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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH BOUNDED DENSITY

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Abstract. This article concerns the three-dimensional compressible magnetohydrodynamic equations on the torus in the presence of vacuum. We establish the existence and uniqueness of strong local solutions for low-regularity initial data without imposing an initial compatibility condition. More precisely, the initial data are assumed to satisfy $0 \le \rho_0 \in L^\infty(\mathbb{T}^3), u_0, b_0 \in H^1(\mathbb{T}^3), \operatorname{div} b_0 = 0$. The main difficulties arise from the strong nonlinear coupling between the velocity and magnetic field and from the low regularity of the density. By introducing the effective viscous flux and deriving suitable time-weighted energy estimates, we establish a uniform local existence theory independent of the initial compatibility condition. For uniqueness, we estimate the density difference in a negative Sobolev space and introduce a coupled backward parabolic system to avoid derivatives of the density. The proof is completed by a logarithmic BMO estimate and an Osgood-type argument.

1. Introduction

1.1. **Compressible magnetohydrodynamic (MHD) equations.** We consider the compressible MHD equations in $\mathbb{T}^3 \times (0, T)$,

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\rho(u_t + u \cdot \nabla u) - \mu\Delta u - (\mu + \lambda)\nabla \operatorname{div} u + \nabla P = (\nabla \times b) \times b, \tag{1.2}$$

$$b_t - \nabla \times (u \times b) = \nu\Delta b, \tag{1.3}$$

$$\operatorname{div} b = 0, \tag{1.4}$$

where the unknowns (ρ, u, b) represent the density, velocity and magnetic field, respectively. The pressure is $P = a\rho$ where a is a positive constant. The viscous coefficients μ and λ satisfy the physical constraints

$$\mu > 0, \quad 2\mu + 3\lambda \ge 0.$$

There is an extensive literature on the mathematical analysis of the compressible MHD equations, which couple the compressible Navier-Stokes equations of fluid dynamics with Maxwell's equations of electromagnetism. When vacuum is allowed, i.e., when the initial density may vanish in some region, many results have been obtained for the compressible MHD equations. For local and global weak solutions to the compressible MHD equations, see [15, 20, 26, 27, 28, 29, 21, 17, 33] and the references therein. Global well-posedness of strong solutions to compressible flows with large oscillations and vacuum was studied by Lv, Shi and Xu [36] and by Li, Xu and Zhang [30], and the latter result was subsequently improved by Hong, Hou, Peng and Zhu [19]. For the non-isentropic case,

Fan and Yu [16] considered the local existence and uniqueness of strong solutions with initial data satisfying $\rho_0 \in W^{1,q}$, $u_0, b_0 \in H^2, \operatorname{div} b_0 = 0$, and the following compatibility conditions

$$-\mu\Delta u_0 - (\mu + \lambda)\nabla \operatorname{div} u_0 + \nabla P_0 - (\nabla \times b_0) \times b_0 = \sqrt{\rho_0} g_1, \quad P_0 = R\rho_0\theta_0, \tag{1.5}$$

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$$\kappa\Delta\theta_0+\frac{\mu}{2}|\nabla u_0+(\nabla u_0)^T|^2+\lambda(\operatorname{div}u_0)^2+\frac{1}{\sigma}|\nabla\times b_0|^2=\sqrt{\rho_0}g_2, \tag{1.6}$$

for some $g_1, g_2 \in L^2$. This result thereafter was extended by Liu and Zhong [34] to the global existence result. Recently, Lu and Huang [35] investigated the existence of strong local solutions for the full compressible MHD equations with vacuum and zero heat conduction, see also Xi and Hao [39] and Zhong [42]. For local and global well-posedness to the incompressible MHD equations, one refers to [2, 3, 4, 5, 7, 32, 40] and the references therein.

The MHD system is concerned with the interaction between fluid flow and magnetic field. When the electromagnetic effect is absent, i.e., $b = 0$, the MHD system reduces to the compressible Navier-Stokes equations, which have been extensively studied, see [9, 10, 11, 18, 23] and the references therein. Typically, local well-posedness to the Navier-Stokes equations requires suitably regular initial data satisfying the following compatibility conditions

$$-\mu\Delta u_0-(\mu+\lambda)\nabla\operatorname{div}u_0+\nabla(R\rho_0\theta_0)=\sqrt{\rho_0}h_1, \tag{1.7}$$

$$\kappa\Delta\theta_0+\frac{\mu}{2}|\nabla u_0+(\nabla u_0)^T|^2+\lambda(\operatorname{div}u_0)^2=\sqrt{\rho_0}h_2, \tag{1.8}$$

for $h_1, h_2 \in L^2$. It should be pointed out that these compatibility conditions (1.7) and (1.8) impose some restrictive constraints on the initial data in the vacuum region and also in the neighborhood of the vacuum-nonvacuum interface. It is therefore of both mathematical and physical importance to establish an alternative well-posedness theory without requiring any initial compatibility conditions like (1.7) and (1.8). The first study towards this direction was made by Li [24] for the inhomogeneous incompressible Navier-Stokes equations, where the local well-posedness was successfully established without any compatibility conditions on the initial data, see Danchin and Mucha [14] for further developments aiming to relax the smoothness of the initial density. Local well-posedness theory to the compressible Navier-Stokes equations without any initial compatibility conditions was later established by Gong, Li, Liu and Zhang [18] and Huang [22] for the isentropic case, and thereafter extended by Li and Zheng [31] to the heat conductive case. Similar results have been pursued for the MHD equations. Cheng and Dong [8] and Zhang, Guo and Fang [41] established local well-posedness theory to the non-isentropic MHD equations in several space dimensions without imposing the corresponding initial compatibility conditions (1.5)–(1.6). Their results, however, require stronger spatial regularity of the initial density than that assumed in the present paper.

The aim of this paper is to establish local existence and uniqueness of strong solutions to the compressible MHD equations under low-regularity assumptions and without any initial compatibility conditions. In particular, we allow the initial density to be merely bounded and nonnegative, $\rho_0 \in L^\infty$, which is substantially weaker than the regularity assumptions used in the preceding local theories. This is achieved by successfully extending and refining the approach of Danchin and Mucha [14] for the compressible Navier-Stokes equations. Moreover, we introduce a coupled backward parabolic system and apply a logarithmic Grönwall-type inequality to prove the uniqueness of solutions.

System (1.1)–(1.4) is supplemented with the initial data

$$(\rho,\rho u,b)|_{t=0}=(\rho_0,\rho_0u_0,b_0), \tag{1.9}$$

where ρ_0, u_0 and b_0 are given functions. For sufficiently regular solutions of system (1.1)–(1.4), it is straightforward to verify that both the total mass and momentum are conserved through the evolution. Specifically, for all $t \geq 0$,

$$\int_{\mathbb{T}^3}\rho(x,t)\,\mathrm{d}x=\int_{\mathbb{T}^3}\rho_0(x)\,\mathrm{d}x \quad\text{and}\quad \int_{\mathbb{T}^3}(\rho u)(x,t)\,\mathrm{d}x=\int_{\mathbb{T}^3}(\rho_0u_0)(x)\,\mathrm{d}x.$$

Without loss of generality, we always suppose

$$\int_{\mathbb{T}^3}\rho_0(x)\,\mathrm{d}x=1 \quad\text{and}\quad \int_{\mathbb{T}^3}(\rho_0u_0)(x)\,\mathrm{d}x=0. \tag{1.10}$$

It should be pointed out that the actual values of u_0 that we need are only those in the non-vacuum region $\mathbb{T}^3_+ := \{x \in \mathbb{T}^3|\rho_0(x) > 0\}$ but not in the vacuum region $\mathbb{T}^3_0 := \{x \in \mathbb{T}^3|\rho_0(x) = 0\}$. More precisely, we denote by $u_0^{\rm real}$ the initial velocity in the non-vacuum region $\mathbb{T}^3_+$, and define

$$\mathscr{S}_{\rm ext} = \big\{\tilde{u}_0 : \tilde{u}_0 = u_0^{\rm real} \text{ on } \mathbb{T}^3_+, \tilde{u}_0 \in H^1(\mathbb{T}^3)\big\}.$$

Then any $u_0 \in \mathscr{S}_{\rm ext}$ can be chosen as the "initial" velocity without changing the initial condition (1.9) and normalization (1.10). In fact, for any $u_0 \in \mathscr{S}_{\rm ext}$, it is clear that

$$\rho_0 u_0 = \begin{cases} \rho_0 u_0^{\rm real} & \text{in } \mathbb{T}^3_+, \\ 0 & \text{in } \mathbb{T}^3_0. \end{cases}$$

Because of the above explanation, throughout this paper, we assume that the "initial" velocity u_0 is defined on the whole domain such that $u_0 \in H^1(\mathbb{T}^3)$.

Before stating the main results, we first clarify some necessary notations used throughout this paper and state the definition of solutions to be established.

For $1 \le p \le \infty$ and positive integer m, we write $L^p = L^p(\mathbb{T}^3)$ and $W^{m,p} = W^{m,p}(\mathbb{T}^3)$ to denote the standard Lebesgue and Sobolev spaces, respectively, and we use H^m to replace $W^{m,2}$. For simplicity, we also use notations L^p and H^m to denote the N product spaces $(L^p)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_p$ to stand for the L^p norm of u, while the L^2 norm is further simplified as $\|\cdot\|$. To shorten notation, we sometimes use $\|(f_1, f_2, \cdots, f_N)\|_X$ to denote the norm $\sum_{i=1}^N \|f_i\|_X$ or the equivalent product norm $(\sum_{i=1}^N \|f_i\|_X^2)^{1/2}$.

We introduce the material derivative, effective viscous flux and vorticity as follows:

$$\begin{aligned} \dot{u} &:= u_t + u\cdot\nabla u, \\ G &:= (2\mu+\lambda)\operatorname{div} u - P, \\ \omega &:= \nabla\times u. \end{aligned} \tag{1.11}$$

Thanks to this, applying divergence and curl to the momentum equation (1.2) yields the elliptic systems

$$\Delta G = \operatorname{div}(\rho\dot{u} - (\nabla\times b)\times b), \quad \mu\Delta\omega = \nabla\times(\rho\dot{u} - (\nabla\times b)\times b). \tag{1.12}$$

In what follows, we shall frequently use the Hodge estimate on $\mathbb{T}^3$. For any $1 < p < \infty$, it follows from (1.11)–(1.12) that

$$\|\nabla u\|_{L^p} \le C\left(\|\operatorname{div} u\|_p + \|\omega\|_p\right) \le C\left(\|P\|_p + \|G\|_p + \|\omega\|_p\right). \tag{1.13}$$

To be more precise, the strong solutions to be established in this paper are defined as follows.

Definition 1.1. Let $T > 0$ and $q \in (3,6)$, and assume that

$$0 \le \rho_0 \in L^\infty, \quad u_0,\ b_0 \in H^1, \quad \operatorname{div} b_0 = 0.$$

A triple (ρ, u, b) is called a strong solution to system (1.1)–(1.4) in $\mathbb{T}^3\times(0,T)$, if it has the regularity

$$\begin{gathered} 0 \le \rho \in L^\infty(0,T;L^\infty) \cap C([0,T];L^q), \\ (u,b) \in L^\infty(0,T;H^1), \quad \sqrt{\rho}u \in C([0,T];L^2), \quad b \in C([0,T];L^2) \\ \nabla u \in L^2(0,T;L^6), \quad b \in L^2(0,T;H^2), \quad (\nabla G, \sqrt{\rho}\dot{u}, b_t) \in L^2(0,T;L^2), \\ (\sqrt{t}\sqrt{\rho}\dot{u}, \sqrt{t}b_t) \in L^\infty(0,T;L^2), \quad (\sqrt{t}\nabla\dot{u}, \sqrt{t}\nabla b_t) \in L^2(0,T;L^2), \end{gathered}$$

satisfies (1.1)–(1.4) in the sense of distributions, in $\mathbb{T}^3\times(0,T)$, and fulfills the initial conditions (1.9) and normalization (1.10).

Remark 1.2. The term "strong solution" refers to the regularity of ρ, u, b specified in Definition 1.1 and to the strong attainment of the initial data. However, since the density is only bounded, the equations (1.1)–(1.4) are understood in the sense of distributions.

We now proceed to state the main result of this paper.

Theorem 1.3. *Let $q \in (3,6)$, and assume that*

$$0 \le \rho_0 \in L^\infty, \quad u_0,\ b_0 \in H^1, \quad \operatorname{div} b_0 = 0\,,$$

and that the normalization (1.10) *holds. Then, there exists a positive time T_0 depending only on a, μ, λ, ν, q, and Φ_0, such that system* (1.1)–(1.4) *has a unique strong solution in $\mathbb{T}^3 \times (0, T_0)$, where $\Phi_0 := \|\rho_0\|_\infty + \|\nabla u_0\|^2 + \|b_0\|_{H^1}^2$.*

The key to the existence part of Theorem 1.3 is to derive some suitable a priori estimates independent of the compatibility condition for the quantity

$$\Phi(t) := 1 + \sup_{0\le s\le t} \left(\|\rho\|_\infty + \|\nabla u\|^2 + \|b\|_{H^1}^2\right) + \int_0^t \|(\sqrt{\rho}\dot{u}, \nabla^2 b, \sqrt{s}\nabla \dot{u})\|^2 \,\mathrm{d}s, \tag{1.14}$$

for any approximate solution (ρ, u, b) to system (1.1)–(1.4) subject to (1.9)–(1.10). To this end, we introduce the effective viscous flux and the vorticity, which allow us to estimate the velocity without the gradient of density $\nabla\rho$. The magnetic coupling produces additional nonlinear terms, and these are handled through the resistive structure of the magnetic equation together with carefully chosen time-weighted energy estimates. More precisely, we prove a smallness condition of the form $T^{\frac{6-q}{4q}} \Phi^{\frac{7q-6}{4q}}(T) \le \epsilon_0$, which implies a uniform bound for $\Phi(T) \le C$, where ϵ_0 and C are two positive constants independent of the compatibility condition, see Corollary 2.8. Moreover, due to the lower regularities on the initial data and the absence of an initial compatibility condition, the regularities of the strong solutions obtained in this paper are weaker than those required in [16, 39] to prove the uniqueness. In order to overcome this difficulty, we first estimate the density difference in a negative Sobolev space and use a logarithmic estimate involving the BMO-norm of ∇u. To avoid the appearance of $\nabla\rho$, we introduce a coupled backward parabolic system, and then carry out some suitable weighted energy differential inequalities, which fit into the framework of a logarithmic Grönwall inequality, to prove the uniqueness.

Next, we present some important lemmas. The following Poincaré inequality proved in [14] will be used frequently.

Lemma 1.4 ([14, Lemma A.2])**.** *Let ρ be in L^s with $\frac{1}{p} + \frac{1}{s} = 1$ and $2 \le p \le 6$. Assume that*

$$\int_{\mathbb{T}^3} \rho v \,\mathrm{d}x = 0 \quad \text{and} \quad M := \int_{\mathbb{T}^3} \rho \,\mathrm{d}x > 0.$$

Then there exists a constant C_p depending on p, such that

$$\|v\| \le \Big(1 + \frac{C_p}{M}\|\rho - c\|_s\Big)\|\nabla v\|, \quad \textit{for a real number } c.$$

Lemma 1.4 immediately yields

$$\|v\|_p \le C_p\Big(1 + \frac{1}{M}\|\rho - c\|_{L^s}\Big)\|\nabla v\|, \quad \forall 2 \le p \le 6.$$

The next lemma gives basic estimates for G and ω.

Lemma 1.5. *Let (ρ, u, b) be a smooth solution of problem* (1.1)–(1.4)*. Then there exists a positive constant C such that for any $q \in (3, 6)$,*

$$\|(\nabla G, \nabla\omega)\| \le C(\|\rho\|_\infty^{1/2}\|\sqrt{\rho}\dot{u}\| + \|b\|_{H^1}^{3/2}\|\nabla^2 b\|^{1/2}),$$
$$\|(\nabla G, \nabla\omega)\|_q \le C(\|\rho\|_\infty^{\frac{5q-6}{4q}}\|\sqrt{\rho}\dot{u}\|^{\frac{6-q}{2q}}\|\nabla\dot{u}\|^{\frac{3q-6}{2q}} + \|b\|_{H^1}^2 + \|b\|_{H^1}^{3/q}\|\nabla^2 b\|^{\frac{2q-3}{q}}).$$

Proof. From the momentum equation (1.2), one has

$$\int_{\mathbb{T}^3} \rho\dot{u} \,\mathrm{d}x = \int_{\mathbb{T}^3} (\nabla \times b) \times b \,\mathrm{d}x.$$

By (1.4), it follows

$$\int_{\mathbb{T}^3} (\nabla \times b) \times b \,\mathrm{d}x = \int_{\mathbb{T}^3} \Big((b\cdot\nabla)b - \frac{1}{2}\nabla|b|^2\Big) \,\mathrm{d}x = 0,$$

which implies

$$\int_{\mathbb{T}^3} \rho\dot{u} \,\mathrm{d}x = 0.$$

The standard L^2-estimate for elliptic system (1.12) directly yields

$$\begin{aligned}\|\nabla G\|+\|\nabla\omega\| &\le C(\|\rho\|_\infty^{1/2}\|\sqrt{\rho}\dot u\|+\|\nabla b\|_3\|b\|_6)\\ &\le C(\|\rho\|_\infty^{1/2}\|\sqrt{\rho}\dot u\|+\|b\|_{H^1}^{3/2}\|\nabla^2 b\|^{1/2}).\end{aligned}$$

Similarly, the Gagliardo-Nirenberg inequality shows that for each $q\in(3,6)$,

$$\begin{aligned}\|\nabla G\|_q+\|\nabla\omega\|_q &\le C(\|\rho\dot u\|_q+\|(\nabla\times b)\times b\|_q)\\ &\le C(\|\rho\dot u\|^{\frac{6-q}{2q}}\|\rho\dot u\|_6^{\frac{3q-6}{2q}}+\|b\|_\infty\|\nabla b\|_q)\\ &\le C(\|\rho\|_\infty^{\frac{5q-6}{4q}}\|\sqrt{\rho}\dot u\|^{\frac{6-q}{2q}}\|\nabla\dot u\|^{\frac{3q-6}{2q}}+\|b\|_{H^1}^2+\|b\|_{H^1}^{3/q}\|\nabla^2 b\|^{\frac{2q-3}{q}}).\end{aligned}$$
$\square$

We also state the following Osgood type lemma to be used in proving uniqueness.

Lemma 1.6 ([1, Lemma 3.4]). *Let $b>0$, and $\mu:[0,b]\to[0,\infty)$ be a continuous nondecreasing function such that $\mu(r)>0$ for $r\in(0,b]$. We say that μ is an Osgood modulus of continuity on $[0,b]$ if*

$$\int_0^b\frac{dr}{\mu(r)}=+\infty.$$

Given $t_0<T$, let $\Gamma\in L^1(t_0,T)$ be a nonnegative function, and $Y:[t_0,T]\to[0,b]$ be a nonnegative measurable function. Assume that, for some real number $c\ge 0$,

$$Y(t)\le c+\int_{t_0}^t\Gamma(s)\mu(Y(s))\,\mathrm{d}s\quad\text{for a.e. } t\in[t_0,T].$$

If c is positive, then we have, for a.e. $t\in[t_0,T]$,

$$-M(Y(t))+M(c)\le\int_{t_0}^t\Gamma(s)\,\mathrm{d}s,\quad M(x):=\int_x^b\frac{\mathrm{d}r}{\mu(r)}.$$

In particular, if $c=0$ and μ is an Osgood modulus of continuity, then

$$Y(t)=0\quad\text{for a.e. } t\in[t_0,T].$$

2. A priori estimates independent of the compatibility condition

In this section, we aim to derive some a priori estimates for strong solutions to system (1.1)–(1.4), with initial data satisfying (1.9)–(1.10) and the compatibility condition. In particular, the a priori estimates established in this section do not depend on the initial compatibility condition. We begin with the following local existence result for the compressible MHD system, which can be proved in the same way as in [16], where the natural compatibility condition is required.

Proposition 2.1. *Let $q\in(3,6]$ and assume that (ρ_0,u_0,b_0) satisfies*

$$0\le\rho_0\in W^{1,q}(\mathbb{T}^3),\quad(u_0,b_0)\in H^2(\mathbb{T}^3),\quad\operatorname{div}b_0=0,$$

and the compatibility condition

$$\mu\Delta u_0+(\mu+\lambda)\nabla\operatorname{div}u_0-\nabla P_0+(\nabla\times b_0)\times b_0=\sqrt{\rho_0}g,$$

for some $g\in L^2(\mathbb{T}^3)$. Then, there exists a positive time T_* depending on a, μ, λ, ν, q, $\|\nabla\rho_0\|_q$, $\|\nabla^2u_0\|$, $\|\nabla^2b_0\|$ and $\|g\|$, such that system (1.1)–(1.4) admits a unique strong solution (ρ,u,b) in $(\mathbb{T}^3)\times(0,T_*)$, satisfying

$$\begin{gathered}\rho\in C([0,T_*];W^{1,q}),\quad\rho_t\in C([0,T_*];L^q),\\ (u,b)\in C([0,T_*];H^1)\cap L^2(0,T_*;W^{2,q}),\\ (u_t,b_t)\in L^2(0,T_*;H^1),\quad(\sqrt{\rho}u_t,b_t)\in L^\infty(0,T_*;L^2).\end{gathered}$$

It will be shown in this section that the existence time T_* in the above proposition can be chosen depending only on a, μ, λ, ν, q, and the upper bound of

$$\Phi_0 := \|\rho_0\|_\infty + \|\nabla u_0\|^2 + \|b_0\|_{H^1}^2.$$

In particular, T_* can be chosen independent of $\|\nabla\rho_0\|_q$, $\|\nabla^2 u_0\|$, $\|\nabla^2 b_0\|$ and $\|g\|$. Let Φ be the quantity given by (1.14). The main content of this section is to obtain the local-in-time estimate of Φ independent of $\|\nabla\rho_0\|_q$, $\|\nabla^2 u_0\|$, $\|\nabla^2 b_0\|$ and $\|g\|$, and therefore independent of the initial compatibility condition.

From here to the last proposition of this section, we assume that (ρ, u, b) is a solution to system (1.1)–(1.4) in $\mathbb{T}^3 \times (0, T)$, for some positive time $T \le 1$, satisfying the regularities in Proposition 2.1 with T_* there replaced by T. We emphasize again that C, which may vary from place to place, is a generic constant depending only on a, μ, λ, ν, q, and the upper bound of Φ_0.

Proposition 2.2. *It holds that*

$$\int_0^T (\|\operatorname{div} u\|_\infty + \|(G,\omega)\|_\infty + \|(\nabla G, \nabla\omega)\| + \|(\nabla G, \nabla\omega)\|_q)\, \mathrm{d}t \le CT^{\frac{6-q}{4q}} \Phi^{\frac{7q-6}{4q}}(T).$$

Proof. It follows from Lemma 1.5 that

$$\begin{aligned}
\int_0^T ||(\nabla G, \nabla\omega)||\, \mathrm{d}t &\le C\int_0^T \Big(\|\rho\|_\infty^{1/2}\|\sqrt{\rho}\dot{u}\| + \|b\|_{H^1}^{3/2}\|\nabla^2 b\|^{1/2}\Big)\, \mathrm{d}t \\
&\le C\Big(T^{1/2}\Phi(T) + T^{\frac{3}{4}}\Phi(T)\Big) \\
&\le CT^{1/2}\Phi(T),
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
\int_0^T ||(\nabla G, \nabla\omega)||_q\, \mathrm{d}t &\le C\int_0^T \|\rho\|_\infty^{\frac{5q-6}{4q}}\|\sqrt{\rho}\dot{u}\|^{\frac{6-q}{2q}}\|\nabla\dot{u}\|^{\frac{3q-6}{2q}}\, \mathrm{d}t \\
&\quad + C\int_0^T \big[\|b\|_{H^1}^2 + \|b\|_{H^1}^{3/q}\|\nabla^2 b\|^{\frac{2q-3}{q}}\big]\, \mathrm{d}t \\
&\le C\Big(T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T) + T\Phi(T) + T^{\frac{3}{2q}}\Phi(T)\Big) \\
&\le CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T),
\end{aligned} \tag{2.2}$$

where we used $\frac{3}{2q} > \frac{6-q}{4q}$ for $q \in (3,6)$, $T \le 1$, and $\Phi(T) \ge 1$. From relation (1.11), one has

$$\|G\| \le C(\|\operatorname{div} u\| + \|\rho\|) \le C(\|\nabla u\| + \|\rho\|_\infty). \tag{2.3}$$

Therefore, by the Gagliardo-Nirenberg and Young inequalities, one obtains

$$\begin{aligned}
\int_0^T \|G\|_\infty\, \mathrm{d}t &\le C\int_0^T (\|G\| + \|\nabla G\|_q)\, \mathrm{d}t \\
&\le C\Big(T\Phi^{1/2}(T) + T\Phi(T) + T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)\Big) \\
&\le CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T),
\end{aligned}$$

and

$$\int_0^T \|\omega\|_\infty\, \mathrm{d}t \le C\int_0^T (\|\omega\| + \|\nabla\omega\|_q)\, \mathrm{d}t \le CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T).$$

Moreover, using again (1.11) yields

$$\int_0^T \|\operatorname{div} u\|_\infty\, \mathrm{d}t \le C\int_0^T (\|G\|_\infty + \|\rho\|_\infty)\, \mathrm{d}t \le CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T).$$

This completes the proof. $\square$

Proposition 2.3. *It holds that*

$$\sup_{0\le t\le T}\|\rho\|_\infty \le \|\rho_0\|_\infty \exp\big\{CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)\big\}.$$

Proof. For any given $x \in \mathbb{T}^3$ and $s \in [0,T]$, we define $U(x,t;s)$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}U(x,t;s) = u(U(x,t;s),t), \quad \forall\, t \in [0,T],$$
$$U(x,s;s) = x.$$

Then, it follows from transport equation (1.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(U(x,t;s),t) = -\operatorname{div} u(U(x,t;s),t)\rho(U(x,t;s),t), \quad \forall\, t \in (0,T).$$

Solving the above ordinary differential equation yields

$$\rho(U(x,t;s),t) = \rho(x,s)e^{-\int_s^t \operatorname{div} u(U(x,\tau;s),\tau)\,\mathrm{d}\tau}, \quad \forall\, s,t \in [0,T].$$

Choosing $t = 0$ in the above, one gets

$$\rho(x,s) = \rho_0(U(x,0;s))e^{-\int_0^s \operatorname{div} u(U(x,\tau;s),\tau)\,\mathrm{d}\tau}, \quad \forall (x,s) \in \mathbb{T}^3 \times [0,T].$$

Therefore, by Proposition 2.2, it holds that

$$\sup_{0\le t\le T} \|\rho\|_\infty(t) \le \|\rho_0\|_\infty e^{\int_0^T \|\operatorname{div} u\|_\infty(\tau)\,\mathrm{d}\tau} \le \|\rho_0\|_\infty e^{CT^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)},$$

which completes the proof. □

As a direct corollary of Proposition 2.3, one obtains:

Corollary 2.4. *There is a sufficiently small positive constant $\epsilon_0 \le 1$ depending only on a, μ, λ, ν, q, and Φ_0, such that*

$$\sup_{0\le t\le T} \|\rho\|_\infty \le \bar{\rho},$$

as long as

$$T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T) \le \epsilon_0, \tag{2.4}$$

where the positive constant $\bar{\rho}$ denotes the upper bound of density.

Since $\Phi(T) \ge 1$, it is easy to check that the following relations hold by the assumption (2.4):

$$T\Phi^3(T) = \Big(T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)\Big)^{\frac{12q}{7q-6}} T^{\frac{10q-24}{7q-6}} \le \epsilon_0^{\frac{12q}{7q-6}} \le 1, \tag{2.5}$$

$$T^{\frac{1}{2}}\Phi^2(T) = \Big(T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)\Big)^{\frac{8q}{7q-6}} T^{\frac{11q-30}{2(7q-6)}} \le \epsilon_0^{\frac{8q}{7q-6}} \le 1, \tag{2.6}$$

which will be frequently used in the rest of this section.

Proposition 2.5. *Let ϵ_0 be the number stated in Corollary 2.4 and assume that* (2.4) *holds. Then, the following estimate holds*

$$\sup_{0\le t\le T} \|b\|^2 + \int_0^T \|\nabla b\|^2\,\mathrm{d}t \le C.$$

Proof. Multiplying (1.3) with b and integrating over $\mathbb{T}^3$, one deduces by the Hölder, Sobolev, and Young inequalities

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\|^2 + \nu\|\nabla b\|^2 = \int_{\mathbb{T}^3} (u\times b)\cdot(\nabla\times b)\,\mathrm{d}x \le C\|u\|_6\|b\|_3\|\nabla b\| \le \frac{\nu}{2}\|\nabla b\|^2 + C\|\nabla u\|^2\|b\|_{H^1}^2.$$

By (2.5), integrating the above inequality over $(0,T)$ gives

$$\begin{aligned}\sup_{0\le t\le T}\|b\|^2 + \nu\int_0^T \|\nabla b\|^2\,\mathrm{d}t &\le \Big(\|b_0\|^2 + C\int_0^T \|\nabla u\|^2\|b\|_{H^1}^2\,\mathrm{d}t\Big)\\ &\le C\Big(1 + T\Phi^2(T)\Big) \le C.\end{aligned}$$

This completes the proof. □

Proposition 2.6. *Under the conditions of Proposition 2.5, it holds that*

$$\sup_{0\le t\le T}\|(\nabla u,\nabla b)\|^2+\int_0^T\left(\|(\nabla G,\sqrt{\rho}\dot u,b_t,\nabla^2 b)\|^2+\|\nabla u\|_6^2\right)\,\mathrm{d}t\le C.$$

Proof. Applying the elliptic estimates to (1.3) and using the Young and Gagliardo-Nirenberg inequalities, one gets

$$\begin{aligned}\|\nabla^2 b\|^2&\le C(\|b_t\|^2+\|\nabla\times(u\times b)\|^2)\\&\le C(\|b_t\|^2+\|b\|_\infty^2\|\nabla u\|^2+\|u\|_6^2\|\nabla b\|_3^2)\\&\le C(\|b_t\|^2+(\|b\|_{H^1}^2+\|b\|_{H^1}\|\nabla^2 b\|)\|\nabla u\|^2+\|\nabla b\|\|\nabla^2 b\|\|\nabla u\|^2)\\&\le\frac12\|\nabla^2 b\|^2+C\|b_t\|^2+C\|b\|_{H^1}^2(1+\|\nabla u\|^4),\end{aligned}$$

and, thus

$$\|\nabla^2 b\|^2\le C\|b_t\|^2+C\|b\|_{H^1}^2(1+\|\nabla u\|^4).\tag{2.7}$$

Multiplying (1.3) with b_t, and integrating it over $\mathbb{T}^3$, one obtains by the Young, Sobolev and Gagliardo-Nirenberg inequalities

$$\begin{aligned}\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b\|^2+\|b_t\|^2&=\int_{\mathbb{T}^3}\nabla\times(u\times b)\cdot b_t\,\mathrm{d}x\\&\le C(\|u\|_6\|\nabla b\|_3+\|b\|_\infty\|\nabla u\|)\|b_t\|\\&\le\eta_1(\|b_t\|^2+\|\nabla^2 b\|^2)+C\|b\|_{H^1}^2(1+\|\nabla u\|^4),\end{aligned}$$

for any positive number η_1. Combining the above estimates with (2.7) multiplied with a small positive number ϵ_1, choosing η_1 sufficiently small, and integrating the resultant over $(0,T)$, one has by Proposition 2.5 and (2.5)

$$\begin{aligned}\nu\sup_{0\le t\le T}\|\nabla b\|^2+\int_0^T(\|b_t\|^2+\epsilon_1\|\nabla^2 b\|^2)\,\mathrm{d}t&\le C\Big(\|\nabla b_0\|^2+\int_0^T\|b\|_{H^1}^2(1+\|\nabla u\|^4)\,\mathrm{d}t\Big)\\&\le C\left(1+T\Phi^3(T)\right)\le C.\end{aligned}\tag{2.8}$$

Since $\int_{\mathbb{T}^3}\omega\,\mathrm{d}x=0$, one deduces by (1.13), Lemma 1.5 and Corollary 2.4 that

$$\begin{aligned}\|\nabla u\|_6^2&\le C(\|P\|_6^2+\|G\|_6^2+\|\omega\|_6^2)\\&\le C(\|P\|_\infty^2+\|G\|^2+\|\nabla G\|^2+\|\nabla\omega\|^2)\\&\le C(1+\|\nabla u\|^2+\|\sqrt{\rho}\dot u\|^2+\|b\|_{H^1}^3\|\nabla^2 b\|).\end{aligned}\tag{2.9}$$

For the pressure term, integrating by parts yields

$$\begin{aligned}\int_{\mathbb{T}^3}\nabla P u_t\,\mathrm{d}x&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}P\mathrm{div}u\,\mathrm{d}x+\int_{\mathbb{T}^3}P_t\mathrm{div}u\,\mathrm{d}x\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}P\,\mathrm{div}\,u\,\mathrm{d}x+\frac{1}{2\mu+\lambda}\int_{\mathbb{T}^3}P_t(G+P)\,\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2(2\mu+\lambda)}\|P\|^2-\int_{\mathbb{T}^3}P\,\mathrm{div}\,u\,\mathrm{d}x\Big)+\frac{a}{2\mu+\lambda}\int_{\mathbb{T}^3}\rho u\cdot\nabla G\,\mathrm{d}x.\end{aligned}$$

By the same token, we deal with the Lorentz force term as follows:

$$\begin{aligned}\int_{\mathbb{T}^3}(\nabla\times b)\times b\cdot u_t\,\mathrm{d}x&=\int_{\mathbb{T}^3}\Big((b\cdot\nabla)b-\frac12\nabla|b|^2\Big)\cdot u_t\,\mathrm{d}x,\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\Big((b\cdot\nabla)u\cdot b-\frac12|b|^2\,\mathrm{div}\,u\Big)\,\mathrm{d}x\\&\quad+\int_{\mathbb{T}^3}\Big((b_t\cdot\nabla)u\cdot b+(b\cdot\nabla)u\cdot b_t-\frac12\partial_t|b|^2\,\mathrm{div}\,u\Big)\,\mathrm{d}x\end{aligned}$$

Multiplying (1.2) with u_t, integrating over $\mathbb{T}^3$, and using the above two identities, it follows from (1.11) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mu}{2}\|\nabla u\|^2+\frac{\mu+\lambda}{2}\|\operatorname{div} u\|^2+\frac{1}{2(2\mu+\lambda)}\|P\|^2\Big)\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\Big((b\cdot\nabla)u\cdot b-\frac{1}{2}|b|^2\operatorname{div} u-P\operatorname{div} u\Big)\,\mathrm{d}x+\|\sqrt{\rho}\dot u\|^2\\
&=-\frac{a}{2\mu+\lambda}\int_{\mathbb{T}^3}\rho u\cdot\nabla G\,\mathrm{d}x+\int_{\mathbb{T}^3}\rho\dot u\cdot(u\cdot\nabla)u\,\mathrm{d}x-\frac{1}{2}\int_{\mathbb{T}^3}\partial_t|b|^2\operatorname{div} u\,\mathrm{d}x\\
&\quad+\int_{\mathbb{T}^3}\Big((b_t\cdot\nabla)u\cdot b+(b\cdot\nabla)u\cdot b_t\Big)\,\mathrm{d}x.
\end{aligned}\tag{2.10}
$$

By Lemmas 1.4 and 1.5 and Corollary 2.4, it follows from the Gagliardo-Nirenberg, Sobolev, and Young inequalities that

$$
\begin{aligned}
\Big|\frac{a}{2\mu+\lambda}\int_{\mathbb{T}^3}\rho u\cdot\nabla G\,\mathrm{d}x\Big| &\le C\|\rho\|_3\|u\|_6\|\nabla G\|\\
&\le C\bar\rho\|\nabla u\|(\|\sqrt{\rho}\dot u\|+\|b\|_{H^1}^{3/2}\|\nabla^2 b\|^{1/2})\\
&\le \frac{1}{8}\|\sqrt{\rho}\dot u\|^2+C(\|\nabla u\|^2+\|b\|_{H^1}^3\|\nabla^2 b\|),\\
\Big|\int_{\mathbb{T}^3}\rho\dot u\cdot(u\cdot\nabla)u\,\mathrm{d}x\Big| &\le C\sqrt{\bar\rho}\|\sqrt{\rho}\dot u\|\|u\|_6\|\nabla u\|_3\\
&\le C\|\sqrt{\rho}\dot u\|\|\nabla u\|^{3/2}\|\nabla u\|_6^{1/2}\\
&\le \frac{1}{8}\|\sqrt{\rho}\dot u\|^2+\eta_2\|\nabla u\|_6^2+C\|\nabla u\|^6,\\
\Big|\frac{1}{2}\int_{\mathbb{T}^3}\partial_t|b|^2\operatorname{div} u\,\mathrm{d}x\Big| &\le C\|b_t\|\|\nabla u\|_3\|b\|_6\\
&\le \eta_2\|\nabla u\|_6^2+C(\|b_t\|^2+\|\nabla u\|^2\|b\|_{H^1}^4),\\
\Big|\int_{\mathbb{T}^3}\Big((b_t\cdot\nabla)u\cdot b+(b\cdot\nabla)u\cdot b_t\Big)\,\mathrm{d}x\Big| &\le \eta_2\|\nabla u\|_6^2+C(\|b_t\|^2+\|\nabla u\|^2\|b\|_{H^1}^4),
\end{aligned}
$$

for any positive number η_2. Plugging the above estimates into (2.10), adding the resulting inequality and (2.9) multiplied with a small positive number ϵ_2, and choosing η_2 sufficiently small, one obtains

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mu}{2}\|\nabla u\|^2+\frac{\mu+\lambda}{2}\|\operatorname{div} u\|^2+\frac{1}{2(2\mu+\lambda)}\|P\|^2\Big)\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^3}\Big((b\cdot\nabla)u\cdot b-\frac{1}{2}|b|^2\operatorname{div} u-P\operatorname{div} u\Big)\,\mathrm{d}x+\frac{1}{2}\|\sqrt{\rho}\dot u\|^2+\frac{\epsilon_2}{2}\|\nabla u\|_6^2\\
&\le C\left(1+\|b_t\|^2+\|\nabla u\|^6+\|\nabla u\|^2\|b\|_{H^1}^4+\|b\|_{H^1}^3\|\nabla^2 b\|\right).
\end{aligned}
$$

Integrating the above inequality over $(0,T)$, one deduces by Proposition 2.5 and (2.8) that

$$
\begin{aligned}
&\mu\sup_{0\le t\le T}\|\nabla u\|^2+\int_0^T\left(\|\sqrt{\rho}\dot u\|^2+\epsilon_2\|\nabla u\|_6^2\right)\mathrm{d}t\\
&\le 2\sup_{0\le t\le T}\Big(\Big|\int_{\mathbb{T}^3}(b\cdot\nabla)u\cdot b\,\mathrm{d}x\Big|+\frac{1}{2}\Big|\int_{\mathbb{T}^3}|b|^2\operatorname{div} u\,\mathrm{d}x\Big|+\Big|\int_{\mathbb{T}^3}P\operatorname{div} u\,\mathrm{d}x\Big|\Big)\\
&\quad+C\Big(\|\nabla u_0\|^2+\|\rho_0\|^2+\Big|\int_{\mathbb{T}^3}(b_0\cdot\nabla)u_0\cdot b_0\,\mathrm{d}x\Big|+\Big|\int_{\mathbb{T}^3}|b_0|^2\operatorname{div} u_0\,\mathrm{d}x\Big|+\Big|\int_{\mathbb{T}^3}P_0\operatorname{div} u_0\,\mathrm{d}x\Big|\Big)\\
&\quad+C\int_0^T\left(1+\|b_t\|^2+\|\nabla u\|^6+\|\nabla u\|^2\|b\|_{H^1}^4+\|b\|_{H^1}^3\|\nabla^2 b\|\right)\mathrm{d}t\\
&\le \frac{\mu}{2}\sup_{0\le t\le T}\|\nabla u\|^2+C\Big[1+\sup_{0\le t\le T}\left(\|\rho\|^2+\|b\|_{H^1}^4\right)+T\Phi^3(T)+T^{1/2}\Phi^2(T)\Big]
\end{aligned}
$$

$$\le \frac{\mu}{2} \sup_{0\le t\le T} \|\nabla u\|^2 + C\left(1 + T\Phi^3(T) + T^{1/2}\Phi^2(T)\right),$$

by (2.5)–(2.6), which implies

$$\sup_{0\le t\le T} \|\nabla u\|^2 + \int_0^T \left(\|\sqrt{\rho}\dot{u}\|^2 + \|\nabla u\|_6^2\right) \mathrm{d}t \le C.$$

Moreover, from Lemma 1.5, it follows that

$$\int_0^T \|\nabla G\|^2 \,\mathrm{d}t \le C\int_0^T (\|\rho\|_\infty \|\sqrt{\rho}\dot{u}\|^2 + \|b\|_{H^1}^3 \|\nabla^2 b\|)\,\mathrm{d}t \le C\Big(1 + T^{1/2}\Phi^2(T)\Big) \le C.$$

Combining the above two estimates and (2.8) yields the desired conclusion. □

Proposition 2.7. *Under the conditions of Proposition 2.5, it holds that*

$$\sup_{0\le t\le T} \Big(\|(\sqrt{t}\sqrt{\rho}\dot{u}, \sqrt{t}b_t, \sqrt{t}\nabla^2 b)\|^2 + \|\sqrt{t}\nabla u\|_6^2\Big) + \int_0^T \|(\sqrt{t}\nabla\dot{u}, \sqrt{t}\nabla b_t)\|^2 \,\mathrm{d}t \le C.$$

Proof. Differentiating (1.3) with respect to t yields

$$b_{tt} - \nu\Delta b_t = \nabla\times(u_t\times b) + \nabla\times(u\times b_t).$$

Multiplying the above identity with tb_t and integrating over $\mathbb{T}^3\times(0,T)$ yields

$$\begin{aligned} &\frac{1}{2}\sup_{0\le t\le T} \|\sqrt{t}b_t\|^2 + \nu\int_0^T \|\sqrt{t}\nabla b_t\|^2 \,\mathrm{d}t \\ &= \frac{1}{2}\int_0^T \|b_t\|^2 \,\mathrm{d}t + \int_0^T t\int_{\mathbb{T}^3} (u_t\times b)\cdot(\nabla\times b_t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T t\int_{\mathbb{T}^3} (u\times b_t)\cdot(\nabla\times b_t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.11}$$

By the Young and Gagliardo-Nirenberg inequalities, and Propositions 2.5–2.6, we obtain

$$\begin{aligned} &\Big|\int_0^T t\int_{\mathbb{T}^3} (u_t\times b)\cdot(\nabla\times b_t)\,\mathrm{d}x\,\mathrm{d}t\Big| \\ &\le \int_0^T t\|u_t\|_6\|b\|_3\|\nabla b_t\|\,\mathrm{d}t \\ &\le \int_0^T (\|\sqrt{t}\dot{u}\|_6 + \|\sqrt{t}u\|_\infty\|\nabla u\|_6)\|b\|_{H^1}\|\sqrt{t}\nabla b_t\|\,\mathrm{d}t \\ &\le \frac{\nu}{4}\int_0^T \|\sqrt{t}\nabla b_t\|^2 \,\mathrm{d}t + C\int_0^T (\|\sqrt{t}\nabla\dot{u}\|^2 + \|\nabla u\|\|\sqrt{t}\nabla u\|_6\|\nabla u\|_6^2)\|b\|_{H^1}^2 \,\mathrm{d}t \\ &\le \frac{\nu}{4}\int_0^T \|\sqrt{t}\nabla b_t\|^2 \,\mathrm{d}t + C\int_0^T \|\sqrt{t}\nabla\dot{u}\|^2 \,\mathrm{d}t + C\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6. \end{aligned}$$

Since

$$\int_{\mathbb{T}^3} b_t \,\mathrm{d}x = \int_{\mathbb{T}^3} \nabla\times(u\times b)\,\mathrm{d}x + \nu\int_{\mathbb{T}^3} \Delta b\,\mathrm{d}x = 0,$$

one obtains

$$\begin{aligned} \Big|\int_0^T t\int_{\mathbb{T}^3} (u\times b_t)\cdot(\nabla\times b_t)\,\mathrm{d}x\,\mathrm{d}t\Big| &\le \int_0^T t\|u\|_6\|b_t\|_3\|\nabla b_t\|\,\mathrm{d}t \\ &\le C\int_0^T t^{\frac{1}{4}}\|\nabla u\|\|b_t\|^{1/2}\|\sqrt{t}\nabla b_t\|^{3/2}\,\mathrm{d}t \\ &\le \frac{\nu}{4}\int_0^T \|\sqrt{t}\nabla b_t\|^2 \,\mathrm{d}t + C\int_0^T t\|\nabla u\|^4\|b_t\|^2 \,\mathrm{d}t \\ &\le \frac{\nu}{4}\int_0^T \|\sqrt{t}\nabla b_t\|^2 \,\mathrm{d}t + CT\Phi^3(T). \end{aligned}$$

By (2.5) and Proposition 2.6, combining these above estimates with (2.11) entails

$$\sup_{0\le t\le T}\|\sqrt{t}b_t\|^2+\nu\int_0^T\|\sqrt{t}\nabla b_t\|^2\,\mathrm{d}t\le C\int_0^T\|\sqrt{t}\nabla\dot u\|^2\,\mathrm{d}t+C\Big(\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6+1\Big). \tag{2.12}$$

Recalling (2.7), it follows from Proposition 2.6 and the Young inequality that

$$\begin{aligned}\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 b\|^2&\le C\sup_{0\le t\le T}\Big(\|\sqrt{t}b_t\|^2+t\|b\|_{H^1}^2(1+\|\nabla u\|^4)\Big)\\&\le C\sup_{0\le t\le T}\Big(\|\sqrt{t}b_t\|^2+T\Phi^3(T)+1\Big)\\&\le C\int_0^T\|\sqrt{t}\nabla\dot u\|^2\,\mathrm{d}t+C\Big(\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6+1\Big).\end{aligned}\tag{2.13}$$

Direct computations yield

$$\begin{aligned}\partial_t(\rho\dot u_i)+\operatorname{div}(u\rho\dot u_i)&=\rho(\partial_t\dot u_i+u\cdot\nabla\dot u_i)\\\partial_t(\partial_i P)+\operatorname{div}(u\partial_i P)&=-\operatorname{div}((\nabla u)^T P),\\\partial_t(\Delta u_i)+\operatorname{div}(u\Delta u_i)&=\Delta\dot u_i-\Delta(u\cdot\nabla u_i)+\operatorname{div}(u\Delta u_i)\\&=\Delta\dot u_i+\operatorname{div}[\nabla u(\operatorname{div}uI-\nabla u-(\nabla u)^T)],\\\partial_t(\partial_i\operatorname{div}u)+\operatorname{div}(u\partial_i\operatorname{div}u)&=\partial_i\operatorname{div}\dot u-\partial_i\operatorname{div}(u\cdot\nabla u)+\operatorname{div}(u\partial_i\operatorname{div}u),\\&\partial_t((\nabla\times b)\times b)_i+\partial_j[((\nabla\times b)\times b)_i u_j]\\&=\partial_t\Big(b_j\partial_j b_i-\frac12\partial_i|b|^2\Big)+\partial_k\Big[\Big(b_j\partial_j b_i-\frac12\partial_i|b|^2\Big)u_k\Big].\end{aligned}$$

Hence, applying the operator $\partial_t(\cdot)+\operatorname{div}(u\cdot)$ to (1.2), we rewrite (1.2) as the following identity:

$$\begin{aligned}&\rho(\dot u_t+u\cdot\nabla\dot u)-\mu\Delta\dot u-(\mu+\lambda)\nabla\operatorname{div}\dot u\\&=\operatorname{div}((\nabla u)^T P)+\mu\operatorname{div}[\nabla u(\operatorname{div}uI-\nabla u-(\nabla u)^T)]\\&\quad+(\mu+\lambda)\operatorname{div}[((\operatorname{div}u)^2-\nabla u:(\nabla u)^T)I-\operatorname{div}u(\nabla u)^T]\\&\quad+\partial_t\Big(b\cdot\nabla b-\frac12\nabla|b|^2\Big)+\operatorname{div}\Big[\Big(b\cdot\nabla b-\frac12\nabla|b|^2\Big)\otimes u\Big].\end{aligned}\tag{2.14}$$

Multiplying (2.14) with $t\dot u$, and integrating over $\mathbb{T}^3\times(0,T)$ yields

$$\begin{aligned}&\frac12\sup_{0\le t\le T}\|\sqrt{t}\sqrt{\rho}\dot u\|^2+\mu\int_0^T\|\sqrt{t}\nabla\dot u\|^2\,\mathrm{d}t+(\mu+\lambda)\int_0^T\|\sqrt{t}\operatorname{div}\dot u\|^2\,\mathrm{d}t\\&\le\frac12\int_0^T\|\sqrt{\rho}\dot u\|^2\,\mathrm{d}t+\int_0^T t\int_{\mathbb{T}^3}\operatorname{div}[(\nabla u)^T P+\mu(\nabla u(\operatorname{div}uI-\nabla u-(\nabla u)^T))]\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t\\&\quad+(\mu+\lambda)\int_0^T t\int_{\mathbb{T}^3}\operatorname{div}[((\operatorname{div}u)^2-\nabla u:(\nabla u)^T)I-\operatorname{div}u(\nabla u)^T]\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T t\int_{\mathbb{T}^3}\partial_t\Big(b\cdot\nabla b-\frac12\nabla|b|^2\Big)\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T t\int_{\mathbb{T}^3}\operatorname{div}\Big[\Big(b\cdot\nabla b-\frac12\nabla|b|^2\Big)\otimes u\Big]\cdot\dot u\,\mathrm{d}x\,\mathrm{d}t=:\sum_{i=1}^5 J_i.\end{aligned}\tag{2.15}$$

By the Cauchy and Gagliardo-Nirenberg inequalities, it follows from Corollary 2.4 and Propositions 2.5–2.6 that $J_1 \le C$, and

$$
\begin{aligned}
|J_2| &\le C\int_0^T t(\|P\|_\infty\|\nabla u\| + \|\nabla u\|_3\|\nabla u\|_6)\|\nabla\dot u\|\,\mathrm{d}t\\
&\le C\int_0^T (\sqrt{t}\bar\rho\|\nabla u\| + \|\nabla u\|^{1/2}\|\nabla u\|_6^{1/2}\|\sqrt{t}\nabla u\|_6)\|\sqrt{t}\nabla\dot u\|\,\mathrm{d}t\\
&\le C\Big(T\Phi(T) + T^{\frac14}\Phi(T)\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6\Big),
\end{aligned}
$$

$$
|J_3| \le C\int_0^T \|\nabla u\|_3\|\sqrt{t}\nabla u\|_6\|\sqrt{t}\nabla\dot u\|\,\mathrm{d}t \le CT^{\frac14}\Phi(T)\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6,
$$

$$
\begin{aligned}
|J_4| &\le C\int_0^T t\|b_t\|_3\|b\|_6\|\nabla\dot u\|\,\mathrm{d}t\\
&\le C\int_0^T t^{\frac14}\|b_t\|^{1/2}\|\sqrt{t}\nabla b_t\|^{1/2}\|b\|_{H^1}\|\sqrt{t}\nabla\dot u\|\,\mathrm{d}t\\
&\le \frac{\mu}{4}\int_0^T\|\sqrt{t}\nabla\dot u\|^2\,\mathrm{d}t + \eta_3\int_0^T\|\sqrt{t}\nabla b_t\|^2\,\mathrm{d}t + \int_0^T t\|b\|_{H^1}^4\|b_t\|^2\,\mathrm{d}t\\
&\le \frac{\mu}{4}\int_0^T\|\sqrt{t}\nabla\dot u\|^2\,\mathrm{d}t + \eta_3\int_0^T\|\sqrt{t}\nabla b_t\|^2\,\mathrm{d}t + C,
\end{aligned}
$$

$$
\begin{aligned}
|J_5| &\le C\int_0^T t\|u\|_6\|b\|_6\|\nabla b\|_6\|\nabla\dot u\|\,\mathrm{d}t\\
&\le C\int_0^T \sqrt{t}\|\nabla u\|(\|b\|_{H^1}^2 + \|b\|_{H^1}\|\nabla^2 b\|)\|\sqrt{t}\nabla\dot u\|\,\mathrm{d}t\\
&\le CT^{\frac12}\Phi^2(T),
\end{aligned}
$$

for any positive number η_3. Plugging the above estimates into (2.15), and choosing η_3 small enough, it follows from Proposition 2.6, (2.5)–(2.6) and (2.12) that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\sqrt{t}\sqrt{\rho}\dot u\|^2 + \mu\int_0^T\|\sqrt{t}\nabla\dot u\|^2\,\mathrm{d}t\\
&\le C\Big(1 + T^{\frac14}\Phi(T)\Big)\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6 + C\Big(1 + T\Phi(T) + T^{\frac12}\Phi^2(T)\Big)\\
&\le C\Big(\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6 + 1\Big).
\end{aligned} \tag{2.16}
$$

Recalling (2.9), it follows from Propositions 2.5–2.6 and (2.13), and the Young inequality that

$$
\begin{aligned}
\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6^2 &\le C\sup_{0\le t\le T}\Big(t + t\|\nabla u\|^2 + \|\sqrt{t}\sqrt{\rho}\dot u\|^2 + t\|b\|_{H^1}^6 + \|\sqrt{t}\nabla^2 b\|^2\Big)\\
&\le C\sup_{0\le t\le T}\Big(\|\sqrt{t}\sqrt{\rho}\dot u\|^2 + \|\sqrt{t}\nabla u\|_6 + T\Phi^3(T) + 1\Big)\\
&\le C\sup_{0\le t\le T}\Big(\|\sqrt{t}\nabla u\|_6 + 1\Big)\\
&\le \frac12\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6^2 + C,
\end{aligned}
$$

and, thus, $\sup_{0\le t\le T}\|\sqrt{t}\nabla u\|_6^2 \le C$. With the aid of this, the conclusion follows from (2.12)–(2.13) and (2.16). $\square$

As a direct corollary of Corollary 2.4 and Propositions 2.5–2.7, we have the following result.

Corollary 2.8. *Under the assumptions of Proposition 2.5, it holds that*

$$\Phi(T)+\sup_{0\le t\le T}\Big(\|(G,\sqrt{t}\sqrt{\rho}\dot u,\sqrt{t}b_t,\sqrt{t}\nabla^2 b)\|^2+\|\sqrt{t}\nabla u\|_6^2\Big)$$
$$+\int_0^T\Big(\|(b_t,\sqrt{t}\nabla b_t,\nabla G)\|^2+\|\nabla u\|_6^2\Big)\,\mathrm{d}t\le C,$$

that is

$$\sup_{0\le t\le T}\Big(\|\rho\|_\infty+\|(\nabla u,\nabla b,G,\sqrt{t}\sqrt{\rho}\dot u,\sqrt{t}\nabla^2 b,\sqrt{t}b_t)\|^2+\|\sqrt{t}\nabla u\|_6^2\Big)$$
$$+\int_0^T\Big(\|(\sqrt{\rho}\dot u,\sqrt{t}\nabla\dot u,b_t,\sqrt{t}\nabla b_t,\nabla^2 b,\nabla G)\|^2+\|\nabla u\|_6^2\Big)\,\mathrm{d}t\le C.$$

We end this section by proving in the next proposition that the existence time T_0 depends only on a,μ,λ,ν,q, and the upper bound of Φ_0, but is independent of the quantities $\|\nabla\rho_0\|_q$, $\|\nabla^2 u_0\|$, $\|\nabla^2 b_0\|$ and $\|g\|$.

Proposition 2.9. *Let $q\in(3,6)$ and assume that (ρ_0,u_0,b_0) satisfies*

$$0<\underline{\rho}\le\rho_0\in W^{1,q},\quad u_0,b_0\in H^2,\quad \operatorname{div} b_0=0,$$
$$\int_{\mathbb{T}^3}\rho_0\,\mathrm{d}x=1,\quad \int_{\mathbb{T}^3}\rho_0u_0\,\mathrm{d}x=0.$$

Then, there exist two positive constants T_0 and C depending only on a,μ,λ,ν,q, and the upper bound of Φ_0, such that system (1.1)–(1.4)*, subject to* (1.9)–(1.10)*, admits a unique solution (ρ,u,b), in $\mathbb{T}^3\times(0,T_0)$, satisfying*

$$\sup_{0\le t\le T_0}\Big(\|\rho\|_\infty+\|(\nabla u,\nabla b,G,\sqrt{t}\sqrt{\rho}\dot u,\sqrt{t}\nabla^2 b,\sqrt{t}b_t)\|^2+\|\sqrt{t}\nabla u\|_6^2\Big)\le C,$$

and

$$\int_0^{T_0}\Big(\|(\sqrt{\rho}\dot u,\sqrt{t}\nabla\dot u,b_t,\sqrt{t}\nabla b_t,\nabla^2 b,\nabla G)\|^2+\|\nabla u\|_6^2\Big)\,\mathrm{d}t\le C.$$

Proof. By Proposition 2.1, there is a unique local strong solution (ρ,u,b) in $\mathbb{T}^3\times(0,T_*)$ satisfying the regularities stated in Proposition 2.1. By applying Proposition 2.1 inductively, one can extend the local solution uniquely to the maximal time of existence $T_{\max}$. Then, the following holds

$$\sup_{T_*\le t<T_{\max}}\Big(\|\frac{1}{\rho}\|_\infty+\|\rho\|_{W^{1,q}}+\|u\|_{H^2}+\|b\|_{H^2}\Big)=\infty. \tag{2.17}$$

For any $T\in(0,T_{\max})$, (ρ,u,b) satisfies the regularities in Proposition 2.1 with T_* there replaced by T. Let ϵ_0 be the constant stated in Corollary 2.4, Φ the function given by (1.14), and set

$$T_0=\sup\Big\{T\in(0,T_{\max}):T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)\le\epsilon_0\Big\}.$$

Claim: $T_0<T_{\max}$. Assume by contradiction that $T_0=T_{\max}$. Then, by definition

$$T^{\frac{6-q}{4q}}\Phi^{\frac{7q-6}{4q}}(T)\le\epsilon_0,\quad \forall T\in(0,T_{\max}). \tag{2.18}$$

Since $\Phi(T)\ge 1$, it follows from (2.18) that $T_{\max}\le\epsilon_0^{\frac{4q}{6-q}}$. Thanks to (2.18), it follows from Corollary 2.8 that

$$\Phi(T)+\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 b\|^2\le C,\quad \forall T\in(0,T_{\max}), \tag{2.19}$$

for a positive constant C independent of $T\in(0,T_{\max})$. By following the arguments in Proposition 2.3, one deduces by Proposition 2.2 and (2.18) that

$$\inf_{\mathbb{T}^3\times(0,T)}\rho\ge\underline{\rho}e^{-\int_0^T\|\operatorname{div}u\|_\infty dt}\ge\underline{\rho}e^{-C\epsilon_0},\quad \forall T\in(0,T_{\max}), \tag{2.20}$$

for a positive constant C independent of $T\in(0,T_{\max})$. Differentiating the continuity equation gives

$$\partial_t\nabla\rho+(u\cdot\nabla)\nabla\rho+(\nabla u)^T\nabla\rho+(\operatorname{div}u)\nabla\rho+\rho\nabla\operatorname{div}u=0.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_q \le C\|\nabla u\|_\infty\|\nabla\rho\|_q + C\|\rho\|_\infty\|\nabla\operatorname{div} u\|_q.$$

Since

$$G = (2\mu+\lambda)\operatorname{div} u - a\rho,$$

we have

$$\|\nabla\operatorname{div} u\|_q \le C\|\nabla G\|_q + C\|\nabla\rho\|_q.$$

By the logarithmic and elliptic estimates

$$\begin{aligned}
\|\nabla u\|_\infty &\le C\Big(1 + \|\operatorname{div} u\|_\infty + \|\omega\|_\infty\Big)\log\left(e + \|\nabla^2 u\|_q\right) \\
&\le C\Big(1 + \|\nabla u\| + \|\nabla G\|_q + \|\nabla\omega\|_q\Big)\log\left(e + \|\nabla^2 u\|_q\right), \\
\|\nabla^2 u\|_q &\le C\left(\|\nabla G\|_q + \|\nabla\omega\|_q + \|\nabla\rho\|_q\right).
\end{aligned}$$

By the bounds on ∇G and $\nabla\omega$ following from Lemma 1.5, we obtain

$$Y'(t) \le CA(t)Y(t)\log Y(t).$$

We set

$$Y(t) := e + \|\nabla\rho(t)\|_q, \quad A_0(t) := 1 + \|\nabla u\| + \|\nabla G\|_q + \|\nabla\omega\|_q.$$

Since $Y(t) \ge e$, it follows that

$$\log\big(e + \|\nabla^2 u\|_q\big) \le C\left(1 + \log(e + A_0(t)) + \log Y(t)\right).$$

Consequently,

$$Y'(t) \le CA_0(t)\big(1 + \log(e + A_0(t))\big)Y(t)\log Y(t).$$

Defining

$$A(t) := A_0(t)\big(1 + \log(e + A_0(t))\big),$$

it follows from Proposition 2.2 that $A(t) \in L^1(0, T_{\max})$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\log Y(t) = \frac{Y'(t)}{Y(t)\log Y(t)} \le CA(t).$$

By Proposition 2.2, $A \in L^1(0, T_{\max})$. Integrating the preceding inequality gives

$$\log\log Y(t) \le \log\log Y(0) + C\int_0^t A(s)\,\mathrm{d}s.$$

Exponentiating once, we obtain

$$\log Y(t) \le \log Y(0)\exp\Big(C\int_0^t A(s)\,\mathrm{d}s\Big).$$

Therefore

$$\sup_{0<t<T_{\max}} \|\rho\|_{W^{1,q}} < \infty. \tag{2.21}$$

Next, we control u in H^2. The elliptic estimate applied to (1.2) entails

$$\begin{aligned}
\|u\|_{H^2} &\le C\left(\|\rho\dot{u}\| + \|\nabla P\| + \|(\nabla\times b)\times b\| + \|u\|\right) \\
&\le C\left(\|\rho\|_\infty^{1/2}\|\sqrt{\rho}\dot{u}\| + \|\nabla\rho\| + \|b\|_{H^1}^{3/2}\|\nabla^2 b\|^{1/2} + \|\nabla u\|\right), \quad \forall t \ge T_*,
\end{aligned}$$

by (2.19) and (2.21), which gives

$$\sup_{T_*<t<T_{\max}} \|u\|_{H^2} < \infty. \tag{2.22}$$

Therefore, combining (2.19)–(2.22) yields

$$\sup_{T_*\le t<T_{\max}} \Big(\|\frac{1}{\rho}\|_\infty + \|\rho\|_{W^{1,q}} + \|u\|_{H^2} + \|b\|_{H^2}\Big) \le C,$$

which contradicts (2.17). This contradiction proves the claim.

Since $T_0 < T_{\max}$ and noticing that $\Phi(T)$ is continuous on $[0, T_{\max})$, one gets by the definition of T_0 such that

$$T_0^{\frac{6-q}{4q}} \Phi^{\frac{7q-6}{4q}}(T_0) = \epsilon_0. \tag{2.23}$$

Thanks to this and recalling that $\Phi(T) \geq 1$, it follows from Corollary 2.8 that $T_0 \leq \epsilon_0^{\frac{4q}{6-q}}$ and $\Phi(T_0) \leq C_0$ for a positive constant C_0 depending only on a, μ, λ, ν, q, and the upper bound of Φ_0. Therefore, it follows from (2.23) that $T_0 \geq \big(\frac{\epsilon_0}{C_0^{\frac{7q-6}{4q}}}\big)^{\frac{4q}{6-q}}$. The corresponding estimates follow from Corollary 2.8. This completes the proof. $\square$

3. Proof of Theorem 1.3

3.1. **Existence.** The proof is divided into 6 steps:

Step 1. Construction of the initial data. We proceed to construct the initial data in such a way that the initial density is bounded away from zero, and the approximate initial data $(\rho_{0n}, u_{0n}, b_{0n})$ satisfy (1.9) and (1.10). Choose a sequence $\varepsilon_n \to 0^+$ and let j_{ε_n} be a standard mollifier on $\mathbb{T}^3$. We assume, without loss of generality, that $|\mathbb{T}^3| = 1$. We define $\tilde{\rho}_{0n} = j_{\varepsilon_n} * \rho_0$. Then we have

$$\tilde{\rho}_{0n} \in C^\infty, \quad \tilde{\rho}_{0n} \geq 0 \quad \text{and} \quad \|\tilde{\rho}_{0n}\|_\infty \leq \|\rho_0\|_\infty, \tag{3.1}$$

and for $1 \leq p < \infty$,

$$\int_{\mathbb{T}^3} \tilde{\rho}_{0n} \, \mathrm{d}x = \int_{\mathbb{T}^3} \rho_0 \, \mathrm{d}x = 1 \quad \text{and} \quad \tilde{\rho}_{0n} \to \rho_0 \quad \text{in } L^p. \tag{3.2}$$

Let $\alpha_n > 0$ satisfy $\alpha_n \to 0$ as $n \to \infty$, and define

$$\rho_{0n} := \frac{\tilde{\rho}_{0n} + \alpha_n}{1 + \alpha_n}.$$

Then $\rho_{0n} \in C^\infty \subset W^{1,q}$ for any $q \in (3, 6)$, and

$$\rho_{0n} \geq \frac{\alpha_n}{1 + \alpha_n} > 0.$$

Moreover, relation (3.2) gives

$$\int_{\mathbb{T}^3} \rho_{0n} \, \mathrm{d}x = \frac{1}{1 + \alpha_n} \int_{\mathbb{T}^3} (\tilde{\rho}_{0n} + \alpha_n) \, \mathrm{d}x = 1. \tag{3.3}$$

Also, for all $1 \leq p < \infty$,

$$\rho_{0n} \to \rho_0 \quad \text{in } L^p. \tag{3.4}$$

Indeed,

$$\rho_{0n} - \rho_0 = \frac{\tilde{\rho}_{0n} - \rho_0}{1 + \alpha_n} + \Big(\frac{1}{1 + \alpha_n} - 1\Big)\rho_0 + \frac{\alpha_n}{1 + \alpha_n},$$

by (3.2), hence

$$\|\rho_{0n} - \rho_0\|_p \leq \|\tilde{\rho}_{0n} - \rho_0\|_p + C\alpha_n \|\rho_0\|_p + C\alpha_n \to 0.$$

Furthermore, since $\|\rho_0\|_{L^\infty} \geq 1$, we have $\|\rho_{0n}\|_\infty \leq \|\rho_0\|_\infty$, provided that n is large enough.

Next set

$$v_{0n} := j_{\varepsilon_n} * u_0, \quad \ell_n := \int_{\mathbb{T}^3} \rho_{0n} v_{0n} \, \mathrm{d}x,$$

and define

$$u_{0n} := v_{0n} - \ell_n.$$

Then $u_{0n} \in C^\infty \subset H^2$ and $v_{0n} \to u_0$ in H^1 as $n \to \infty$. It follows from (3.3) that

$$\int_{\mathbb{T}^3} \rho_{0n} u_{0n} \, \mathrm{d}x = \int_{\mathbb{T}^3} \rho_{0n} v_{0n} \, \mathrm{d}x - \ell_n \int_{\mathbb{T}^3} \rho_{0n} \, \mathrm{d}x = 0.$$

It remains to prove that $u_{0n} \to u_0$ in H^1 as $n \to \infty$. Since $\int_{\mathbb{T}^3} \rho_0 u_0 \, \mathrm{d}x = 0$, we write

$$|\ell_n| = \Big| \int_{\mathbb{T}^3} \rho_{0n} v_{0n} \, \mathrm{d}x - \int_{\mathbb{T}^3} \rho_0 u_0 \, \mathrm{d}x \Big|$$

$$
\begin{aligned}
&= \Big| \int_{\mathbb{T}^3} \rho_{0n}(v_{0n} - u_0)\,\mathrm{d}x + \int_{\mathbb{T}^3} (\rho_{0n} - \rho_0) u_0 \,\mathrm{d}x \Big| \\
&\le \|\rho_{0n}\|_{6/5} \|v_{0n} - u_0\|_6 + \|\rho_{0n} - \rho_0\|_{6/5} \|u_0\|_6,
\end{aligned}
$$

from which, using Sobolev embedding $H^1 \hookrightarrow L^6$ and $v_{0n} \to u_0$ in H^1 as $n \to \infty$, we obtain $\ell_n \to 0$. Therefore,

$$
u_{0n} = v_{0n} - \ell_n \to u_0 \quad \text{in } H^1. \tag{3.5}
$$

Moreover,

$$
\nabla u_{0n} = \nabla(j_{\varepsilon_n} * u_0) = j_{\varepsilon_n} * \nabla u_0,
$$

and thus

$$
\|\nabla u_{0n}\| \le \|\nabla u_0\|.
$$

Finally, let $\mathbb{P}_\sigma$ denote the Leray projection onto divergence-free vector fields on $\mathbb{T}^3$, and define

$$
b_{0n} := \mathbb{P}_\sigma(j_{\varepsilon_n} * b_0).
$$

Then $b_{0n} \in C^\infty \subset H^2$, $\operatorname{div} b_{0n} = 0$, and, since $\operatorname{div} b_0 = 0$,

$$
b_{0n} \to b_0 \quad \text{in } H^1.
$$

In addition, the low-order quantities are uniformly controlled:

$$
\|\rho_{0n}\|_\infty + \|\nabla u_{0n}\|^2 + \|b_{0n}\|_{H^1}^2 \le C\left(\|\rho_0\|_\infty + \|\nabla u_0\|^2 + \|b_0\|_{H^1}^2\right). \tag{3.6}
$$

Since $\rho_{0n} > 0$, the approximate initial data $(\rho_{0n}, u_{0n}, b_{0n})$ trivially satisfy the following compatibility condition

$$
g_n := \rho_{0n}^{-1/2}\left[\mu \Delta u_{0n} + (\mu + \lambda)\nabla \operatorname{div} u_{0n} - \nabla P_{0n} + (\nabla \times b_{0n}) \times b_{0n}\right].
$$

Step 2. Uniform estimates and weak compactness. By (3.6) and Proposition 2.9, there are two positive constants T_0 and C independent of n such that system (1.1)–(1.4) admits a unique solution (ρ_n, u_n, b_n), in $\mathbb{T}^3 \times (0, T_0)$, and the following a priori estimates hold

$$
\begin{gathered}
\int_0^{T_0} \left(\|(\sqrt{\rho_n}\dot{u}_n, \partial_t b_n, \nabla^2 b_n, \nabla G_n)\|^2 + \|\nabla u_n\|_6^2\right) \mathrm{d}t \le C, \\
\sup_{0\le t\le T_0} \left(\|\rho_n\|_\infty + \|(\nabla u_n, \nabla b_n, G_n)\|^2\right) \le C, \\
\int_0^{T_0} \|(\sqrt{t}\nabla \dot{u}_n, \sqrt{t}\nabla \partial_t b_n)\|^2 \,\mathrm{d}t \le C, \\
\sup_{0\le t\le T_0} \Big(\|(\sqrt{t}\sqrt{\rho_n}\dot{u}_n, \sqrt{t}\partial_t b_n, \sqrt{t}\nabla^2 b_n)\|^2 + \|\sqrt{t}\nabla u_n\|_6^2\Big) \le C,
\end{gathered} \tag{3.7}
$$

for large n. Then, by the Banach-Alaoglu theorem and a diagonal argument, there exists a subsequence, still denoted by (ρ_n, u_n, b_n), and (ρ, u, b) satisfying

$$
\rho \in L^\infty(0, T_0; L^\infty), \quad u \in L^\infty(0, T_0; H^1), \quad \nabla u \in L^2(0, T_0; L^6), \tag{3.8}
$$

$$
b \in L^\infty(0, T_0; H^1) \cap L^2(0, T_0; H^2), \quad b_t \in L^2(0, T_0; L^2), \tag{3.9}
$$

$$
\sqrt{t}\nabla u \in L^\infty(0, T_0; L^6), \quad \sqrt{t}\nabla^2 b, \sqrt{t} b_t \in L^\infty(0, T_0; L^2), \tag{3.10}
$$

$$
\sqrt{t}\dot{u}, \sqrt{t} b_t \in L^2(0, T_0; H^1), \quad G \in L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1), \tag{3.11}
$$

such that

$$
\rho_n \overset{*}{\rightharpoonup} \rho, \quad \text{in } L^\infty(0, T_0; L^\infty), \tag{3.12}
$$

$$
u_n \overset{*}{\rightharpoonup} u, \quad \text{in } L^\infty(0, T_0; H^1), \tag{3.13}
$$

$$
\nabla u_n \rightharpoonup \nabla u, \quad \text{in } L^2(0, T_0; L^6), \tag{3.14}
$$

$$
\dot{u}_n \rightharpoonup v, \quad \text{in } L^2(\delta, T_0; H^1), \tag{3.15}
$$

$$
b_n \overset{*}{\rightharpoonup} b, \quad \text{in } L^\infty(0, T_0; H^1), \tag{3.16}
$$

$$
b_n \rightharpoonup b, \quad \text{in } L^2(0, T_0; H^2), \tag{3.17}
$$

$$\partial_t b_n \rightharpoonup b_t, \quad \text{in } L^2(\delta, T_0; H^1), \tag{3.18}$$

for any $\delta \in (0, T_0)$.

Step 3. Strong compactness of b_n , u_n and G_n. Since $H^2 \hookrightarrow\hookrightarrow H^1 \hookrightarrow\hookrightarrow L^2$, it follows from the Aubin-Lions lemma and (3.16)–(3.18) that

$$b_n \to b, \quad \text{in } C([0, T_0]; L^2) \cap L^2(0, T_0; H^1). \tag{3.19}$$

Next, we derive the compactness of u_n. By Sobolev embedding $H^1 \hookrightarrow L^6$ and interpolation, we estimate the convection term

$$\begin{aligned}\int_0^{T_0} \|u_n \cdot \nabla u_n\|^2 \,\mathrm{d}t &\le C\|u_n\|^2_{L^\infty(0,T_0;H^1)} \int_0^{T_0} \|\nabla u_n\|_3^2 \,\mathrm{d}t \\ &\le C\|u_n\|^2_{L^\infty(0,T_0;H^1)} \|\nabla u_n\|_{L^\infty(0,T_0;L^2)} \int_0^{T_0} \|\nabla u_n\|_6 \,\mathrm{d}t \le C_{T_0}.\end{aligned}$$

Therefore,

$$\sqrt{t}\,\partial_t u_n = \sqrt{t}\,\dot{u}_n - \sqrt{t}\,u_n \cdot \nabla u_n$$

is bounded in $L^2(0, T_0; L^2)$. By the Hölder inequality, it follows from Lemma 1.4 and (3.7) that $u_n \in L^\infty(0, T_0; L^2)$ and $\sqrt{t}\partial_t u_n \in L^2(0, T_0; L^2)$. Applying the time-regularity result [13, Lemma 3.4], we infer that for all $\alpha \in (0, \frac{1}{2})$,

$$u_n \text{ is bounded in } H^{\frac{1}{2}-\alpha}(0, T_0; L^2).$$

By the compact embedding

$$L^2(0, T_0; H^1) \cap H^{1/2-\alpha}(0, T_0; L^2) \hookrightarrow\hookrightarrow L^2(0, T_0; L^2),$$

we obtain, up to a subsequence,

$$u_n \to u, \quad \text{in } L^2(0, T_0; L^2). \tag{3.20}$$

We next identify the weak limit v. For every $\phi \in C_c^\infty\big((\delta, T_0) \times \mathbb{T}^3; \mathbb{R}^3\big)$,

$$\begin{aligned}&\int_\delta^{T_0} \int_{\mathbb{T}^3} \big(u_n \cdot \nabla u_n - u \cdot \nabla u\big) \cdot \phi \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_\delta^{T_0} \int_{\mathbb{T}^3} \big((u_n - u) \cdot \nabla u_n\big) \cdot \phi \,\mathrm{d}x \,\mathrm{d}t + \int_\delta^{T_0} \int_{\mathbb{T}^3} \big(u \cdot (\nabla u_n - \nabla u)\big) \cdot \phi \,\mathrm{d}x \,\mathrm{d}t \to 0,\end{aligned}$$

where the first term tends to zero because

$$\|u_n - u\|_{L^2(\delta,T_0;L^2)} \|\nabla u_n\|_{L^2(\delta,T_0;L^2)} \to 0,$$

and the second term tends to zero by the weak convergence of ∇u_n. Therefore,

$$u_n \cdot \nabla u_n \to u \cdot \nabla u \quad \text{in } \mathcal{D}'\big(\mathbb{T}^3 \times (\delta, T_0)\big).$$

Since $u_n \to u$ in the sense of distributions,

$$\partial_t u_n \to \partial_t u \quad \text{in } \mathcal{D}'(\mathbb{T}^3 \times (\delta, T_0)).$$

Passing to the limit in $\partial_t u_n = \dot{u}_n - u_n \cdot \nabla u_n$ gives

$$\partial_t u = v - u \cdot \nabla u,$$

which implies $v = \dot{u}$.

To pass to the limit in the pressure term of the momentum equation, we prove the strong convergence of the effective viscous flux G_n. From the uniform estimates (3.7), we have

$$G_n \quad \text{is bounded in } L^\infty(0, T_0; L^2) \cap L^2(0, T_0; H^1). \tag{3.21}$$

It remains to obtain a time compactness estimate for $\partial_t G_n = (2\mu + \lambda) \operatorname{div} \partial_t u_n - \partial_t P_n$. For the linear pressure law, the continuity equation (1.1) gives

$$\partial_t P_n = -\operatorname{div}(P_n u_n).$$

Therefore,

$$\partial_t G_n = \operatorname{div}\big((2\mu + \lambda)\dot{u}_n - (2\mu + \lambda)u_n \cdot \nabla u_n + P_n u_n\big).$$

From (3.13)–(3.15), we know that

$$\nabla u_n \in L^\infty(0,T_0;L^2)\cap L^2(0,T_0;L^6),$$

which implies $\nabla u_n \in L^2(0,T_0;L^3)$. By the Hölder and Sobolev inequalities, we have

$$u_n\cdot\nabla u_n \text{ is bounded in } L^2(0,T_0;L^2).$$

Since $P_n \in L^\infty(0,T_0;L^\infty)$ and $u_n \in L^\infty(0,T_0;H^1)$,

$$\sqrt{t}P_n u_n \text{ is bounded in } L^2(0,T_0;L^2).$$

Consequently,

$$\sqrt{t}\partial_t G_n \text{ is bounded in } L^2(0,T_0;H^{-1}).$$

The time-regularity lemma [13, Lemma 3.4] then gives, for every $\alpha\in(0,1/2)$,

$$G_n \text{ is bounded in } H^{1/2-\alpha}(0,T_0;H^{-1}).$$

Since $H^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow H^{-1}$, Aubin-Lions lemma yields

$$G_n \to G, \quad \text{in } L^2(0,T_0;L^2). \tag{3.22}$$

From the identity

$$G_n = (2\mu+\lambda)\operatorname{div} u_n - a\rho_n,$$

the weak convergence of ∇u_n and the weak-* convergence of ρ_n imply

$$G = (2\mu+\lambda)\operatorname{div} u - a\rho \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)). \tag{3.23}$$

Moreover,

$$\operatorname{div} u_n \rightharpoonup \operatorname{div} u \quad \text{in } L^2(0,T_0;L^2).$$

Consider the extended-valued functional

$$\mathscr{J}(f) := \int_0^{T_0}\|f(t)\|_\infty\,\mathrm{d}t, \quad f\in L^2((0,T_0)\times\mathbb{T}^3).$$

The functional $\mathscr{J}$ is convex and lower semicontinuous with respect to the strong L^2-topology, and hence it is weakly lower semicontinuous on L^2. Therefore,

$$\int_0^{T_0}\|\operatorname{div} u\|_\infty\,\mathrm{d}t \le \liminf_{n\to\infty}\int_0^{T_0}\|\operatorname{div} u_n\|_\infty\,\mathrm{d}t \le C,$$

where the last inequality follows from Proposition 2.2. Consequently,

$$\operatorname{div} u \in L^1(0,T_0;L^\infty). \tag{3.24}$$

Step 4. The continuity equation and strong convergence of the density. Following Danchin-Mucha [14], we prove the strong convergence of the density. We first identify its weak initial trace. For every scalar test function $\varphi \in C_c^\infty\big(\mathbb{T}^3\times[0,T_0)\big)$, the approximate continuity equation gives

$$-\int_0^{T_0}\int_{\mathbb{T}^3}\rho_n\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^{T_0}\int_{\mathbb{T}^3}\rho_n u_n\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^3}\rho_{0n}\varphi(0)\,\mathrm{d}x.$$

Since

$$\rho_n \overset{*}{\rightharpoonup} \rho \quad \text{in } L^\infty(0,T_0;L^\infty), \quad u_n\to u \quad \text{in } L^2(0,T_0;L^2),$$
$$\rho_{0n}\to\rho_0 \quad \text{in } L^p \quad \text{for every } 1\le p<\infty,$$

we may pass to the limit and obtain

$$-\int_0^{T_0}\int_{\mathbb{T}^3}\rho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^{T_0}\int_{\mathbb{T}^3}\rho u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}^3}\rho_0\varphi(0)\,\mathrm{d}x. \tag{3.25}$$

Thus, the limit density satisfies

$$\partial_t\rho + \operatorname{div}(\rho u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)),$$

with initial value ρ_0 in the weak sense. Since $u \in L^1(0,T_0;H^1)$ and (3.24) holds, the renormalization theorem applies to the limit continuity equation. For smooth approximate solutions, the renormalized continuity equation with $\beta(z) = z\log z$ gives

$$\partial_t(\rho_n \log\rho_n) + \operatorname{div}(\rho_n \log\rho_n\, u_n) + \rho_n \operatorname{div} u_n = 0. \tag{3.26}$$

Since

$$\operatorname{div} u_n = \frac{1}{2\mu+\lambda}(G_n + a\rho_n),$$

we rewrite (3.26) as

$$\partial_t(\rho_n \log\rho_n) + \operatorname{div}(\rho_n \log\rho_n\, u_n) + \frac{1}{2\mu+\lambda}\rho_n G_n + \frac{a}{2\mu+\lambda}\rho_n^2 = 0. \tag{3.27}$$

Let

$$\overline{\rho\log\rho} := \text{weak-*}\lim_{n\to\infty} \rho_n\log\rho_n, \quad \overline{\rho^2} := \text{weak-*}\lim_{n\to\infty}\rho_n^2.$$

We also have

$$\rho_n \log\rho_n\, u_n \rightharpoonup \overline{\rho\log\rho}\, u \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)). \tag{3.28}$$

Indeed, for every $\phi \in C_c^\infty((0,T_0)\times\mathbb{T}^3;\mathbb{R}^3)$, we write

$$\begin{aligned}
&\int_0^{T_0}\int_{\mathbb{T}^3} \big(\rho_n\log\rho_n\, u_n - \overline{\rho\log\rho}\, u\big)\cdot\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^{T_0}\int_{\mathbb{T}^3} \rho_n\log\rho_n\,(u_n-u)\cdot\phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T_0}\int_{\mathbb{T}^3}\big(\rho_n\log\rho_n - \overline{\rho\log\rho}\big)\, u\cdot\phi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The first term tends to zero because $\sup_n \|\rho_n\log\rho_n\|_{L^\infty} < \infty$ and $u_n \to u$ in $L^2(0,T_0;L^2)$, while the second term tends to zero by the weak-* convergence of $\rho_n\log\rho_n$ in L^∞. Since $G_n \to G$ strongly in L^2 and ρ_n is bounded in L^∞, we have

$$\rho_n G_n \to \rho G \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)).$$

Passing to the limit in (3.27), we obtain

$$\partial_t\overline{\rho\log\rho} + \operatorname{div}(\overline{\rho\log\rho}\, u) + \frac{1}{2\mu+\lambda}\rho G + \frac{a}{2\mu+\lambda}\overline{\rho^2} = 0. \tag{3.29}$$

On the other hand, the limit density is a renormalized solution of the continuity equation:

$$\partial_t(\rho\log\rho) + \operatorname{div}(\rho\log\rho\, u) + \rho\operatorname{div} u = 0.$$

Consequently, it follows from the relation $\operatorname{div} u = \frac{1}{2\mu+\lambda}(G + a\rho)$ that

$$\partial_t(\rho\log\rho) + \operatorname{div}(\rho\log\rho\, u) + \frac{1}{2\mu+\lambda}\rho G + \frac{a}{2\mu+\lambda}\rho^2 = 0. \tag{3.30}$$

Subtracting (3.30) from (3.29), integrating over $\mathbb{T}^3\times(0,t)$, and using

$$\rho_{0n} \to \rho_0 \quad \text{in } L^p,\ 1\le p<\infty,$$

we obtain

$$\int_{\mathbb{T}^3}\big(\overline{\rho\log\rho} - \rho\log\rho\big)(t)\,\mathrm{d}x + \frac{a}{2\mu+\lambda}\int_0^t\int_{\mathbb{T}^3}\Big(\overline{\rho^2} - \rho^2\Big)\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{3.31}$$

By the convexity of functions $z\mapsto z\log z$ and $z\mapsto az^2$, we have

$$\overline{\rho\log\rho} \ge \rho\log\rho, \quad \overline{\rho^2}\ge\rho^2.$$

Hence both nonnegative terms in (3.31) vanish. Thus

$$\overline{\rho\log\rho} = \rho\log\rho.$$

Since $z\mapsto z\log z$ is strictly convex on $[0,\infty)$, it follows that

$$\rho_n \to \rho \quad \text{a.e. in } \mathbb{T}^3\times(0,T_0).$$

Together with the uniform L^∞-bound, this gives

$$\rho_n \to \rho \quad \text{in } L^p(0,T_0;L^p), \quad 1\le p<\infty. \tag{3.32}$$

We now prove the time continuity of the density. Since $\rho \in L^\infty(0,T_0;L^\infty)$ and $\partial_t\rho = -\operatorname{div}(\rho u)$ in the sense of distributions, we have

$$\rho \in C_w([0,T_0];L^p), \quad 1 < p < \infty.$$

The renormalized equation with $\beta(z) = z^p$ yields

$$\partial_t\rho^p + \operatorname{div}(\rho^p u) + (p-1)\rho^p \operatorname{div} u = 0.$$

Since $\operatorname{div} u \in L^1(0,T_0;L^\infty)$, we infer that

$$t \mapsto \|\rho\|_p^p$$

is continuous. Combining the weak continuity with the continuity of the L^p-norm gives

$$\rho \in C([0,T_0];L^p), \quad 1 < p < \infty.$$

The continuity in L^1 then follows from the finite measure of $\mathbb{T}^3$ and the continuity in any fixed L^p with $p > 1$, that is

$$\|\rho(t) - \rho(s)\|_1 \le |\mathbb{T}^3|^{1/2}\|\rho(t) - \rho(s)\|.$$

Consequently,

$$\rho \in C([0,T_0];L^p), \quad 1 \le p < \infty. \tag{3.33}$$

Step 5. Passage to the limit in the nonlinear terms. We use the conservative form of the momentum equation:

$$\partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - \mu\Delta u_n - (\mu+\lambda)\nabla \operatorname{div} u_n + \nabla P_n = (\nabla \times b_n) \times b_n.$$

We first observe that

$$\rho_n u_n \to \rho u, \quad \rho_n u_n \otimes u_n \to \rho u \otimes u \quad \text{in } L^1(0,T_0;L^1).$$

Indeed, since $u \in L^\infty(0,T_0;H^1) \hookrightarrow L^\infty(0,T_0;L^6)$, it follows from (3.20) and (3.32) that

$$\begin{aligned}
&\|\rho_n u_n - \rho u\|_{L^1(0,T_0;L^1)}\\
&\le \|\rho_n(u_n - u)\|_{L^1(0,T_0;L^1)} + \|(\rho_n - \rho)u\|_{L^1(0,T_0;L^1)}\\
&\le \|\rho_n\|_{L^2(0,T_0;L^2)}\|u_n - u\|_{L^2(0,T_0;L^2)} + \|\rho_n - \rho\|_{L^2(0,T_0;L^2)}\|u\|_{L^2(0,T_0;L^2)} \to 0.
\end{aligned}$$

Moreover, we decompose

$$\rho_n u_n \otimes u_n - \rho u \otimes u = \rho_n(u_n - u) \otimes u_n + \rho_n u \otimes (u_n - u) + (\rho_n - \rho)u \otimes u.$$

Using (3.20) and (3.32) again, and the fact that ρ_n is bounded in $L^\infty(0,T_0;L^\infty)$ and u_n is bounded in $L^\infty(0,T_0;H^1)$, we have

$$\|\rho_n(u_n - u) \otimes u_n\|_{L^1(0,T_0;L^1)} \le \|\rho_n\|_{L^\infty}\|u_n - u\|_{L^2(0,T_0;L^2)}\|u_n\|_{L^2(0,T_0;L^2)} \to 0,$$

and similarly,

$$\|\rho_n u \otimes (u_n - u)\|_{L^1(0,T_0;L^1)} \to 0.$$

Since $\rho_n \to \rho$ in $L^2(0,T_0;L^2)$ and $u \otimes u \in L^\infty(0,T_0;L^2)$, we obtain

$$\|(\rho_n - \rho)u \otimes u\|_{L^1(0,T_0;L^1)} \le \|\rho_n - \rho\|_{L^2(0,T_0;L^2)}\|u \otimes u\|_{L^2(0,T_0;L^2)} \to 0.$$

Therefore,

$$\rho_n u_n \otimes u_n \to \rho u \otimes u \quad \text{in } L^1(0,T_0;L^1).$$

From the relation $P_n = a\rho_n$, the strong convergence of ρ_n gives

$$P_n \to P \quad \text{in } L^p(0,T_0;L^p),\ 1 \le p < \infty.$$

Therefore,

$$\nabla P_n \to \nabla P \quad \text{in } \mathcal{D}'(\mathbb{T}^3 \times (0,T_0)).$$

The viscous terms pass to the limit by weak convergence:

$$\Delta u_n \to \Delta u, \quad \nabla \operatorname{div} u_n \to \nabla \operatorname{div} u \quad \text{in } \mathcal{D}'(\mathbb{T}^3 \times (0,T_0)).$$

For any test function $\phi \in C_c^\infty(\mathbb{T}^3 \times (0,T_0);\mathbb{R}^3)$, recalling that (3.16)–(3.17) and $b_n \to b$ in L^2 as $n \to \infty$, we obtain

$$\begin{aligned}
&\int_0^{T_0}\int_{\mathbb{T}^3} [(\nabla \times b_n)\times b_n - (\nabla \times b)\times b]\cdot \phi \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^{T_0}\int_{\mathbb{T}^3} (\nabla \times b_n)\cdot \big((b_n - b)\times \phi\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^{T_0}\int_{\mathbb{T}^3} (\nabla \times b_n - \nabla \times b)\cdot (b\times\phi)\,\mathrm{d}x\,\mathrm{d}t \to 0.
\end{aligned}$$

Hence

$$(\nabla \times b_n)\times b_n \to (\nabla\times b)\times b \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)).$$

For the magnetic induction equation, it follows from (3.19)–(3.20) that

$$\begin{aligned}
&\|u_n \times b_n - u\times b\|_{L^1(0,T_0;L^1)} \\
&\le \|(u_n-u)\times b_n\|_{L^1(0,T_0;L^1)} + \|u\times(b_n-b)\|_{L^1(0,T_0;L^1)} \\
&\le \|u_n-u\|_{L^2(0,T_0;L^2)}\|b_n\|_{L^2(0,T_0;L^2)} + \|u\|_{L^2(0,T_0;L^2)}\|b_n-b\|_{L^2(0,T_0;L^2)} \to 0,
\end{aligned}$$

which yields

$$u_n\times b_n \to u\times b \quad \text{in } L^1(0,T_0;L^1).$$

Consequently,

$$\nabla\times(u_n\times b_n) \to \nabla\times(u\times b) \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)).$$

Since $b_n \rightharpoonup b$ weakly in $L^2(0,T_0;H^2)$, we have

$$\Delta b_n \to \Delta b \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)).$$

Thus the limit b satisfies

$$b_t - \nabla\times(u\times b) = \nu\Delta b \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)).$$

Since $\operatorname{div} b_n = 0$ for every n, passing to the limit gives

$$\operatorname{div} b = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}^3\times(0,T_0)).$$

Moreover, by the weak lower semicontinuity of norms, it follows from (3.7), (3.15) and (3.32)

$$\int_\delta^{T_0}\|\sqrt{\rho}\dot u\|^2 dt \le \liminf_{n\to\infty}\int_\delta^{T_0}\|\sqrt{\rho_n}\dot u_n\|^2 dt \le C.$$

Step 6. Attainment of the initial data and completion of the proof. Combining (3.25) with (3.33) gives

$$\rho(t)\to\rho_0 \quad \text{in } L^p \text{ as } t\to 0^+.$$

Since $b_{0n}\to b_0$ in L^2, it follows from (3.19) that $b(t)\to b_0$ in L^2 as $t\to 0^+$.

It remains to pass to the limit in the momentum equation together with its initial value. For every $\psi\in C_c^\infty\big(\mathbb{T}^3\times[0,T_0);\mathbb{R}^3\big)$, passing to the limit in the weak formulation of the approximate momentum equations gives

$$\begin{aligned}
&-\int_0^{T_0}\int_{\mathbb{T}^3}\rho u\cdot\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^{T_0}\int_{\mathbb{T}^3}\rho u\otimes u:\nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\
&+\mu\int_0^{T_0}\int_{\mathbb{T}^3}\nabla u:\nabla\psi\,\mathrm{d}x\,\mathrm{d}t + (\mu+\lambda)\int_0^{T_0}\int_{\mathbb{T}^3}\operatorname{div} u\operatorname{div}\psi\,\mathrm{d}x\,\mathrm{d}t \\
&-\int_0^{T_0}\int_{\mathbb{T}^3}P\operatorname{div}\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^{T_0}\int_{\mathbb{T}^3}\big((\nabla\times b)\times b\big)\cdot\psi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\mathbb{T}^3}\rho_0 u_0\cdot\psi(0)\,\mathrm{d}x.
\end{aligned} \tag{3.34}$$

Since $\rho u\in L^\infty(0,T_0;L^2)$ and the conservative momentum equation gives

$$\partial_t(\rho u)\in L^1(0,T_0;W^{-1,3/2}),$$

the standard weak-continuity argument yields

$$\rho u \in C_w([0,T_0];L^2), \quad \rho u(0) = \rho_0 u_0. \tag{3.35}$$

In particular,

$$\rho u(t) \rightharpoonup \rho_0 u_0 \quad \text{in } L^2 \quad \text{as } t \to 0^+.$$

We first establish the time continuity of the velocity away from the initial time. For all $t \in [\delta, T_0]$,

$$\int_\delta^{T_0} \|\dot u\|^2 \,\mathrm{d}t \le C \int_\delta^{T_0} \|\nabla \dot u\|^2 \,\mathrm{d}t \le \frac{C}{\delta} \int_\delta^{T_0} \|\sqrt{t}\nabla \dot u\|^2 \,\mathrm{d}t < \infty,$$

thus $\dot u \in L^2(\delta, T_0; L^2)$. By Hölder inequality, we have

$$\begin{aligned} \int_\delta^{T_0} \|u\cdot\nabla u\|_2^2 \,\mathrm{d}t &\le \|u\|^2_{L^\infty(\delta,T_0;L^6)} \int_\delta^{T_0} \|\nabla u\|_3^2 \,\mathrm{d}t \\ &\le \|u\|^2_{L^\infty(\delta,T_0;L^6)} \|\nabla u\|_{L^\infty(\delta,T_0;L^2)} \int_\delta^{T_0} \|\nabla u\|_6 \,\mathrm{d}t < \infty, \end{aligned}$$

which implies $u\cdot\nabla u \in L^2(\delta,T_0;L^2)$. It follows from the material derivative that

$$u_t \in L^2(\delta, T_0; L^2).$$

Together with $u \in L^2(\delta,T_0;L^2)$, this gives

$$u \in W^{1,2}(\delta,T_0;L^2) \hookrightarrow C([\delta,T_0];L^2). \tag{3.36}$$

Indeed, for $\delta \le s < t \le T_0$,

$$\|u(t)-u(s)\| \le |t-s|^{1/2} \|u_t\|_{L^2(s,t;L^2)}.$$

Combining (3.36) with

$$\rho \in C([0,T_0];L^{3/2}) \quad u \in L^\infty(0,T_0;L^6),$$

we obtain

$$\sqrt{\rho}\, u \in C((0,T_0];L^2). \tag{3.37}$$

We define the pressure potential by

$$\mathcal{H}(r) := \begin{cases} ar\log r, & r>0, \\ 0, & r=0, \end{cases}$$

and set

$$\mathcal{F}(t) := \int_{\mathbb{T}^3} \Big(\frac12 \rho|u|^2 + \mathcal{H}(\rho) + \frac12 |b|^2\Big)(t) \,\mathrm{d}x, \quad \mathcal{F}_0 := \int_{\mathbb{T}^3} \Big(\frac12 \rho_0|u_0|^2 + \mathcal{H}(\rho_0) + \frac12 |b_0|^2\Big) \,\mathrm{d}x.$$

Each smooth approximate solution satisfies the standard total-energy equality associated with (1.1)–(1.4). Let $\mathcal{F}_n$ be defined analogously, with (ρ,u,b) replaced by (ρ_n,u_n,b_n). The construction of the approximate initial data implies

$$\mathcal{F}_n(0) \to \mathcal{F}_0.$$

Therefore, by weak lower semicontinuity, for almost every $t \in (0,T_0)$,

$$\mathcal{F}(t) + \int_0^t \int_{\mathbb{T}^3} \big(\mu|\nabla u|^2 + (\mu+\lambda)|\operatorname{div} u|^2 + \nu|\nabla b|^2\big) \,\mathrm{d}x\,\mathrm{d}s \le \mathcal{F}_0. \tag{3.38}$$

The map

$$t \longmapsto \int_{\mathbb{T}^3} \mathcal{H}(\rho(t)) \,\mathrm{d}x$$

is continuous, because $\rho \in C([0,T_0];L^p)$ for every $1 \le p < \infty$, the density is uniformly bounded, and $\mathcal{H}$ is continuous on every bounded interval. Together with (3.19) and (3.37), this shows that the left-hand side of (3.38) is continuous on $(0,T_0]$. Hence (3.38) holds for every $t \in (0,T_0]$.

We set

$$K(t) := \int_{\mathbb{T}^3} \rho(t)|u(t)|^2 \,\mathrm{d}x, \quad K_0 := \int_{\mathbb{T}^3} \rho_0|u_0|^2 \,\mathrm{d}x.$$

By the weighted Cauchy-Schwarz inequality,

$$\Big|\int_{\mathbb{T}^3} \rho u(t)\cdot u_0 \,\mathrm{d}x\Big|^2 \le K(t)\int_{\mathbb{T}^3} \rho(t)|u_0|^2\,\mathrm{d}x. \tag{3.39}$$

It follows from (3.35) that

$$\int_{\mathbb{T}^3} \rho u(t)\cdot u_0\,\mathrm{d}x \to \int_{\mathbb{T}^3} \rho_0|u_0|^2\,\mathrm{d}x = K_0.$$

Moreover, since $\rho(t)\to\rho_0$ in $L^{3/2}$ and $u_0\in L^6$,

$$\int_{\mathbb{T}^3} \rho(t)|u_0|^2\,\mathrm{d}x \to K_0.$$

Taking the lower limit in (3.39), we obtain

$$K_0 \le \liminf_{t\to 0^+} K(t). \tag{3.40}$$

On the other hand, (3.38), together with $\rho(t)\to\rho_0$ in L^p and $b(t)\to b_0$ in L^2, gives

$$\limsup_{t\to 0^+} K(t)\le K_0. \tag{3.41}$$

Combining (3.40) and (3.41), we obtain

$$K(t)\to K_0 \quad \text{as } t\to 0^+. \tag{3.42}$$

Therefore,

$$\int_{\mathbb{T}^3} \rho(t)|u(t)-u_0|^2\,\mathrm{d}x = K(t) + \int_{\mathbb{T}^3}\rho(t)|u_0|^2\,\mathrm{d}x - 2\int_{\mathbb{T}^3}\rho u(t)\cdot u_0\,\mathrm{d}x \to 0.$$

Moreover,

$$\|(\sqrt{\rho(t)}-\sqrt{\rho_0})u_0\|_2^2 \le \|\rho(t)-\rho_0\|_{3/2}\|u_0\|_6^2 \to 0.$$

Therefore,

$$\|\sqrt{\rho(t)}u(t)-\sqrt{\rho_0}u_0\| \le \Big(\int_{\mathbb{T}^3}\rho(t)|u(t)-u_0|^2\,\mathrm{d}x\Big)^{1/2} + \|(\sqrt{\rho(t)}-\sqrt{\rho_0})u_0\| \to 0.$$

Thus

$$\sqrt{\rho(t)}u(t)\to\sqrt{\rho_0}u_0 \quad \text{in } L^2 \text{ as } t\to 0^+.$$

Combining this strong initial trace with (3.37), we conclude that

$$\sqrt{\rho}\,u \in C([0,T_0];L^2). \tag{3.43}$$

3.2. **Uniqueness.** We prove the uniqueness of the strong solutions established in Subsection 3.1. The low regularity creates two main difficulties. The first one is that the conventional energy estimate for the density difference is not available, because the condition $\nabla\rho\in L^q$, $q\in(3,6)$, used in [18], is not assumed. The other difficulty is that $\operatorname{div} u, \operatorname{curl} u \in L^1(0,T;L^\infty)$ cannot ensure that $\nabla u\in L^1(0,T;L^\infty)$. Inspired by [14], the available regularity of the solution allows us to show that $\nabla u\in L^1(0,T;\mathrm{BMO})$. With the aid of this, we introduce the backward parabolic system to construct the difference of the velocity and magnetic field without introducing derivatives of density. The following proposition is crucial for proving the uniqueness.

Proposition 3.1. *Assume that u is the velocity component of a strong solution in the sense of Definition 1.1. Then we have*

$$\int_0^{T_0} (1+|\log t|)\,\|\nabla u\|_{\mathrm{BMO}}\,\mathrm{d}t < \infty.$$

Proof. Since each component of ∇u has zero mean on $\mathbb{T}^3$, the Hodge decomposition gives

$$\|\nabla u\|_{\mathrm{BMO}} \le C\left(\|\operatorname{div} u\|_{\mathrm{BMO}} + \|\omega\|_{\mathrm{BMO}}\right).$$

Thanks to this, by the boundedness of Riesz transforms on BMO, the Gagliardo-Nirenberg inequality and Lemma 1.5, we have

$$\|\nabla u\|_{\mathrm{BMO}} \le C(\|\rho\|_\infty + \|\nabla G\| + \|\nabla\omega\| + \|\nabla G\|_q + \|\nabla\omega\|_q)$$

$$\begin{aligned}&\le C(1+\|\rho\|_\infty^{1/2}\|\sqrt{\rho}\dot u\|+\|b\|_{H^1}^{3/2}\|\nabla^2 b\|^{1/2})\\&\quad+C(\|\sqrt{\rho}\dot u\|^{\frac{6-q}{2q}}\|\nabla\dot u\|^{\frac{3q-6}{2q}}+\|b\|_{H^1}^2+\|b\|_{H^1}^{\frac{3}{q}}\|\nabla^2 b\|^{\frac{2q-3}{q}}),\end{aligned}$$

from which, by Corollary 2.8 and the Hölder inequality, we deduce that

$$\begin{aligned}&\int_0^{T_0}(1+|\log t|)\,\|\nabla u(t)\|_{\mathrm{BMO}}\,\mathrm{d}t\\&\le C\int_0^{T_0}(1+|\log t|)\,(1+\|\sqrt{\rho}\dot u\|^{\frac{6-q}{2q}}\|\nabla\dot u\|^{\frac{3q-6}{2q}}+\|b\|_{H^1}^2+\|b\|_{H^1}^{\frac{3}{q}}\|\nabla^2 b\|^{\frac{2q-3}{q}})\,\mathrm{d}t\le C. \qquad \square\end{aligned}$$

Let (ρ_1,u_1,b_1) and (ρ_2,u_2,b_2) be two solutions to system (1.1)–(1.4) in $\mathbb{T}^3\times(0,T)$ with the same initial data (ρ_0,u_0,b_0). Denote

$$\delta\rho=\rho_1-\rho_2,\quad \delta u=u_1-u_2,\quad \delta b=b_1-b_2.$$

One can check by direct calculations that $(\delta\rho,\delta u,\delta b)$ satisfies:

$$\delta\rho_t+\operatorname{div}(\rho_1\delta u+\delta\rho u_2)=0, \tag{3.44}$$

$$\begin{aligned}&\rho_1(\delta u_t+u_1\cdot\nabla\delta u)-\mu\Delta\delta u-(\mu+\lambda)\nabla\operatorname{div}\delta u+a\nabla\delta\rho\\&=-\delta\rho\dot u_2-\rho_1\delta u\cdot\nabla u_2+\delta b\cdot\nabla b_2+b_1\cdot\nabla\delta b-\frac12\nabla(\delta b\cdot(b_1+b_2)),\end{aligned} \tag{3.45}$$

$$\delta b_t-\nabla\times(u_1\times\delta b+\delta u\times b_2)=\nu\Delta\delta b,\quad \operatorname{div}\delta b=0. \tag{3.46}$$

To prove $(\delta\rho,\delta u,\delta b)=0$, we need to perform estimates for $\delta\rho$ in $L^\infty(0,T;H^{-1})$, and for $\sqrt{\rho_1}\delta u,\delta b$ in $L^2(0,T;L^2)$.

Step 1. Estimates for $\delta\rho$. Since $\int_{\mathbb{T}^3}\delta\rho\,\mathrm{d}x=0$, we set $\phi:=-(-\Delta)^{-1}\delta\rho$ such that

$$\|\nabla\phi\|=\|\delta\rho\|_{\dot H^{-1}}\simeq\|\delta\rho\|_{H^{-1}}.$$

Multiplying (3.44) with ϕ yields

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2\le\Big|\int_{\mathbb{T}^3}(\rho_1\delta u\cdot\nabla\phi+\delta\rho u_2\cdot\nabla\phi)\,\mathrm{d}x\Big|.$$

By the Hölder inequality,

$$\Big|\int_{\mathbb{T}^3}\rho_1\delta u\cdot\nabla\phi\,\mathrm{d}x\Big|\le\sqrt{\bar\rho}\|\sqrt{\rho_1}\delta u\|\|\nabla\phi\|.$$

It follows from integration by parts

$$\int_{\mathbb{T}^3}\delta\rho u_2\cdot\nabla\phi\,\mathrm{d}x=\int_{\mathbb{T}^3}u_2^i\partial_i\phi\Delta\phi\,\mathrm{d}x=-\int_{\mathbb{T}^3}\partial_k u_2^i\partial_i\phi\partial_k\phi\,\mathrm{d}x+\frac12\int_{\mathbb{T}^3}\operatorname{div}u_2|\nabla\phi|^2\,\mathrm{d}x. \tag{3.47}$$

To bound the first term in (3.47), we use the following important inequality in Hardy space (see e.g.[37, Theorem. D]):

$$\|f\|_{\mathcal{H}^1}\le C\|f\|_1(|\log\|f\|_1|+\log(e+\|f\|_\infty)).$$

Thus, combining with the fact that $\|\nabla\phi(t)\|_\infty\le C\bar\rho$, for all $t\in(0,T_0)$, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{T}^3}\delta\rho u_2\cdot\nabla\phi\,\mathrm{d}x\Big|&\le C\|\nabla u_2\|_{\mathrm{BMO}}\|\nabla\phi\otimes\nabla\phi\|_{\mathcal{H}^1}\\&\le C\|\nabla u_2\|_{\mathrm{BMO}}\|\nabla\phi\|^2(|\log\|\nabla\phi\|^2|+\log(e+\|\nabla\phi\|_\infty^2))\\&\le C\|\nabla u_2\|_{\mathrm{BMO}}\|\nabla\phi\|^2(1+|\log\|\nabla\phi\||).\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2\le C\,(\|\sqrt{\rho_1}\delta u\|+\|\nabla u_2\|_{\mathrm{BMO}}\|\nabla\phi\|(1+|\log\|\nabla\phi\||))\,\|\nabla\phi\|.$$

Since $\|\nabla\phi(0)\|=0$, one gets after integration that for all $t\in(0,T_0)$

$$\|\nabla\phi\|\le C\Big(\int_0^t\|\sqrt{\rho_1}\delta u\|\,\mathrm{d}\tau+\int_0^t\|\nabla u_2\|_{\mathrm{BMO}}\|\nabla\phi\|(1+|\log\|\nabla\phi\||)\,\mathrm{d}\tau\Big).$$

Denoting $Z(t) := \sup_{\tau\le t}\tau^{-1/2}\|\delta\rho(\tau)\|_{H^{-1}}$ and $Z(0) := 0$, one deduces by the Cauchy inequality that for all $T\in[0,T_0]$

$$Z(T) \le C\Big(\int_0^T \|\nabla u_2\|_{\mathrm{BMO}} Z(\tau)(1+|\log\tau|+|\log Z(\tau)|)\,\mathrm{d}\tau + \|\sqrt{\rho_1}\delta u\|_{L^2(0,T;L^2)}\Big). \tag{3.48}$$

Step 2. Coupled backward parabolic system. In this part, we intend to control the differences for velocity and magnetic field. To this end, we introduce the solution (w,z) to the following coupled backward system:

$$\rho_1(w_t+u_1\cdot\nabla w)+\mu\Delta w+(\mu+\lambda)\nabla\operatorname{div} w+b_2\times(\nabla\times z) = -\rho_1\delta u, \tag{3.49}$$

$$z_t+\nu\Delta z = -\delta b,\quad \operatorname{div} z=0, \tag{3.50}$$

$$w(T)=0,\quad z(T)=0. \tag{3.51}$$

We emphasize that the above system fails for classical linear parabolic theory when the coefficients are rough and may vanish. However, if ρ_1, u_1, b_2 are smooth and ρ_1 is bounded away from zero, the well-posedness is standard. For this system, the existence theory can be achieved by a regularizing procedure together with the uniform estimates (3.58).

Note that the spatial mean of each magnetic field is conserved,

$$\int_{\mathbb{T}^3}\delta b\,\mathrm{d}x = 0.$$

Integrating (3.50) over $\mathbb{T}^3$ and using $z(T)=0$, we obtain

$$\int_{\mathbb{T}^3} z(t)\,\mathrm{d}x = 0.$$

Since $\int_{\mathbb{T}^3}\rho_1 w\,\mathrm{d}x$ need not be zero, we set

$$N := \int_{\mathbb{T}^3}\rho_1\,\mathrm{d}x = 1,\quad \bar w := \int_{\mathbb{T}^3}\rho_1 w\,\mathrm{d}x,$$

then

$$\int_{\mathbb{T}^3}\rho_1(w-\bar w)\,\mathrm{d}x = 0.$$

Using the Poincaré inequality in Lemma 1.4 yields

$$\|w-\bar w\| \le C(\bar\rho)\|\nabla w\|.$$

Therefore,

$$\|w\| \le C(\bar\rho)\|\nabla w\| + \Big|\int_{\mathbb{T}^3}\rho_1 w\,\mathrm{d}x\Big| \le C(\|\sqrt{\rho_1}w\|+\|\nabla w\|). \tag{3.52}$$

First, testing equation of (w,z) with w and z respectively, one deduces

$$\begin{aligned}
&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\sqrt{\rho_1}w\|^2+\|z\|^2\big)+\mu\|\nabla w\|^2+(\mu+\lambda)\|\operatorname{div} w\|^2+\nu\|\nabla z\|^2\\
&= \int_{\mathbb{T}^3}\rho_1\delta u\cdot w\,\mathrm{d}x + \int_{\mathbb{T}^3}\delta b\cdot z\,\mathrm{d}x + \int_{\mathbb{T}^3} b_2\times(\nabla\times z)\cdot w\,\mathrm{d}x.
\end{aligned}$$

It follows from the Cauchy inequality that

$$\Big|\int_{\mathbb{T}^3}\rho_1\delta u\cdot w\,\mathrm{d}x + \int_{\mathbb{T}^3}\delta b\cdot z\,\mathrm{d}x\Big| \le C(\|\sqrt{\rho_1}\delta u\|^2+\|\delta b\|^2+\|\sqrt{\rho_1}w\|^2+\|z\|^2).$$

We set

$$I(t) := \int_{\mathbb{T}^3} b_2\times(\nabla\times z)\cdot w\,\mathrm{d}x.$$

Using the periodic boundary condition, we have

$$I(t) = \int_{\mathbb{T}^3}(\nabla\times z)\cdot(w\times b_2)\,\mathrm{d}x = \int_{\mathbb{T}^3} z\cdot\nabla\times(w\times b_2)\,\mathrm{d}x.$$

Since $\operatorname{div} b_2 = 0$,

$$\nabla\times(w\times b_2) = -b_2\operatorname{div} w + (b_2\cdot\nabla)w - (w\cdot\nabla)b_2,$$

hence, one deduces by the Hölder, Gagliardo-Nirenberg and Young inequalities

$$
\begin{aligned}
|I(t)| &\le \|z\|_3(\|b_2\|_6\|\nabla w\| + \|\nabla b_2\|\|w\|_6) \\
&\le \|z\|^{1/2}\|\nabla z\|^{1/2}\|b_2\|_{H^1}(\|\sqrt{\rho_1}w\| + \|\nabla w\|) \\
&\le \frac{\nu}{2}\|\nabla z\|^2 + \frac{\mu}{2}\|\nabla w\|^2 + C(1+\|b_2\|_{H^1}^4)(\|\sqrt{\rho_1}w\|^2 + \|z\|^2).
\end{aligned} \tag{3.53}
$$

Combining the above estimates entails

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho_1}w\|^2 + \|z\|^2\right) + \mu\|\nabla w\|^2 + \nu\|\nabla z\|^2 \\
&\le C\left(\|\sqrt{\rho_1}\delta u\|^2 + \|\delta b\|^2\right) + C(1+\|b_2\|_{H^1}^4)(\|\sqrt{\rho_1}w\|^2 + \|z\|^2).
\end{aligned} \tag{3.54}
$$

Next, applying the elliptic estimates to (3.49) yields

$$
\|\nabla^2 w\|^2 \le C(\|\rho_1\delta u\|^2 + \|\rho_1 w_t\|^2 + \|\rho_1(u_1\cdot\nabla)w\|^2 + \|b_2\times(\nabla\times z)\|^2),
$$

from which, by the Hölder, Sobolev and Young inequalities, one deduces that

$$
\begin{aligned}
\|\rho_1(u_1\cdot\nabla)w\|^2 &\le C\|u_1\|_6^2\|\nabla w\|_3^2 \le C\|\nabla u_1\|^2\|\nabla w\|\|\nabla^2 w\| \\
&\le \frac{1}{2}\|\nabla^2 w\|^2 + C\|\nabla u_1\|^4\|\nabla w\|^2, \\
\|b_2\times(\nabla\times z)\|^2 &\le C\|b_2\|_6^2\|\nabla z\|_3^2 \le \|b_2\|_{H^1}^2\|\nabla z\|\|\nabla^2 z\| \\
&\le \frac{1}{2}\|\nabla^2 z\|^2 + C\|b_2\|_{H^1}^4\|\nabla z\|^2;
\end{aligned}
$$

thus,

$$
\frac{1}{2}\|\nabla^2 w\|^2 \le \frac{1}{2}\|\nabla^2 z\|^2 + C\|(\sqrt{\rho_1}\delta u, \sqrt{\rho_1}w_t)\|^2 + C\left(\|\nabla u_1\|^4\|\nabla w\|^2 + \|b_2\|_{H^1}^4\|\nabla z\|^2\right).
$$

Similarly, applying the elliptic estimates to (3.50) gives

$$
\|\nabla^2 z\|^2 \le C(\|\delta b\|^2 + \|z_t\|^2).
$$

Combining the above two estimates, we have

$$
\begin{aligned}
\|(\nabla^2 w, \nabla^2 z)\|^2 &\le C\left(\|(\sqrt{\rho_1}w_t, z_t)\|^2 + \|(\sqrt{\rho_1}\delta u, \delta b)\|^2\right) \\
&\quad + C\left(\|\nabla u_1\|^4\|\nabla w\|^2 + \|b_2\|_{H^1}^4\|\nabla z\|^2\right).
\end{aligned} \tag{3.55}
$$

Then, multiplying (3.49) and (3.50) with w_t and z_t respectively gives

$$
\begin{aligned}
&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\nabla w\|^2 + (\mu+\lambda)\|\operatorname{div} w\|^2 + \nu\|\nabla z\|^2\right) + \|\sqrt{\rho_1}w_t\|^2 + \|z_t\|^2 \\
&= -\int_{\mathbb{T}^3}\delta b\cdot z_t\,\mathrm{d}x - \int_{\mathbb{T}^3}\rho_1\delta u\cdot w_t\,\mathrm{d}x - \int_{\mathbb{T}^3}\rho_1(u_1\cdot\nabla)w\cdot w_t\,\mathrm{d}x - \int_{\mathbb{T}^3} b_2\times(\nabla\times z)\cdot w_t\,\mathrm{d}x.
\end{aligned}
$$

By the Hölder, Gagliardo-Nirenberg and Young inequalities, one obtains that for any positive number ζ,

$$
\begin{aligned}
\Big|\int_{\mathbb{T}^3}\rho_1(u_1\cdot\nabla)w\cdot w_t\,\mathrm{d}x\Big| &\le \sqrt{\bar\rho}\|\sqrt{\rho_1}w_t\|\|u_1\|_6\|\nabla w\|_3 \\
&\le \|\sqrt{\rho_1}w_t\|\|\nabla u_1\|\|\nabla w\|^{1/2}\|\nabla^2 w\|^{1/2} \\
&\le \zeta(\|\sqrt{\rho_1}w_t\|^2 + \|\nabla^2 w\|^2) + C\|\nabla u_1\|^4\|\nabla w\|^2.
\end{aligned}
$$

Since

$$
\int_{\mathbb{T}^3} b_2\times(\nabla\times z_t)\cdot w\,\mathrm{d}x = \int_{\mathbb{T}^3}(\nabla\times z_t)\cdot(w\times b_2)\,\mathrm{d}x = \int_{\mathbb{T}^3} z_t\cdot\nabla\times(w\times b_2)\,\mathrm{d}x,
$$

integration by parts gives

$$
\int_{\mathbb{T}^3} b_2\times(\nabla\times z)\cdot w_t\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}I(t) - \int_{\mathbb{T}^3}\partial_t b_2\times(\nabla\times z)\cdot w\,\mathrm{d}x - \int_{\mathbb{T}^3} z_t\cdot\nabla\times(w\times b_2)\,\mathrm{d}x.
$$

It follows from the Gagliardo-Nirenberg and Cauchy inequalities that

$$\begin{aligned}\Big|\int_{\mathbb{T}^3}\partial_t b_2\times(\nabla\times z)\cdot w\mathrm{d}x\Big| &\le \|\partial_t b_2\|\|\nabla z\|_6\|w\|_3\\ &\le \|\partial_t b_2\|\|\nabla^2 z\|(\|\sqrt{\rho_1}w\|+\|\nabla w\|)\\ &\le \zeta\|\nabla^2 z\|^2+C\|\partial_t b_2\|^2(\|\sqrt{\rho_1}w\|^2+\|\nabla w\|^2),\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{T}^3} z_t\cdot\nabla\times(w\times b_2)\,\mathrm{d}x\Big|\\ &\le \|z_t\|\,(\|b_2\|_6\|\nabla w\|_3+\|\nabla b_2\|_3\|w\|_6)\\ &\le C\|z_t\|\,\Big(\|b_2\|_{H^1}\|\nabla w\|^{1/2}\|\nabla^2 w\|^{1/2}+\|\nabla b_2\|^{1/2}\|\nabla^2 b_2\|^{1/2}(\|\sqrt{\rho_1}w\|+\|\nabla w\|)\Big)\\ &\le \zeta\left(\|z_t\|^2+\|\nabla^2 w\|^2\right)+C\left(\|b_2\|_{H^1}^4+\|\nabla b_2\|\|\nabla^2 b_2\|\right)(\|\sqrt{\rho_1}w\|^2+\|\nabla w\|^2).\end{aligned}$$

Therefore, plugging the above estimates, adding the resulting inequalities to (3.55) multiplied with a positive number ϵ_3, and choosing ζ sufficiently small shows that

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\nabla w\|^2+(\mu+\lambda)\|\operatorname{div}w\|^2+\nu\|\nabla z\|^2-2I(t)\right)\\ &+\|\sqrt{\rho_1}w_t\|^2+\|z_t\|^2+\epsilon_3\|\nabla^2 w\|^2+\epsilon_3\|\nabla^2 z\|^2\\ &\le C\|(\sqrt{\rho_1}\delta u,\delta b)\|^2+CB(t)(\|\sqrt{\rho_1}w\|^2+\|\nabla w\|^2+\|\nabla z\|^2),\end{aligned}\tag{3.56}$$

where

$$B(t):=1+\|\partial_t b_2\|^2+\|\nabla u_1\|^4+\|b_2\|_{H^1}^4+\|\nabla b_2\|\|\nabla^2 b_2\|\in L^1((0,T)).$$

We define

$$E(t):=\mu\|\nabla w\|^2+(\mu+\lambda)\|\operatorname{div}w\|^2+\nu\|\nabla z\|^2+\gamma\left(\|\sqrt{\rho_1}w\|^2+\|z\|^2\right)-2I(t),$$

and

$$\mathcal{E}(t):=\|\nabla w\|^2+\|\operatorname{div}w\|^2+\|\nabla z\|^2+\|\sqrt{\rho_1}w\|^2+\|z\|^2,$$

for a large number γ. Similar to (3.53), we have

$$|I(t)|\le\varepsilon(\|\nabla z\|^2+\|\nabla w\|^2)+C_\varepsilon(1+\|b_2\|_{H^1}^4)(\|\sqrt{\rho_1}w\|^2+\|z\|^2).$$

Choosing $\varepsilon>0$ sufficiently small and then $\gamma>0$ sufficiently large, we obtain

$$c_0\mathcal{E}(t)\le E(t)\le C_0\mathcal{E}(t),\tag{3.57}$$

where c_0 and C_0 are two positive constants depending only on $\varepsilon,\mu,\lambda,\nu,\gamma$ and on the norms of solutions (ρ_i,u_i,b_i), $i=1,2$. Therefore, multiplying (3.54) by a sufficiently large number γ and adding the result to (3.56) gives

$$-\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\|\sqrt{\rho_1}w_t\|^2+\|z_t\|^2+\|\nabla^2 w\|^2+\|\nabla^2 z\|^2\le C\left(\|\sqrt{\rho_1}\delta u\|^2+\|\delta b\|^2\right)+CB(t)E(t).$$

Since $B(t)\in L^1(0,T)$ and $w(T)=z(T)=0$, the Grönwall inequality and (3.57) give

$$\begin{aligned}&\sup_{0\le t\le T}\|(\sqrt{\rho_1}w,z,\nabla w,\nabla z)\|^2+\int_0^T\|(\sqrt{\rho_1}w_t,z_t,\nabla^2 w,\nabla^2 z)\|^2\,\mathrm{d}t\\ &\le C_T\|(\sqrt{\rho_1}\delta u,\delta b)\|_{L^2(0,T;L^2)}^2.\end{aligned}\tag{3.58}$$

Step 3. Uniqueness. We next estimate $\sqrt{\rho_1}\delta u, \delta b$ in $L^2(0,T;L^2)$. Testing (3.45) and (3.49), respectively, with w and δu, and adding the resulting identities together, one obtains

$$\begin{aligned}\|\sqrt{\rho_1}\delta u\|^2_{L^2(0,T;L^2)} &= a\int_0^T\int_{\mathbb{T}^3}\delta\rho\operatorname{div} w\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{T}^3}(\delta\rho\dot{u}_2 + \rho_1(\delta u\cdot\nabla)u_2)\cdot w\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\int_{\mathbb{T}^3}\Big((\delta b\cdot\nabla)b_2 + (b_1\cdot\nabla)\delta b - \frac{1}{2}\nabla(\delta b\cdot(b_1+b_2))\Big)\cdot w\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_0^T\int_{\mathbb{T}^3} b_2\times(\nabla\times z)\cdot\delta u\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.59}$$

Similarly, multiplying (3.46) by z and (3.50) by δb leads to

$$\|\delta b\|^2_{L^2(0,T;L^2)} = \int_0^T\int_{\mathbb{T}^3}\nabla\times(\delta u\times b_2)\cdot z\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^3}\nabla\times(u_1\times\delta b)\cdot z\,\mathrm{d}x\,\mathrm{d}t. \tag{3.60}$$

Since

$$\int_0^T\int_{\mathbb{T}^3}\nabla\times(\delta u\times b_2)\cdot z\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^3}\delta u\cdot\big(b_2\times(\nabla\times z)\big)\,\mathrm{d}x\,\mathrm{d}t,$$

combining (3.59) and (3.60) implies

$$\begin{aligned}&\|(\sqrt{\rho_1}\delta u, \delta b)\|^2_{L^2(0,T;L^2)}\\ &= a\int_0^T\int_{\mathbb{T}^3}\delta\rho\operatorname{div} w\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{T}^3}(\delta\rho\dot{u}_2 + \rho_1(\delta u\cdot\nabla)u_2)\cdot w\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\int_{\mathbb{T}^3}\Big((\delta b\cdot\nabla)b_2 + (b_1\cdot\nabla)\delta b - \frac{1}{2}\nabla(\delta b\cdot(b_1+b_2))\Big)\cdot w\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_0^T\int_{\mathbb{T}^3}\nabla\times(u_1\times\delta b)\cdot z\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.61}$$

The terms on the right-hand side can be estimated by the Hölder, Gagliardo-Nirenberg and Young inequalities as follows:

$$\Big|\int_0^T\int_{\mathbb{T}^3}\delta\rho\operatorname{div} w\,\mathrm{d}x\,\mathrm{d}t\Big| \le T\|t^{-1/2}\delta\rho\|_{L^\infty(0,T;H^{-1})}\|\nabla\operatorname{div} w\|_{L^2(0,T;L^2)},$$

$$\begin{aligned}&\Big|\int_0^T\int_{\mathbb{T}^3}\delta\rho\dot{u}_2\cdot w\,\mathrm{d}x\,\mathrm{d}t\Big|\\ &\le \|t^{-1/2}\delta\rho\|_{L^\infty(0,T;H^{-1})}\int_0^T\|\sqrt{t}\nabla(\dot{u}_2 w)\|\,\mathrm{d}t\\ &\le C\|t^{-1/2}\delta\rho\|_{L^\infty(0,T;H^{-1})}\int_0^T\Big(\|\sqrt{t}\nabla\dot{u}_2\|\|w\|_\infty + \|\sqrt{t}\dot{u}_2\|_6\|\nabla w\|_3\Big)\,\mathrm{d}t\\ &\le CT^{\frac14}\|t^{-1/2}\delta\rho\|_{L^\infty(0,T;H^{-1})}\|\sqrt{t}\nabla\dot{u}_2\|_{L^2(0,T;L^2)}\,\big(\|(\sqrt{\rho_1}w,\nabla w)\|_{L^\infty(0,T;L^2)}\\ &\quad + \|(\sqrt{\rho_1}w,\nabla w)\|^{1/2}_{L^\infty(0,T;L^2)}\|\nabla^2 w\|^{1/2}_{L^2(0,T;L^2)}\,\big),\end{aligned}$$

$$\begin{aligned}&\Big|\int_0^T\int_{\mathbb{T}^3}\rho_1(\delta u\cdot\nabla)u_2\cdot w\,\mathrm{d}x\,\mathrm{d}t\Big|\\ &\le \int_0^T\sqrt{\bar\rho}\|\sqrt{\rho_1}\delta u\|\|\nabla u_2\|_3\|w\|_6\,\mathrm{d}t\\ &\le C\int_0^T\|\sqrt{\rho_1}\delta u\|\|\nabla u_2\|^{1/2}\|\nabla u_2\|_6^{1/2}\,(\|\sqrt{\rho_1}w\| + \|\nabla w\|)\,\mathrm{d}t\\ &\le CT^{1/4}\|\sqrt{\rho_1}\delta u\|_{L^2(0,T;L^2)}\|\nabla u_2\|^{1/2}_{L^\infty(0,T;L^2)}\|\nabla u_2\|^{1/2}_{L^2(0,T;L^6)}\|(\sqrt{\rho_1}w,\nabla w)\|_{L^\infty(0,T;L^2)},\end{aligned}$$

$$
\begin{aligned}
&\Big|\int_0^T\int_{\mathbb{T}^3}\Big(((\delta b\cdot\nabla)b_2+(b_1\cdot\nabla)\delta b-\frac12\nabla(\delta b\cdot(b_1+b_2))\Big)\cdot w\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\le C\int_0^T\left(\|\nabla b_2\|_3\|w\|_6+\|b_1\|_\infty\|\nabla w\|+\|b_2\|_\infty\|\nabla w\|\right)\|\delta b\|\,\mathrm{d}t\\
&\le C\int_0^T\Big(\|(b_1,b_2)\|_{H^1}+\|(b_1,b_2)\|_{H^1}^{1/2}\|\nabla^2(b_1,b_2)\|^{1/2}\Big)\left(\|\sqrt{\rho_1}w\|+\|\nabla w\|\right)\|\delta b\|\,\mathrm{d}t\\
&\le CT^{1/4}\|\delta b\|_{L^2(0,T;L^2)}\Big(\|(b_1,b_2)\|_{L^\infty(0,T;H^1)}\\
&\quad+\|(b_1,b_2)\|_{L^\infty(0,T;H^1)}^{1/2}\|\nabla^2(b_1,b_2)\|_{L^2(0,T;L^2)}^{1/2}\Big)\|(\sqrt{\rho_1}w,\nabla w)\|_{L^\infty(0,T;L^2)},\\
&\quad\Big|\int_0^T\int_{\mathbb{T}^3}\nabla\times(u_1\times\delta b)\cdot z\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\quad\le C\int_0^T\|u_1\|_\infty\|\delta b\|\|\nabla z\|\,\mathrm{d}t\\
&\quad\le CT^{1/4}\|\delta b\|_{L^2(0,T;L^2)}\|\nabla u_1\|_{L^\infty(0,T;L^2)}^{1/2}\|\nabla u_1\|_{L^2(0,T;L^6)}^{1/2}\|\nabla z\|_{L^\infty(0,T;L^2)}.
\end{aligned}
$$

Plugging the above estimates in (3.61), one deduces by Corollary 2.8 and (3.58)

$$
\begin{aligned}
\|(\sqrt{\rho_1}\delta u,\delta b)\|_{L^2(0,T;L^2)}^2&\le CT^{1/4}\|t^{-1/2}\delta\rho\|_{L^\infty(0,T;H^{-1})}\|(\sqrt{\rho_1}\delta u,\delta b)\|_{L^2(0,T;L^2)}\\
&\quad+CT^{1/4}\|(\sqrt{\rho_1}\delta u,\delta b)\|_{L^2(0,T;L^2)}^2.
\end{aligned}
$$

Therefore, we choose T sufficiently small such that

$$
\|(\sqrt{\rho_1}\delta u,\delta b)\|_{L^2(0,T;L^2)}\le CT^{1/4}\|t^{-1/2}\delta\rho\|_{L^\infty(0,T;H^{-1})}\le CT^{1/4}Z(T),\tag{3.62}
$$

which, together with (3.48), leads to

$$
Z(t)\le C\int_0^t\|\nabla u_2(\tau)\|_{\mathrm{BMO}}Z(\tau)(1+|\log\tau|+|\log Z(\tau)|)\,\mathrm{d}\tau,\quad\text{for all } t\in[0,T].
$$

Since $T_0\le 1$,

$$
1+|\log s|+|\log Z(s)|\le(1+|\log s|)(1+|\log Z(s)|).
$$

We set

$$
F(s):=C(1+|\log s|)\|\nabla u_2(s)\|_{\mathrm{BMO}},\quad \mu(r):=\begin{cases} r(1+|\log r|), & r>0,\\ 0, & r=0.\end{cases}
$$

Hence, by Lemma 1.6, the above inequality implies $Z=0$ on $[0,T]$, that is, $\delta\rho=0$. Thanks to this, it follows from (3.62) that $\sqrt{\rho_1}\delta u=0$ and $\delta b=0$. Consequently, recalling (3.45), one has

$$
\rho_1(\delta u_t+u_1\cdot\nabla\delta u)-\mu\Delta\delta u-(\mu+\lambda)\nabla\operatorname{div}\delta u=0;
$$

therefore,

$$
\frac12\|\sqrt{\rho_1}\delta u\|_{L^\infty(0,T;L^2)}^2+\int_0^T\left(\mu\|\nabla\delta u\|^2+(\mu+\lambda)\|\operatorname{div}\delta u\|^2\right)\,\mathrm{d}t=0,
$$

which implies that $\delta u=0$ on $[0,T]$. The preceding argument proves uniqueness on some interval $[0,T_1]$. Since the two solutions coincide at $t=T_1$, the same argument can be restarted from T_1. At every positive restart time, the time-weighted estimates imply the corresponding local unweighted estimates. A standard open–closed continuation argument therefore yields uniqueness on the whole interval $[0,T_0]$.

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