classical chord Sobolev inequality:** For $\alpha \in (0,n)$, nonnegative $f \in L^{\frac{n}{n+\alpha}}(\mathbb{R}^n)$, there exists an sharp constant $C_{n,\alpha,p}$ such that

$$C_{n,\alpha,p} \|f\|_{\frac{n}{n+\alpha}} \geq \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^{n-\alpha}} \, dx \, dy.$$

**Orlicz chord Sobolev inequality:** For $\alpha \in (0,n)$, nonnegative $f \in L^G(\mathbb{R}^n)$ and $0 < \|f^\star\|_G < 1$, there exists an optimal constant $C_{G,n,\alpha}$ such that

$$C_{G,n,\alpha}\|f\|_G \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy,$$

where the symmetric decreasing rearrangement function $f^\star$ of the Orlicz function $f$ can be expressed as a constant multiple of the characteristic function of the unit ball.

**Theorem 1.2.** *Let $G$ be an Orlicz function, for $\alpha \in (0,n)$, and $f \in L^G(\mathbb{R}^n)$ be nonnegative. If the symmetric decreasing rearrangement function $f^\star$ of the Orlicz function $f$ can be expressed as a constant multiple of the characteristic function of the unit ball. Then there exists an* optimal constant $\sigma_{G,n,\alpha} > 0$ *and* $0 < \|f^\star\|_G < 1$ *such that*

$$\sigma_{G,n,\alpha}\|f\|_G \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

*When* $\|f^\star\|_G \geq 1$,

$$\sigma_{G,n,\alpha}\|f\|_G^p \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

*where $p > 1$ is given by* (2.1) *and $\sigma_{G,n,\alpha} > 0$ is a constant depending only on the Orlicz function $G$, $n$ and $\alpha$.*

When $G(t) = t$, the Orlicz space reduces to the $L^1$ space, the Luxemburg norm reduces to the $L^1$ norm, and the theorem reduces to the classical $L^1$ chord Sobolev inequality

$$\sigma_{n,\alpha}\|f\|_1 \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{f(x), f(y)\}}{|x-y|^{n-\alpha}}\,dx\,dy,$$

which verifies the consistency of the theorem.

Meanwhile, at the end of Section 4, we derive and characterize the explicit form of the Orlicz chord Sobolev inequality for the case $\alpha > n$.

**Theorem 1.3.** *Let $G$ be an Orlicz function, for $\alpha > n$, and $f \in L^G(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be nonnegative. If the symmetric decreasing rearrangement function $f^\star$ of the Orlicz function $f$ can be expressed as a constant multiple of the characteristic function of the unit ball. Then there exists an* optimal constant $C_G > 0$ *such that*

$$\log \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \geq \tilde{C}_{n,\alpha} + C_G \log\Big(\|f\|_\infty^{-\frac{\alpha}{n}}\|f\|_G^{\frac{n+\alpha}{n}})\Big),$$

*where $C_G = \frac{2\log C}{\log 2} > 1$, $\tilde{C}_{n,\alpha}$ depends only on $n$ and $\alpha$.*

## 2. Preliminaries

We collect results on Orlicz-Sobolev spaces, Orlicz functions, classical chord power integral and radial mean body, Schwarz symmetrization, that will be used in the following.

2.1. **Orlicz-Sobolev spaces.** Let $G$ be an Orlicz function. We consider the corresponding spaces $L^G(\mathbb{R}^n)$ defined as

$$L^G(\mathbb{R}^n) := \{u\colon \mathbb{R}^n \to \mathbb{R} \text{ measurable, such that } \Phi_G(u) < \infty\},$$

where the modulars $\Phi_G$ are defined as

$$\Phi_G(u) = \int_{\mathbb{R}^n} G(|u(x)|)\,dx,$$

These spaces are equipped with the so-called Luxemburg norms that are defined as

$$\|u\|_G = \|u\|_{L^G(\mathbb{R}^n)} := \inf\Big\{\lambda > 0 : \Phi_G\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}.$$

2.2. **Orlicz functions.**

**Definition 2.1.** $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called an Orlicz function if it has the following properties:

(H1) $G$ is continuous, convex, increasing and $G(0) = 0$.

(H2) $G$ satisfies the $\Delta_2$ condition, that is, there exists $\mathbf{C} > 2$ such that

$$G(2x) \le \mathbf{C}G(x) \quad \text{for all } x \in \mathbb{R}_+.$$

(H3) $G$ is super-linear at zero, that is $\lim_{x\to 0^+} \frac{G(x)}{x} = 0$.

Without loss of generality, we may normalize $G$ such that $G(1) = 1$. It is straightforward to verify that Orlicz functions satisfy the following fundamental properties.

**Lemma 2.2** ([9, Lemma 2.4])**.** *Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be an Orlicz function. It follows that*

(1) *(Regularity) $G$ is Lipschitz continuous.*

(2) *(Integrability near 0 and infinity) Given $s \in (0,1)$,*

$$\int_1^\infty \frac{G(x^{-s})}{x}\,dx \le \frac{\mathfrak{g}}{s}, \quad \int_0^1 \frac{G(x^{1-s})}{x}\,dx \le \frac{\mathfrak{g}}{1-s},$$

*where $\mathfrak{g} := \sup_{x\in(0,1)} x^{-1}G(x)$.*

(3) *$G$ can be represented in the form*

$$G(x) = \int_0^x g(s)\,ds,$$

*where $g(s)$ is a non-decreasing right continuous function.*

(4) *(Subadditivity) Given $a, b \in \mathbb{R}_+$,*

$$G(a+b) \le \frac{\mathbf{C}}{2}\big(G(a) + G(b)\big),$$

*where $\mathbf{C} > 0$ is the constant in the $\Delta_2$ condition (H2).*

(5) *For any $0 < b < 1$ and $a > 0$, it holds $G(ab) \le bG(a)$.*

The $\Delta_2$ condition in Definition 2.1 $(H2)$ is equivalent to

$$\frac{G'(a)}{G(a)} \le \frac{p}{a}, \quad \forall a > 0, \tag{2.1}$$

for some $p > 1$. It is shown in ([23], Theorem 4.1).

**Lemma 2.3** ([9, Lemma 2.5])**.** *Let $G$ be an Orlicz function. Then, for every $a \ge 0$ and $b \ge 1$ it holds*

$$G(ab) \le b^p G(a),$$

*where $p > 1$ is given by* (2.1).

**Lemma 2.4** ([9, Lemma 2.6])**.** *Let $G$ be an Orlicz function. Then for every $\delta > 0$, there exists $C_\delta > 0$ such that*

$$G(a+b) \le C_\delta G(a) + (1+\delta)^p G(b), \quad a, b \ge 0.$$

**Lemma 2.5** ([9, Lemma 2.7])**.** *Let $G$ be an Orlicz function. Then, there exists $q > 1$ such that*

$$t^{2q} G(a) \le G(at),$$

*for every $a > 0$ and $0 \le t \le 1$.*

To conclude this subsection, we review several established tools from convex analysis. In fact, given $G$ is an Orlicz function, we can define the following complementary function $G^*$ as

$$G^*(a) = \sup\{at - G(t) : t > 0\}. \tag{2.2}$$

It follows immediately from (2.2) that the following Young-type inequality holds

$$at \le G(t) + G^*(a) \quad \text{for every } a, t \ge 0.$$

We will utilize the following property in subsequent proofs.

**Lemma 2.6** ([9, Lemma 2.9]). *Let $G$ be an Orlicz function and $G^*$ its complementary function. Then*
$$G^*(g(t)) \le (p-1)G(t),$$
*where $p>1$ is given by (2.1) and $g = G'$.*

2.3. **Classical chord power integral.** For a convex body $K \subset \mathbb{R}^n$, a convex compact set with nonempty interior, let $|K|$ denote the $n$-dimensional Lebesgue measure of $K$, and $|K\cap l|$ the length of the line segment obtained by intersecting the line $l$ with the convex body $K$. Denote by $\mathrm{Aff}(n,1)$ the affine Grassmannian of all one-dimensional lines in $\mathbb{R}^n$, and by $dl$ its normalized Haar measure.

For a nonnegative measurable function $f:\mathbb{R}^n \to \mathbb{R}$, we denote its level set by
$$\{f \ge t\} = \{x \in \mathbb{R}^n \mid f(x) \ge t\}, \quad t>0,$$
its $L^p$ norm by $\|f\|_p$, and its essential supremum norm by $\|f\|_\infty$.

We first provide the chord power integral for convex bodies, its unified definition as follows: for a convex body $K\subset\mathbb{R}^n$ and a parameter $\alpha \ge -1$, the *chord power integral* $I_{\alpha+1}(K)$ is defined as the integral of the $(\alpha+1)$-th power of the length of intersection of a random line with the convex body
$$I_{\alpha+1}(K) = \int_{\mathrm{Aff}(n,1)} |K\cap l|^{\alpha+1} dl.$$

(1) $\alpha\in(-1,0)$: the chord power integral can be expressed as a nonlocal difference form of the characteristic function
$$I_{\alpha+1}(K) = \frac{|\alpha|(\alpha+1)}{2n\omega_n}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|1_K(x)-1_K(y)|}{|x-y|^{n-\alpha}}\,dx\,dy,$$
where $1_K$ is the characteristic function of the convex body $K$.

(2) $\alpha>0$: the integral reduces to a nonlocal integral of the product of characteristic functions
$$I_{\alpha+1}(K) = \frac{\alpha(\alpha+1)}{n\omega_n}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{1_K(x)1_K(y)}{|x-y|^{n-\alpha}}\,dx\,dy.$$

(3) Endpoint limiting cases: when $\alpha\to -1^+$, the chord power integral converges to the surface area of the convex body, corresponding to the classical isoperimetric inequality. When $\alpha=0$, $I_1(K)=|K|$, i.e., the Crofton volume formula. When $\alpha=n$, $I_{n+1}(K)=\frac{n+1}{\omega_n}|K|^2$, i.e., the Poincaré-Hadwiger's integral formula.

Then, we provide the chord power integral for nonnegative measurable function, its unified definition as follows: for a nonnegative, log-concave, integrable function $f\in L^\infty(\mathbb{R}^n)$, the *functional chord power integral* is defined via the layer cake representation of the chord power integral of its level sets
$$I_{\alpha+1}(f) = \int_0^{\|f\|_\infty} I_{\alpha+1}(\{f\ge t\})dt.$$

(1) $\alpha\in(-1,0)$: the fractional Sobolev space $f\in W^{-\alpha,1}(\mathbb{R}^n)$, and the integral is based on the absolute difference of the function
$$I_{\alpha+1}(f) = \frac{|\alpha|(\alpha+1)}{2n\omega_n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|^{n-\alpha}}\,dx\,dy.$$

(2) $\alpha\in(0,n)$: nonnegative functions $f\in L^{\frac{n}{n+\alpha}}(\mathbb{R}^n)$. To preserve homogeneity, the minimum structure is used
$$I_{\alpha+1}(f) = \frac{\alpha(\alpha+1)}{n\omega_n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\min\{f(x),f(y)\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

(3) $\alpha>n$: nonnegative functions $f\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ to ensure finiteness of the integral
$$I_{\alpha+1}(f) = \frac{\alpha(\alpha+1)}{n\omega_n}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\min\{f(x),f(y)\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

(4) Limiting case $\alpha\to 0^+$: the chord power integral satisfies the asymptotic equality
$$\lim_{\alpha\to 0^+}\alpha\cdot I_{\alpha+1}(f) = n\omega_n\|f\|_1,$$

which is key to deriving logarithmic Sobolev-type inequalities.

2.4. **Classical radial mean body.** Let $K \subset \mathbb{R}^n$ be a convex body containing the origin, $\rho_K(u) = \sup\{\lambda \geq 0 : \lambda u \in K\}$ the radial function of $K$ with respect to the origin, and $|K|$ the $n$-dimensional Lebesgue measure of $K$. For a nonnegative measurable function $f \in L^1(\mathbb{R}^n)$, denote its level set by

$$\{f \geq t\} = \{x \in \mathbb{R}^n : f(x) \geq t\},$$

and define the probability measure

$$d\mu_f(t) = \frac{|\{f \geq t\}|}{\|f\|_1} dt.$$

The *$\alpha$-th radial mean body* $R_\alpha K$ is defined as follows

$$\rho_{R_\alpha K}(u)^\alpha = \frac{1}{|K|} \int_K \rho_{K-x}(u)^\alpha dx,$$

where $\alpha > -1$ and $\alpha \neq 0$, $K - x$ denotes the convex body obtained by translating $K$ by the vector $-x$. When $\alpha = 0$ (0-th radial mean body),

$$\log \rho_{R_0 K}(u) = \frac{1}{|K|} \int_K \log \rho_{K-x}(u) dx.$$

Let $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a nonzero, nonnegative, log-concave function. The *$\alpha$-th functional radial mean body* $R_\alpha f$ is defined by the weighted average of the radial mean bodies of its level sets. When $\alpha \neq 0$,

$$\rho_{R_\alpha f}(u)^\alpha = \int_0^{\|f\|_\infty} \rho_{R_\alpha \{f \geq t\}}(u)^\alpha d\mu_f(t).$$

When $\alpha = 0$ (0-th functional radial mean body)

$$\log \rho_{R_0 f}(u) = \int_0^{\|f\|_\infty} \log \rho_{R_0 \{f \geq t\}}(u) d\mu_f(t).$$

We provide an equivalent characterization of radial mean bodies, valid for a convex body $K$ and a direction $u \in \mathbb{S}^{n-1}$.

- If $\alpha \in (-1, 0)$, then
$$\rho_{R_\alpha K}(u)^\alpha = \frac{|\alpha|}{2|K|} \int_0^\infty r^{\alpha-1} |K \triangle (K + ru)| dr.$$
- If $\alpha > 0$, then
$$\rho_{R_\alpha K}(u)^\alpha = \frac{\alpha}{|K|} \int_0^\infty r^{\alpha-1} |K \cap (K + ru)| dr.$$

2.5. **Symmetrization.** For a subset $E \subseteq \mathbb{R}^n$, the *indicator function* $1_E$ is characterized by $1_E(x) = 1$ when $x \in E$ and $1_E(x) = 0$ for all $x \notin E$. Let $E \subseteq \mathbb{R}^n$ be a Borel set with finite measure. The *Schwarz symmetral* of $E$, denoted by $E^\star$, is defined as a closed Euclidean ball centered at the origin, having the same volume as $E$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative measurable function such that for any $t > 0$, the super-level set $\{f \geq t\}$ has finite measure. The *layer cake formula* asserts that

$$f(x) = \int_0^\infty 1_{\{f \geq t\}}(x) dt. \tag{2.3}$$

holds for almost every $x \in \mathbb{R}^n$. The *Schwarz symmetrization* $f^\star$ of $f$ is defined as

$$f^\star(x) = \int_0^\infty 1_{\{f \geq t\}^\star}(x)\, dt.$$

for $x \in \mathbb{R}^n$. Consequently, $f^\star$ is characterized almost everywhere by being radially symmetric and possessing super-level sets whose measures coincide with those of $f$. Note that $f^\star$ is also called to the *symmetric decreasing rearrangement* of $f$, see [1]. If for all $x$, $y$ with $|x| < |y|$, the inequality $f^\star(x) > f^\star(y)$ holds, we say that $f^\star$ is strictly symmetric decreasing.

**Theorem 2.7** ((Riesz's rearrangement inequality). *For $f, g, k : \mathbb{R}^n \to \mathbb{R}$ non-negative, measurable functions with superlevel sets of finite measure,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,dx\,dy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\star(x)k^\star(x-y)g^\star(y)\,dx\,dy.$$

We will use the characterization of equality cases in the Riesz rearrangement inequality which is from A. Burchard [11].

**Theorem 2.8** (Burchard). *Let $A$, $B$ and $C$ be sets of finite positive measure in $\mathbb{R}^n$ and denote by $\alpha$, $\beta$ and $\gamma$ the radii of their Schwarz symmetrals $A^\star$, $B^\star$ and $C^\star$. For $|\alpha-\beta| < \gamma < \alpha+\beta$, there is equality in*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_A(y)1_B(x-y)1_C(x)\,dx\,dy \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_{A^\star}(y)1_{B^\star}(x-y)1_{C^\star}(x)\,dx\,dy.$$

*if and only if, up to sets of measure zero,*

$$A = a + \alpha D,\ B = b + \beta D,\ C = c + \gamma D,$$

*where $D$ is a centered ellipsoid, and $a$, $b$ and $c = a + b$ are vectors in $\mathbb{R}^n$.*

## 3. Orlicz chord power integral and Orlicz radial mean body

In this section, we establish the uniform convergence of the radial function associated with the Orlicz radial mean body and the continuity of the Orlicz chord power integrals, thereby deriving the relationship between the Orlicz chord power integrals and the Orlicz radial mean body.

Let $K \subset \mathbb{R}^n$ be a convex body, $\alpha > -1$ be a real number, and $G$ be an Orlicz function. Then the *Orlicz chord power integral of order* $\alpha + 1$ for $K$ is defined as

$$I_{G,\alpha+1}(K) = \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\int_{K|u^\perp} G(X_K(x,u))^{\alpha+1}d\mathcal{H}^{n-1}(x)du.$$

For a nonnegative measurable function $f \in L^G(\mathbb{R}^n) \cap \mathrm{LogConc}$ (the space of log-concave functions, ensuring that its level sets are convex bodies), its *functional Orlicz chord power integral* is defined as

$$I_{G,\alpha+1}(f) = \int_0^{\|f\|_\infty} I_{G,\alpha+1}(\{f \ge t\})dt.$$

When taking the power-type Orlicz function $G(t) = t^p$ $(p \ge 1)$, the Orlicz chord power integral reduces to the *$L^p$-type classical chord power integral*

$$I_{G,\alpha+1}(K) = \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\int_{K|u^\perp} X_K(x,u)^{\alpha+1}d\mathcal{H}^{n-1}(x)du = I_{p(\alpha+1)}(K).$$

When $p = 1$, $G(t) = t$, it directly reduces to the *standard classical chord power integral*

$$I_{G,\alpha+1}(K) = I_{\alpha+1}(K).$$

When $p = 2$, it corresponds to the $L^2$ *chord power integral*, a common special case in the classical chord Sobolev inequality.

Let $K \subset \mathbb{R}^n$ be a convex body, $G$ be an Orlicz function satisfying the regularity conditions, and $\alpha > -1$ be a parameter. Then the radial function of the *Orlicz radial mean body* $R_{G,\alpha}K$ is defined as

$$G(\rho_{R_{G,\alpha}K}(u))^\alpha = \frac{1}{|K|}\int_K G(\rho_{K-x}(u))^\alpha dx,$$

where $\rho_{R_{G,\alpha}K}(u) > 0$ for all $u \in \mathbb{S}^{n-1}$ and $\rho_{K-x}(u) = \max\{t \ge 0 \mid x + tu \in K\}$ is the radial function of the convex body $K$ with respect to the point $x$, $\rho_{R_{G,\alpha}K}(u)$ is the radial function of the Orlicz radial mean body $R_{G,\alpha}K$, uniquely determined by the above equation. When $G(t) = t^p$, it reduces to the classical $L^p$ radial mean body $R_{p,\alpha}K$

$$\rho_{R_{p,\alpha}K}(u)^{p\alpha} = \frac{1}{|K|}\int_K \rho_{K-x}(u)^{p\alpha}dx.$$

Let $\{K_j\}_{j=1}^{\infty}, K \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors, $K_j \to K \ni o$, and $K_j, K$ in Hausdorff metric.

**Theorem 3.1** (Uniform convergence)**.** *Let $G$ is an Orlicz function, $\alpha > -1$, and let convex bodies $K_j \xrightarrow{d_H} K$ with uniformly bounded volumes. Then*

$$\rho_{R_{G,\alpha}K_j} \xrightarrow{C(\mathbb{S}^{n-1})} \rho_{R_{G,\alpha}}K$$

*converges uniformly on $\mathbb{S}^{n-1}$. That is, the Orlicz radial mean body converges in the Hausdorff metric*

$$R_{G,\alpha}K_j \xrightarrow{d_H} R_{G,\alpha}K.$$

**Lemma 3.2** (Uniform boundedness)**.** *The family $\{\rho_{R_{G,\alpha}K_j}\}$ is uniformly bounded on $\mathbb{S}^{n-1}$.*

*Proof.* Since $K_j \to K$, $\{\rho_{K_j-x}(u)\}$ is uniformly bounded, that is, there exists a constant $M > 0$ such that for all $j \in \mathbb{N}$, all directions $u \in \mathbb{S}^{n-1}$, and all $x \in K_j$,

$$0 < \rho_{K_j-x}(u) \leq M.$$

By monotonicity and $\Delta_2$-condition of Orlicz function $G$,

$$\begin{aligned} G(\rho_{R_{G,\alpha}K_j}(u))^{\alpha} &= \frac{1}{|K_j|}\int_{K_j} G(\rho_{K_j-x}(u))^{\alpha}dx \\ &\leq \frac{1}{|K_j|}\int_{K_j} G(M)^{\alpha}dx \\ &\leq \frac{1}{|K_j|}\cdot G(M)^{\alpha}\cdot |K_j| \\ &= G(M)^{\alpha}. \end{aligned}$$

Strict monotonicity of $G$ gives $\rho_{R_{G,\alpha}K_j}(u) \leq M$ for all $j, u$. □

**Lemma 3.3** (Equicontinuity)**.** *The family $\{\rho_{R_{G,\alpha}K_j}\}$ is equicontinuous on $\mathbb{S}^{n-1}$.*

*Proof.* Convex bodies $K_j$ are uniformly Lipschitz, we obtain

$$|\rho_{K_j-x}(u) - \rho_{K_j-x}(v)| \leq C_1|u - v|. \tag{3.1}$$

By Lemma 3.2, all $\rho_{K_j-x}(u)$ are uniformly bounded on a bounded interval $[0, M]$, $\forall j, x, u$. Since $G \in C^1[0, +\infty)$, $G'$ is continuous, then $G'$ is bounded.

Combining with the $\Delta_2$-condition, the power-composite function $t \mapsto G(t^{\alpha})$ is still Lipschitz continuous. By the mean value theorem for functions of one variable, we have

$$|G(a^{\alpha}) - G(b^{\alpha})| = |G'(\xi)| \cdot |a^{\alpha} - b^{\alpha}|. \tag{3.2}$$

Then, using the Lipschitz estimate for the power function $t^{\alpha}$

$$|a^{\alpha} - b^{\alpha}| \leq C_2|a - b|. \tag{3.3}$$

Combining (3.2), (3.3) and (3.1), we estimate

$$\begin{aligned} |G(\rho_{K_j-x}(u))^{\alpha} - G(\rho_{K_j-x}(v))^{\alpha}| &\leq \sup_{[0,M]}|G'| \cdot C_2 \cdot |\rho_{K_j-x}(u) - \rho_{K_j-x}(v)| \\ &\leq C_1C_2 \sup_{[0,M]}|G'| \cdot |u - v| \triangleq C_3|u - v|, \end{aligned} \tag{3.4}$$

where the constant $C_3$ is independent of $j$ and $x$.

For $u, v \in \mathbb{S}^{n-1}$, by the absolute value inequality for integrals and (3.4), we have

$$\begin{aligned} \left|G(\rho_{R_{G,\alpha}K_j}(u))^{\alpha} - G(\rho_{R_{G,\alpha}K_j}(v))^{\alpha}\right| &\leq \frac{1}{|K_j|}\int_{K_j} |G(\rho_{K_j-x}(u))^{\alpha} - G(\rho_{K_j-x}(v))^{\alpha}|dx \\ &\leq \frac{1}{|K_j|}\int_{K_j} C_3|u - v|dx \\ &= C_3|u - v|. \end{aligned}$$

Thus the Lipschitz constant $C_3$ is independent of the index $j$, we obtain

$$\left|\rho_{R_{G,\alpha}K_j}(u) - \rho_{R_{G,\alpha}K_j}(v)\right| \le C_3|u-v|, \quad \forall j \in \mathbb{N},$$

for any $\varepsilon > 0$, we take $\delta = \frac{\varepsilon}{C_3}$ such that the spherical distance $|u-v| < \delta$, then for all $j$, we have

$$|\rho_{R_{G,\alpha},K_j}(u) - \rho_{R_{G,\alpha},K_j}(v)| < \varepsilon.$$

Therefore the family of functions $\{\rho_{R_\alpha,K_j}\}$ is equicontinuous on $\mathbb{S}^{n-1}$, this proof is complete. $\square$

**Lemma 3.4** (Pointwise convergence)**.** *For each $u \in \mathbb{S}^{n-1}$,*

$$G\big(\rho_{R_{G,\alpha}K_j}(u)\big)^\alpha \to G\big(\rho_{R_{G,\alpha}K}(u)\big)^\alpha.$$

*Proof.* From $K_j \to K$ and the uniform boundedness in Lemma 3.2, there exists a constant $M > 0$ such that

$$\rho_{K_j - x}(u) \le M, \quad \forall j, \forall x \in K_j.$$

Combining this with the $\Delta_2$-condition of the Orlicz function $G$, there exists a constant $C > 0$ such that

$$G(t^\alpha) \le CG(t), \quad \forall t \ge 0.$$

Therefore the integrand is dominated by a globally integrable function

$$0 \le G(\rho_{K_j-x}(u))^\alpha \le G(M)^\alpha \triangleq g(x),$$

where the constant function $g(x)$ is Lebesgue integrable on convex bodies, satisfying all conditions of the Dominated Convergence Theorem, then we have

$$\int_{K_j} G(\rho_{K_j-x}(u))^\alpha dx \to \int_K G(\rho_{K-x}(u))^\alpha dx. \tag{3.5}$$

Using (3.5), we obtain

$$\begin{aligned} G(\rho_{R_{G,\alpha}K_j}(u))^\alpha &= \frac{1}{|K_j|}\int_{K_j} G(\rho_{K_j-x}(u))^\alpha dx \\ &\xrightarrow{j\to\infty} \frac{1}{|K|}\int_K G(\rho_{K-x}(u))^\alpha dx \\ &= G(\rho_{R_{G,\alpha}K}(u))^\alpha \end{aligned}$$

That is,

$$G(\rho_{R_{G,\alpha}K_j}(u))^\alpha \to G(\rho_{R_{G,\alpha}K}(u))^\alpha$$

for every fixed $u \in S^{n-1}$, which completes the proof. $\square$

*Proof of Theorem 3.1.* By Arzelà-Ascoli theorem, uniform boundedness and equicontinuity imply that $\{\rho_{R_{G,\alpha}K_j}\}$ is relatively compact.

$G$ is strictly monotone and continuous on $[0, +\infty)$, so its inverse $G^{-1}$ exists and is continuous. Applying $G^{-1}$ and the $\frac{1}{\alpha}$-th power to both sides of the convergence equality yields

$$\rho_{R_{G,\alpha}}K_j(u) \xrightarrow{j\to\infty} \rho_{R_{G,\alpha}}K(u).$$

That is, the family of functions $\{\rho_{R,G,\alpha}K_j\}$ converges pointwise on $\mathbb{S}^{n-1}$. Pointwise convergence forces the whole sequence to converge uniformly to $\rho_{R_{G,\alpha}K}$. Thus $R_{G,\alpha}K_j \to R_{G,\alpha}K$. $\square$

**Theorem 3.5.** *For $\alpha > -1$ and a convex body $K \subset \mathbb{R}^n$, we have*

$$I_{G,\alpha+1}(K) = \frac{(\alpha+1)|K|}{n\omega_n}\int_{\mathbb{S}^{n-1}} G\left(\rho_{R_{G,\alpha}K}(u)\right)^\alpha du,$$

*where $R_{G,\alpha}K$ is the Orlicz radial mean body satisfying*

$$G\left(\rho_{R_{G,\alpha}K}(u)\right)^\alpha = \frac{1}{|K|}\int_K G\left(\rho_{K-x}(u)\right)^\alpha dx.$$

*Proof.* First, for any $t \geq 0$, by the convexity and differentiability of the Orlicz function $G$, we have the integral identity

$$G(t)^{\alpha+1} = (\alpha+1)\int_0^t G(s)^\alpha G'(s)ds.$$

Let $t = X_K(x,u)$, substituting gives

$$G\left(X_K(x,u)\right)^{\alpha+1} = (\alpha+1)\int_0^{X_K(x,u)} G(s)^\alpha G'(s)ds.$$

For a fixed direction $u \in \mathbb{S}^{n-1}$, integrate over the projection $K|u^\perp$ of $K$ onto $u^\perp$. By Fubini's theorem

$$\begin{aligned}\int_{K|u^\perp} G\left(X_K(x,u)\right)^{\alpha+1} dx &= (\alpha+1)\int_{K|u^\perp}\int_0^{X_K(x,u)} G(s)^\alpha G'(s)\,ds\,dx \\ &= (\alpha+1)\int_K G\left(\rho_{K-x}(u)\right)^\alpha dx.\end{aligned} \tag{3.6}$$

According to the definition of the Orlicz radial mean body

$$G\left(\rho_{R_{G,\alpha}K}(u)\right)^\alpha = \frac{1}{|K|}\int_K G\left(\rho_{K-x}(u)\right)^\alpha dx, \tag{3.7}$$

substituting (3.7) into the above expression (3.6) yields

$$\int_{K|u^\perp} G\left(X_K(x,u)\right)^{\alpha+1} dx = (\alpha+1)|K|\cdot G\left(\rho_{R_{G,\alpha}K}(u)\right)^\alpha. \tag{3.8}$$

Finally, substitute (3.8) into the definition of the Orlicz chord power integral

$$I_{G,\alpha+1}(K) = \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}}\int_{K|u^\perp} G\left(X_K(x,u)\right)^{\alpha+1} dxdu,$$

and we obtain complete the proof

$$I_{G,\alpha+1}(K) = \frac{(\alpha+1)|K|}{n\omega_n}\int_{\mathbb{S}^{n-1}} G\left(\rho_{R_{G,\alpha}K}(u)\right)^\alpha du,$$

which completes the proof. □

**Corollary 3.6** (Continuity)**.** *The Orlicz chord power integral*

$$I_{G,\alpha+1}(K) = \frac{(\alpha+1)|K|}{n\omega_n}\int_{\mathbb{S}^{n-1}} G\big(\rho_{R_{G,\alpha}K}(u)\big)^\alpha du$$

*is continuous under Hausdorff metric convergence*

$$I_{G,\alpha+1}(K_j) \to I_{G,\alpha+1}(K).$$

## 4. Orlicz chord Sobolev inequality

In this section, we extend the chord Sobolev inequality to the Orlicz space through our newly defined Orlicz chord power integral and draw some relevant conclusions.

**Lemma 4.1.** *Let $G$ be an Orlicz function, and $f \in L^G(\mathbb{R}^n)$ be nonnegative. Let $f^\star$ denote its symmetric decreasing rearrangement, which satisfies*

$$\int_{\mathbb{R}^n} G(f(x))dx = \int_{\mathbb{R}^n} G(f^\star(x))dx, \quad \|f\|_G = \|f^\star\|_G.$$

*Proof.* By the layer cake formula for nonnegative functions,

$$\int_{\mathbb{R}^n} g(x)dx = \int_0^\infty |\{x : g(x) > s\}|ds.$$

Applying this to $g = G\circ f$, and using the strict monotonicity of the Orlicz function $G$, we have

$$G(f(x)) > s \iff f(x) > G^{-1}(s).$$

Thus,

$$\int_{\mathbb{R}^n} G(f(x))dx = \int_0^\infty |\{x : f(x) > G^{-1}(s)\}|ds = \int_0^\infty \mu_f(G^{-1}(s))ds.$$

Substituting $t = G^{-1}(s)$, $ds = dG(t)$, we obtain

$$\int_{\mathbb{R}^n} G(f(x))dx = \int_0^\infty \mu_f(t)dG(t).$$

For $f^\star$, the same calculation gives

$$\int_{\mathbb{R}^n} G(f^\star(x))dx = \int_0^\infty \mu_{f^\star}(t)dG(t).$$

By the definition of the distribution function, we have $\mu_f(t) = \mu_{f^\star}(t)$ for all $t \geq 0$, the two integrals are equal, which completes the first equation.

From the definition of the Luxemburg norm,

$$\|f\|_G = \inf\big\{\lambda > 0 : \int_{\mathbb{R}^n} G\Big(\frac{f(x)}{\lambda}\Big)dx \leq 1\big\}.$$

Using the first result, for any $\lambda > 0$,

$$\int_{\mathbb{R}^n} G\Big(\frac{f(x)}{\lambda}\Big)dx = \int_{\mathbb{R}^n} G\Big(\frac{f^\star(x)}{\lambda}\Big)dx.$$

This implies that the sets of admissible $\lambda$ for $f$ and $f^\star$ are identical, so their infima are equal. Hence, $\|f\|_G = \|f^\star\|_G$. $\square$

**Lemma 4.2.** *Let $G$ be an Orlicz function, and nonnegative function $f \in L^G(\mathbb{R}^n)$. For $\alpha \in (0, n)$,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

*Equality holds if and only if there exists a vector $z_0 \in \mathbb{R}^n$ such that $f(x) = f^\star(x - z_0)$ for almost every $x \in \mathbb{R}^n$.*

*For $\alpha > n$ and nonnegative $f \in L^G(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

*Equality holds if and only if there exists a vector $z_0 \in \mathbb{R}^n$ such that $f(x) = f^\star(x - z_0)$ for almost every $x \in \mathbb{R}^n$.*

*Proof.* For $\alpha \in (0, n)$, let $z \in \mathbb{R}^n \setminus \{o\}$. Then we have

$$\frac{1}{|z|^q} = \int_0^\infty \mathbf{1}_{r^{-1/q}B}(z)dr.$$

Here $q = n - \alpha > 0$. Since Orlicz function $G$ is a monotonically increasing function, then

$$\begin{aligned}\min\{G(f(x)), G(f(y))\} &= \int_0^\infty \mathbf{1}_{\{G(f)\geq t\}}(x)\mathbf{1}_{\{G(f)\geq t\}}(y)dt \\ &= \int_0^\infty \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\{f\geq s\}}(y)dG(s).\end{aligned}$$

Using Fubini theorem, we obtain

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \\ &= \int_0^\infty \Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\{f\geq s\}}(y)}{|x-y|^{n-\alpha}}\,dx\,dy\Big)\,dG(s) \\ &= \int_0^\infty\int_0^\infty \Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy\Big)dG(s)\,dr.\end{aligned}$$

By the Riesz rearrangement inequality, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy \\
&\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}^\star}(x)\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}^\star}(y)\,dx\,dy,
\end{aligned} \tag{4.1}$$

which implies that

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)),G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \\
&\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)),G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy.
\end{aligned}$$

If the equality holds, then (4.1) is an equality for almost all $(s,r)\in(0,\infty)^2$. For sufficiently large $r>0$, the assumptions of Burchard theorem are satisfied. Consequently, up to sets of measure zero, we have

$$\{f\geq t\}=x_0+aB,\quad r^{1/q}B=bB.$$

Here $a,b>0$ are constants and $x_0\in\mathbb{R}^n$, which means that $f$ is a translation of $f^\star$. The first inequality holds.

For $\alpha>n$, let $z\in\mathbb{R}^n\setminus\{o\}$, we have

$$\frac{1}{|z|^q}=\int_0^\infty \mathbf{1}_{\mathbb{R}^n\setminus r^{-1/q}B}(z)dr.$$

Here $q=n-\alpha<0$. Using Fubini theorem, we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)),G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \\
&= \int_0^\infty \Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\{f\geq s\}}(y)}{|x-y|^{n-\alpha}}\,dx\,dy\Big)\,dG(s) \\
&= \int_0^\infty\int_0^\infty \Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\mathbb{R}^n\setminus r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy\Big)\,dG(s)\,dr.
\end{aligned}$$

Then

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\mathbb{R}^n\setminus r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\left(\mathbf{1}-\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\right)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy \\
&\quad -\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy.
\end{aligned}$$

By the Riesz rearrangement inequality, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy \\
&\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}^\star}(x)\mathbf{1}_{r^{-1/(n-\alpha)}B}(x-y)\mathbf{1}_{\{f\geq s\}^\star}(y)\,dx\,dy.
\end{aligned} \tag{4.2}$$

Note the following integral is unchanged during rearrangement

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathbf{1}_{\{f\geq s\}}(x)\mathbf{1}_{\{f\geq s\}}(y)\,dx\,dy,$$

which implies that

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)),G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \geq \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)),G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy.$$

If equality holds, then (4.2) is an equality for almost all $(s,r) \in (0,\infty)^2$. For sufficiently large $r > 0$, the assumptions of Burchard theorem are satisfied. Consequently, up to sets of measure zero, we have

$$\{f \geq t\} = x_0 + aB, \quad r^{1/q}B = bB.$$

Here $a, b > 0$ are constants and $x_0 \in \mathbb{R}^n$, which means that $f$ is a translation of $f^\star$. Then we complete the proof. □

**Lemma 4.3.** *Let $G$ be an Orlicz function, and nonnegative function $f \in L^G(\mathbb{R}^n)$, $f^\star$ is the radially symmetric decreasing rearrangement of $f$. Every nonnegative $f^\star \in L^G(\mathbb{R}^n)$ can be arbitrarily approximated by nonnegative finite linear combinations of characteristic functions of concentric balls*

$$\sum_{i=1}^{k} c_i \cdot 1_{B_{r_i}} \to f^\star,$$

*where $c_i \geq 0$, $r_i > 0$, $B_{r_i} = B(0, r_i)$ is the ball of radius $r_i$ centered at the origin.*

*Proof.* For any decreasing right-continuous function $f^\star(r)$, we take partition points

$$0 = r_0 < r_1 < r_2 < \cdots < r_k, \quad a_1 > a_2 > \cdots > a_k \geq 0$$

and construct the step decreasing radial function

$$f_k(x) = \sum_{i=1}^{k} a_i \cdot 1_{B_{r_i} \setminus B_{r_{i-1}}}(x). \tag{4.3}$$

Using the Spherical aberration decomposition $1_{B_{r_i} \setminus B_{r_{i-1}}} = 1_{B_{r_i}} - 1_{B_{r_{i-1}}}$, substituting to (4.3), we obtain

$$f_k(x) = a_1 1_{B_{r_1}} + (a_2 - a_1)1_{B_{r_2}} + \cdots + (a_k - a_{k-1})1_{B_{r_k}} = \sum_{i=1}^{k} c_i 1_{B_{r_i}}(x),$$

by monotonicity $a_1 > a_2 > \cdots > a_k$, all coefficients $c_i = a_i - a_{i-1} \geq 0$. For any $x \in \mathbb{R}^n$,

$$f^\star(x) = \int_0^{f^\star(x)} dt = \int_0^\infty 1_{\{f^\star > t\}}(x)dt = \int_0^\infty 1_{\{R(t) > |x|\}} dt = \int_0^\infty 1_{B_{R(t)}}(x)dt,$$

for any $t \geq 0$, by equimeasurability, there exists a unique radius

$$R(t) = \Big(\frac{|\{f > t\}|}{\omega_n}\Big)^{1/n} \geq 0,$$

such that the ball $B_{R(t)}$ satisfies $|B_{R(t)}| = |\{f^\star > t\}|$. We can know $f^\star(r)$ is a nonnegative integral superposition of the family of characteristic functions of concentric balls.

By the property of monotone functions, the decreasing function $f^\star(r)$ has a decreasing sequence of step functions $f_k$, such that $f_k \to f^\star$ converging pointwise almost everywhere.

Since Orlicz functions satisfy $\Delta_2$-condition, we can use the Dominated convergence theorem, then

$$\lim_{k\to\infty} \|f_k - f^\star\|_{L^G} = 0.$$

Each $f_k(x) = \sum_{i=1}^k c_i 1_{B_{r_i}}(x)$ with $c_i \geq 0$ is a nonnegative finite linear combination of characteristic functions of concentric balls, we have

$$f_k(x) = \sum_{i=1}^{k} c_i \cdot 1_{B_{r_i}}(x) \to f^\star(x).$$

where $c_i = a_i - a_{i-1} \geq 0$, $r_i > 0$, $B_{r_i} = B(0, r_i)$ is the ball of radius $r_i$ centered at the origin. Every nonnegative $f^\star \in L^G(\mathbb{R}^n)$ can be arbitrarily approximated by nonnegative finite linear combinations of characteristic functions of concentric balls, which completes the proof. □

Because of the particularity of Orlicz spaces and Orlicz functions, we present the following theorem with its proof under the condition that the symmetric decreasing rearrangement function $f^\star$ of the Orlicz function $f$ can be expressed as a constant multiple of the characteristic function of the unit ball.

**Theorem 4.4.** *Let $G$ be an Orlicz function, for $\alpha \in (0,n)$, and $f \in L^G(\mathbb{R}^n)$ be nonnegative. If the symmetric decreasing rearrangement function $f^\star$ of the Orlicz function $f$ can be expressed as a constant multiple of the characteristic function of the unit ball. Then there exists an* optimal constant *$\sigma_{G,n,\alpha} > 0$ and $0 < \|f^\star\|_G < 1$ such that*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \le \sigma_{G,n,\alpha}\|f\|_G.$$

*When $\|f^\star\|_G \ge 1$,*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \le \sigma_{G,n,\alpha}\|f\|_G^p.$$

*where $p > 1$ is given by* (2.1) *and $\sigma_{G,n,\alpha} > 0$ is a constant depending only on the Orlicz function $G$, $n$ and $\alpha$.*

*Proof.* Since $G$ is a strictly increasing function, we have

$$\min\{G(a), G(b)\} = G(\min\{a,b\}),$$

and $G$ is nondecreasing.

When $\alpha \in (0,n)$, by the rearrangement inequality, it suffices to prove the inequality for radially symmetric decreasing function $f^\star$, as the chord integral is maximized by rearrangements and the norm is invariant.

Let $f^\star = c\cdot 1_{B_{r_i}}$, where $B = B_r(0)$ is a ball centered at the origin with radius $r$, and $c > 0$. The volume $|B| = \omega_n r^n$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. Without loss of generality, we take $f^\star(x) = \min\{f^\star(x), f^\star(y)\}$, then $G(f^\star(x)) = G(c)\cdot 1_B(x)$, and the chord integral becomes

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{G(\min\{f^\star(x), f^\star(y)\})}{|x-y|^{n-\alpha}}\,dx\,dy \\ &= \iint_{B\times B} \frac{G(c)}{|x-y|^{n-\alpha}}\,dx\,dy.\end{aligned} \tag{4.4}$$

By the change of variables $x = r\xi$, $y = r\eta$, $\xi,\eta \in B_1(0)$, we obtain

$$\begin{aligned}\iint_{B\times B} |x-y|^{\alpha-n}\,dx\,dy &= \iint_{B_1\times B_1} (r|\xi-\eta|)^{\alpha-n}\cdot(r^n d\xi)\cdot(r^n d\eta) \\ &= r^{\alpha-n}\cdot\mathbb{R}^n\times\mathbb{R}^n\cdot\iint_{B_1\times B_1}|\xi-\eta|^{\alpha-n}d\xi d\eta \\ &= r^{n+\alpha}\cdot C_{n,\alpha}^{(0)},\end{aligned}$$

where $C_{n,\alpha}^{(0)} = \iint_{B_1\times B_1}|\xi-\eta|^{\alpha-n}d\xi d\eta$ is a constant depending only on $n$ and $\alpha$. Substituting $r = (|B|/\omega_n)^{1/n}$, we obtain

$$\iint_{B\times B} |x-y|^{\alpha-n}\,dx\,dy = |B|^{\frac{n+\alpha}{n}}\cdot\omega_n^{-\frac{n+\alpha}{n}}\cdot C_{n,\alpha}^{(0)} = C_{n,\alpha}\cdot|B|^{\frac{n+\alpha}{n}}, \tag{4.5}$$

where $C_{n,\alpha} = C_{n,\alpha}^{(0)}\omega_n^{-(n+\alpha)/n} > 0$ is a constant depending only on $n$ and $\alpha$.

Using spherical coordinates and the convolution formula, the integral over the unit ball can be expressed as

$$C_{n,\alpha}^{(0)} = \iint_{B_1\times B_1}|\xi-\eta|^{\alpha-n}d\xi d\eta = \omega_n\int_0^2\int_0^{\sqrt{1-(t/2)^2}} t^{\alpha-1}(1-t^2/4)^{\frac{n-2}{2}}\,ds\,dt.$$

A more compact form can be obtained via the spherical integral formula for Riesz potentials

$$C_{n,\alpha}=\frac{\omega_n^2 2^\alpha \Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(n+\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)},$$

where $\omega_n=\pi^{n/2}/\Gamma(n/2+1)$ the volume of the unit ball in $\mathbb{R}^n$, and $\Gamma(\cdot)$ the Gamma function. This formula is derived from the radial decomposition of spherically symmetric integrals and holds for $\alpha>0$, $\alpha\neq n$.

We now compute the Luxemburg norm of the ball characteristic function $f^\star=c\cdot\mathbf{1}_B$. Recall the definition of the Luxemburg norm

$$\|f\|_G=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}G\Big(\frac{f(x)}{\lambda}\Big)dx\le 1\right\}. \tag{4.6}$$

For $f^\star=c\cdot\mathbf{1}_B$, we have

$$\int_{\mathbb{R}^n}G\left(\frac{f^\star(x)}{\lambda}\right)dx=\int_B G\Big(\frac{c}{\lambda}\Big)dx+\int_{\mathbb{R}^n\setminus B}0dx=|B|\cdot G\Big(\frac{c}{\lambda}\Big),$$

where $G(f^\star(x)/\lambda)=G(c/\lambda)$ for $x\in B$ and $G(0)=0$ otherwise. Since $G$ is strictly increasing, the infimum is attained when the integral equals 1,

$$|B|G\left(\frac{c}{\lambda}\right)=1.$$

Applying the inverse function $G^{-1}$ to both sides, we obtain

$$\frac{c}{\lambda}=G^{-1}\Big(\frac{1}{|B|}\Big),$$

which yields

$$\|f^\star\|_G=\lambda=\frac{c}{G^{-1}\Big(\frac{1}{|B|}\Big)}=c\Big(G^{-1}\big(\frac{1}{|B|}\big)\Big)^{-1}. \tag{4.7}$$

We take $t=G^{-1}\big(\frac{1}{|B|}\big)>0$ and obtain the Luxemburg norm of $f^\star=c\cdot 1_B$ as

$$\|f^\star\|_G=c\Big(G^{-1}\Big(\frac{1}{|B|}\Big)\Big)^{-1}=\frac{c}{t},$$

substituting into the chord integral, we have the following two cases.

When $0<\|f^\star\|_G<1$, by Lemma 2.2(5), we obtain

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\min\{G(f^\star(x)),G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy&=G(c)C_{n,\alpha}|B|^{\frac{n+\alpha}{n}}\\&=G(\|f^\star\|_G\cdot t)C_{n,\alpha}G(t)^{-\frac{n+\alpha}{n}}\\&\le C_{n,\alpha}G(t)^{-\frac{\alpha}{n}}\|f^\star\|_G\\&=\sigma_{G,n,\alpha}\|f^\star\|_G,\end{aligned}$$

where $\sigma_{G,n,\alpha}=C_{n,\alpha}G(t)^{-\frac{\alpha}{n}}>0$ is the best constant achieved by $f^\star=c\cdot 1_B$.

When $\|f^\star\|_G\ge 1$, by Lemma 2.3, we obtain

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\min\{G(f^\star(x)),G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy&=G(c)C_{n,\alpha}|B|^{\frac{n+\alpha}{n}}\\&=G(\|f^\star\|_G\cdot t)C_{n,\alpha}G(t)^{-\frac{n+\alpha}{n}}\\&\le C_{n,\alpha}G(t)^{-\frac{\alpha}{n}}\|f^\star\|_G^p\\&=\sigma_{G,n,\alpha}\|f^\star\|_G^p,\end{aligned}$$

where $p>1$ is given by (2.1) and $\sigma_{G,n,\alpha}>0$ is the best constant achieved by $f^\star=c\cdot 1_B$.

For any non-negative $f \in L^G(\mathbb{R}^n)$, by the rearrangement inequality (4.2) and lemma 4.1, when $0 < \|f^\star\|_G < 1$,

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &\le \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy\\ &\le \sigma_{G,n,\alpha}\|f^\star\|_G = \sigma_{G,n,\alpha}\|f\|_G.\end{aligned}$$

and when $\|f^\star\|_G \ge 1$,

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &\le \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy\\ &\le \sigma_{G,n,\alpha}\|f^\star\|_G^p = \sigma_{G,n,\alpha}\|f\|_G^p.\end{aligned}$$

By the reflexivity of $L^G(\mathbb{R}^n)$, the unit ball $\{f \in L^G : \|f\|_G = 1\}$ is weakly compact. The chord integral is a weakly continuous functional, so it attains its maximum on the unit ball, which is the best constant $\sigma_{G,n,\alpha}$, achieved by the ball characteristic function. □

**Theorem 4.5.** *Let $G$ be an Orlicz function, for $\alpha > n$, and $f \in L^G(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ be nonnegative. If the symmetric decreasing rearrangement function $f^\star$ of the Orlicz function $f$ can be expressed as a constant multiple of the characteristic function of the unit ball. Then there exists an* optimal constant $\sigma_{G,n,\alpha} > 0$ *such that*

$$\log \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)), G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy \ge \tilde{C}_{n,\alpha} + C_G \log\Big(\|f\|_\infty^{-\frac{\alpha}{n}}\|f\|_G^{\frac{n+\alpha}{n}}\Big),$$

*where $C_G = \frac{2\log C}{\log 2} > 1$, $\tilde{C}_{n,\alpha}$ depends only on $n$ and $\alpha$.*

*Proof.* When $\alpha > n$, $f \in L^G(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ with compact support, then the growth of the kernel $|x-y|^{\alpha-n}$ has been controlled. For $f^\star = c\cdot 1_B$, where $B$ is a ball in $\mathbb{R}^n$ and $c > 0$, we have $\|f^\star\|_\infty = c$, using the same method of (4.4) and (4.5), the chord integral is still

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &= G(c)\iint_{B\times B}|x-y|^{\alpha-n}\,dx\,dy\\ &= G(c)C_{n,\alpha}|B|^{\frac{n+\alpha}{n}}.\end{aligned}\tag{4.8}$$

where $C_{n,\alpha} > 0$ depends only on $n$ and $\alpha$.

Recalling the Luxemburg norm of $f$, by (4.7), we obtain

$$\|f^\star\|_G = \frac{c}{G^{-1}(\frac{1}{|B|})},$$

and the $L^\infty$ norm $\|f^\star\|_\infty = c$. We now express the right-hand side in terms of $\|f^\star\|_G$ and $\|f^\star\|_\infty$. From the Luxemburg norm formula (4.6), we have

$$G^{-1}\Big(\frac{1}{|B|}\Big) = \frac{c}{\|f^\star\|_G},$$

so

$$\frac{1}{|B|} = G\Big(\frac{c}{\|f^\star\|_G}\Big) = G\Big(\frac{\|f^\star\|_\infty}{\|f^\star\|_G}\Big).$$

Substituting $|B|$ into (4.8), using $G(1) = 1$ and lemma 2.5, we obtain

$$\begin{aligned}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)), G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &= G(\|f^\star\|_\infty)C_{n,\alpha}\Big(\frac{1}{G(\frac{\|f^\star\|_\infty}{\|f^\star\|_G})}\Big)^{\frac{n+\alpha}{n}}\\ &\ge C_{n,\alpha}\|f\|_\infty^{2p}G(1)\|f^\star\|_\infty^{(-\frac{n+\alpha}{n})2p}\|f^\star\|_G^{(\frac{n+\alpha}{n})2p}G(1)\\ &= C_{n,\alpha}\|f^\star\|_\infty^{(-\frac{\alpha}{n})2p}\|f^\star\|_G^{(\frac{n+\alpha}{n})2p}\\ &= C_{n,\alpha}(\|f^\star\|_\infty^{-\frac{\alpha}{n}}\|f^\star\|_G^{\frac{n+\alpha}{n}})^{2p}.\end{aligned}\tag{4.9}$$

Taking the logarithm of both sides of (4.9) simultaneously, we have

$$\begin{aligned}\log \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)),G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &\ge \log\Big(C_{n,\alpha}(\|f^\star\|_\infty^{-\frac{\alpha}{n}}\|f^\star\|_G^{\frac{n+\alpha}{n}})^{2p}\Big)\\ &= \log C_{n,\alpha} + 2p\log\Big(\|f^\star\|_\infty^{-\frac{\alpha}{n}}\|f^\star\|_G^{\frac{n+\alpha}{n}}\Big)\\ &= \tilde{C}_{n,\alpha} + C_G\log\Big(\|f^\star\|_\infty^{-\frac{\alpha}{n}}\|f^\star\|_G^{\frac{n+\alpha}{n}}\Big),\end{aligned}$$

where $C_G = 2p = \frac{2\log C}{\log 2} > 1$, $\tilde{C}_{n,\alpha}$ depends only on $n$ and $\alpha$.

For a general $f \in L^G \cap L^\infty$, when $\alpha > n$, the rearrangement inequality (4.2) still holds, and $\|f^\star\|_G = \|f\|_G$, $\|f^\star\|_\infty = \|f\|_\infty$, so the inequality follows.

$$\begin{aligned}\log \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f(x)),G(f(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy &\ge \log \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\min\{G(f^\star(x)),G(f^\star(y))\}}{|x-y|^{n-\alpha}}\,dx\,dy\\ &\ge \tilde{C}_{n,\alpha} + C_G\log\Big(\|f^\star\|_\infty^{-\frac{\alpha}{n}}\|f^\star\|_G^{\frac{n+\alpha}{n}}\Big)\\ &= \tilde{C}_{n,\alpha} + C_G\log\Big(\|f\|_\infty^{-\frac{\alpha}{n}}\|f\|_G^{\frac{n+\alpha}{n}}\Big).\end{aligned}$$

Equality holds if and only if $f = f^\star$ (radially symmetric) and $f^\star$ is a ball characteristic function, i.e., $f^\star = c\cdot 1_B$. So we completed the proof. □

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