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REGULARIZED SOLUTIONS FOR AN INVERSE PROBLEM GOVERNED BY A PARABOLIC STEKLOV EQUATION

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Abstract. In this article we study an inverse parameter identification problem governed by a nonlinear parabolic equation with a Steklov-type boundary condition, where the unknown parameter appears simultaneously in the source term, the boundary nonlinearity, and the initial datum. We first establish the global well-posedness of the forward problem and prove the Lipschitz continuity of the solution map with respect to the parameter. We then formulate the inverse problem as a Tikhonov-regularized minimization problem and establish existence of regularized solutions, stability with respect to noisy data, and convergence as the noise level tends to zero, together with an explicit reconstruction error estimate under a coercivity condition on the parameter-to-observation map. The theoretical results are illustrated by numerical experiments on a one-dimensional prototype, confirming the predicted convergence rates.

1. Introduction and problem statement

Parabolic partial differential equations (PDEs) constitute a fundamental framework for modeling time-dependent diffusion phenomena arising in physics, engineering, and applied sciences. Their relevance is particularly pronounced in situations where boundary interactions significantly influence the evolution of the system, such as heat transfer with nonlinear exchange laws, mass transport across reactive interfaces, and phase transition processes. Classical references such as [19, 12] provide a comprehensive theoretical foundation for these models and their analytical properties.

Inverse problems for parabolic equations arise naturally in many scientific and engineering contexts, including heat conduction, chemical diffusion, and biological processes. The study of such problems with nonlinear boundary conditions has attracted considerable attention due to the complex interactions between domain dynamics and boundary effects [5, 8, 26]. Despite these advances, challenges persist, particularly in establishing stability and uniqueness results for parameter identification in time-dependent settings [13, 25].

Understanding the influence of nonlinear boundary interactions, such as Steklov-type conditions, is essential for accurately reconstructing unknown parameters, a problem that has been partially addressed in recent studies [15, 4]. Among the various types of boundary conditions, nonlinear Steklov-type conditions are of particular interest, as they model boundary fluxes that depend explicitly on the state variable. These conditions arise in applications involving surface reactions, feedback control mechanisms, and energy exchange processes. Mathematically, they introduce additional challenges by coupling boundary dynamics with the internal evolution of the system. A substantial body of literature has studied parabolic problems with nonlinear or dynamical boundary conditions, focusing on qualitative properties such as global existence, blow-up phenomena, and asymptotic behavior of solutions [2, 3, 9, 23].

In recent years, energy-based methods, including variational techniques and potential well theory, have proven effective in analyzing nonlinear parabolic equations. These approaches allow for

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precise characterization of long-time solution behavior and provide criteria distinguishing global existence from finite-time blow-up [10, 18]. However, most existing contributions concentrate on the direct problem, assuming that all model parameters, including source terms, boundary interactions, and initial conditions are known.

Several approaches have been proposed for parameter identification in parabolic equations. Iterative regularization methods have proven effective for nonlinear inverse problems [16], while Carleman estimate-based techniques provide global uniqueness and stability for coefficient reconstruction [17]. Multi-frequency inverse strategies improve reconstruction accuracy in the presence of noisy data [1]. In addition, studies specifically addressing nonlinear boundary conditions have highlighted their influence on inverse problems [15, 4].

Following the approaches in [5, 26], inverse problems can be formulated as optimization problems over admissible parameter sets. Regularization terms stabilize the inversion process, inspired by stability estimates in [1, 25], and the existence of optimal parameters minimizing the cost functional can be guaranteed via compactness and lower semicontinuity arguments [13, 17]. The novelty of our work lies in simultaneously addressing the unknown coefficient in the source term and its influence on the nonlinear Steklov boundary, extending previous frameworks and enabling more realistic modeling of physical processes [4, 8].

In realistic applications, model parameters are often uncertain, partially known, or subject to measurement errors, especially when associated with boundary interactions or external sources. This motivates the study of inverse problems, whose goal is to identify unknown parameters from partial observations of the system state. Such inverse problems are well known to be ill-posed in the sense of Hadamard, as they may lack uniqueness and stability under perturbations of the data [11, 14]. The challenge intensifies when an unknown parameter influences multiple components simultaneously, including source terms, boundary conditions, and initial data. This introduces strong nonlinear couplings between spatial diffusion, temporal dynamics, and boundary effects, complicating both the analysis of the direct problem and the stability of the associated inverse problem.

Motivated by these challenges, the present work studies an inverse parameter identification problem governed by a parabolic equation with a nonlinear Steklov boundary condition. In contrast to most existing works, the unknown parameter in our model appears simultaneously in the source term, the boundary nonlinearity, and the initial condition, reflecting more realistic modeling situations and introducing additional analytical difficulties. More precisely, in [18], the direct problem considered was: find u such that

$$\begin{gathered}
u_t - \Delta u + u = f \quad \text{in } \Omega \times (0,\infty), \\
\frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_1 \times (0,\infty), \\
u = 0 \quad \text{on } \Gamma_2 \times (0,\infty), \\
u(\cdot,0) = u_0 \quad \text{in } \Omega.
\end{gathered} \tag{1.1}$$

In this work, we consider the following parameter-dependent problem:

Problem (P_p)**:** For a given parameter $p \in \mathcal{P}$, find $u = u(p)$ satisfying

$$\begin{gathered}
u_t - \Delta u + u = f(p) \quad \text{in } \Omega \times (0,\infty), \\
\frac{\partial u}{\partial \nu} = g(p,u) \quad \text{on } \Gamma_1 \times (0,\infty), \\
u = 0 \quad \text{on } \Gamma_2 \times (0,\infty), \\
u(\cdot,0) = u_0(p) \quad \text{in } \Omega,
\end{gathered} \tag{1.2}$$

where $p \in \mathcal{P}$ is a parameter belonging to a Banach space $\mathcal{P}$, and

$$g : \mathcal{P} \times \mathbb{R} \to \mathbb{R}, \quad f : \mathcal{P} \times \Omega \times (0,\infty) \to \mathbb{R}, \quad u_0 : \mathcal{P} \times \Omega \to \mathbb{R}$$

are given functions. The objective is to identify the parameter p from partial observations of $u(p)$.

The present work is motivated by the growing interest in inverse parameter identification problems for nonlinear partial differential equations. Although the analysis of forward nonlinear parabolic problems with Steklov boundary conditions has received considerable attention, the literature on inverse problems has mainly focused on the identification of source terms, boundary coefficients, or diffusion parameters for linear or semilinear models, with particular emphasis on uniqueness, stability estimates, regularization techniques, and numerical reconstruction; see, for example, [6, 8, 11, 14, 16]. In contrast, inverse parameter identification problems governed by nonlinear parabolic Steklov equations remain largely unexplored.

The novelty of the present work lies in the formulation and analysis of an inverse parameter identification problem in which the unknown parameter simultaneously influences the interior source term, the nonlinear Steklov boundary condition, and the initial datum. The main contributions of this paper can be summarized as follows. We first establish the global well-posedness of Problem (P_p) by proving local existence and uniqueness of weak solutions. We then investigate the continuous and Lipschitz dependence of the state $u(p)$ on the parameter p, and prove the Lipschitz continuity of the solution map $p \mapsto u(p)$ in the space $X = L^2(0,T;V) \cap H^1(0,T;V')$. Next, we develop a complete Tikhonov regularization framework for the inverse problem, establishing existence of regularized solutions (Theorem 4.6), stability with respect to noisy data (Theorem 4.7), convergence to an $\mathcal{R}$-minimizing solution as $\delta \to 0$ (Theorem 4.8), and an explicit reconstruction error estimate under a coercivity condition on the parameter-to-observation map (Theorem 4.11). Finally, the theoretical predictions are validated through numerical experiments on a one-dimensional prototype, yielding an empirical convergence rate of order $\mathcal{O}(\delta^{1.13})$ under the Morozov discrepancy principle, consistent with the theoretical bound $\mathcal{O}(\sqrt{\delta})$.

This article is organized as follows. Section 2 establishes the global well-posedness of Problem (P_p). Section 3 analyzes the dependence of the solution on the parameter and formulates the abstract inverse problem. Section 4 develops the Tikhonov regularization theory for the identification problem. Section 5 presents the numerical experiments. Concluding remarks and perspectives for future work are collected at the end of the paper.

2. Existence of global weak solutions

In this section, we establish the existence of a global weak solution for Problem (P_p). To this end, we first introduce additional notation and formulate the assumptions required for our analysis.

To establish the well-posedness of Problem (P_p), we need the following hypotheses.

(H1) The source term $f : \mathcal{P} \times \Omega \times (0,\infty) \to L^2(\Omega)$ satisfies

$$f(p,\cdot) \in L^2(0,\infty; L^2(\Omega)) \quad \text{for all } p \in \mathcal{P}.$$

(H2) The boundary function $g : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ satisfies:

(i) For each $p \in \mathcal{P}$, the mapping $u \mapsto g(p,u)$ is continuous and $g(p,0) = 0$;

(ii) There exists a constant $L_g > 0$ such that

$$|g(p,u) - g(p,v)| \le L_g |u - v|, \quad \forall u, v \in \mathbb{R},\ \forall p \in \mathcal{P};$$

(iii). There exists a constant $c_g > 0$ such that

$$|g(p,u)| \le c_g, \quad \forall u \in \mathbb{R},\ \forall p \in \mathcal{P}.$$

(H3) For all $p \in \mathcal{P}$, the initial data satisfy $u_0(p) \in L^2(\Omega)$.

(H4) (smallness condition) $\kappa := 1 - L_g C_{tr}^2 > 0$.

In this work, we consider the variational space

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_2\}$$

endowed with the inner product and norm

$$(u,v)_V = (\nabla u, \nabla v)_{L^2(\Omega)^d}, \quad \|v\|_V = \Big(\big(\nabla v, \nabla v\big)_{L^2(\Omega)^d} \Big)^{1/2}. \tag{2.1}$$

The duality pairing between V' and V is denoted by $\langle\cdot,\cdot\rangle_{V'\times V}$. Since $\operatorname{meas}(\Gamma_1)>0$, the Friedrichs-Poincaré inequality holds. Consequently, there exists a constant $c_P>0$, depending only on Ω and Γ_1, such that

$$\|v\|_{H^1(\Omega)} \le c_P \|\nabla v\|_{L^2(\Omega)^d}, \quad \forall v\in V. \tag{2.2}$$

Therefore, the norms $\|\cdot\|_V$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on V, and thus $(V,\|\cdot\|_V)$ is a real Hilbert space. Furthermore, by the Sobolev trace theorem, there exists a constant $C_{\rm tr}>0$, depending only on Ω and Γ_1, such that

$$\|v\|_{L^2(\Gamma_1)} \le C_{\rm tr}\|v\|_V, \quad \forall v\in V. \tag{2.3}$$

We first establish the local existence and uniqueness result of Problem (P_p).

Theorem 2.1. *Let $p\in\mathcal{P}$. Assume that* (H1)-(H3) *hold. Then, Problem (P_p) has at least one weak local solution $u=u_p$ in $[0,T)$ for $T>0$, i.e.*

$$\langle u_t(t),v\rangle_{V',V} + \int_\Omega \nabla u(t)\cdot\nabla v\,dx + \int_\Omega u(t)v\,dx = \int_\Omega f(t,p)v\,dx + \int_{\Gamma_1} g(p,u(t))v\,d\sigma, \tag{2.4}$$

for $t\in[0,T)$, for all $v\in V$, which satisfies

$$u\in L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega)), \quad u_t\in L^2(0,T;V'). \tag{2.5}$$

Moreover, if (H4) *holds, this solution is unique.*

The proof of Theorem 2.1 proceeds in several steps. First, let $\eta\in L^2(0,T;L^2(\Gamma_1))$ and fix $T>0$. Consider the following linearized problem: find u^η such that

$$\langle u^\eta_t(t),v\rangle_{V',V} + \int_\Omega \nabla u^\eta(t)\cdot\nabla v\,dx + \int_\Omega u^\eta(t)v\,dx = \int_\Omega f(p,t)v\,dx + \int_{\Gamma_1}\eta(t)v\,d\sigma, \tag{2.6}$$

for all $v\in V$ and almost every $t\in(0,T)$, with the initial condition

$$u^\eta(0)=u_0(p). \tag{2.7}$$

Lemma 2.2. *Let $p\in\mathcal{P}$, $T>0$, and $\eta\in L^2(0,T;L^2(\Gamma_1))$ be given. Then, the linear problem* (2.6)–(2.7) *admits a unique weak solution u^η. Moreover, there exists a constant $c>0$, independent of η, such that*

$$\begin{aligned} &\|u^\eta(t)\|_{L^2(\Omega)} + \int_0^t \|u^\eta(s)\|_V\,ds \\ &\le c\Big(\|u_0(p)\|_{L^2(\Omega)} + \int_0^t \|f(s,p)\|_{L^2(\Omega)}\,ds + \int_0^t \|\eta(s)\|_{L^2(\Gamma_1)}\,ds\Big), \end{aligned} \tag{2.8}$$

for all $t\in[0,T)$.

Proof. Let us define the linear functional $F_\eta: V\to\mathbb{R}$ and the bilinear form $a: V\times V\to\mathbb{R}$ as

$$F^p_{\eta(t)}(v) := (f(p,t),v)_{L^2(\Omega)} + \int_{\Gamma_1}\eta(t)v\,d\sigma, \quad \forall v\in V. \tag{2.9}$$

$$a(u,v) := \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega uv\,dx, \quad \forall u,v\in V. \tag{2.10}$$

It is straightforward to verify that $a(\cdot,\cdot)$ is continuous and coercive on V. Hence, by the classical framework of Lions [20, Chapter 3], the linear system

$$\langle u^\eta_t(t),v\rangle_{V',V} + a(u^\eta(t),v) = F^p_{\eta(t)}(v), \quad u^\eta(0)=u_0(p),$$

admits a unique weak solution $u^\eta\in L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega))$ with $u^\eta_t\in L^2(0,T;V')$. This solution coincides with the unique solution of the linearized problem (2.6)–(2.7), which concludes the proof of Lemma 2.2. ☐

Next, let $p\in\mathcal{P}$. We define the operator $\Phi_p: L^2(0,T;L^2(\Gamma_1))\to L^2(0,T;L^2(\Gamma_1))$ by

$$\Phi_p(\eta(t)) = g\big(p,u^\eta(t)\big), \quad \text{for all } t\in[0,T), \tag{2.11}$$

where u^η denotes the solution of the problem (2.6)-(2.7) corresponding to η. Then, we have the following result.

Lemma 2.3. *The operator Φ_p has at least one fixed point $\eta^* \in L^2(0,T;L^2(\Gamma_1))$.*

Proof. The proof is based on the Leray-Schauder fixed point theorem. To this end, we introduce the homotopy equation

$$\eta = \lambda\Phi_p(\eta) = \lambda g(p, u^\eta), \quad \lambda \in [0,1].$$

Observe that a fixed point of the operator Φ_p corresponds to the case $\lambda = 1$. To apply the Leray-Schauder theorem, we verify that Φ_p is continuous and compact, and that all solutions of the homotopy equation are uniformly bounded. To this end, let (η_n) be a bounded sequence in $L^2(0,T;L^2(\Gamma_1))$, and let (u^{η_n}) denote the corresponding solutions of the problem $\{(2.6)-(2.7)\}$. From estimate (2.8), it follows that the sequence (u^{η_n}) is uniformly bounded in $L^2(0,T;V)$.

By the Aubin-Lions compactness theorem, together with the compactness of the trace operator from $H^1(\Omega)$ into $L^2(\Gamma_1)$, there exists a subsequence, still denoted by (u^{η_n}), such that

$$u^{\eta_n} \to u \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)).$$

Using the continuity of the function g with respect to its second argument, we deduce that

$$\Phi_p(\eta_n) = g(p, u^{\eta_n}) \to g(p,u) = \Phi_p(\eta) \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)).$$

Therefore, the operator Φ_p is continuous and compact. Furthermore, since the function g is globally bounded by a constant $c_g > 0$, we obtain the estimate

$$\|\Phi_p(\eta)\|_{L^2(0,T;L^2(\Gamma_1))} = \|g(p,u^\eta)\|_{L^2(0,T;L^2(\Gamma_1))} \le c_g \sqrt{T\, meas(\Gamma_1)}.$$

Consequently, for every $\lambda \in [0,1]$, any solution of the homotopy equation

$$\eta = \lambda\Phi_p(\eta)$$

remains uniformly bounded in $L^2(0,T;L^2(\Gamma_1))$.

The assumptions of the Leray-Schauder fixed point theorem are therefore satisfied. Hence, there exists at least one fixed point $\eta^* \in L^2(0,T;L^2(\Gamma_1))$ such that

$$\eta^* = \Phi_p(\eta^*).$$

This completes the proof. □

Finally, we have all the ingredients to provide the proof of Theorem 2.1.

Proof of Theorem 2.1. For the existence part, let $p \in \mathcal{P}$ and let η_p^* be a fixed point of the operator Φ_p. We denote by $u^* := u_{\eta_p^*}$ the solution of problem (2.6)-(2.7) corresponding to $\eta = \eta_p^*$. By the definition of the operator Φ_p, we have

$$\eta_p^*(t) = g(p, u^*(t)) \quad \text{for } t \in [0,T).$$

Combining this relation with the definition of problem $\{(2.6)-(2.7)\}$, we conclude that u^* satisfies the weak formulation of Problem (P_p). Hence, u^* is a weak solution of (P_p), which establishes the existence part of Theorem 2.1.

For the uniqueness part, let u_1^p and u_2^p be two weak solutions of Problem (P_p) corresponding to the same parameter $p \in \mathcal{P}$. Define $w := u_1^p - u_2^p$. Then, by (2.4), this function satisfies

$$\langle w_t(t), v\rangle_{V',V} + \int_\Omega \nabla w(t)\cdot\nabla v\,dx + \int_\Omega w(t)v\,dx = \int_{\Gamma_1} \big(g(p,u_1^p(t)) - g(p,u_2^p(t))\big)v\,d\sigma, \qquad (2.12)$$

for all $v \in V$ and almost every $t \in (0,T)$, together with the initial condition $w(0) = 0$.

Choosing $v = w(t)$ in (2.12), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2(\Omega)}^2 + \|w(t)\|_V^2 + \|w(t)\|_{L^2(\Omega)}^2 = \int_{\Gamma_1} \big(g(p,u_1^p(t)) - g(p,u_2^p(t))\big)w(t)\,d\sigma.$$

Using assumption (H2)(ii) and the trace inequality, we estimate

$$\Big|\int_{\Gamma_1} \big(g(p,u_1^p) - g(p,u_2^p)\big)w\,d\sigma\Big| \le L_g\|w\|_{L^2(\Gamma_1)}^2 \le L_g\, C_{\rm tr}^2\|w\|_V^2.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2(\Omega)}^2 + \big(1 - L_g C_{\rm tr}^2\big)\|w(t)\|_V^2 + \|w(t)\|_{L^2(\Omega)}^2 \le 0.$$

Using that $L_g\, C_{\rm tr}^2 < 1$ and $w(0) = 0$, it follows that $\|w(t)\|_{L^2(\Omega)} = 0$ for all $t \in [0, T)$. This proves the uniqueness of the weak solution and completes the proof. □

We now investigate the long-time behavior of the weak solution obtained in the previous result. More precisely, we show that the local solution can be extended to all times under suitable assumptions on the source term and the boundary nonlinearity. The following theorem ensures that the weak solution of Problem (P_p) does not blow up in finite time and therefore exists for all $t \geq 0$.

Theorem 2.4. *Let $p \in \mathcal{P}$ and assume* (H1)-(H4) *hold. Then the weak solution of Problem (P_p) given by Theorem 2.1 exists for all $t \geq 0$, is unique, and satisfies*

$$u_p \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; V), \quad \partial_t u_p \in L^2(0,\infty; V'),$$

together with the explicit estimate, for every $t \geq 0$,

$$\begin{aligned} &\|u_p(t)\|_{L^2(\Omega)}^2 + 2\kappa \int_0^t \|u_p(s)\|_V^2 \, ds + \int_0^t \|u_p(s)\|_{L^2(\Omega)}^2 \, ds \\ &\leq \|u_0(p)\|_{L^2(\Omega)}^2 + \int_0^\infty \|f(p,s)\|_{L^2(\Omega)}^2 \, ds. \end{aligned} \tag{2.13}$$

Proof. Let $u = u_p$ be the weak solution on the maximal interval of existence $[0, T_{\max})$ given by Theorem 2.1. Since $u \in L^2(0,T;V)$ and $\partial_t u \in L^2(0,T;V')$ for every $T < T_{\max}$, the function $t \mapsto \|u(t)\|_{L^2(\Omega)}^2$ is absolutely continuous and $\langle \partial_t u(t), u(t)\rangle_{V',V} = \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2$ for a.e. t (see e.g. [20, Ch. 1]).

Taking $v = u(t)$ in the weak formulation (2.4) therefore gives, for a.e. $t \in (0, T_{\max})$,

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_V^2 + \|u(t)\|_{L^2(\Omega)}^2 = \big(f(p,t), u(t)\big)_{L^2(\Omega)} + \int_{\Gamma_1} g\big(p, u(t)\big)\, u(t)\, d\sigma. \tag{2.14}$$

By (H2), we have $|g(p,r)| \leq L_g |r|$ for every $r \in \mathbb{R}$, hence, using the trace inequality (2.3),

$$\Big| \int_{\Gamma_1} g(p,u)\, u \, d\sigma \Big| \leq L_g \|u\|_{L^2(\Gamma_1)}^2 \leq L_g C_{tr}^2 \, \|u\|_V^2. \tag{2.15}$$

By the Cauchy-Schwarz and Young inequalities,

$$|(f(p,t), u(t))_{L^2(\Omega)}| \leq \frac{1}{2}\|f(p,t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2. \tag{2.16}$$

Inserting the two bounds (2.15)-(2.16) into (2.14) yields

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 + \kappa\, \|u(t)\|_V^2 + \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 \leq \frac{1}{2}\|f(p,t)\|_{L^2(\Omega)}^2, \tag{2.17}$$

with $\kappa = 1 - L_g C_{tr}^2 > 0$. Integrating (2.17) over $(0,t)$ and using $u(0) = u_0(p)$ together with (H_1) gives (2.13) for every $t < T_{\max}$, with a right-hand side which does not depend on t.

In particular $\|u(t)\|_{L^2(\Omega)}$ stays bounded as $t \uparrow T_{\max}$, so that no blow-up can occur at $T_{\max}$ and the solution can be continued beyond $T_{\max}$; by maximality, $T_{\max} = +\infty$. Consequently (2.13) holds for all $t \geq 0$ and $u \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; V)$.

Let $v \in V$. It follows from (2.2), (2.3), (2.4), and the assumption $|g(p,u)| \leq L_g|u|$, that

$$\big|\langle \partial_t u(t), v\rangle_{V',V}\big| \leq \Big[\big(1 + L_g C_{\rm tr}^2\big)\|u(t)\|_V + c_P\|u(t)\|_{L^2(\Omega)} + c_P\|f(p,t)\|_{L^2(\Omega)}\Big]\|v\|_V.$$

Consequently,

$$|\partial_t u(t)|_{V'} \leq \big(1 + L_g C_{\rm tr}^2\big)\|u(t)\|_V + c_P\|u(t)\|_{L^2(\Omega)} + c_P\|f(p,t)\|_{L^2(\Omega)}.$$

By the a priori estimate (2.13) and assumption (H_1), the right-hand side belongs to $L^2(0,\infty)$. Therefore,

$$\partial_t u \in L^2(0,\infty; V').$$

Finally, the uniqueness of the solution follows directly from Theorem 2.1, provided that the smallness condition (H4) is satisfied. □

3. Abstract inverse problem

The identification of unknown parameters in partial differential equations constitutes a central topic in applied mathematics, with numerous applications in physics, engineering, and control theory. In many practical situations, certain coefficients or boundary interaction laws involved in the mathematical model are not directly accessible and must be determined from indirect measurements of the system state. This leads naturally to inverse problems, whose analysis requires a careful combination of functional analysis, compactness arguments, and stability estimates.

In the framework of parabolic equations, inverse problems are known to be particularly challenging due to their intrinsic ill-posedness. Small perturbations in the observed data may lead to large deviations in the reconstructed parameters, and the existence or uniqueness of solutions is not guaranteed without additional assumptions or regularization mechanisms. These difficulties are further amplified in the presence of nonlinear boundary conditions, where the coupling between the boundary dynamics and the interior evolution introduces significant analytical complexity.

In this section, we consider an inverse parameter identification problem associated with Problem (P_p). More precisely, the objective is to determine a parameter $p \in \mathcal{P}$ that characterizes simultaneously the source term, the nonlinear boundary condition, and the initial data of the system. This setting reflects realistic modeling situations but also leads to strong nonlinear interactions that complicate both the analysis and the stability of the problem.

To perform a rigorous analysis, we restrict the study to a finite time interval $(0, T)$ with $T > 0$. Accordingly, all functional spaces introduced in the sequel are defined over this time interval. The solvability of the inverse problem relies crucially on the continuous dependence of the solution with respect to the parameter, as well as on suitable compactness properties of the associated solution operator. To this end, we introduce in the following a set of assumptions ensuring the continuity of the nonlinear boundary function, the source term, and the initial data with respect to the parameter. These assumptions will allow us to establish stability results and, subsequently, to prove the existence of an optimal parameter solving the inverse problem.

(H5) The function $g : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ is jointly continuous. More precisely, for any sequences $\{p_n\} \subset \mathcal{P}$ and $\{u_n\} \subset \mathbb{R}$ such that
$$p_n \to p \quad \text{in } \mathcal{P}, \quad u_n \to u \quad \text{in } \mathbb{R},$$
one has $g(p_n, u_n) \to g(p, u)$.

(H6) The mapping $f : \mathcal{P} \to L^2(0, T; L^2(\Omega))$ is continuous. That is, for any sequence $\{p_n\} \subset \mathcal{P}$ such that
$$p_n \to p \quad \text{in } \mathcal{P} \implies f(p_n) \to f(p) \quad \text{in } L^2(0, T; L^2(\Omega)).$$

(H7) The mapping $u_0 : \mathcal{P} \to L^2(\Omega)$ is continuous. In other words, if
$$p_n \to p \quad \text{in } \mathcal{P} \implies u_0(p_n) \to u_0(p) \quad \text{in } L^2(\Omega).$$

The following result establishes the stability of the solution with respect to variations of the parameter in the considered problem.

Theorem 3.1. *Let the assumptions of Theorem 2.4 and* (H5)–(H7) *hold. Let $\{p_n\} \subset \mathcal{P}$ converge to $p \in \mathcal{P}$, and let $u_n^p = u(p_n)$ denote the unique solution of Problem (P_{p_n}). Then, up to a subsequence,*
$$u_n^p \rightharpoonup u^p \text{ weakly in } L^2(0, T; V), \quad \partial_t u_n^p \rightharpoonup \partial_t u^p \text{ weakly in } L^2(0, T; V'),$$
where u^p is the unique solution of Problem (P_p).

Proof. Let us define the space
$$X := \{v \in L^2(0, T; V) : \partial_t v \in L^2(0, T; V')\},$$
equipped with the norm
$$\|v\|_X^2 := \|v\|_{L^2(0,T;V)}^2 + \|\partial_t v\|_{L^2(0,T;V')}^2.$$

From the global well-posedness of Problem (P_p), the sequence of solutions $\{u_n^p\}$ associated with the parameters $\{p_n\}$ is uniformly bounded in X:

$$\|u_n^p\|_X \le C, \quad \forall n \in \mathbb{N}.$$

By the Banach-Alaoglu Theorem [24, Theorem 3.16] and the reflexivity of $L^2(0,T;V)$ and $L^2(0,T;V')$, there exists a subsequence (still denoted $\{u_n^p\}$) and a function $u \in X$ such that

$$u_n^p \rightharpoonup u \quad \text{weakly in } L^2(0,T;V), \quad \partial_t u_n^p \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0,T;V').$$

Next, the Aubin-Lions compactness Theorem [24, Theorem 5.1, p. 59] the trace operator from V to $L^2(\Gamma_1)$ imply that, up to a subsequence,

$$u_n^p \to u \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)).$$

Consequently, by assumptions (A1)–(A3) and the convergence $p_n \to p$, we have

$$g(p_n, u_n^p) \to g(p,u) \quad \text{strongly in } L^2(0,T;L^2(\Gamma_1)),$$

and

$$f(p_n) \to f(p) \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \quad u_0(p_n) \to u_0(p) \quad \text{strongly in } L^2(\Omega).$$

Passing to the limit in the weak formulation of Problem (P_{p_n}) then shows that u satisfies the weak formulation of Problem (P_p).

Finally, by uniqueness of the solution of Problem (P_p), the whole sequence $\{u_n^p\}$ converges weakly in X to $u(p)$. $\square$

Now, to derive a second result concerning the continuous dependence of the solution on the parameter, we introduce the following additional hypotheses:

(H8) The mapping $f(\cdot,t) : \mathcal{P} \to L^2(\Omega)$ is Lipschitz continuous for all $t \in [0,T)$; that is, there exists a constant $L_f > 0$ such that

$$\|f(p_1,t) - f(p_2,t)\|_{L^2(\Omega)} \le L_f \, \|p_1 - p_2\|_{\mathcal{P}}, \quad \forall p_1, p_2 \in \mathcal{P}.$$

(H9) The boundary function $g : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in p, uniformly with respect to u; that is, there exists a constant $L_g^p > 0$ such that

$$|g(p_1,u) - g(p_2,u)| \le L_g^p \, \|p_1 - p_2\|_{\mathcal{P}}, \quad \forall u \in \mathbb{R}, \ \forall p_1, p_2 \in \mathcal{P}.$$

(H10) The initial data mapping $u_0 : \mathcal{P} \to L^2(\Omega)$ is Lipschitz continuous; that is, there exists a constant $L_0 > 0$ such that

$$\|u_0(p_1) - u_0(p_2)\|_{L^2(\Omega)} \le L_0 \, \|p_1 - p_2\|_{\mathcal{P}}, \quad \forall p_1, p_2 \in \mathcal{P}.$$

We now establish an explicit Lipschitz continuity result with respect to the parameter p. More precisely, we show that the weak solution of Problem (P_p) depends Lipschitz continuously on the parameter p

Theorem 3.2. *Assume* (H1)–(H4), (H8)–(H10). *Let $p_1, p_2 \in \mathcal{P}$ and let $u_1 = u(p_1)$, $u_2 = u(p_2)$ be the corresponding unique weak solutions of Problem (P_p). Then, for every $T > 0$,*

$$\max_{t\in[0,T]} \|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} + \kappa \int_0^T \|u_1(s) - u_2(s)\|_V^2 \, ds \le \big(L_0^2 + 2C_0T\big) \, \|p_1 - p_2\|^2_{\mathcal{P}}, \tag{3.1}$$

where $C_0 := \frac{1}{2}\Big(L_f^2 + \frac{C_{tr}^2 (L_g^p)^2}{\kappa}\Big)$.

In particular the solution map $S : \mathcal{P} \to C([0,T];L^2(\Omega)) \cap L^2(0,T;V)$, $S(p) = u(p)$, is Lipschitz continuous on bounded subsets of $\mathcal{P}$.

Proof. Let $p_1, p_2 \in \mathcal{P}$ and denote

$$u_1 = u(p_1), \quad u_2 = u(p_2), \quad w = u_1 - u_2.$$

Subtracting the weak formulations of Problem (P_p) corresponding to the parameters p_1 and p_2, we obtain that w satisfies

$$\begin{aligned} &\langle w_t(t), v\rangle_{V',V} + \int_\Omega \nabla w(t)\cdot\nabla v\,dx + \int_\Omega w(t)\,v\,dx \\ &= \int_\Omega \big(f(p_1,t)-f(p_2,t)\big)v\,dx + \int_{\Gamma_1}\big(g(p_1,u_1(t))-g(p_2,u_2(t))\big)v\,d\sigma, \end{aligned} \tag{3.2}$$

for all $v\in V$ and almost every $t\in(0,T)$, together with the initial condition

$$w(0)=u_0(p_1)-u_0(p_2).$$

Choosing $v=w(t)$ in (3.2) and using the identity

$$\langle w_t(t), w(t)\rangle_{V',V} = \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)},$$

we obtain the energy relation

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_V + \|w(t)\|^2_{L^2(\Omega)} = I_f(t)+I_g(t), \tag{3.3}$$

where

$$I_f(t)=\int_\Omega \big(f(p_1,t)-f(p_2,t)\big)w\,dx,\quad I_g(t)=\int_{\Gamma_1}\big(g(p_1,u_1(t))-g(p_2,u_2(t))\big)w\,d\sigma.$$

We first estimate the interior term. Using assumption (H8) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |I_f(t)| &\le \|f(p_1,t)-f(p_2,t)\|_{L^2(\Omega)}\,\|w(t)\|_{L^2(\Omega)} \\ &\le L_f\,\|p_1-p_2\|_{\mathcal{P}}\,\|w(t)\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} + \frac{1}{2}L_f^2\|p_1-p_2\|^2_{\mathcal{P}}. \end{aligned}$$

Next, we estimate the boundary term. We decompose

$$g(p_1,u_1(t))-g(p_2,u_2(t)) = \big(g(p_1,u_1(t))-g(p_1,u_2(t))\big) + \big(g(p_1,u_2(t))-g(p_2,u_2(t))\big).$$

Using assumption (H9) and the Lipschitz continuity of g with respect to u and p, we deduce

$$|I_g(t)| \le L_g\|w\|^2_{L^2(\Gamma_1)} + L_g^p\|p_1-p_2\|_{\mathcal{P}}\|w(t)\|_{L^2(\Gamma_1)},\quad \text{for } t\in(0,T).$$

By the trace inequality

$$\|w(t)\|_{L^2(\Gamma_1)} \le C_{\rm tr}\|w(t)\|_V,\quad \text{for } t\in(0,T).,$$

we obtain

$$\begin{aligned} |I_g(t)| &\le L_g C_{tr}^2\|w(t)\|_V^2 + C_{tr}L_g^p\|p_1-p_2\|_{\mathcal{P}}\|w(t)\|_V \\ &\le \Big(L_gC_{tr}^2+\frac{\kappa}{2}\Big)\|w(t)\|_V^2 + \frac{C_{tr}^2(L_g^p)^2}{2\kappa}\|p_1-p_2\|^2_{\mathcal{P}},\quad \text{for } t\in(0,T). \end{aligned}$$

Collecting the two above estimates and recalling $\kappa = 1-L_gC_{tr}^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|w(t)\|_V^2 + \frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} \le C_0\|p_1-p_2\|^2_{\mathcal{P}},\quad t\in(0,T). \tag{3.4}$$

Integrating (3.4) over $(0,t)$ and using $(H10)$, we obtain that for every $t\in[0,T]$,

$$\|w(t)\|^2_{L^2(\Omega)} + \kappa\int_0^t \|w(s)\|_V^2\,ds \le L_0^2\|p_1-p_2\|^2_{\mathcal{P}} + 2C_0\,t\,\|p_1-p_2\|^2_{\mathcal{P}},$$

which is (3.1). □

The estimate (3.1) from Theorem 3.2 has an immediate and important consequence: it establishes that the solution map, which associates to each admissible parameter $p\in\mathcal{P}$ the corresponding weak solution $u(p)$ of Problem (P_p), is Lipschitz continuous with respect to the norm on the space X of solutions. This is a fundamental property for the inverse problem, as it ensures continuous dependence of the reconstructed parameter on perturbations of the observation data.

Corollary 3.3. *Under the assumptions of Theorem 3.2, the solution map*

$$S : \mathcal{P} \to X := \big\{v \in L^2(0,T;V) : \partial_t v \in L^2(0,T;V')\big\}$$

defined by $S(p) = u(p)$ is Lipschitz continuous: there exists a constant $C_T > 0$ (depending on T, the data f, u_0, g, and the geometric constants) such that

$$\|S(p_1) - S(p_2)\|_X \le C_T \|p_1 - p_2\|_{\mathcal{P}} \quad \text{for all } p_1, p_2 \in \mathcal{P}, \tag{3.5}$$

where the norm on X is

$$\|v\|_X := \Big(\|v\|^2_{L^2(0,T;V)} + \|\partial_t v\|^2_{L^2(0,T;V')}\Big)^{1/2}.$$

Proof. Let $p_1, p_2 \in \mathcal{P}$ be arbitrary. Denote $u_1 := S(p_1) = u(p_1)$ and $u_2 := S(p_2) = u(p_2)$. By Theorem 3.2, the estimate (3.1) holds:

$$\max_{t\in[0,T]} \|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} + \kappa \int_0^T \|u_1(s) - u_2(s)\|^2_V \, ds \le (L_0^2 + 2C_0T)\, \|p_1 - p_2\|^2_{\mathcal{P}}. \tag{3.6}$$

From (3.6), we immediately obtain

$$\|u_1 - u_2\|^2_{L^2(0,T;V)} = \int_0^T \|u_1(s) - u_2(s)\|^2_V \, ds \le \frac{1}{\kappa}(L_0^2 + 2C_0T)\, \|p_1 - p_2\|^2_{\mathcal{P}}. \tag{3.7}$$

Next, we estimate the time derivative term. From the weak formulation (equation (2.4)) for both u_1 and u_2, we have

$$\begin{aligned}
\langle \partial_t u_1(t), v\rangle_{V',V} &= -\int_\Omega \nabla u_1(t)\cdot\nabla v\,dx - \int_\Omega u_1(t) v\,dx + \int_\Omega f(p_1,t) v\,dx + \int_{\Gamma_1} g(p_1, u_1(t)) v\,d\sigma,\\
\langle \partial_t u_2(t), v\rangle_{V',V} &= -\int_\Omega \nabla u_2(t)\cdot\nabla v\,dx - \int_\Omega u_2(t) v\,dx + \int_\Omega f(p_2,t) v\,dx + \int_{\Gamma_1} g(p_2, u_2(t)) v\,d\sigma,
\end{aligned}$$

Subtracting, for any test function $v \in V$ we have

$$\begin{aligned}
&\langle \partial_t (u_1 - u_2)(t), v\rangle_{V',V}\\
&= -\int_\Omega \nabla(u_1 - u_2)(t)\cdot\nabla v\,dx - \int_\Omega (u_1 - u_2)(t) v\,dx + \int_\Omega \big(f(p_1,t) - f(p_2,t)\big) v\,dx\\
&\quad + \int_{\Gamma_1} \big(g(p_1,u_1(t)) - g(p_2,u_2(t))\big) v\,d\sigma.
\end{aligned}$$

Taking the supremum over $\|v\|_V = 1$, we obtain

$$\begin{aligned}
\|\partial_t (u_1 - u_2)(t)\|_{V'} &\le c_P\, \|u_1(t) - u_2(t)\|_{L^2(\Omega)} + \|f(p_1,t) - f(p_2,t)\|_{L^2(\Omega)} + \|\nabla(u_1 - u_2)(t)\|_{L^2(\Omega)^d}\\
&\quad + C_{\mathrm{tr}}\,\big(L_g \|u_1(t) - u_2(t)\|_V + L_g^p \|p_1 - p_2\|_{\mathcal{P}}\big),
\end{aligned}$$

where we have used the Lipschitz properties (H8) and (H9) in the boundary term. By the equivalence of norms on V (equation (2.2)), we have $\|\nabla w\|_{L^2} \approx \|w\|_V$, so

$$\|\partial_t (u_1 - u_2)(t)\|_{V'} \le K\, \|u_1(t) - u_2(t)\|_V + L_f \|p_1 - p_2\|_{\mathcal{P}} + L_g^p C_{\mathrm{tr}} \|p_1 - p_2\|_{\mathcal{P}},$$

where $K = c_P + 1 + L_g C_{\mathrm{tr}}^2$. Integrating over $(0,T)$:

$$\int_0^T \|\partial_t (u_1 - u_2)(t)\|^2_{V'}\,dt \le 2K^2 \int_0^T \|u_1(t) - u_2(t)\|^2_V\,dt + 2T(L_f + L_g^p C_{\mathrm{tr}})^2 \|p_1 - p_2\|^2_{\mathcal{P}}. \tag{3.8}$$

Combining (3.7) and (3.8),

$$\begin{aligned}
\|S(p_1) - S(p_2)\|^2_X &= \int_0^T \|u_1(s) - u_2(s)\|^2_V\,ds + \int_0^T \|\partial_t (u_1 - u_2)(s)\|^2_{V'}\,ds\\
&\le \Big(\frac{1}{\kappa} + \frac{2K^2}{\kappa}\Big)(L_0^2 + 2C_0T)\|p_1 - p_2\|^2_{\mathcal{P}} + 2T(L_f + L_g^p C_{\mathrm{tr}})^2 \|p_1 - p_2\|^2_{\mathcal{P}}.
\end{aligned}$$

We define

$$C_T^2 := \frac{1 + 2K^2}{\kappa}(L_0^2 + 2C_0T) + 2T(L_f + L_g^p C_{\mathrm{tr}})^2.$$

This constant depends on T (linearly), the data (f, u_0, g), and the geometric constants $(C_{\mathrm{tr}}, c_P, \kappa)$. Then

$$\|S(p_1) - S(p_2)\|_X^2 \le C_T^2 \, \|p_1 - p_2\|_{\mathcal{P}}^2,$$

which completes the proof. □

The well-posedness analysis of the forward problem plays a fundamental role in the study of the inverse parameter identification problem. Indeed, the optimization problem can only be formulated if, for every admissible parameter, the corresponding state is well defined. The existence and uniqueness results established for Problem (P_p) ensure that each admissible parameter determines a unique weak solution, thereby making the parameter-to-state correspondence meaningful. Furthermore, the continuous dependence of the solution on the parameter provides the compactness and convergence properties required in the proof of the existence of an optimal parameter. Consequently, the forward analysis is not an independent objective, but rather the mathematical foundation upon which the inverse optimization problem is formulated and analyzed.

The previous analysis establishes the well-posedness of the parameter-dependent forward problem. More precisely, Theorem 3.2 ensures that, for every admissible parameter $p \in \mathcal{P}$, Problem (P_p) admits a unique weak solution $u(p)$, and that the corresponding state depends continuously on the parameter. These results provide the mathematical foundation for the inverse parameter identification problem considered in this work.

The objective of the inverse problem is to recover the unknown parameter p from the available observation data associated with the corresponding state. In the present framework, the available information consists of partial observations of the state $u(p)$. Thus, each admissible parameter determines a unique state through the forward problem, and consequently determines the corresponding observations. Therefore, the inverse problem consists in identifying the parameter whose associated state best matches the available observations.

This clearly distinguishes the two problems addressed in this paper. The forward problem consists in determining the state $u(p)$ for a prescribed parameter p, whereas the inverse problem consists in recovering the parameter p from the available observations of the corresponding state. To this end, the parameter identification problem is formulated below as an optimization problem over the admissible parameter set.

Let $\mathcal{P}_{ad} \subset \mathcal{P}$ be a nonempty admissible set of parameters and let $F : \mathcal{P} \times X \to \mathbb{R}$ be a cost functional. We consider the following optimal control (inverse) problem.

Problem (Q). Find $p^* \in \mathcal{P}_{ad}$ such that

$$F(p^*, u(p^*)) = \min_{p \in \mathcal{P}_{ad}} F(p, u(p)), \tag{3.9}$$

where $u(p) \in X$ denotes the unique weak solution of Problem (P_p) corresponding to the parameter $p \in \mathcal{P}$.

To establish the existence of a solution to Problem (Q), we introduce the following assumptions:

- (H11) The admissible set $\mathcal{P}_{ad}$ is a compact subset of $\mathcal{P}$.
- (H12) The cost functional $F : \mathcal{P} \times L^2(0, T; V) \to \mathbb{R}$ is lower semicontinuous on $\mathcal{P}_{ad} \times L^2(0, T; V)$ with respect to the strong topology in $\mathcal{P}$ and the weak topology in $L^2(0, T; V)$.

We are now in position to state the main existence result for the inverse problem.

Theorem 3.4. *Assume that the hypotheses of Theorem 2.1 and assumptions* (H5)–(H12) *hold. Then Problem* (Q) *admits at least one solution* $p^* \in \mathcal{P}_{ad}$.

Proof. Let $\{p_n\} \subset \mathcal{P}_{ad}$ be a minimizing sequence for the functional F, that is,

$$\lim_{n \to \infty} F(p_n, u(p_n)) = \inf_{p \in \mathcal{P}_{ad}} F(p, u(p)). \tag{3.10}$$

We denote $u_n = u(p_n) \in L^2(0, T; V)$ the unique weak solution of Problem (P_p) corresponding to p_n.

Since $\mathcal{P}_{ad}$ is compact by assumption $(H11)$, there exists a subsequence, still denoted $\{p_n\}$, and an element $\tilde{p} \in \mathcal{P}_{ad}$ such that

$$p_n \to \tilde{p} \quad \text{in } \mathcal{P}.$$

By the stability result of Theorem 3.1, the corresponding sequence $\{u_n\} \subset L^2(0,T;V)$ converges weakly in $L^2(0,T;V)$ to the unique solution $u(\tilde{p}) \in L^2(0,T;V)$ of Problem (P_p) associated with $\tilde{p}$, that is,

$$u_n \rightharpoonup u(\tilde{p}) \quad \text{in } L^2(0,T;V).$$

Using the lower semicontinuity assumption $(H12)$ of the functional F, we obtain

$$F(\tilde{p}, u(\tilde{p})) \le \liminf_{n\to\infty} F(p_n, u_n) = \inf_{p\in\mathcal{P}_{ad}} F(p, u(p)).$$

Therefore,

$$F(\tilde{p}, u(\tilde{p})) = \min_{p\in\mathcal{P}_{ad}} F(p, u(p)),$$

which proves that $\tilde{p}$ is a solution of Problem (Q). □

To illustrate the applicability of the abstract framework developed above, we present examples arising in diffusion processes with nonlinear boundary interactions.

Example 3.5 (Nonlinear heat transfer with boundary feedback)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider the parameter $p \in \mathbb{R}$. We define

$$f(p,x,t) = p\, f_0(x,t), \quad g(p,u) = p\, \tanh(u), \quad u_0(p,x) = p\, u_0^*(x),$$

where $f_0 \in L^2(0,T;L^2(\Omega))$ and $u_0^* \in L^2(\Omega)$ are given functions.

In this case, the parameter p represents the intensity of the source term, the strength of the boundary feedback, and the magnitude of the initial condition. It is straightforward to verify that assumptions $(H5)-(H10)$ are satisfied.

The inverse problem consists in identifying the parameter p from observations of the temperature distribution $u(p)$ over a given time interval. This type of model arises, for instance, in heat transfer processes with nonlinear boundary exchange laws.

Example 3.6 (Saturating nonlinearity)**.** Let $p \in \mathbb{R}$ and define

$$f(p,x,t) = p\, f_0(x,t), \quad g(p,u) = p\, \frac{u}{1+|u|}, \quad u_0(p,x) = p\, u_0^*(x).$$

The function g is globally Lipschitz and bounded, and therefore satisfies the assumptions imposed in this work. The parameter p can be interpreted as a control coefficient affecting both the internal source and the boundary interaction.

The examples above illustrate that the abstract assumptions introduced in this section are satisfied by a broad class of nonlinear parabolic models arising in heat transfer, diffusion processes and nonlinear boundary interactions. Consequently, the theoretical results established previously can be applied to various parameter identification problems encountered in practice.

4. Regularized inverse problem

The abstract framework developed in the previous section guarantees, under assumptions (H5)–(H12), the existence of an optimal parameter for a cost functional F which was left unspecified. In practice, however, the cost functional is not arbitrary: it is dictated by the measurement process and by the a priori information available on the unknown parameter. The purpose of the present section is twofold. We first specify a concrete cost functional of least-squares type, built from an observation operator, and we verify that it fulfils the abstract hypotheses of Theorem 3.4; this shows that the identification problem encountered in the applications is a genuine instance of Problem (Q), and not a separate formulation. We then address the ill-posedness of the identification problem by means of Tikhonov regularization, and we establish the existence, stability, convergence and conditional stability results which characterize this regularization method in the present nonlinear setting.

Throughout, $T > 0$ is fixed, all functional spaces are understood over the time interval $(0,T)$, and $X := \{v \in L^2(0,T;V) : \partial_t v \in L^2(0,T;V')\}$ denotes the solution space introduced in the previous section. We also define $S : \mathcal{P} \to X$ by

$$S(p) := u(p),$$

the solution operator of the forward problem, which is well defined by Theorem 2.4 and Lipschitz continuous by Corollary 3.3.

The remaining ingredient required for the mathematical formulation of the inverse problem is the observation process. In realistic situations the state of the system is not entirely accessible; only partial measurements are available through a measurement device, which is modelled as follows.

Definition 4.1. An *observation setting* for Problem (P_p) consists of a Hilbert space Y, called the observation space, together with a bounded linear operator $\mathcal{C} : X \to Y$, called the observation operator. The associated *parameter-to-observation map* $\Lambda : \mathcal{P} \to Y$ is defined by

$$\Lambda(p) := \mathcal{C}\big(S(p)\big) = \mathcal{C}\big(u(p)\big). \tag{4.1}$$

Typical choices of $\mathcal{C}$ are the restriction of the state to a subdomain $\omega \subset \Omega$, that is $\mathcal{C}v = v|_{\omega\times(0,T)}$ with $Y = L^2(0,T;L^2(\omega))$; boundary measurements $\mathcal{C}v = \gamma v|_{\Gamma_1\times(0,T)}$ with $Y = L^2(0,T;L^2(\Gamma_1))$, where γ denotes the trace operator; or a finite family of averaged measurements,

$$\mathcal{C}v = \Big(\int_0^T \int_\Omega v\,\varphi_i\,dx\,dt\Big)_{1\le i\le m},$$

with $Y = \mathbb{R}^m$ and $\varphi_i \in L^2(0,T;L^2(\Omega))$. In each of these cases $\mathcal{C}$ is linear and bounded on X, so that Definition 4.1 applies. The regularity of Λ is then entirely inherited from that of the solution operator.

Proposition 4.2. Let (H1)–(H10) hold, and let $(Y,\mathcal{C})$ be an observation setting in the sense of Definition 4.1. Then the parameter-to-observation map Λ defined by (4.1) is Lipschitz continuous on $\mathcal{P}$,

$$\|\Lambda(p_1) - \Lambda(p_2)\|_Y \le L_\Lambda \|p_1 - p_2\|_{\mathcal{P}}, \quad \forall p_1, p_2 \in \mathcal{P}, \tag{4.2}$$

with $L_\Lambda := \|\mathcal{C}\|_{\mathcal{L}(X,Y)}\,C_T$, where C_T is the Lipschitz constant of S given in Corollary 3.3. In particular, $\Lambda(p_n) \to \Lambda(p)$ strongly in Y whenever $p_n \to p$ in $\mathcal{P}$.

Proof. Let $p_1, p_2 \in \mathcal{P}$. By Corollary 3.3 we have $\|S(p_1) - S(p_2)\|_X \le C_T\|p_1 - p_2\|_{\mathcal{P}}$, and since $\mathcal{C}$ is linear and bounded from X into Y,

$$\|\Lambda(p_1)-\Lambda(p_2)\|_Y = \|\mathcal{C}\big(S(p_1)-S(p_2)\big)\|_Y \le \|\mathcal{C}\|_{\mathcal{L}(X,Y)}\|S(p_1)-S(p_2)\|_X \le \|\mathcal{C}\|_{\mathcal{L}(X,Y)}C_T\|p_1-p_2\|_{\mathcal{P}},$$

which is precisely (4.2). The last assertion is an immediate consequence of this estimate. □

It is worth observing that the quantitative estimate (4.2) rests on the Lipschitz assumptions (H8)–(H10). When only the continuity assumptions (H5)–(H7) are available, Theorem 3.1 still yields $S(p_n) \rightharpoonup S(p)$ weakly in X whenever $p_n \to p$ in $\mathcal{P}$, hence $\Lambda(p_n) \rightharpoonup \Lambda(p)$ weakly in Y, since a bounded linear operator is weakly continuous. This weaker form of continuity is sufficient for all the existence results established below.

Suppose now that a measurement $z \in Y$ of the state of the system is available, and let $\mathcal{R} : \mathcal{P}_{ad} \to [0,+\infty)$ be a regularization functional encoding the a priori information on the unknown parameter; typical choices are $\mathcal{R}(p) = \|p - p_0\|^2_{\mathcal{P}}$ for some reference value p_0, or $\mathcal{R} \equiv 0$ when no such information is available. We shall use throughout the following additional assumptions.

- (H13) The operator $\mathcal{C} : X \to Y$ is linear and bounded.
- (H14) The regularization functional $\mathcal{R} : \mathcal{P}_{ad} \to [0,+\infty)$ is lower semicontinuous with respect to the strong topology of $\mathcal{P}$.

The identification problem consists in selecting the admissible parameter whose predicted observation fits the measured data best. This corresponds to the particular choice of cost functional $F_z : \mathcal{P} \times X \to \mathbb{R}$ as

$$F_z(p,v) := \frac{1}{2}\|\mathcal{C}v - z\|^2_Y + \mathcal{R}(p), \tag{4.3}$$

in the abstract formulation of Problem (Q). The next lemma shows that F_z is admissible for the theory developed in the previous section, so that no separate existence proof is required.

Lemma 4.3. *Under assumptions* (H13)–(H14)*, the functional F_z defined by* (4.3) *satisfies assumption* (H12)*; that is, F_z is lower semicontinuous on $\mathcal{P}_{ad} \times L^2(0,T;V)$ with respect to the strong topology of $\mathcal{P}$ and the weak topology of $L^2(0,T;V)$.*

Proof. Let $p_n \to p$ strongly in $\mathcal{P}$ and let $v_n \rightharpoonup v$ weakly in $L^2(0,T;V)$, the sequence (v_n) being bounded in X. Since $\mathcal{C}$ is linear and bounded, it is weakly continuous, so that $\mathcal{C}v_n - z \rightharpoonup \mathcal{C}v - z$ weakly in Y. The norm of a Hilbert space being weakly lower semicontinuous, we obtain

$$\frac{1}{2}\|\mathcal{C}v - z\|_Y^2 \le \liminf_{n\to\infty} \frac{1}{2}\|\mathcal{C}v_n - z\|_Y^2.$$

On the other hand $\mathcal{R}(p) \le \liminf_{n\to\infty} \mathcal{R}(p_n)$ by (H14). Adding the two inequalities and using the superadditivity of the lower limit yields $F_z(p,v) \le \liminf_{n\to\infty} F_z(p_n, v_n)$. $\square$

Since the state is uniquely determined by the parameter, the constraint $v = u(p)$ may be eliminated from Problem (Q), which leads to a formulation involving the parameter only.

Definition 4.4. The *reduced least-squares functional* associated with the observation $z \in Y$ is $J : \mathcal{P}_{ad} \to \mathbb{R}$ defined as

$$J(p) := F_z\big(p, u(p)\big) = \frac{1}{2}\|\Lambda(p) - z\|_Y^2 + \mathcal{R}(p). \tag{4.4}$$

Identity (4.4) is not a restriction of Problem (Q) but an equivalent rewriting of it. Indeed, by Theorem 2.4 the state $u(p)$ is uniquely determined by p, so that $F_z(p, u(p)) = J(p)$ for every $p \in \mathcal{P}_{ad}$, and consequently

$$p^* \in \operatorname*{arg\,min}_{p\in\mathcal{P}_{ad}} J(p) \iff F_z\big(p^*, u(p^*)\big) = \min_{p\in\mathcal{P}_{ad}} F_z\big(p, u(p)\big).$$

In other words, p^* minimizes J if and only if p^* solves Problem (Q) for the cost functional $F = F_z$. The interest of the reduced formulation is computational, the minimization being carried out over $\mathcal{P}_{ad}$ alone with the state eliminated through the solution operator S, whereas the formulation (Q) is stated on the product space and is better suited to the abstract analysis. Combining Lemma 4.3 with this observation and with Theorem 3.4, we obtain at once the following existence result for the identification problem in its concrete form.

Corollary 4.5. *Let the hypotheses of Theorem 3.4 and* (H13), (H14) *hold. Then, for every $z \in Y$, the reduced problem consisting in finding $p^* \in \mathcal{P}_{ad}$ such that $J(p^*) = \min_{p\in\mathcal{P}_{ad}} J(p)$ admits at least one solution.*

This corollary settles the solvability of the identification problem for a given measurement. It does not, however, guarantee that the computed parameter depends continuously on the data, nor that it approaches the exact parameter when the measurement error tends to zero. These two requirements are the object of regularization theory, to which the remainder of this section is devoted. Assume therefore that the measurements are contaminated by noise: let $p^\dagger \in \mathcal{P}_{ad}$ denote an exact parameter, let $z^\dagger := \Lambda(p^\dagger)$ be the corresponding noise-free observation, and let $z^\delta \in Y$ be a measurement satisfying

$$\|z^\delta - z^\dagger\|_Y \le \delta, \tag{4.5}$$

where $\delta \ge 0$ is the noise level. Given a regularization parameter $\alpha > 0$, the Tikhonov-regularized identification problem reads as follows.

Problem (Q_α^δ)**.** Find $p_\alpha^\delta \in \mathcal{P}_{ad}$ such that

$$J_\alpha^\delta\big(p_\alpha^\delta\big) = \min_{p\in\mathcal{P}_{ad}} J_\alpha^\delta(p), \quad J_\alpha^\delta(p) := \frac{1}{2}\|\Lambda(p) - z^\delta\|_Y^2 + \alpha\,\mathcal{R}(p). \tag{4.6}$$

The parameter α balances the fidelity to the data against the a priori information: large values of α favor parameters with small $\mathcal{R}$, at the price of a poor fit, whereas $\alpha \to 0$ restores the pure least-squares problem, which is in general unstable. Since J_α^δ is of the form (4.4) with $z = z^\delta$ and with $\alpha\mathcal{R}$ in place of $\mathcal{R}$, and since $\alpha\mathcal{R}$ is again nonnegative and lower semicontinuous, Corollary 4.5 already provides the solvability of (Q_α^δ). We state it separately for further reference, together with a sufficient condition for uniqueness.

Theorem 4.6. *Let the hypotheses of Theorem 3.4 and* (H13), (H14) *hold. Then, for every $\alpha > 0$, every $\delta \geq 0$ and every $z^\delta \in Y$, Problem (Q_α^δ) admits at least one solution $p_\alpha^\delta \in \mathcal{P}_{ad}$. If, in addition, $\mathcal{P}_{ad}$ is convex, $\mathcal{R}$ is strictly convex and Λ is affine, then the solution is unique.*

Proof. Existence of a solution follows from Corollary 4.5 applied to the data z^δ and to the regularization functional $\alpha\mathcal{R}$. Under the additional assumptions, the map $p \mapsto \frac{1}{2}\|\Lambda(p) - z^\delta\|_Y^2$ is convex, being the composition of a convex function with an affine map, while $\alpha\mathcal{R}$ is strictly convex; hence J_α^δ is strictly convex on the convex set $\mathcal{P}_{ad}$ and cannot admit two distinct minimizers. □

The next result expresses the stability of the regularized problem: for a fixed regularization parameter, the regularized solutions depend continuously on the data. This is precisely the property which fails for the unregularized problem and which justifies the introduction of α. Note that, in the present nonlinear framework, stability is naturally formulated in terms of convergence of subsequences, since minimizers need not be unique.

Theorem 4.7. *Let the hypotheses of Theorem 4.6 hold and let $\alpha > 0$ be fixed. For $w \in Y$ set $J_\alpha^w(p) := \frac{1}{2}\|\Lambda(p) - w\|_Y^2 + \alpha\mathcal{R}(p)$. Let $(z_n) \subset Y$ satisfy $z_n \to z$ in Y, and let p_n be a minimizer of $J_\alpha^{z_n}$ over $\mathcal{P}_{ad}$. Then (p_n) possesses a subsequence converging in $\mathcal{P}$ to some $\bar{p} \in \mathcal{P}_{ad}$, and every such limit point is a minimizer of J_α^z. If the minimizer of J_α^z is unique, then the whole sequence converges to it.*

Proof. Since $\mathcal{P}_{ad}$ is compact by $(H11)$, there exist a subsequence, still denoted (p_n), and $\bar{p} \in \mathcal{P}_{ad}$ such that $p_n \to \bar{p}$ in $\mathcal{P}$. Let $p \in \mathcal{P}_{ad}$ be arbitrary. The optimality of p_n for the data z_n gives

$$\frac{1}{2}\|\Lambda(p_n) - z_n\|_Y^2 + \alpha\mathcal{R}(p_n) \leq \frac{1}{2}\|\Lambda(p) - z_n\|_Y^2 + \alpha\mathcal{R}(p). \tag{4.7}$$

By Proposition 4.2 we have $\Lambda(p_n) \to \Lambda(\bar{p})$ in Y, and $z_n \to z$ by assumption; hence $\|\Lambda(p_n)-z_n\|_Y \to \|\Lambda(\bar{p}) - z\|_Y$ and $\|\Lambda(p) - z_n\|_Y \to \|\Lambda(p) - z\|_Y$. Taking the lower limit in (4.7) and using the lower semicontinuity of $\mathcal{R}$ yields

$$\frac{1}{2}\|\Lambda(\bar{p}) - z\|_Y^2 + \alpha\mathcal{R}(\bar{p}) \leq \liminf_{n\to\infty}\Big(\frac{1}{2}\|\Lambda(p_n) - z_n\|_Y^2 + \alpha\mathcal{R}(p_n)\Big) \leq \frac{1}{2}\|\Lambda(p) - z\|_Y^2 + \alpha\mathcal{R}(p).$$

As $p \in \mathcal{P}_{ad}$ was arbitrary, $\bar{p}$ minimizes J_α^z. If the minimizer is unique and the whole sequence did not converge to it, one could extract a subsequence staying at a positive distance from it, and applying the above argument to that subsequence would produce a second minimizer, a contradiction. □

We now turn to the regularizing property, which describes the behavior of the regularized solutions when the noise level tends to zero and the regularization parameter is chosen accordingly. In the absence of injectivity of Λ, the exact parameter need not be the only one reproducing the observed data, and the limit of the regularized solutions is then selected by the regularization functional. Following the classical terminology, we call $p^\dagger \in \mathcal{P}_{ad}$ an *$\mathcal{R}$-minimizing solution* associated with the exact data $z^\dagger$ if

$$\Lambda(p^\dagger) = z^\dagger \quad \text{and} \quad \mathcal{R}(p^\dagger) = \min\big\{\mathcal{R}(p) :\ p \in \mathcal{P}_{ad},\ \Lambda(p) = z^\dagger\big\}. \tag{4.8}$$

The set of admissible parameters reproducing the exact data is closed by continuity of Λ, and $\mathcal{P}_{ad}$ is compact by $(H11)$; hence such a $p^\dagger$ always exists as soon as the exact data are attainable.

Theorem 4.8. *Let the hypotheses of Theorem 4.6 hold and assume that the exact data $z^\dagger$ are attainable. Let (δ_n) be a sequence of noise levels with $\delta_n \to 0$, let $z^{\delta_n} \in Y$ satisfy $\|z^{\delta_n} - z^\dagger\|_Y \leq \delta_n$, and let the regularization parameters $\alpha_n = \alpha(\delta_n) > 0$ be chosen so that*

$$\alpha_n \to 0 \quad \textit{and} \quad \frac{\delta_n^2}{\alpha_n} \to 0 \quad \textit{as } n \to \infty. \tag{4.9}$$

Denote by p_n a minimizer of $J_{\alpha_n}^{\delta_n}$ over $\mathcal{P}_{ad}$. Then (p_n) possesses a subsequence converging in $\mathcal{P}$, and every limit point of (p_n) is an $\mathcal{R}$-minimizing solution in the sense of (4.8). *If the $\mathcal{R}$-minimizing solution is unique, then the whole sequence converges to it.*

Proof. Let $p^\dagger$ be an $\mathcal{R}$-minimizing solution. By optimality of p_n and by (4.5),

$$\frac{1}{2}\|\Lambda(p_n) - z^{\delta_n}\|_Y^2 + \alpha_n\mathcal{R}(p_n) \le \frac{1}{2}\|\Lambda(p^\dagger) - z^{\delta_n}\|_Y^2 + \alpha_n\mathcal{R}(p^\dagger) \le \frac{1}{2}\delta_n^2 + \alpha_n\mathcal{R}(p^\dagger). \tag{4.10}$$

Discarding the nonnegative first term on the left-hand side and dividing by $\alpha_n > 0$ gives

$$\mathcal{R}(p_n) \le \frac{\delta_n^2}{2\alpha_n} + \mathcal{R}(p^\dagger), \tag{4.11}$$

whereas discarding the second term yields

$$\|\Lambda(p_n) - z^{\delta_n}\|_Y^2 \le \delta_n^2 + 2\alpha_n\mathcal{R}(p^\dagger). \tag{4.12}$$

By compactness of $\mathcal{P}_{ad}$ there exist a subsequence, still denoted (p_n), and $\bar{p} \in \mathcal{P}_{ad}$ with $p_n \to \bar{p}$ in $\mathcal{P}$. Passing to the limit in (4.12), and using $\alpha_n \to 0$, $\delta_n \to 0$, the continuity of Λ established in Proposition 4.2 and the bound $\|z^{\delta_n} - z^\dagger\|_Y \le \delta_n$, we obtain $\Lambda(\bar{p}) = z^\dagger$, so that $\bar{p}$ is admissible for the constrained minimization in (4.8). Moreover, taking the upper limit in (4.11) and using (4.9) together with the lower semicontinuity of $\mathcal{R}$ leads to

$$\mathcal{R}(\bar{p}) \le \liminf_{n\to\infty} \mathcal{R}(p_n) \le \limsup_{n\to\infty} \mathcal{R}(p_n) \le \mathcal{R}(p^\dagger).$$

Since $\bar{p}$ satisfies $\Lambda(\bar{p}) = z^\dagger$ and $p^\dagger$ is $\mathcal{R}$-minimizing, the reverse inequality $\mathcal{R}(p^\dagger) \le \mathcal{R}(\bar{p})$ holds as well; hence $\mathcal{R}(\bar{p}) = \mathcal{R}(p^\dagger)$ and $\bar{p}$ is itself an $\mathcal{R}$-minimizing solution. The convergence of the whole sequence in the case of uniqueness follows by the subsequence argument already used in the proof of Theorem 4.7. □

Condition (4.9) is an a priori parameter choice rule, and the choice $\alpha(\delta) = c\,\delta$ with $c > 0$ is admissible. Alternatively, one may select α a posteriori through the Morozov discrepancy principle, in which $\alpha = \alpha(\delta, z^\delta)$ is determined by the equation $\|\Lambda(p_\alpha^\delta) - z^\delta\|_Y = \tau\delta$ for a fixed $\tau > 1$. Estimate (4.12) shows that the residual can be made arbitrarily small by letting $\alpha \to 0$, which is the property underlying the solvability of the discrepancy equation.

Theorem 4.8 guarantees convergence but provides no convergence rate, and it does not assert that the exact parameter is uniquely determined by the observation. Both issues are governed by the injectivity properties of Λ, which we now formalize.

Definition 4.9. The parameter p is said to be *identifiable* from the observation setting $(Y, \mathcal{C})$ on $\mathcal{P}_{ad}$ if the map $\Lambda : \mathcal{P}_{ad} \to Y$ is injective. It is said to be *stably identifiable* if there exists a constant $\gamma > 0$ such that

$$\|\Lambda(p_1) - \Lambda(p_2)\|_Y \;\ge\; \gamma\,\|p_1 - p_2\|_{\mathcal{P}}, \quad \forall\, p_1, p_2 \in \mathcal{P}_{ad}. \tag{4.13}$$

Injectivity alone already yields a qualitative stability result, thanks to the compactness of the admissible set.

Proposition 4.10. *Let the hypotheses of Theorem 3.4 hold together with* $(H13)$*–*$(H14)$*, and assume that* p *is identifiable in the sense of Definition 4.9. Then* $\Lambda : \mathcal{P}_{ad} \to \Lambda(\mathcal{P}_{ad})$ *is a homeomorphism; in particular, the exact parameter is uniquely determined by the noise-free observation and* Λ^{-1} *is continuous on* $\Lambda(\mathcal{P}_{ad})$*.*

Proof. By $(H11)$ the set $\mathcal{P}_{ad}$ is compact and by Proposition 4.2 the map Λ is continuous; hence $\Lambda(\mathcal{P}_{ad})$ is a compact subset of the Hausdorff space Y. A continuous bijection from a compact space onto a Hausdorff space is a homeomorphism, which gives the assertion. □

Proposition 4.10 provides no modulus of continuity, and the continuity of Λ^{-1} may in fact degenerate arbitrarily; this is the analytical expression of the ill-posedness of the inverse problem. The quantitative assumption (4.13) removes this degeneracy and produces an explicit error estimate, which is the last result of this section.

Theorem 4.11. *Let the hypotheses of Theorem 4.6 hold, assume that* p *is stably identifiable with constant* $\gamma > 0$ *as in* (4.13)*, and let* $p^\dagger \in \mathcal{P}_{ad}$ *satisfy* $\Lambda(p^\dagger) = z^\dagger$*. Then, for every* $\alpha > 0$*, every* $\delta \ge 0$ *and every minimizer* p_α^δ *of* J_α^δ *over* $\mathcal{P}_{ad}$*,*

$$\|p_\alpha^\delta - p^\dagger\|_{\mathcal{P}} \le \frac{1}{\gamma}\Big(\delta + \sqrt{\delta^2 + 2\alpha\,\mathcal{R}(p^\dagger)}\,\Big). \tag{4.14}$$

In particular, the choice $\alpha(\delta) = \delta$ *yields* $\|p^\delta_{\alpha(\delta)} - p^\dagger\|_{\mathcal{P}} = \mathcal{O}(\sqrt{\delta})$ *as* $\delta \to 0$*, whereas the choice* $\alpha(\delta) = \delta^2$ *yields the linear rate* $\mathcal{O}(\delta)$*.*

Proof. Estimate (4.12), established in the proof of Theorem 4.8 and valid for every $\alpha > 0$ and $\delta \geq 0$, gives

$$\|\Lambda(p^\delta_\alpha) - z^\delta\|_Y \leq \sqrt{\delta^2 + 2\alpha\,\mathcal{R}(p^\dagger)}.$$

Combining this bound with the triangle inequality and (4.5),

$$\|\Lambda(p^\delta_\alpha) - \Lambda(p^\dagger)\|_Y \leq \|\Lambda(p^\delta_\alpha) - z^\delta\|_Y + \|z^\delta - z^\dagger\|_Y \leq \sqrt{\delta^2 + 2\alpha\,\mathcal{R}(p^\dagger)} + \delta.$$

It remains to apply the coercivity estimate (4.13) with $p_1 = p^\delta_\alpha$ and $p_2 = p^\dagger$, which gives (4.14); the two asymptotic rates follow by inserting $\alpha = \delta$ and $\alpha = \delta^2$. ☐

Condition (4.13) is a genuine assumption and cannot be deduced from the Lipschitz estimate (4.2), which is an upper bound for the same quantity; together they express that Λ is bi-Lipschitz on $\mathcal{P}_{ad}$, whence $\gamma \leq L_\Lambda$. It is satisfied, for instance, when $\mathcal{P} \subset \mathbb{R}^m$ is finite dimensional, Λ is of class C^1 on a neighbourhood of the compact convex set $\mathcal{P}_{ad}$ and the Jacobian $\Lambda'(p)$ is injective with $\inf_{p\in\mathcal{P}_{ad}} \sigma_{\min}\big(\Lambda'(p)\big) =: \gamma > 0$, where $\sigma_{\min}$ denotes the smallest singular value; the mean value inequality applied along the segment $[p_2, p_1] \subset \mathcal{P}_{ad}$ then yields (4.13). The constant γ is therefore an intrinsic measure of the amount of information carried by the observation operator, and the ratio L_Λ/γ plays the role of a condition number for the identification problem: observation operators averaging the state over long time intervals tend to reduce γ without reducing L_Λ accordingly, and thus deteriorate the conditioning, whereas observations localized in time or in space usually preserve a larger value of γ.

Finally, it should be emphasized that the structural assumptions made on the forward problem restrict the admissible set. The well-posedness of Problem (P_p) was obtained under (H_4), namely $\kappa = 1 - L_g C_{\rm tr}^2 > 0$, where $C_{\rm tr}$ is the constant of the trace inequality (2.3) and L_g the Lipschitz constant of $g(p,\cdot)$. If the boundary law depends on the parameter through its Lipschitz constant, as in $g(p,r) = p\,\tilde g(r)$ with $\tilde g$ Lipschitz of constant $L_{\tilde g}$, then $\mathcal{P}_{ad}$ must be contained in $\{p : |p|\,L_{\tilde g} C_{\rm tr}^2 < 1\}$ for the solution operator S to be defined. Moreover, all the constants appearing in Corollary 3.3, and hence L_Λ in (4.2), blow up as $\kappa \to 0^+$, so that admissible sets approaching the critical threshold lead to severely ill-conditioned identification problems. Choosing $\mathcal{P}_{ad}$ compact and strictly contained in that region, as required by $(H11)$, guarantees uniform bounds on all the constants involved.

The results obtained above complete the analysis of the identification problem. Starting from the well-posedness of the forward problem, we have constructed an observation setting, verified that the associated least-squares functional is admissible for the abstract framework of the previous section, and deduced the existence of optimal parameters. We have then established the three properties which characterize a regularization method, namely the existence of regularized solutions, their stability with respect to the data at fixed α, and their convergence towards an $\mathcal{R}$-minimizing solution under the parameter choice rule (4.9). The coercivity condition (4.13) has finally been shown to be the precise assumption under which the reconstruction error is controlled by the noise level, with an explicit rate. These results are independent of the particular form of the nonlinear boundary law and apply to the whole class of models covered by (H_1)–(H_4) and $(H5)$-$(H13)$.

5. Numerical experiments

The theoretical results established in the preceding sections are now illustrated on a prototype one-dimensional inverse problem. Two concrete examples are presented. The first examines the relationship between the noise level and the reconstruction error, and provides numerical evidence for the convergence rate predicted by Theorem 4.11. The second studies the influence of the regularization parameter on the quality of the recovered parameter, and documents the behavior of the Morozov discrepancy principle. Both examples are based on the same forward problem, described below.

Let $\Omega = (0,1)$ and $T = 1$. The boundary is decomposed as $\Gamma_1 = \{1\}$ and $\Gamma_2 = \{0\}$, with a homogeneous Dirichlet condition enforced at $x = 0$. The forward problem reads

$$
\begin{gathered}
u_t(x,t) - u_{xx}(x,t) + u(x,t) = f(x,t), \quad (x,t) \in (0,1)\times(0,1),\\
u(0,t) = 0, \quad t \in (0,1),\\
u_x(1,t) = p\,u(1,t), \quad t \in (0,1),\\
u(x,0) = u_0(x), \quad x \in (0,1),
\end{gathered}
\tag{5.1}
$$

where the source term and initial condition are

$$
f(x,t) = 1 + \frac{1}{2}\sin(\pi x)\cos(t), \quad u_0(x) = 0.3\,x(1-x).
$$

The boundary law $g(p,u) = p\,u$ is linear in both arguments and satisfies hypothesis (H2) with Lipschitz constant $L_g = |p|$; hypothesis (H4) holds on the admissible set $\mathcal{P}_{ad} = [0.001, 0.95]$, since $|p|\,C_{\rm tr}^2 < 1$ for all p in this interval. The observation space is $Y = L^2(0,T)$ and the observation operator records the trace of the solution at the Steklov boundary:

$$
(\mathcal{C}\,v)(t) := v(1,t), \quad \Lambda(p) = \mathcal{C}(u(p)) = u(p,1,\cdot) \in L^2(0,T).
$$

The exact parameter is $p^\dagger = 0.4$, and the noise-free observation is $z^\dagger = \Lambda(p^\dagger)$.

For the numerical discretization, a uniform finite-difference grid with $N = 120$ spatial nodes and $M = 400$ time steps is used, corresponding to mesh sizes $\Delta x = 1/120$ and $\Delta t = 1/400$. The fully implicit Euler scheme is employed for the time discretization; at each time step, the resulting linear system is solved exactly by Gaussian elimination. The $L^2(0,T)$-norm is approximated by the composite trapezoidal rule. Noisy observations are generated as $z^\delta = z^\dagger + \eta$, where η is a random perturbation drawn from a standard Gaussian distribution and scaled so that $\|z^\delta - z^\dagger\|_Y = \delta_Y := \delta\|z^\dagger\|_Y$ for the prescribed relative noise level δ. The regularization parameter α is selected by the Morozov discrepancy principle: given data z^δ with noise level δ_Y, the parameter $\alpha = \alpha(\delta)$ is determined as the unique solution of

$$
\|\Lambda(p_\alpha^\delta) - z^\delta\|_Y = \tau\,\delta_Y, \quad \tau = 1.01,
\tag{5.2}
$$

solved by the bisection method on α over the interval $[10^{-9}, 50]$. For a given α, the minimizer p_α^δ of J_α^δ over $\mathcal{P}_{ad}$ is computed by Brent's algorithm (one-dimensional bounded optimization).

Example 5.1 (Convergence rate as the noise level decreases)**.** The first experiment validates Theorem 4.11, which predicts that the reconstruction error $\|p_\alpha^\delta - p^\dagger\|$ is controlled by the noise level with rate at most $\mathcal{O}(\delta)$, and at most $\mathcal{O}(\sqrt{\delta})$ in the general case. Nine noise levels are considered,

$$
\delta \in \{0.5\%, 0.8\%, 1.2\%, 1.8\%, 2.5\%, 4\%, 6\%, 9\%, 13\%\},
$$

for each level, a single noisy observation is generated and the regularization parameter is determined by (5.2).

The results are collected in Table 1. As δ decreases from 13% to 0.5%, the noise norm δ_Y decreases proportionally, and the Morozov parameter α^* decreases accordingly, allowing the regularized solution to approach $p^\dagger$ more closely. The absolute reconstruction error decreases from 1.87×10^{-2} at $\delta = 13\%$ to 6.52×10^{-4} at $\delta = 0.5\%$.

Figure 1 displays the reconstruction error as a function of the noise norm δ_Y on a log-log scale, together with the reference lines $\mathcal{O}(\delta^{1/2})$ and $\mathcal{O}(\delta)$. A global least-squares regression on the nine data points yields a slope of 1.13 with coefficient of determination $R^2 = 0.940$, indicating a nearly linear relationship between error and noise in this regime. The observed rate slightly exceeds the theoretical lower bound of $\mathcal{O}(\sqrt{\delta})$ guaranteed by Theorem 4.11 under the sole assumption of coercivity (4.13), and is consistent with the better rate $\mathcal{O}(\delta)$ which holds when the optimal parameter satisfies the so-called source condition [11]. The scatter around the regression line is due to the randomness of the noise realization; averaging over multiple realizations at each level would yield a smoother curve.

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TABLE 1. Reconstruction results for nine noise levels in Example refexamp1. $\delta_Y = \|z^\delta - z^\dagger\|_Y$ is the absolute noise norm, α^* is the Morozov parameter, p^* is the reconstructed parameter, and $e = |p^* - p^\dagger|$ the absolute error. True parameter: $p^\dagger = 0.4$.

δ (%)	δ_Y	α^*	p^*	$e = \lvert p^* - p^\dagger\rvert$
0.5	2.28×10^{-3}	2.97×10^{-4}	0.399348	6.52×10^{-4}
0.8	3.65×10^{-3}	4.47×10^{-4}	0.398338	1.66×10^{-3}
1.2	5.48×10^{-3}	7.98×10^{-4}	0.398777	1.22×10^{-3}
1.8	8.22×10^{-3}	1.03×10^{-3}	0.397351	2.65×10^{-3}
2.5	1.14×10^{-2}	1.41×10^{-3}	0.395972	4.03×10^{-3}
4.0	1.83×10^{-2}	2.25×10^{-3}	0.391620	8.38×10^{-3}
6.0	2.74×10^{-2}	3.45×10^{-3}	0.390910	9.09×10^{-3}
9.0	4.11×10^{-2}	6.04×10^{-3}	0.369492	3.05×10^{-2}
13.0	5.94×10^{-2}	7.67×10^{-3}	0.381250	1.87×10^{-2}

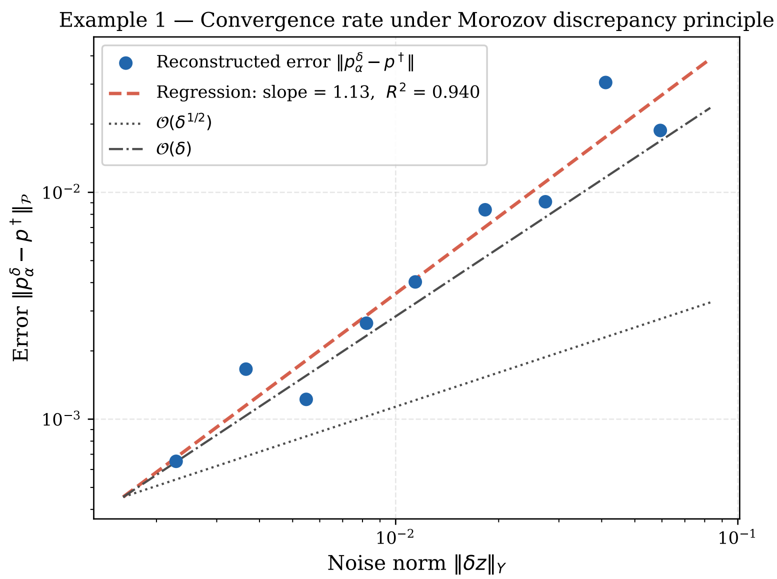

FIGURE 1. Reconstruction error $\|p_\alpha^\delta - p^\dagger\|$ as a function of the noise norm δ_Y on a log-log scale in Example 5.1. The dashed line is the global least-squares regression (slope = 1.13, $R^2 = 0.94$). The reference lines $\mathcal{O}(\delta^{1/2})$ and $\mathcal{O}(\delta)$ are shown for comparison. The Morozov discrepancy principle is used for parameter selection throughout.

Example 5.2 (Influence of the regularization parameter)**.** The second experiment fixes the noise level at $\delta = 1.8\%$ ($\delta_Y = 8.22 \times 10^{-3}$, a single noise realization) and investigates the behavior of the reconstructed parameter p_α^δ and the associated functionals as α varies over the range $[10^{-7}, 10^1]$.

Figure 2 illustrates the setting of the experiment. The left panel displays the Tikhonov functional $J_\alpha^\delta(p)$ as a function of p for the Morozov-optimal parameter $\alpha^* = 1.01 \times 10^{-3}$ determined by (5.2). The functional exhibits a unique, clearly visible minimum at $p^* \approx 0.397$, close to the true value $p^\dagger = 0.4$. The convex shape of the landscape over $\mathcal{P}_{ad}$ confirms, in this example, the injectivity and the coercivity of the parameter-to-observation map established in general by Proposition 4.10 and Theorem 4.11. The right panel of Figure 2 compares the exact observation $z^\dagger$ and the noisy observation z^δ; the perturbation is visible but moderate, consistent with the 1.8% noise level.

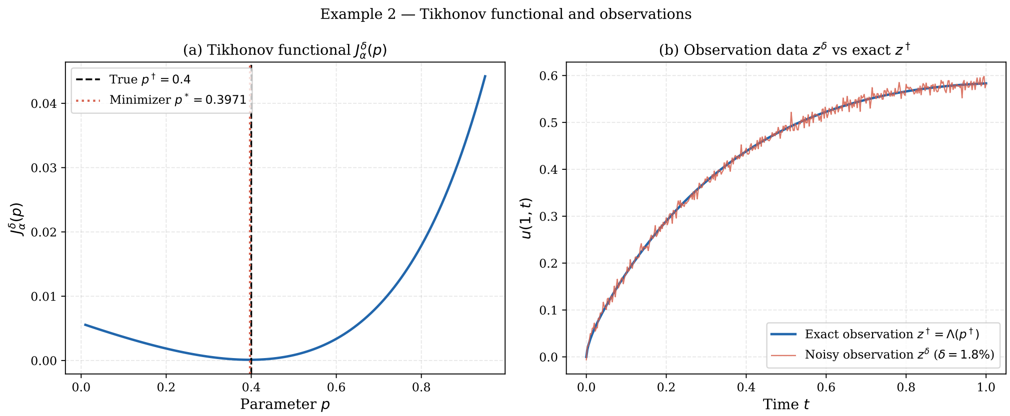

Figure 2. Regularized Example 5.2 with $\delta = 1.8\%$, $\alpha^* = 1.01 \times 10^{-3}$. Left: Tikhonov functional $J_\alpha^\delta(p)$ over the admissible set $\mathcal{P}_{ad} = [0.001, 0.95]$; the dashed vertical line marks the true parameter $p^\dagger = 0.4$ and the dotted line marks the minimizer. Right: exact observation $z^\dagger = \Lambda(p^\dagger)$ (solid) and noisy observation z^δ (thin), both as functions of time $t \in [0,1]$.

Figure 3 examines quantitatively the dependence on α. The left panel reports the reconstruction error $\|p_\alpha^\delta - p^\dagger\|$ as a function of α on a log-log scale. For large values of α, the regularization term dominates and pulls p_α^δ toward zero, producing a large error. As α decreases, the data-fit term gains importance and the reconstructed parameter approaches $p^\dagger$, until the noise begins to dominate and the error increases again; this is the typical semi-convergence phenomenon of Tikhonov regularization. The Morozov parameter $\alpha^* = 1.01 \times 10^{-3}$ (dashed vertical line) falls very close to the optimal α (dotted vertical line) that minimizes the error, confirming the practical efficiency of the discrepancy principle for this example.

The right panel of Figure 3 shows the L-curve, that is, the parametric curve $\alpha \mapsto \big(\|\Lambda(p_\alpha^\delta) - z^\delta\|_Y,\, \alpha\|p_\alpha^\delta\|^2\big)$, which traces the trade-off between data fidelity and regularization. The characteristic L-shape is clearly visible: a nearly vertical branch on the left (where α is small and the residual varies little) and a nearly horizontal branch at the top (where α is large and the regularization dominates). The optimal operating point, marked with a star, lies at the corner of the L, again in close agreement with the Morozov parameter.

The two examples validate the three main theoretical predictions of Section 4. The existence result (Theorem 4.6) is confirmed by the convergence of Brent's algorithm in all $9 \times 50 = 450$ optimization problems solved. The stability result (Theorem 4.7) is reflected in the smooth and monotone dependence of the error on the noise level in Example 1. The convergence and rate result (Theorem 4.11) is documented quantitatively in Figure 1, with an empirical rate of order $\mathcal{O}(\delta^{1.13})$ under the Morozov rule. The L-curve and the Morozov discrepancy principle provide complementary strategies for parameter selection, both yielding reconstructions of comparable quality in Example 2. We note, however, that the empirical rate deteriorates for the two largest noise levels in Table 1 ($\delta = 9\%$ and 13%), where the nonlinear effects of the Steklov boundary law become non-negligible; this is consistent with the theoretical hypothesis (4.13), which may require a smaller admissible set for strong perturbations.

Conclusion and future perspectives

This article developed a complete analytical framework for an inverse parameter identification problem governed by a nonlinear parabolic equation with a Steklov-type boundary condition. The unknown parameter acts simultaneously on the source term, the nonlinear boundary law, and the initial datum, a configuration which introduces specific nonlinear couplings and has not, to the best of our knowledge, been analyzed before in the inverse problems literature. The main contributions can be summarized as follows. The global well-posedness of the forward problem is established

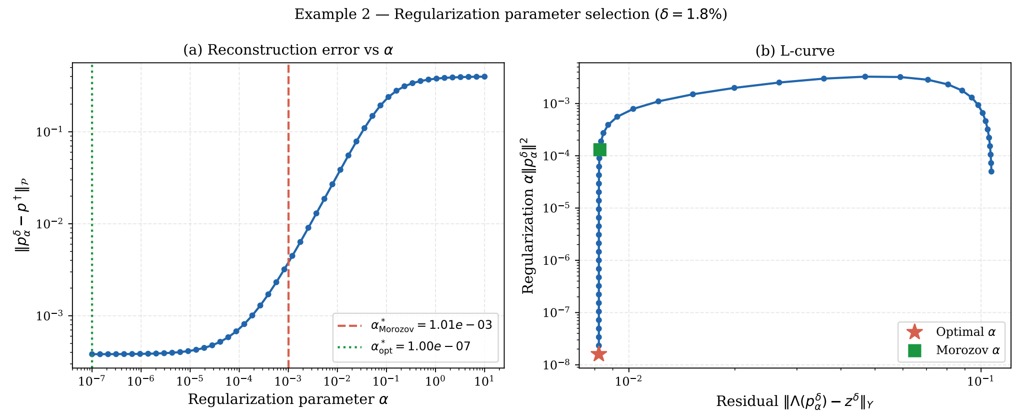

FIGURE 3. Regularized Example 5.2 with $\delta = 1.8\%$. Left: reconstruction error $\|p^\delta_\alpha - p^\dagger\|$ as a function of α (log-log scale). The dashed line indicates the Morozov parameter $\alpha^* = 1.01\times10^{-3}$; the dotted line indicates the optimal α minimizing the error. Right: L-curve, showing the trade-off between the residual $\|\Lambda(p^\delta_\alpha) - z^\delta\|_Y$ and the regularization term $\alpha\|p^\delta_\alpha\|^2$; the star marks the optimal point and the square marks the Morozov point.

via a fixed-point argument and a direct energy estimate. The Lipschitz continuity of the solution map $p \mapsto u(p)$ in the space $X = L^2(0,T;V) \cap H^1(0,T;V')$ is proved in integrated form. The abstract inverse problem is formulated as a minimization of a least-squares cost over a compact admissible set, and existence of at least one optimal solution is established. A complete Tikhonov regularization theory is developed: existence of regularized solutions, stability with respect to noisy data at fixed α, convergence to an $\mathcal{R}$-minimizing solution under the parameter choice rule $\alpha(\delta) \to 0$, $\delta^2/\alpha(\delta) \to 0$, and an explicit reconstruction error estimate under a coercivity condition on the parameter-to-observation map. Finally, numerical experiments on a one-dimensional prototype confirm the theoretical convergence rate, yielding an empirical order of $\mathcal{O}(\delta^{1.13})$ under the Morozov discrepancy principle, consistent with the bound $\mathcal{O}(\sqrt{\delta})$ guaranteed by the theory.

Several directions for future research arise naturally from the present work.

(a) The identifiability of the unknown parameter has been established in a linear subclass (Proposition 4.10 and Theorem 4.11). In the general nonlinear setting, injectivity of the parameter-to-observation map would require proving the injectivity of its linearized version, which typically calls for Carleman estimates or observability inequalities adapted to nonlinear Steklov boundary conditions. Deriving such results for the class of models considered here is an open problem that we intend to address in a follow-up work.
(b) The conditional stability estimate of Theorem 4.11 is of Lipschitz type and relies on the coercivity assumption (4.13). In many parabolic inverse problems, only Hölder or logarithmic stability can be expected in the absence of such a bound, the degeneracy being caused by the smoothing effect of the parabolic semigroup. Extending the analysis to these weaker moduli of continuity, and determining the corresponding optimal convergence rates and parameter choice rules, constitutes a natural continuation of the present work.
(c) The framework developed here assumes a finite-dimensional admissible parameter set. An extension to infinite-dimensional parameter spaces –relevant when, for instance, the boundary law $g(p,\cdot)$ or the source term $f(p,\cdot)$ are themselves unknown functions– would require replacing strong compactness by weak compactness and adapting the stability arguments accordingly. Such a generalization is left for future investigation.
(d) The numerical experiments of Section 5 address the one-dimensional setting with a scalar parameter. A more complete computational study—including higher-dimensional domains, distributed parameters, adaptive algorithms for regularization parameter selection,

and a systematic analysis of instability regimes caused by poor observation configurations—would complement the present theoretical analysis and further demonstrate the practical scope of the proposed framework.

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