Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 68, pp. 1–29.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.68

GLOBAL WELL-POSEDNESS AND INVARIANT REGIONS FOR A ONE-DIMENSIONAL RD-ARD GLIOBLASTOMA MODEL

ZHIPENG YANG, ZHENYU ZHAN

Abstract. We study a one-dimensional reaction-diffusion and advection-reaction-diffusion system arising from a heterogeneous glioblastoma model with shared crowding effects. The two components represent latent subpopulation densities that both diffuse and proliferate; the second is additionally subject to a constant advective bias, with crowding coupled through the total density. On a bounded interval, under either componentwise Neumann conditions or the natural zero-flux condition for the advective component, we establish a global well-posedness theory, positivity preservation, invariant regions, and uniform absorbing bounds in the supremum norm. Under the zero-flux condition, the invariant and absorbing upper barriers for the second component carry the exponential spatial weight $e^{(a/D_v)x}$, in contrast to the constant barriers available under componentwise Neumann conditions. We further formulate the associated semiflow and prove the existence of a global attractor.

1. Introduction and main results

Reaction-diffusion equations play a central role in the mathematical description of tumor growth and invasion. A classical starting point is the Fisher-KPP equation
$$U_t = D\Delta U + \rho U\Big(1 - \frac{U}{K}\Big), \tag{1.1}$$
which combines random motility and logistic proliferation. In glioma modeling, this equation and its variants have been extensively used to quantify tumor invasion and growth kinetics; see, for example, [2, 15, 16]. From the mathematical side, (1.1) has a well-developed theory including comparison principles, invariant regions, global existence, traveling waves, and spreading speeds. In particular, in homogeneous media the quantity $2\sqrt{D\rho}$ plays the role of the minimal wave speed, and the ratio D/ρ provides a natural measure of diffuseness versus proliferation. These features make (1.1) a natural first model, but they also reveal its limitations when intratumoral heterogeneity becomes important.

A number of experimental and theoretical studies indicate that glioblastoma populations are far from homogeneous. In particular, tumor spheroids often exhibit a relatively dense core and a more invasive outer rim, and the two regions may propagate at markedly different speeds. Such phenomena are difficult to reconcile with a single scalar Fisher-KPP equation. This has led to more structured PDE models incorporating migration-proliferation dichotomies, phenotypic switching, chemotactic bias, or density-dependent motility; see, for instance, [4, 11, 13, 14]. One important example is the reaction-cross-diffusion system studied by Burger, Friele, and Pietschmann [3],
$$\partial_t p = D_\alpha(1-\rho)\Delta p + R_p(p,m), \tag{1.2}$$
$$\partial_t m = D_\nu\big((1-\rho)\Delta m + m\Delta\rho\big) + R_m(p,m), \tag{1.3}$$
where $\rho = p + m$. In that model, size exclusion leads to a degenerate diffusion structure and a nonlinear cross-diffusion term, and the one-dimensional existence theory requires a delicate combination of implicit time discretization, partial entropy dissipation, variational arguments,

2020 *Mathematics Subject Classification.* 35K57, 35B40, 92C50.
Key words and phrases. Reaction-diffusion-advection system; glioblastoma model; global well-posedness.

Submitted June 25 2026. Published September 9, 2026.

and compactness methods [3]. This illustrates that even in one space dimension, glioblastoma systems with multiple subpopulations may lead to substantial analytical difficulties.

This paper is motivated by the recent work of Malik et al. [8], where the authors studied multicellular tumor spheroids obtained from patient-derived glioblastoma cell lines. Biologically, these spheroids are three-dimensional cell aggregates embedded in extracellular matrix and monitored through longitudinal brightfield imaging. The experimentally accessible quantity is not the density of each subpopulation separately, but rather a radially averaged total density profile extracted from the images as a function of distance from the spheroid center and time. Malik et al. observed that a single-population model often fails to reproduce two characteristic features of the data: first, the core region and the invasive rim may expand at different effective speeds; second, some samples display a pronounced hump-type structure near the invasion front. To capture these effects, they introduced a two-population PDE model in which one subpopulation is governed by reaction-diffusion dynamics and the other by advection-reaction-diffusion dynamics. More precisely, their original three-dimensional model reads

$$\partial_t u_1 = \nabla\cdot(D_1\nabla u_1) + \rho_1 u_1\Big(1-\frac{u_1+u_2}{K_1}\Big), \tag{1.4}$$

$$\partial_t u_2 = \nabla\cdot(D_2\nabla u_2) + \rho_2 u_2\Big(1-\frac{u_1+u_2}{K_2}\Big) - \nabla\cdot(A_2u_2), \tag{1.5}$$

where u_1 and u_2 denote the two subpopulation densities, $D_1, D_2 > 0$ are diffusion coefficients, $\rho_1, \rho_2 > 0$ are proliferation rates, $K_1, K_2 > 0$ are carrying capacities, and A_2 describes a directed outward bias in the motion of the second population [8]. Since the measurements are radial averages, Malik et al. reduced the model to the spherically symmetric form

$$\partial_t u_1 = \frac{1}{r^2}\partial_r\Big(r^2D_1\partial_r u_1\Big) + \rho_1 u_1\Big(1-\frac{u_1+u_2}{K_1}\Big), \tag{1.6}$$

$$\partial_t u_2 = \frac{1}{r^2}\partial_r\Big(r^2D_2\partial_r u_2\Big) + \rho_2 u_2\Big(1-\frac{u_1+u_2}{K_2}\Big) - \partial_r(A_2u_2). \tag{1.7}$$

Because only the total density is observable, they further assumed a proportional splitting of the initial profile,

$$u_1(r,0) = \alpha U_0(r), \quad u_2(r,0) = (1-\alpha)U_0(r), \quad 0 \le \alpha \le 1,$$

which introduces an additional latent parameter into the model. Their analysis was mainly numerical and data-driven: they fitted the model to spheroid data, compared it with simpler PDE models, and introduced wave-parameters based on tracked density thresholds. In particular, the propagation quantities used in [8] are not classical traveling-wave speeds in the PDE sense, but threshold-based numerical descriptors.

The analytical question addressed in the present paper is therefore a preliminary but fundamental one. Before studying threshold propagation, wave selection, or identifiability from aggregate data, one needs a satisfactory PDE framework for the deterministic bounded domain problem itself. We focus on the simplest rigorous setting in which the main effects of the model are already present, namely the one-dimensional system

$$\begin{gathered} u_t = D_u u_{xx} + \rho_u u\Big(1-\frac{u+v}{K_u}\Big), \quad x\in\Omega,\ t>0, \\ v_t = D_v v_{xx} - av_x + \rho_v v\Big(1-\frac{u+v}{K_v}\Big), \quad x\in\Omega,\ t>0, \end{gathered} \tag{1.8}$$

posed on the bounded interval $\Omega = (0, L)$ $L > 0$, with initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x\in\Omega. \tag{1.9}$$

Here $u = u(x,t)$ and $v = v(x,t)$ denote two subpopulations, and

$$w(x,t) = u(x,t) + v(x,t)$$

is the total density. The coefficients $D_u, D_v, \rho_u, \rho_v, K_u, K_v, a$ are assumed positive throughout. The first component u describes a more standard diffusive-proliferative population, while the second component v includes, in addition, a constant directed drift of magnitude a. Mathematically, (1.8) is a semilinear and uniformly parabolic system, but it is not completely standard. The two equations are coupled through the total density $u+v$, the second component contains a first-order advection term, and the natural upper barriers for v depend strongly on the boundary operator. These features make the distinction between different boundary closures essential already at the level of invariant regions and dissipativity.

In this paper we consider the following two boundary conditions.

Assumption 1.1. *We consider one of the following boundary conditions.*

(1) *Componentwise Neumann conditions*

$$u_x(0,t) = u_x(L,t) = 0, \quad v_x(0,t) = v_x(L,t) = 0, \quad t > 0; \tag{1.10}$$

(2) *Mixed Neumann and zero-flux conditions*

$$\begin{aligned} u_x(0,t) &= u_x(L,t) = 0, \\ D_v v_x(0,t) - av(0,t) &= 0, \\ D_v v_x(L,t) - av(L,t) &= 0, \quad t > 0. \end{aligned} \tag{1.11}$$

Condition (1.11) is the natural homogeneous flux condition for the advective-diffusive operator

$$D_v\partial_{xx} - a\partial_x,$$

since the associated one-dimensional flux is

$$J_v = -D_v v_x + av.$$

Condition (1.10) is included as an alternative closure because it is frequently used in numerical and data-driven studies of radially reduced tumor-spheroid models.

We work in the phase space

$$\mathcal{X} = C(\overline{\Omega}) \times C(\overline{\Omega}), \quad \|(u,v)\|_{\mathcal{X}} = \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)},$$

with positive cone

$$\mathcal{X}_+ = \{(u,v) \in \mathcal{X} : \ u \geq 0, \ v \geq 0 \text{ on } \overline{\Omega}\}.$$

Definition 1.2. Let $T > 0$. A pair (u,v) is called a classical solution of (1.8)-(1.9) on $[0,T]$ if

$$u, v \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T]),$$

the equations in (1.8) hold pointwise in $\Omega \times (0,T]$, the initial conditions (1.9) hold on $\overline{\Omega}$, and the chosen boundary conditions from Assumption 1.1 hold for every $t \in (0,T]$.

Definition 1.3. Let $p > 1$. Initial data $(u_0, v_0) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ are called strongly compatible if they satisfy the chosen boundary conditions in the trace sense, namely

$$u_0'(0) = u_0'(L) = 0,$$

and either

$$v_0'(0) = v_0'(L) = 0$$

under (1.10), or

$$D_v v_0'(0) - av_0(0) = 0, \quad D_v v_0'(L) - av_0(L) = 0$$

under (1.11).

The purpose of this paper is to establish the basic PDE and dynamical framework for (1.8) on a bounded interval. Our first theorem gives global well-posedness and positivity in the phase space $\mathcal{X}_+$.

Theorem 1.4. *Assume that $D_u, D_v, \rho_u, \rho_v, K_u, K_v, a > 0$, and let one of the boundary conditions in Assumption 1.1 be fixed.*

(a) *For every $(u_0, v_0) \in \mathcal{X}_+$, there exists a unique global mild solution $(u,v) \in C([0,\infty); \mathcal{X}_+)$, which is classical on every strip $\overline{\Omega} \times [\tau, T]$ with $0 < \tau < T < \infty$.*

(b) *If, in addition,* $(u_0, v_0) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ *for some* $p > 1$, *and* (u_0, v_0) *are strongly compatible, then the corresponding solution is a global* L^p*-strong solution, namely,*
$$u, v \in C([0,\infty); W^{2,p}(\Omega)) \cap C^1([0,\infty); L^p(\Omega)).$$
In particular, it is classical on $[0,\infty)$.

(c) *The solution depends continuously on the initial data in* $\mathcal{X}_+$.

Our second result concerns invariant regions. Because the second component is subject to two different boundary operators, the natural invariant geometry depends on which boundary condition is imposed.

Theorem 1.5. *Under the assumptions of Theorem 1.4, the following assertions hold.*

(i) *If the boundary condition* (1.10) *is imposed and*
$$0 \le u_0(x) \le K_u, \quad 0 \le v_0(x) \le K_v \quad \text{for all } x \in \overline{\Omega},$$
then the unique global solution satisfies
$$0 \le u(x,t) \le K_u, \quad 0 \le v(x,t) \le K_v \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty),$$
so that the rectangle $[0, K_u] \times [0, K_v]$ *is positively invariant.*

(ii) *If the boundary condition* (1.11) *is imposed and, for some* $M_u \ge K_u$ *and* $M_v \ge K_v$,
$$0 \le u_0(x) \le M_u, \quad 0 \le v_0(x) \le M_v e^{\beta x} \quad \text{for all } x \in \overline{\Omega},$$
where $\beta = a/D_v$, *then the unique global solution satisfies*
$$0 \le u(x,t) \le M_u, \quad 0 \le v(x,t) \le M_v e^{\beta x} \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty).$$

Our third result concerns dissipativity in the phase space $\mathcal{X}$.

Theorem 1.6. *Under the assumptions of Theorem 1.4, let* $B \subset \mathcal{X}_+$ *be bounded and let* $\varepsilon > 0$.

(i) *If the boundary condition* (1.10) *is imposed, then there exists a time* $T_B^{\mathrm{N}}(\varepsilon) \ge 0$ *such that every solution with initial datum in* B *satisfies*
$$0 \le u(x,t) \le K_u + \varepsilon, \quad 0 \le v(x,t) \le K_v + \varepsilon$$
for all $x \in \overline{\Omega}$ *and all* $t \ge T_B^{\mathrm{N}}(\varepsilon)$.

(ii) *If the boundary condition* (1.11) *is imposed, then there exists a time* $T_B^{\mathrm{F}}(\varepsilon) \ge 0$ *such that every solution with initial datum in* B *satisfies*
$$0 \le u(x,t) \le K_u + \varepsilon, \quad 0 \le v(x,t) \le e^{\beta x}(K_v + \varepsilon) \le e^{\beta L}(K_v + \varepsilon)$$
for all $x \in \overline{\Omega}$ *and all* $t \ge T_B^{\mathrm{F}}(\varepsilon)$, *where* $\beta = a/D_v$.

In particular, for each fixed choice of boundary conditions in Assumption 1.1, the corresponding semiflow is point dissipative on $\mathcal{X}_+$.

The final theorem states the existence of a compact global attractor.

Theorem 1.7. *For each fixed choice of boundary conditions in Assumption 1.1, let* $\{S(t)\}_{t\ge 0}$ *denote the semiflow on* $\mathcal{X}_+$ *generated by* (1.8). *Then* $\{S(t)\}_{t\ge0}$ *possesses a compact global attractor* $\mathscr{A} \subset \mathcal{X}_+$. *Moreover,* $\mathscr{A}$ *is fully invariant and bounded in* $C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$ *for some* $\alpha \in (0,1)$.

Theorems 1.4-1.7 provide a basic PDE framework for a Malik-type RD-ARD model on bounded intervals. They supply the deterministic framework needed before one can study the more refined questions suggested by [8], including equilibrium selection, threshold propagation speeds, and the relation between numerical wave descriptors and genuine PDE propagation mechanisms. They also show that even for a semilinear heterogeneous glioblastoma model, the distinction between Neumann and flux boundary closures has a genuine analytical effect on the shape of invariant and absorbing regions.

The remainder of the paper is organized as follows. Section 2 collects scalar comparison tools and logistic barriers. Section 3 establishes local solvability and the continuation criterion. Section 4 proves positivity preservation and invariant regions. Section 5 derives global existence, uniform bounds, and parabolic smoothing for positive times. Section 6 is devoted to absorbing regions and dissipativity, and Section 7 develops the semiflow structure and proves the existence of the global attractor.

2. Auxiliary scalar problems and comparison principles

This section collects the scalar tools needed in the later analysis.

2.1. Logistic ordinary differential equations.

Lemma 2.1. *Let $\rho > 0$, $K > 0$, and $U_0 \geq 0$. Then the problem*

$$U'(t) = \rho U(t)\Big(1 - \frac{U(t)}{K}\Big), \quad U(0) = U_0, \tag{2.1}$$

admits a unique global solution $U \in C^1([0,\infty))$, and the following hold:

(i) *$U(t) \geq 0$ for all $t \geq 0$,*
(ii) *if $0 \leq U_0 \leq K$, then $0 \leq U(t) \leq K$ for all $t \geq 0$,*
(iii) *for every $M \geq 0$ and every $\varepsilon > 0$, the family of solutions with $0 \leq U_0 \leq M$ enters $[0, K+\varepsilon]$ after a time depending only on M and ε.*

Proof. The vector field in (2.1) is locally Lipschitz, hence the problem has a unique local C^1-solution. For $U_0 = 0$ this solution is identically zero. If $U_0 > 0$, separation of variables gives

$$U(t) = \frac{K}{1 + \big(\frac{K}{U_0} - 1\big)e^{-\rho t}} = \frac{KU_0 e^{\rho t}}{K + U_0(e^{\rho t} - 1)}. \tag{2.2}$$

The denominator in (2.2) stays positive for all $t \geq 0$, so the solution is global.

Assertions (i) and (ii) are immediate from (2.2): one has $U(t) \geq 0$ for all $t \geq 0$, and if $0 \leq U_0 \leq K$, then $0 \leq U(t) \leq K$ for all $t \geq 0$.

For (iii), fix $M \geq 0$ and $\varepsilon > 0$. If $M \leq K + \varepsilon$, we may take $T = 0$. Otherwise $M > K + \varepsilon$, and (2.2) shows that the solution is monotone with respect to the initial value. It is therefore sufficient to consider the solution with initial data $U_0 = M$. Solving the inequality $U_M(t) \leq K + \varepsilon$ gives the entry time

$$T(M,\varepsilon) = \frac{1}{\rho} \log\Big(\frac{(K+\varepsilon)(M-K)}{\varepsilon M}\Big). \tag{2.3}$$

Hence every solution with $0 \leq U_0 \leq M$ satisfies $U(t) \leq K + \varepsilon$ for all $t \geq T(M,\varepsilon)$. This completes the proof. $\square$

2.2. Maximum and comparison principles.

For the applications below we only need the maximum principle in the two boundary configurations naturally associated with our model, namely the homogeneous Neumann condition

$$z_x(0,t) = z_x(L,t) = 0$$

and the homogeneous flux condition

$$dz_x(0,t) - bz(0,t) = 0, \quad dz_x(L,t) - bz(L,t) = 0.$$

Lemma 2.2. *Let $d > 0$, $b \in \mathbb{R}$, $T > 0$, and $c, f \in L^\infty(\Omega \times (0,T))$. Assume that*

$$z \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T))$$

satisfies

$$z_t - dz_{xx} + bz_x + c(x,t)z \leq f(x,t) \quad \text{in } \Omega \times (0,T), \tag{2.4}$$

together with

$$z(x,0) \leq 0 \quad \text{for all } x \in \overline{\Omega},$$

and either

$$z_x(0,t) = z_x(L,t) = 0 \quad \text{for } t \in (0,T), \tag{2.5}$$

or

$$dz_x(0,t) - bz(0,t) = 0, \quad dz_x(L,t) - bz(L,t) = 0 \quad \text{for } t \in (0,T). \tag{2.6}$$

If $f(x,t) \leq 0$ for almost every $(x,t) \in \Omega \times (0,T)$, then

$$z(x,t) \leq 0 \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T].$$

Proof. Assume first that (2.5) holds. Choose $\lambda > \|c\|_{L^\infty(\Omega\times(0,T))}$ and set $y(x,t) = e^{-\lambda t}z(x,t)$. Then

$$y_t - dy_{xx} + by_x + (c(x,t)+\lambda)y \le e^{-\lambda t}f(x,t) \le 0 \quad \text{in } \Omega\times(0,T),$$

with $y(x,0) \le 0$ and $y_x(0,t) = y_x(L,t) = 0$. Since $c+\lambda \ge \lambda - \|c\|_{L^\infty} > 0$, the standard weak maximum principle for linear parabolic equations with homogeneous Neumann boundary conditions implies $y \le 0$ on $\overline{\Omega}\times[0,T]$; see, for example, [9, 12]. Hence $z \le 0$.

Assume next that (2.6) holds and define $y(x,t) = e^{-bx/d}z(x,t)$. A direct computation gives

$$y_t - dy_{xx} - by_x + c(x,t)y \le e^{-bx/d}f(x,t) \le 0 \quad \text{in } \Omega\times(0,T),$$

while the boundary condition becomes $y_x(0,t) = y_x(L,t) = 0$, and still $y(x,0) \le 0$. Applying the first part to y yields $y \le 0$, and therefore $z = e^{bx/d}y \le 0$. This completes the proof. □

Lemma 2.3. *Let $d > 0$, $b \in \mathbb{R}$, $T > 0$, and let $G \in C(\overline{\Omega}\times[0,T]\times\mathbb{R})$ be locally Lipschitz continuous with respect to the third variable, uniformly on bounded sets. Assume that*

$$z_-, z_+ \in C(\overline{\Omega}\times[0,T]) \cap C^{2,1}(\overline{\Omega}\times(0,T))$$

satisfy

$$(z_-)_t - d(z_-)_{xx} + b(z_-)_x \le G(x,t,z_-) \quad \text{in } \Omega\times(0,T), \tag{2.7}$$
$$(z_+)_t - d(z_+)_{xx} + b(z_+)_x \ge G(x,t,z_+) \quad \text{in } \Omega\times(0,T), \tag{2.8}$$

together with

$$z_-(x,0) \le z_+(x,0) \quad \text{for all } x \in \overline{\Omega},$$

and one of the following boundary conditions:

(i)
$$(z_-)_x(0,t) = (z_-)_x(L,t) = 0, \quad (z_+)_x(0,t) = (z_+)_x(L,t) = 0;$$

(ii)
$$d(z_-)_x(0,t) - bz_-(0,t) = 0, \quad d(z_-)_x(L,t) - bz_-(L,t) = 0,$$
$$d(z_+)_x(0,t) - bz_+(0,t) = 0, \quad d(z_+)_x(L,t) - bz_+(L,t) = 0.$$

Then $z_-(x,t) \le z_+(x,t)$ for all $(x,t) \in \overline{\Omega}\times[0,T]$.

Proof. Set $w = z_- - z_+$. Then $w(x,0) \le 0$, and subtracting (2.8) from (2.7) gives

$$w_t - dw_{xx} + bw_x \le G(x,t,z_-) - G(x,t,z_+) \quad \text{in } \Omega\times(0,T).$$

Because z_- and z_+ are bounded on $\overline{\Omega}\times[0,T]$, local Lipschitz continuity of G in the third variable yields a function $c \in L^\infty(\Omega\times(0,T))$ such that

$$G(x,t,z_-) - G(x,t,z_+) = -c(x,t)w.$$

Hence

$$w_t - dw_{xx} + bw_x + c(x,t)w \le 0 \quad \text{in } \Omega\times(0,T),$$

and w satisfies the same homogeneous boundary condition as in Lemma 2.2. Applying that lemma with $f \equiv 0$ yields $w \le 0$ on $\overline{\Omega}\times[0,T]$, that is, $z_- \le z_+$. □

Lemma 2.4. *The constant functions 0, K_u, K_v serve as natural lower and upper barriers for the reaction terms in (1.8), in the following sense:*

$$\rho_u u\Big(1 - \frac{u+v}{K_u}\Big) \le \rho_u u\Big(1 - \frac{u}{K_u}\Big), \quad \rho_v v\Big(1 - \frac{u+v}{K_v}\Big) \le \rho_v v\Big(1 - \frac{v}{K_v}\Big)$$

whenever $u, v \ge 0$, and

$$\rho_u K_u\Big(1 - \frac{K_u+v}{K_u}\Big) \le 0, \quad \rho_v K_v\Big(1 - \frac{u+K_v}{K_v}\Big) \le 0.$$

Proof. If $u, v \geq 0$, then

$$1 - \frac{u+v}{K_u} \leq 1 - \frac{u}{K_u}, \quad 1 - \frac{u+v}{K_v} \leq 1 - \frac{v}{K_v},$$

and multiplication by $\rho_u u \geq 0$ and $\rho_v v \geq 0$ gives the first two inequalities. The last two follow from the identities

$$\rho_u K_u\Big(1 - \frac{K_u+v}{K_u}\Big) = -\rho_u v \leq 0, \quad \rho_v K_v\Big(1 - \frac{u+K_v}{K_v}\Big) = -\rho_v u \leq 0.$$

The constant function 0 is a lower barrier because both reaction terms vanish at zero. ☐

3. Local solvability and continuation criterion

In this section we establish the local well-posedness of (1.8) in the product space

$$\mathcal{X} = C(\overline{\Omega}) \times C(\overline{\Omega}), \quad \Omega = (0, L),$$

and prove the continuation criterion and continuous dependence on the initial data.

3.1. **Linear operators and semigroups.**

We first introduce the linear operators associated with the two components. For the first equation we define

$$A_u\phi = D_u\phi'',$$

with domain

$$D(A_u) = \{\phi \in C^2(\overline{\Omega}) : \phi'(0) = \phi'(L) = 0\}.$$

For the second equation, depending on the choice of boundary condition in Assumption 1.1, we define

$$A_v\psi = D_v\psi'' - a\psi'$$

with one of the following domains:

$$D(A_v^{\mathrm{N}}) = \left\{\psi \in C^2(\overline{\Omega}) : \psi'(0) = \psi'(L) = 0\right\},$$

or

$$D(A_v^{\mathrm{F}}) = \left\{\psi \in C^2(\overline{\Omega}) : D_v\psi'(0) - a\psi(0) = 0,\ D_v\psi'(L) - a\psi(L) = 0\right\}.$$

Accordingly, A_v denotes either A_v^{N} or A_v^{F}.

Finally, on $\mathcal{X}$ we set

$$\mathcal{A}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}A_u\phi\\ A_v\psi\end{pmatrix}, \quad D(\mathcal{A}) = D(A_u) \times D(A_v).$$

Lemma 3.1. *The operators A_u and A_v generate analytic C_0-semigroups on $C(\overline{\Omega})$. Consequently, $\mathcal{A}$ generates an analytic C_0-semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ on $\mathcal{X}$.*

Proof. The operator A_u is the one-dimensional diffusion operator with homogeneous Neumann boundary conditions. The operator A_v is a uniformly elliptic second-order operator with constant coefficients and either homogeneous Neumann boundary conditions or homogeneous Robin boundary conditions. In each case, the boundary operator is separated and satisfies the standard complementing condition.

It is therefore standard that both A_u and A_v are sectorial on $C(\overline{\Omega})$ and generate analytic C_0-semigroups. One may invoke, for instance, the general results on one-dimensional elliptic operators with boundary conditions in [7, 10]. Since $\mathcal{A}$ is diagonal, its semigroup is simply the product semigroup

$$e^{t\mathcal{A}}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}e^{tA_u}\phi\\ e^{tA_v}\psi\end{pmatrix},$$

which is again analytic on $\mathcal{X}$. This completes the proof. ☐

We next rewrite the nonlinear part of (1.8) as a mapping on $\mathcal{X}$. For $U = (u, v) \in \mathcal{X}$, define

$$\mathcal{F}(U) = \mathcal{F}(u, v) = \begin{pmatrix}\rho_u u\big(1 - \frac{u+v}{K_u}\big)\\ \rho_v v\big(1 - \frac{u+v}{K_v}\big)\end{pmatrix}.$$

Lemma 3.2. *The mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is locally Lipschitz continuous. More precisely, for every $R > 0$, there exist constants $L_R > 0$ and $C_R > 0$ such that*
$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_{\mathcal{X}} \le L_R \|U - V\|_{\mathcal{X}}$$
for all $U, V \in \mathcal{X}$ with $\|U\|_{\mathcal{X}} \le R$ and $\|V\|_{\mathcal{X}} \le R$, and
$$\|\mathcal{F}(U)\|_{\mathcal{X}} \le C_R$$
for all $U \in \mathcal{X}$ with $\|U\|_{\mathcal{X}} \le R$.

Proof. Write $U = (u, v)$ and $V = (\tilde{u}, \tilde{v})$. On the ball $\|U\|_{\mathcal{X}}, \|V\|_{\mathcal{X}} \le R$, the polynomial identities
$$|u^2 - \tilde{u}^2| \le 2R|u - \tilde{u}|, \quad |uv - \tilde{u}\tilde{v}| \le R\big(|u - \tilde{u}| + |v - \tilde{v}|\big),$$
and the analogous estimate for $v^2 - \tilde{v}^2$ give
$$\begin{aligned}
|\mathcal{F}_1(U) - \mathcal{F}_1(V)| &\le \rho_u\Big(1 + \frac{4R}{K_u}\Big)\big(|u - \tilde{u}| + |v - \tilde{v}|\big),\\
|\mathcal{F}_2(U) - \mathcal{F}_2(V)| &\le \rho_v\Big(1 + \frac{4R}{K_v}\Big)\big(|u - \tilde{u}| + |v - \tilde{v}|\big).
\end{aligned}$$
Taking suprema over $x \in \overline{\Omega}$ yields the claimed local Lipschitz bound. The growth estimate follows in the same way from the quadratic form of $\mathcal{F}$ on the ball $\|U\|_{\mathcal{X}} \le R$. $\square$

3.2. **Local existence.** We say that $U = (u, v) \in C([0, T); \mathcal{X})$ is a mild solution of (1.8)-(1.9) on $[0, T)$ if
$$U(t) = e^{t\mathcal{A}}U_0 + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{F}(U(s))\,ds \quad \text{for all } t \in [0, T), \tag{3.1}$$
where
$$U_0 = (u_0, v_0) \in \mathcal{X}.$$

Proposition 3.3. *Let one of the boundary conditions in Assumption 1.1 be fixed. For every $(u_0, v_0) \in \mathcal{X}$, there exist $T_{\max} \in (0, \infty]$ and a unique maximal mild solution*
$$(u, v) \in C([0, T_{\max}); \mathcal{X})$$
to (1.8)-(1.9)*. Moreover, the solution is classical on every strip $\overline{\Omega} \times [\tau, T]$ with $0 < \tau < T < T_{\max}$.*

Proof. We divide the proof into several steps.

Step 1. Fixed-point setting. Let $U_0 = (u_0, v_0) \in \mathcal{X}$. Fix $T > 0$, and set $E_T = C([0, T]; \mathcal{X})$ with the norm
$$\|U\|_{E_T} = \sup_{0 \le t \le T} \|U(t)\|_{\mathcal{X}}.$$
For $U \in E_T$, define
$$(\Phi U)(t) = e^{t\mathcal{A}}U_0 + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{F}(U(s))\,ds, \quad 0 \le t \le T.$$
By Lemma 3.1, $\{e^{t\mathcal{A}}\}_{t \ge 0}$ is a C_0-semigroup on $\mathcal{X}$. Hence for every fixed $T > 0$,
$$M_T := \sup_{0 \le t \le T} \|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{X})} < \infty.$$

Step 2. Choice of the ball. Let $R := 2M_1\big(\|U_0\|_{\mathcal{X}} + 1\big)$. Consider the closed ball
$$\mathbb{B}_{R,T} := \{U \in E_T : \|U\|_{E_T} \le R\}.$$
By Lemma 3.2, there exist constants C_R and L_R such that
$$\|\mathcal{F}(U)\|_{\mathcal{X}} \le C_R \quad \text{for all } U \in \mathbb{B}_{R,T},$$
and
$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_{\mathcal{X}} \le L_R\|U - V\|_{\mathcal{X}} \quad \text{for all } U, V \in \mathbb{B}_{R,T}.$$
Choose $T_0 \in (0, 1]$ so small that
$$M_1 T_0 C_R \le \frac{R}{2}, \tag{3.2}$$

and

$$M_1 T_0 L_R \le \frac{1}{2}. \tag{3.3}$$

We claim that Φ maps $\mathbb{B}_{R,T_0}$ into itself and is a contraction there.

Step 3. Φ maps $\mathbb{B}_{R,T_0}$ into itself. Let $U \in \mathbb{B}_{R,T_0}$. For $0 \le t \le T_0$,

$$\begin{aligned}
\|(\Phi U)(t)\|_{\mathcal{X}} &\le \|e^{t\mathcal{A}} U_0\|_{\mathcal{X}} + \int_0^t \|e^{(t-s)\mathcal{A}} \mathcal{F}(U(s))\|_{\mathcal{X}}\, ds \\
&\le M_1 \|U_0\|_{\mathcal{X}} + M_1 \int_0^t \|\mathcal{F}(U(s))\|_{\mathcal{X}}\, ds \\
&\le M_1 \|U_0\|_{\mathcal{X}} + M_1 T_0 C_R.
\end{aligned}$$

By the definition of R,

$$M_1 \|U_0\|_{\mathcal{X}} \le \frac{R}{2},$$

and by (3.2),

$$M_1 T_0 C_R \le \frac{R}{2}.$$

Hence

$$\|(\Phi U)(t)\|_{\mathcal{X}} \le R \quad \text{for all } 0 \le t \le T_0.$$

Thus $\Phi(\mathbb{B}_{R,T_0}) \subset \mathbb{B}_{R,T_0}$.

Step 4. Φ is a contraction. Let $U, V \in \mathbb{B}_{R,T_0}$. Then for $0 \le t \le T_0$,

$$\begin{aligned}
\|(\Phi U)(t) - (\Phi V)(t)\|_{\mathcal{X}} &\le \int_0^t \|e^{(t-s)\mathcal{A}} (\mathcal{F}(U(s)) - \mathcal{F}(V(s)))\|_{\mathcal{X}}\, ds \\
&\le M_1 \int_0^t \|\mathcal{F}(U(s)) - \mathcal{F}(V(s))\|_{\mathcal{X}}\, ds \\
&\le M_1 L_R \int_0^t \|U(s) - V(s)\|_{\mathcal{X}}\, ds \\
&\le M_1 L_R T_0 \|U - V\|_{E_{T_0}}.
\end{aligned}$$

Taking the supremum over $t \in [0, T_0]$ and using (3.3), we obtain

$$\|\Phi U - \Phi V\|_{E_{T_0}} \le \frac{1}{2} \|U - V\|_{E_{T_0}}.$$

Hence Φ is a contraction on $\mathbb{B}_{R,T_0}$.

Step 5. Existence and uniqueness on a short interval. By the Banach fixed-point theorem, Φ has a unique fixed point $U \in \mathbb{B}_{R,T_0}$. This fixed point is a mild solution of (1.8)-(1.9) on $[0, T_0]$.

Moreover, the construction shows that the lifespan T_0 depends only on $\|U_0\|_{\mathcal{X}}$. More precisely, for every $M > 0$, there exists

$$\tau(M) > 0$$

such that every initial datum $U_0 \in \mathcal{X}$ with $\|U_0\|_{\mathcal{X}} \le M$ generates a unique mild solution on $[0, \tau(M)]$.

Step 6. Maximal extension. Starting from the solution at time T_0, we may repeat the same argument. By standard patching, there exists a maximal existence time $T_{\max} \in (0, \infty]$ and a unique mild solution

$$U \in C([0, T_{\max}); \mathcal{X})$$

satisfying (3.1) on every compact subinterval of $[0, T_{\max})$.

Step 7. Classical regularity for positive times. Fix $0 < \tau < T < T_{\max}$, and choose

$$0 < \eta < \frac{\tau}{2} < T < T_1 < T_{\max}.$$

Since $U \in C([0,T_1];\mathcal{X})$, both components and the polynomial reaction terms are bounded on $\overline{\Omega} \times [\eta, T_1]$. Viewing each equation as a linear uniformly parabolic equation with bounded right-hand side, the boundary Hölder estimates for homogeneous Neumann or homogeneous oblique derivative conditions yield

$$u, v \in C^{\alpha,\alpha/2}\big(\overline{\Omega} \times [\tau/2, T_1]\big)$$

for some $\alpha \in (0,1)$. The reaction terms are therefore Hölder continuous on the same strip. Boundary Schauder estimates then give

$$u, v \in C^{2+\alpha,1+\alpha/2}\big(\overline{\Omega} \times [\tau, T]\big);$$

see, for example, [5, 6]. Hence the original mild solution is classical on every positive-time strip. This completes the proof. ☐

Proposition 3.4. *Let $p > 1$, and assume $(u_0, v_0) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$ are strongly compatible. Then there exist $T > 0$ and a unique L^p-strong solution $U = (u, v)$ on $[0,T]$, satisfying*

$$u, v \in C([0,T]; W^{2,p}(\Omega)) \cap C^1([0,T]; L^p(\Omega)).$$

In particular, (u,v) is classical on $[0,T]$ in the sense of Definition 1.2.

Proof. We again fix one of the boundary conditions in Assumption 1.1. Consider the system

$$\partial_t U = \mathcal{A}_p U + \mathcal{F}(U), \quad U(0) = U_0, \tag{3.4}$$

on the product space $X_p = L^p(\Omega) \times L^p(\Omega)$, where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

and $\mathcal{A}_p$ is the diagonal realization of the same differential operators in L^p, namely

$$A_{u,p}\phi = D_u \phi'', \quad A_{v,p}\psi = D_v \psi'' - a\psi',$$

with domains

$$D(A_{u,p}) = \left\{\phi \in W^{2,p}(\Omega) : \phi'(0) = \phi'(L) = 0\right\},$$

and either

$$D(A^{\mathrm{N}}_{v,p}) = \left\{\psi \in W^{2,p}(\Omega) : \psi'(0) = \psi'(L) = 0\right\},$$

or

$$D(A^{\mathrm{F}}_{v,p}) = \left\{\psi \in W^{2,p}(\Omega) : D_v\psi'(0) - a\psi(0) = 0,\ D_v\psi'(L) - a\psi(L) = 0\right\}.$$

By the strong compatibility assumption, $U_0 \in D(\mathcal{A}_p)$.

Since $\mathcal{A}_p$ is a uniformly elliptic diagonal operator with separated boundary conditions, it generates an analytic semigroup on X_p; see, for example, [1, 7]. Choose $\omega > 0$ such that

$$B_p := \omega - \mathcal{A}_p$$

is sectorial, and choose $\theta \in \left(\frac{1+1/p}{2}, 1\right)$. The fractional domain interpolation for this elliptic realization gives

$$D(B_p^\theta) \hookrightarrow W^{2\theta,p}(\Omega) \times W^{2\theta,p}(\Omega) \hookrightarrow W^{1,p}(\Omega) \times W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega}).$$

See [7, Sections 2.2.2 and 3.1] for these fractional domain and elliptic realization properties. Consequently, the polynomial mapping

$$\mathcal{F} : D(B_p^\theta) \to X_p$$

is locally Lipschitz. Since $U_0 \in D(\mathcal{A}_p)$, the standard regularity theorem for semilinear analytic equations, as in [7, Chapter 7], yields a time $T > 0$ and a unique solution

$$U \in C([0,T]; D(\mathcal{A}_p)) \cap C^1([0,T]; X_p).$$

Equivalently,

$$u, v \in C([0,T]; W^{2,p}(\Omega)) \cap C^1([0,T]; L^p(\Omega)).$$

Since $p > 1$ and $\Omega \subset \mathbb{R}$ is one-dimensional, the Sobolev embedding gives

$$W^{2,p}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$$

for some $\alpha \in (0,1)$. Hence the nonlinearities are continuous on $\overline{\Omega} \times [0,T]$. Standard positive-time parabolic regularity then implies that

$$u, v \in C(\overline{\Omega} \times [0,T]) \cap C^{2,1}(\overline{\Omega} \times (0,T]),$$

and the boundary conditions hold pointwise. Thus (u,v) is a classical solution on $[0,T]$ in the sense of Definition 1.2.

The uniqueness follows from the uniqueness in the semilinear L^p-theory, or alternatively from Proposition 3.3. This completes the proof. ☐

3.3. **Continuation and dependence on data.**

Lemma 3.5. *Let (u,v) be the maximal mild solution on $[0,T_{\max})$. If*

$$\sup_{0 \le t < T_{\max}} \big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \big) < \infty,$$

then $T_{\max} = \infty$.

Proof. Set

$$U(t) = \begin{pmatrix} u(\cdot,t) \\ v(\cdot,t) \end{pmatrix}, \quad M := \sup_{0 \le t < T_{\max}} \|U(t)\|_{\mathcal{X}} < \infty.$$

By Proposition 3.3, there exists a time $\tau(M) > 0$ depending only on M, such that every initial datum in $\mathcal{X}$ with norm at most M generates a unique mild solution on $[0,\tau(M)]$.

Assume by contradiction that $T_{\max} < \infty$. Choose

$$t_0 \in \Big(T_{\max} - \frac{\tau(M)}{2}, T_{\max} \Big).$$

Since $\|U(t_0)\|_{\mathcal{X}} \le M$, the local existence result applied with initial time t_0 and initial datum $U(t_0)$ yields a mild solution $\widetilde{U} \in C([0,\tau(M)];\mathcal{X})$ of the shifted problem

$$\widetilde{U}'(t) = \mathcal{A}\widetilde{U}(t) + \mathcal{F}(\widetilde{U}(t)), \quad \widetilde{U}(0) = U(t_0).$$

Define

$$\widehat{U}(t) = \widetilde{U}(t - t_0), \quad t \in [t_0, t_0 + \tau(M)].$$

Then $\widehat{U}$ is a mild solution of the original problem on $[t_0, t_0 + \tau(M)]$ with initial value $U(t_0)$ at time t_0.

By uniqueness of mild solutions, $\widehat{U}$ must coincide with the original solution U on the overlap interval $[t_0, T_{\max})$. Hence U extends beyond $T_{\max}$ up to time $t_0 + \tau(M)$. Since

$$t_0 + \tau(M) > T_{\max},$$

this contradicts the maximality of $T_{\max}$. Therefore $T_{\max} = \infty$. The proof is complete. ☐

Lemma 3.6. *Let $(u_0^{(n)}, v_0^{(n)}) \to (u_0, v_0)$ in $\mathcal{X}$. Then the corresponding maximal mild solutions converge to the solution generated by (u_0, v_0) uniformly on compact time intervals contained in the common interval of existence.*

Proof. Let

$$U_0^{(n)} = \begin{pmatrix} u_0^{(n)} \\ v_0^{(n)} \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

and let U_n and U denote the corresponding maximal mild solutions. Fix a compact interval $[0,T]$ with $0 < T < T_{\max}(U_0)$. We shall prove that

$$\sup_{0 \le t \le T} \|U_n(t) - U(t)\|_{\mathcal{X}} \to 0 \quad \text{as } n \to \infty,$$

and in particular U_n is defined on $[0,T]$ for all sufficiently large n.

Since $U \in C([0,T];\mathcal{X})$, we may choose $R > 0$ such that

$$\sup_{0 \le t \le T} \|U(t)\|_{\mathcal{X}} \le R - 1.$$

By Lemma 3.2, there exists $L_R > 0$ such that $\mathcal{F}$ is Lipschitz on the closed ball of radius R in $\mathcal{X}$. Also define

$$M_T := \sup_{0 \le t \le T} \|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{X})}.$$

For all sufficiently large n, we have

$$\|U_0^{(n)} - U_0\|_{\mathcal{X}} < 1.$$

For such n, set $\tau_n := \min\{T, T_{\max}(U_n)\}$ and define

$$T_n^* := \sup\{S \in [0, \tau_n) : \ \|U_n(t) - U(t)\|_{\mathcal{X}} \le 1 \text{ for all } t \in [0, S]\}.$$

By continuity, $T_n^* > 0$. The admissible set in this definition is an initial interval. Hence, for every $S \in (0, T_n^*)$,

$$\|U_n(t)\|_{\mathcal{X}} \le \|U(t)\|_{\mathcal{X}} + 1 \le R \quad \text{for all } t \in [0, S],$$

so both $U_n(t)$ and $U(t)$ stay in the closed ball of radius R on $[0, S]$.

Using the mild formulation, for $0 \le t \le S$,

$$U_n(t) - U(t) = e^{t\mathcal{A}}(U_0^{(n)} - U_0) + \int_0^t e^{(t-s)\mathcal{A}}\big(\mathcal{F}(U_n(s)) - \mathcal{F}(U(s))\big)\, ds.$$

Therefore,

$$\|U_n(t) - U(t)\|_{\mathcal{X}} \le M_T\|U_0^{(n)} - U_0\|_{\mathcal{X}} + M_T L_R \int_0^t \|U_n(s) - U(s)\|_{\mathcal{X}}\, ds.$$

Gronwall's inequality gives

$$\sup_{0 \le t \le S} \|U_n(t) - U(t)\|_{\mathcal{X}} \le M_T e^{M_T L_R T}\|U_0^{(n)} - U_0\|_{\mathcal{X}}.$$

Letting $S \uparrow T_n^*$, we obtain

$$\sup_{0 \le t < T_n^*} \|U_n(t) - U(t)\|_{\mathcal{X}} \le M_T e^{M_T L_R T}\|U_0^{(n)} - U_0\|_{\mathcal{X}}.$$

For all sufficiently large n, the right-hand side is strictly smaller than $1/2$.

We claim that $T_n^* = \tau_n$. Indeed, if $T_n^* < \tau_n$, then both solutions are defined and continuous at T_n^*, and the preceding estimate gives

$$\|U_n(T_n^*) - U(T_n^*)\|_{\mathcal{X}} \le \frac{1}{2}.$$

By continuity, the inequality

$$\|U_n(t) - U(t)\|_{\mathcal{X}} \le 1$$

then persists on $[0, T_n^* + \varepsilon]$ for some $\varepsilon > 0$ with $T_n^* + \varepsilon < \tau_n$, contradicting the definition of T_n^*. Thus $T_n^* = \tau_n$.

It remains to show that $T_{\max}(U_n) > T$ for all sufficiently large n. Otherwise, along a subsequence, $T_{\max}(U_n) \le T$. On this subsequence,

$$T_n^* = \tau_n = T_{\max}(U_n),$$

and hence

$$\sup_{0 \le t < T_{\max}(U_n)} \|U_n(t)\|_{\mathcal{X}} \le \sup_{0 \le t \le T} \|U(t)\|_{\mathcal{X}} + \frac{1}{2} \le R - \frac{1}{2}.$$

Lemma 3.5 would then imply $T_{\max}(U_n) = \infty$, contradicting $T_{\max}(U_n) \le T$. Consequently, $T_{\max}(U_n) > T$ for all sufficiently large n. Thus $\tau_n = T_n^* = T$, and continuity at $t = T$ yields

$$\sup_{0 \le t \le T} \|U_n(t) - U(t)\|_{\mathcal{X}} \le M_T e^{M_T L_R T}\|U_0^{(n)} - U_0\|_{\mathcal{X}} \to 0.$$

This proves the asserted uniform convergence on $[0, T]$. Since $T < T_{\max}(U_0)$ was arbitrary, the lemma follows. $\square$

4. Positivity and invariant regions

In this section we prove positivity preservation for both boundary configurations in Assumption 1.1. We then establish constant upper barriers and forward invariant rectangles under the componentwise Neumann boundary condition (1.10). Finally, for the zero-flux boundary condition (1.11), we record the natural weighted upper barrier for the second component.

4.1. **Positivity preservation.**

Lemma 4.1. *Let $(u_0, v_0) \in \mathcal{X}_+$, and let (u, v) be the corresponding maximal mild solution. Then*

$$u(x,t) \geq 0 \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Proof. Fix an arbitrary $T \in (0, T_{\max})$. Freeze the second component and define

$$G_u(x,t,z) := \rho_u z\Big(1 - \frac{z + v(x,t)}{K_u}\Big), \quad (x,t,z) \in \overline{\Omega} \times [0,T] \times \mathbb{R}.$$

The continuity of v on the closed cylinder implies that G_u is continuous and locally Lipschitz in z, uniformly on bounded sets. By Proposition 3.3, u has the positive-time regularity required in Lemma 2.3; it also satisfies the homogeneous Neumann condition for every $t > 0$. The constant function 0 is a subsolution, satisfies the same boundary condition, and obeys $0 \leq u_0$. Lemma 2.3, applied on $[0,T]$, therefore gives $u \geq 0$ on $\overline{\Omega} \times [0,T]$. Since $T < T_{\max}$ was arbitrary, the conclusion follows. $\square$

Lemma 4.2. *Let $(u_0, v_0) \in \mathcal{X}_+$, and let (u, v) be the corresponding maximal mild solution. Then*

$$v(x,t) \geq 0 \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Proof. Fix an arbitrary $T \in (0, T_{\max})$, freeze the first component, and define

$$G_v(x,t,z) := \rho_v z\Big(1 - \frac{u(x,t) + z}{K_v}\Big), \quad (x,t,z) \in \overline{\Omega} \times [0,T] \times \mathbb{R}.$$

The function G_v is continuous and locally Lipschitz in z, uniformly on bounded sets. Proposition 3.3 supplies the positive-time regularity required in Lemma 2.3. The constant function 0 is a subsolution, obeys $0 \leq v_0$, and satisfies either homogeneous Neumann data or the homogeneous flux condition, exactly as does v. Applying Lemma 2.3 on $[0,T]$ gives $v \geq 0$ on $\overline{\Omega} \times [0,T]$. Since $T < T_{\max}$ was arbitrary, the claim follows. $\square$

Corollary 4.3. *The positive cone $\mathcal{X}_+$ is preserved on every maximal existence interval.*

4.2. **Upper bounds and invariant rectangles under componentwise Neumann conditions.** In this subsection we assume that the componentwise Neumann boundary condition (1.10) is imposed. Under this boundary condition, constant functions are admissible barriers for both components.

Lemma 4.4. *Assume* (1.10) *or* (1.11) *for the system* (1.8)*. Let (u, v) be a nonnegative solution of* (1.8)*. Then u is dominated by the solution U of*

$$U'(t) = \rho_u U(t)\Big(1 - \frac{U(t)}{K_u}\Big), \quad U(0) = \|u_0\|_{L^\infty(\Omega)}.$$

In particular,

$$0 \leq u(x,t) \leq U(t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Proof. By Corollary 4.3, both u and v are nonnegative. Let U be the scalar logistic solution from Lemma 2.1. Since U depends only on t, it satisfies

$$U_x = U_{xx} = 0.$$

Define $w_u(x,t) := U(t)$. Then

$$(w_u)_t - D_u (w_u)_{xx} = U'(t) = \rho_u U(t)\Big(1 - \frac{U(t)}{K_u}\Big).$$

On the other hand, for every (x,t),

$$G_u(x,t,z) := \rho_u z\Big(1 - \frac{z + v(x,t)}{K_u}\Big)$$

satisfies

$$G_u(x,t,U(t)) = \rho_u U(t)\Big(1 - \frac{U(t) + v(x,t)}{K_u}\Big) \le \rho_u U(t)\Big(1 - \frac{U(t)}{K_u}\Big) = (w_u)_t - D_u(w_u)_{xx},$$

because $v(x,t) \ge 0$. Thus w_u is a supersolution of the scalar equation satisfied by u.

Fix an arbitrary $T \in (0, T_{\max})$. The functions u and w_u have the regularity required in Lemma 2.3 on $[0,T]$, satisfy the same homogeneous Neumann boundary condition, and obey

$$u(x,0) \le \|u_0\|_{L^\infty(\Omega)} = U(0) = w_u(x,0).$$

The comparison principle therefore yields $u(x,t) \le U(t)$ on $\overline{\Omega} \times [0,T]$. Since $T < T_{\max}$ was arbitrary, this estimate holds throughout the maximal existence interval. The lower bound $u \ge 0$ is already known from Corollary 4.3. This completes the proof. □

Lemma 4.5. *Assume that the boundary condition* (1.10) *is imposed. Let* (u,v) *be a nonnegative solution of* (1.8)*. Then* v *is dominated by the solution* V *of*

$$V'(t) = \rho_v V(t)\Big(1 - \frac{V(t)}{K_v}\Big), \quad V(0) = \|v_0\|_{L^\infty(\Omega)}.$$

In particular,

$$0 \le v(x,t) \le V(t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Proof. By Corollary 4.3, $u, v \ge 0$. Let V be the scalar logistic solution from Lemma 2.1, and set $w_v(x,t) := V(t)$. Then

$$(w_v)_x = (w_v)_{xx} = 0,$$

and

$$(w_v)_t - D_v(w_v)_{xx} + a(w_v)_x = V'(t) = \rho_v V(t)\Big(1 - \frac{V(t)}{K_v}\Big).$$

We define

$$G_v(x,t,z) := \rho_v z\Big(1 - \frac{u(x,t) + z}{K_v}\Big).$$

Since $u(x,t) \ge 0$,

$$\begin{aligned} G_v(x,t,V(t)) &= \rho_v V(t)\left(1 - \frac{u(x,t) + V(t)}{K_v}\right) \\ &\le \rho_v V(t)\left(1 - \frac{V(t)}{K_v}\right) \\ &= (w_v)_t - D_v(w_v)_{xx} + a(w_v)_x. \end{aligned}$$

Thus w_v is a supersolution of the scalar equation satisfied by v.

The initial inequality

$$v(x,0) \le \|v_0\|_{L^\infty(\Omega)} = V(0) \quad \text{for all } x \in \overline{\Omega}$$

holds by definition. Since under (1.10) both v and w_v satisfy

$$\partial_x(\cdot)(0,t) = \partial_x(\cdot)(L,t) = 0,$$

Lemma 2.3 yields

$$v(x,t) \le V(t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Together with $v \ge 0$, this proves the claim. □

Proposition 4.6. *Assume that the boundary condition* (1.10) *is imposed. For every* $M_u \ge K_u$ *and* $M_v \ge K_v$*, the rectangle* $[0, M_u] \times [0, M_v]$ *is forward invariant for* (1.8)*, provided the initial data satisfy*

$$0 \le u_0(x) \le M_u, \quad 0 \le v_0(x) \le M_v \quad \text{for all } x \in \overline{\Omega}.$$

Proof. Let (u, v) be the maximal mild solution corresponding to such initial data. By Corollary 4.3,

$$u(x,t) \geq 0, \quad v(x,t) \geq 0 \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

We first control u. Let U be the logistic solution

$$U'(t) = \rho_u U(t)\Big(1 - \frac{U(t)}{K_u}\Big), \quad U(0) = M_u.$$

Since $M_u \geq K_u$, Lemma 2.1 implies $0 \leq U(t) \leq M_u$ for all $t \geq 0$. Because $u_0(x) \leq M_u = U(0)$, the comparison argument in the proof of Lemma 4.4, applied with this larger initial barrier, yields

$$0 \leq u(x,t) \leq U(t) \leq M_u \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Next we control v. Let V be the logistic solution

$$V'(t) = \rho_v V(t)\Big(1 - \frac{V(t)}{K_v}\Big), \quad V(0) = M_v.$$

Since $M_v \geq K_v$, Lemma 2.1 gives $0 \leq V(t) \leq M_v$ for all $t \geq 0$. Because $v_0(x) \leq M_v = V(0)$, the comparison argument in the proof of Lemma 4.5, applied with this larger initial barrier, implies

$$0 \leq v(x,t) \leq V(t) \leq M_v \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

The preceding estimates yield

$$\sup_{0 \leq t < T_{\max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \Big) \leq M_u + M_v < \infty.$$

Lemma 3.5 therefore gives $T_{\max} = \infty$. Consequently,

$$(u(x,t), v(x,t)) \in [0, M_u] \times [0, M_v] \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, \infty),$$

which proves the forward invariance. $\square$

Corollary 4.7. *Assume that the boundary condition* (1.10) *is imposed. Then the rectangle* $[0, K_u] \times [0, K_v]$ *is positively invariant.*

4.3. **Weighted upper barriers under mixed Neumann and zero-flux conditions.** We now assume that the mixed boundary condition (1.11) is imposed. In this case constant functions are no longer adapted to the second boundary operator

$$\mathcal{B}_v[\phi] := D_v \phi_x - a\phi.$$

The natural barriers are instead exponential profiles.

Set

$$\beta := \frac{a}{D_v}.$$

Lemma 4.8. *Assume that the boundary condition* (1.11) *is imposed. Let* (u, v) *be a nonnegative solution of* (1.8). *Let* Q *be the solution of*

$$Q'(t) = \rho_v Q(t)\Big(1 - \frac{Q(t)}{K_v}\Big), \quad Q(0) = \big\| e^{-\beta x} v_0(x) \big\|_{L^\infty(\Omega)}.$$

We define $\overline{v}(x,t) := e^{\beta x} Q(t)$. *Then*

$$0 \leq v(x,t) \leq \overline{v}(x,t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

In particular,

$$v(x,t) \leq e^{\beta L} Q(t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Proof. Since $Q \geq 0$ by Lemma 2.1, we have $\overline{v} \geq 0$. Moreover,

$$\overline{v}_x = \beta e^{\beta x} Q(t) = \beta \overline{v}, \quad \overline{v}_{xx} = \beta^2 \overline{v}.$$

Hence

$$\overline{v}_t - D_v \overline{v}_{xx} + a \overline{v}_x = e^{\beta x} Q'(t) - D_v \beta^2 e^{\beta x} Q(t) + a\beta e^{\beta x} Q(t).$$

Since $\beta = \frac{a}{D_v}$,

$$-D_v\beta^2 + a\beta = -D_v\frac{a^2}{D_v^2} + a\frac{a}{D_v} = 0,$$

and therefore

$$\overline{v}_t - D_v\overline{v}_{xx} + a\overline{v}_x = e^{\beta x}Q'(t) = e^{\beta x}\rho_v Q(t)\Big(1 - \frac{Q(t)}{K_v}\Big).$$

On the other hand,

$$\rho_v\overline{v}\Big(1 - \frac{u+\overline{v}}{K_v}\Big) = e^{\beta x}\rho_v Q(t)\Big(1 - \frac{u(x,t) + e^{\beta x}Q(t)}{K_v}\Big).$$

Subtracting, we obtain

$$\begin{aligned}&\overline{v}_t - D_v\overline{v}_{xx} + a\overline{v}_x - \rho_v\overline{v}\Big(1 - \frac{u+\overline{v}}{K_v}\Big)\\ &= e^{\beta x}\rho_v Q(t)\Big(\frac{u(x,t) + e^{\beta x}Q(t) - Q(t)}{K_v}\Big).\end{aligned}$$

Since $u \geq 0$, $Q \geq 0$, and $e^{\beta x} \geq 1$ on $[0, L]$, the right-hand side is nonnegative. Thus $\overline{v}$ is a supersolution of the v-equation.

Next we verify the boundary condition. We have

$$D_v\overline{v}_x - a\overline{v} = D_v\beta\overline{v} - a\overline{v} = a\overline{v} - a\overline{v} = 0 \quad \text{at } x = 0, L.$$

So $\overline{v}$ satisfies the same homogeneous flux boundary condition as v.

At $t = 0$,

$$\overline{v}(x,0) = e^{\beta x}Q(0) = e^{\beta x}\left\|e^{-\beta\xi}v_0(\xi)\right\|_{L^\infty(\Omega)} \geq v_0(x) \quad \text{for all } x \in \overline{\Omega}.$$

Applying Lemma 2.3 to the scalar v-equation, with v as subsolution and $\overline{v}$ as supersolution, yields

$$v(x,t) \leq \overline{v}(x,t) \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

The lower bound $v \geq 0$ is already known from Lemma 4.2. Finally,

$$\overline{v}(x,t) = e^{\beta x}Q(t) \leq e^{\beta L}Q(t),$$

which gives the last estimate. □

Proposition 4.9. *Assume that the boundary condition* (1.11) *is imposed. Let $M_u \geq K_u$, $M_v \geq K_v$, and assume that*

$$0 \leq u_0(x) \leq M_u, \quad 0 \leq v_0(x) \leq M_v e^{\beta x} \quad \text{for all } x \in \overline{\Omega},$$

where $\beta = \frac{a}{D_v}$. Then the corresponding maximal mild solution is global and

$$0 \leq u(x,t) \leq M_u, \quad 0 \leq v(x,t) \leq M_v e^{\beta x} \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,\infty).$$

Proof. The bounds $u \geq 0$ and $v \geq 0$ follow from Corollary 4.3. Since the first component always has Neumann boundary conditions, the same argument as in Proposition 4.6 yields

$$u(x,t) \leq M_u \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

For the second component, define

$$\overline{v}(x,t) := M_v e^{\beta x}, \quad \beta = \frac{a}{D_v}.$$

Then

$$\overline{v}_t = 0, \quad \overline{v}_x = \beta\overline{v}, \quad \overline{v}_{xx} = \beta^2\overline{v},$$

and hence

$$\overline{v}_t - D_v\overline{v}_{xx} + a\overline{v}_x = 0.$$

Moreover,

$$\rho_v\overline{v}\Big(1 - \frac{u+\overline{v}}{K_v}\Big) \leq 0,$$

because $u \geq 0$ and

$$\overline{v}(x,t) = M_v e^{\beta x} \geq M_v \geq K_v.$$

Therefore

$$\overline{v}_t - D_v\overline{v}_{xx} + a\overline{v}_x \geq \rho_v\overline{v}\Big(1 - \frac{u+\overline{v}}{K_v}\Big),$$

so $\overline{v}$ is a supersolution for the v-equation. Also,

$$D_v\overline{v}_x - a\overline{v} = 0 \quad \text{at } x = 0, L,$$

and by assumption,

$$v_0(x) \leq \overline{v}(x,0) \quad \text{for all } x \in \overline{\Omega}.$$

Applying Lemma 2.3, we conclude that

$$v(x,t) \leq M_v e^{\beta x} \quad \text{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

Together with the preceding bounds, this gives

$$\sup_{0\leq t<T_{\max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}\Big) \leq M_u + M_v e^{\beta L} < \infty.$$

Lemma 3.5 therefore implies $T_{\max} = \infty$. Hence the bounds asserted in the statement hold for every $t \geq 0$, which proves the claim. □

Remark 4.10. The simple constant rectangle $[0, M_u] \times [0, M_v]$ is therefore the natural invariant region under the componentwise Neumann condition (1.10), whereas under the zero-flux condition (1.11) the natural upper barrier for the second component is exponential in space. This distinction comes from the fact that constants satisfy the Neumann boundary operator, but not the operator

$$\phi \mapsto D_v\phi_x - a\phi.$$

Proof of Theorem 1.5. Under the componentwise Neumann boundary condition (1.10), Proposition 4.6 with $M_u = K_u$ and $M_v = K_v$, equivalently Corollary 4.7, shows that the corresponding maximal mild solution is global and satisfies

$$0 \leq u(x,t) \leq K_u, \quad 0 \leq v(x,t) \leq K_v$$

for all $x \in \overline{\Omega}$ and $t \geq 0$. This proves part (i).

Under the zero-flux boundary condition (1.11), Proposition 4.9 shows that the corresponding maximal mild solution is global and satisfies

$$0 \leq u(x,t) \leq M_u, \quad 0 \leq v(x,t) \leq M_v e^{\beta x}, \quad \beta = \frac{a}{D_v},$$

for all $x \in \overline{\Omega}$ and $t \geq 0$, provided the initial data satisfy the same pointwise bounds with $M_u \geq K_u$ and $M_v \geq K_v$. This proves part (ii). □

5. Global existence and uniform bounds

In this section we derive the global L^∞ bounds needed for continuation, and then record the standard smoothing properties of the uniformly parabolic system for positive times.

5.1. Global L^∞ control.

Proposition 5.1. *Let $(u_0, v_0) \in \mathcal{X}_+$, and let (u,v) be the corresponding maximal mild solution of* (1.8)-(1.9) *on* $[0, T_{\max})$.

- *(i)] If the boundary condition* (1.10) *is imposed, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq U(t), \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq V(t) \quad \textit{for all } t \in [0, T_{\max}),$$

where U and V are the logistic barriers from Lemmas 4.4 and 4.5.

(ii) *If the boundary condition* (1.11) *is imposed, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq U(t) \quad \textit{for all } t \in [0, T_{\max}),$$

where U is the logistic barrier from Lemma 4.4, and

$$0 \leq v(x,t) \leq e^{\beta x}Q(t) \leq e^{\beta L}Q(t) \quad \textit{for all } (x,t) \in \overline{\Omega} \times [0, T_{\max}),$$

where $\beta = a/D_v$, and Q is the logistic solution from Lemma 4.8, namely

$$Q'(t) = \rho_v Q(t)\Big(1 - \frac{Q(t)}{K_v}\Big), \quad Q(0) = \|e^{-\beta x} v_0(x)\|_{L^\infty(\Omega)}.$$

Proof. If (1.10) is imposed, then the conclusion follows directly from Lemmas 4.4 and 4.5.

If (1.11) is imposed, then the estimate for u is again given by Lemma 4.4, while the estimate for v follows from Lemma 4.8. This proves the proposition. □

Corollary 5.2. *Let $(u_0, v_0) \in \mathcal{X}_+$, and let (u, v) be the corresponding maximal mild solution.*

(i) *If* (1.10) *is imposed, then*

$$0 \le u(x,t) \le \max\left\{\|u_0\|_{L^\infty(\Omega)}, K_u\right\}, \quad 0 \le v(x,t) \le \max\left\{\|v_0\|_{L^\infty(\Omega)}, K_v\right\}$$

for all $(x,t) \in \overline{\Omega} \times [0, T_{\max})$.

(ii) *If* (1.11) *is imposed, then*

$$0 \le u(x,t) \le \max\left\{\|u_0\|_{L^\infty(\Omega)}, K_u\right\}$$

and

$$0 \le v(x,t) \le e^{\beta L} \max\left\{\left\|e^{-\beta x} v_0(x)\right\|_{L^\infty(\Omega)}, K_v\right\} \le e^{\beta L} \max\left\{\|v_0\|_{L^\infty(\Omega)}, K_v\right\}$$

for all $(x,t) \in \overline{\Omega} \times [0, T_{\max})$.

Proof. By Lemma 2.1, every logistic solution

$$Y'(t) = \gamma Y(t)\left(1 - \frac{Y(t)}{K}\right), \quad Y(0) = Y_0 \ge 0,$$

satisfies

$$0 \le Y(t) \le \max\{Y_0, K\} \quad \text{for all } t \ge 0.$$

Applying this to the functions U, V, and Q from Proposition 5.1 yields the claimed estimates. □

Corollary 5.3. *For every $(u_0, v_0) \in \mathcal{X}_+$, the maximal mild solution is global, that is, $T_{\max} = \infty$.*

Proof. By Corollary 5.2, the quantity

$$\sup_{0 \le t < T_{\max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}\Big)$$

is finite. Therefore the continuation criterion of Lemma 3.5 implies that $T_{\max} = \infty$. This completes the proof. □

Corollary 5.4. *Let $p > 1$, and assume that $(u_0, v_0) \in \mathcal{X}_+ \cap \big(W^{2,p}(\Omega) \times W^{2,p}(\Omega)\big)$ are strongly compatible. Then the unique global mild solution furnished by Corollary 5.3 is a global L^p-strong solution. More precisely, writing $U = (u, v)$,*

$$U \in C([0,\infty); D(\mathcal{A}_p)) \cap C^1([0,\infty); X_p).$$

In particular, it is classical on $[0,\infty)$ in the sense of Definition 1.2.

Proof. By Proposition 3.4, there exists $T_0 > 0$ and a unique L^p-strong solution on $[0, T_0]$. The L^p-semigroup and the semigroup on $C(\overline{\Omega})$ are consistent on their common domain. Hence, by uniqueness, this local strong solution coincides with the unique global mild solution furnished by Corollary 5.3.

Let $T > T_0$. The positive-time Schauder regularity established in Proposition 3.3, applied on $[T_0/2, T]$, gives

$$U \in C([T_0/2, T]; D(\mathcal{A}_p)) \cap C^1([T_0/2, T]; X_p).$$

Combining this with the strong regularity on $[0, T_0]$, we obtain

$$U \in C([0,T]; D(\mathcal{A}_p)) \cap C^1([0,T]; X_p).$$

Since $T > T_0$ was arbitrary, the solution is global L^p-strong. Its classical regularity follows from Proposition 3.4 near $t = 0$ and from Proposition 3.3 for positive times. □

5.2. **Regularization for positive times.** From now on, all solutions are global by Corollary 5.3. We now prove the standard parabolic smoothing properties of the system on time intervals away from $t = 0$.

Lemma 5.5. *Let $B \subset \mathcal{X}_+$ be bounded, and let $0 < \tau < T < \infty$. Then there exist $\alpha \in (0,1)$ and a constant $C_{B,\tau,T} > 0$ such that every global solution (u, v) of* (1.8)-(1.9) *with initial datum $(u_0, v_0) \in B$ satisfies*

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[\tau,T])} + \|v\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[\tau,T])} \le C_{B,\tau,T}.$$

Proof. We split the argument into several steps.

Step 1. Uniform L^∞ bounds. Since $B \subset \mathcal{X}_+$ is bounded, there exists $M_B > 0$ such that

$$\|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{L^\infty(\Omega)} \le M_B \quad \text{for all } (u_0, v_0) \in B.$$

By Corollary 5.2, there exists a constant $K_B > 0$, depending only on B and the fixed coefficients of the system, such that every corresponding global solution satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le K_B \quad \text{for all } t \ge 0. \tag{5.1}$$

Step 2. Uniform bounds for the reaction terms. Set

$$f_u(x,t) := \rho_u u(x,t)\Big(1 - \frac{u(x,t)+v(x,t)}{K_u}\Big), \quad f_v(x,t) := \rho_v v(x,t)\Big(1 - \frac{u(x,t)+v(x,t)}{K_v}\Big).$$

By (5.1), there exists $C_B > 0$ such that

$$\|f_u\|_{L^\infty(\Omega\times[0,T+1])} + \|f_v\|_{L^\infty(\Omega\times[0,T+1])} \le C_B. \tag{5.2}$$

Step 3. Hölder bounds away from the initial time. We first claim that there exist $\alpha_0 \in (0,1)$ and a constant $C_1 = C_1(B, \tau, T)$ such that

$$\|u\|_{C^{\alpha_0,\alpha_0/2}(\overline{\Omega}\times[\tau/2,T+1])} + \|v\|_{C^{\alpha_0,\alpha_0/2}(\overline{\Omega}\times[\tau/2,T+1])} \le C_1. \tag{5.3}$$

Indeed, both equations are linear uniformly parabolic equations with bounded coefficients and bounded right-hand sides, once the other component is viewed as part of the forcing term. Moreover, the boundary conditions are either homogeneous Neumann or homogeneous oblique derivative conditions with constant coefficients. Standard global Hölder estimates for linear parabolic equations up to the boundary therefore yield (5.3); see, for example, [5, 6].

Step 4. Hölder bounds for the nonlinearities. Since the nonlinearities are polynomial expressions in u and v, the Hölder estimate (5.3) implies that $f_u, f_v \in C^{\alpha_0,\alpha_0/2}(\overline{\Omega} \times [\tau/2, T + 1])$, and there exists $C_2 = C_2(B, \tau, T)$ such that

$$\|f_u\|_{C^{\alpha_0,\alpha_0/2}(\overline{\Omega}\times[\tau/2,T+1])} + \|f_v\|_{C^{\alpha_0,\alpha_0/2}(\overline{\Omega}\times[\tau/2,T+1])} \le C_2. \tag{5.4}$$

Step 5. Schauder estimates on $[\tau, T]$. We now apply the global Schauder estimate to each component on the smaller strip $\overline{\Omega} \times [\tau, T]$. Since $[\tau, T] \Subset [\tau/2, T + 1]$, the standard estimate gives

$$\|u\|_{C^{2+\alpha_0,1+\alpha_0/2}(\overline{\Omega}\times[\tau,T])} \le C_3\Big(\|u\|_{C^0(\overline{\Omega}\times[\tau/2,T+1])} + \|f_u\|_{C^{\alpha_0,\alpha_0/2}(\overline{\Omega}\times[\tau/2,T+1])}\Big),$$

and likewise

$$\|v\|_{C^{2+\alpha_0,1+\alpha_0/2}(\overline{\Omega}\times[\tau,T])} \le C_4\Big(\|v\|_{C^0(\overline{\Omega}\times[\tau/2,T+1])} + \|f_v\|_{C^{\alpha_0,\alpha_0/2}(\overline{\Omega}\times[\tau/2,T+1])}\Big),$$

where C_3 and C_4 depend only on τ, T, the coefficients of the system, the domain, and the type of boundary condition. Using (5.1) and (5.4), we obtain

$$\|u\|_{C^{2+\alpha_0,1+\alpha_0/2}(\overline{\Omega}\times[\tau,T])} + \|v\|_{C^{2+\alpha_0,1+\alpha_0/2}(\overline{\Omega}\times[\tau,T])} \le C_{B,\tau,T}.$$

Setting $\alpha = \alpha_0$, the proof is complete. $\square$

Lemma 5.6. *For every bounded set $B \subset \mathcal{X}_+$ and every $\tau > 0$, there exist $\alpha \in (0,1)$ and a constant $C_{B,\tau} > 0$ such that*

$$\sup_{(u_0,v_0)\in B} \sup_{t\ge\tau}\Big(\|u(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})}\Big) \le C_{B,\tau}.$$

Proof. Fix $B \subset \mathcal{X}_+$ bounded and $\tau > 0$, and define the shifted-data set

$$B_\tau := \Big\{(u(\cdot,r),v(\cdot,r)) : (u_0,v_0) \in B,\ r \geq \frac{\tau}{2}\Big\},$$

where (u,v) is the solution issued from (u_0,v_0). By Corollary 5.2, B_τ is a bounded subset of $\mathcal{X}_+$. Apply Lemma 5.5 to B_τ on the fixed strip $\overline{\Omega} \times \big[\frac{\tau}{4},\frac{\tau}{2}\big]$. It furnishes an exponent $\alpha \in (0,1)$ and a constant $C_{B,\tau} > 0$ that depend only on B and τ.

Let $(u_0,v_0) \in B$, and let (u,v) be the associated global solution. Fix $t \geq \tau$, and define the time-shifted functions

$$\widetilde{u}(x,s) := u\left(x, t - \frac{\tau}{2} + s\right), \quad \widetilde{v}(x,s) := v\left(x, t - \frac{\tau}{2} + s\right), \quad (x,s) \in \overline{\Omega} \times \big[0, \frac{\tau}{2}\big].$$

Because the system is autonomous, $(\widetilde{u},\widetilde{v})$ solves the same PDE system on the interval $0 \leq s \leq \tau/2$, with initial data

$$\widetilde{u}(x,0) = u\big(x, t - \frac{\tau}{2}\big), \quad \widetilde{v}(x,0) = v\big(x, t - \frac{\tau}{2}\big).$$

Since $t - \tau/2 \geq \tau/2$, these shifted initial data belong to B_τ.

Applying Lemma 5.5 to the shifted problem on the fixed strip $\overline{\Omega} \times [\frac{\tau}{4},\frac{\tau}{2}]$, we obtain, uniformly in both the original initial datum in B and the time $t \geq \tau$,

$$\|\widetilde{u}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[\frac{\tau}{4},\frac{\tau}{2}])} + \|\widetilde{v}\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[\tau/4,\tau/2])} \leq C_{B,\tau}.$$

In particular, evaluating at the final shifted time $s = \tau/2$, we obtain

$$\|\widetilde{u}(\cdot,\tau/2)\|_{C^{1+\alpha}(\overline{\Omega})} + \|\widetilde{v}(\cdot,\tau/2)\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_{B,\tau}.$$

Since

$$\widetilde{u}(\cdot,\tau/2) = u(\cdot,t), \quad \widetilde{v}(\cdot,\tau/2) = v(\cdot,t),$$

this shows that

$$\|u(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_{B,\tau} \quad \text{for all } t \geq \tau.$$

Taking the supremum over $t \geq \tau$ and $(u_0,v_0) \in B$, we obtain the asserted estimate. □

Proof of Theorem 1.4. For part (a), Proposition 3.3 yields a unique local mild solution in $\mathcal{X}_+$, and Corollary 4.3 shows that both components remain nonnegative as long as the solution exists. Corollary 5.3, which combines the continuation criterion from Lemma 3.5 with the a priori sup norm control from Proposition 5.1, extends this local solution to all times $t \geq 0$. Finally, Lemma 5.5 implies that every global mild solution becomes classical on each strip $\overline{\Omega} \times [\tau, T]$ with $0 < \tau < T < \infty$.

Part (b) is exactly Corollary 5.4, which gives the asserted global L^p-strong regularity.

Part (c) follows from Lemma 3.6, which gives continuous dependence of the solution map on the initial data in the phase space $\mathcal{X}$, uniformly on compact time intervals. This completes the proof of Theorem 1.4. □

6. Absorbing sets and dissipativity

In this section we prove the dissipative nature of the semiflow generated by (1.8). As in the previous sections, the precise form of the absorbing region depends on the boundary condition imposed on the second component. Whenever $S(t)$ is used below, one of the two boundary conditions in Assumption 1.1 is understood to be fixed.

6.1. **Absorbing regions.** For a bounded set $B \subset \mathcal{X}_+$, we introduce the quantities

$$M_u(B) := \sup_{(u_0,v_0)\in B} \|u_0\|_{L^\infty(\Omega)}, \quad M_v(B) := \sup_{(u_0,v_0)\in B} \|v_0\|_{L^\infty(\Omega)}.$$

When the flux boundary condition (1.11) is imposed, we also set

$$M_{v,\beta}(B) := \sup_{(u_0,v_0)\in B} \|e^{-\beta x} v_0(x)\|_{L^\infty(\Omega)}, \quad \beta := \frac{a}{D_v}.$$

Lemma 6.1. *Let $B \subset \mathcal{X}_+$ be bounded and let $\varepsilon > 0$.*

(i) *There exists a time $T_{u,B}(\varepsilon) \geq 0$, depending only on B and ε, such that every logistic solution U of*

$$U'(t) = \rho_u U(t)\Big(1 - \frac{U(t)}{K_u}\Big), \quad U(0) \in [0, M_u(B)],$$

satisfies

$$0 \leq U(t) \leq K_u + \varepsilon \quad \textit{for all } t \geq T_{u,B}(\varepsilon).$$

(ii) *If (1.10) is imposed, then there exists a time $T^{\mathrm{N}}_{v,B}(\varepsilon) \geq 0$, depending only on B and ε, such that every logistic solution V of*

$$V'(t) = \rho_v V(t)\Big(1 - \frac{V(t)}{K_v}\Big), \quad V(0) \in [0, M_v(B)],$$

satisfies $0 \leq V(t) \leq K_v + \varepsilon$ for all $t \geq T^{\mathrm{N}}_{v,B}(\varepsilon)$.

(iii) *If (1.11) is imposed, then there exists a time $T^{\mathrm{F}}_{v,B}(\varepsilon) \geq 0$, depending only on B and ε, such that every logistic solution Q of*

$$Q'(t) = \rho_v Q(t)\Big(1 - \frac{Q(t)}{K_v}\Big), \quad Q(0) \in [0, M_{v,\beta}(B)],$$

satisfies $0 \leq Q(t) \leq K_v + \varepsilon$ for all $t \geq T^{\mathrm{F}}_{v,B}(\varepsilon)$.

Proof. This is a direct consequence of Lemma 2.1(iii) applied to the scalar logistic equation.

Indeed, the set of admissible initial values in each case is a bounded interval of the form $[0, M]$. Lemma 2.1(iii) therefore yields an entry time depending only on M and ε, after which every corresponding logistic solution enters the interval $[0, K + \varepsilon]$, where K is the corresponding carrying capacity.

Applying this 3 times, once with $(\rho, K, M) = (\rho_u, K_u, M_u(B))$, a second time with $(\rho, K, M) = (\rho_v, K_v, M_v(B))$, and a third time with $(\rho, K, M) = (\rho_v, K_v, M_{v,\beta}(B))$, we obtain the three assertions. $\square$

Proposition 6.2. *Let $B \subset \mathcal{X}_+$ be bounded and let $\varepsilon > 0$.*

(i) *If the boundary condition (1.10) is imposed, then there exists a time $T^{\mathrm{N}}_B(\varepsilon) \geq 0$ such that every solution starting from B satisfies*

$$0 \leq u(x,t) \leq K_u + \varepsilon, \quad 0 \leq v(x,t) \leq K_v + \varepsilon$$

for all $x \in \overline{\Omega}$ and all $t \geq T^{\mathrm{N}}_B(\varepsilon)$. Equivalently,

$$S(t)B \subset [0, K_u + \varepsilon] \times [0, K_v + \varepsilon] \quad \textit{for all } t \geq T^{\mathrm{N}}_B(\varepsilon).$$

(ii) *If the boundary condition (1.11) is imposed, then there exists a time $T^{\mathrm{F}}_B(\varepsilon) \geq 0$ such that every solution starting from B satisfies*

$$0 \leq u(x,t) \leq K_u + \varepsilon, \quad 0 \leq v(x,t) \leq e^{\beta x}(K_v + \varepsilon) \leq e^{\beta L}(K_v + \varepsilon)$$

for all $x \in \overline{\Omega}$ and all $t \geq T^{\mathrm{F}}_B(\varepsilon)$, where $\beta = a/D_v$. In particular,

$$S(t)B \subset [0, K_u + \varepsilon] \times [0, e^{\beta L}(K_v + \varepsilon)] \quad \textit{for all } t \geq T^{\mathrm{F}}_B(\varepsilon)$$

as a subset of $\mathcal{X}$.

Proof. Let $(u_0, v_0) \in B$, and let (u, v) be the corresponding global solution.

Step 1. Estimate for the first component. By Proposition 5.1, the first component is dominated by the logistic barrier U solving

$$U'(t) = \rho_u U(t)\Big(1 - \frac{U(t)}{K_u}\Big), \quad U(0) = \|u_0\|_{L^\infty(\Omega)}.$$

Since $(u_0, v_0) \in B$, we have $U(0) \leq M_u(B)$. Hence Lemma 6.1(i) implies that

$$0 \leq U(t) \leq K_u + \varepsilon \quad \text{for all } t \geq T_{u,B}(\varepsilon).$$

Therefore

$$0 \leq u(x,t) \leq K_u + \varepsilon \quad \text{for all } x \in \overline{\Omega},\ t \geq T_{u,B}(\varepsilon). \tag{6.1}$$

Step 2. Second component under (1.10). Assume first that (1.10) is imposed. By Proposition 5.1, the second component is dominated by the logistic barrier V solving

$$V'(t)=\rho_v V(t)\Big(1-\frac{V(t)}{K_v}\Big),\quad V(0)=\|v_0\|_{L^\infty(\Omega)}.$$

Since $V(0)\le M_v(B)$, Lemma 6.1(ii) yields

$$0\le V(t)\le K_v+\varepsilon\quad\text{for all } t\ge T^{\mathrm{N}}_{v,B}(\varepsilon).$$

Consequently,

$$0\le v(x,t)\le K_v+\varepsilon\quad\text{for all } x\in\overline{\Omega},\ t\ge T^{\mathrm{N}}_{v,B}(\varepsilon).$$

Combining this with (6.1), we see that the conclusion holds with

$$T^{\mathrm{N}}_B(\varepsilon):=\max\{T_{u,B}(\varepsilon),\,T^{\mathrm{N}}_{v,B}(\varepsilon)\}.$$

Step 3. Second component under (1.11). Assume now that (1.11) is imposed. By Proposition 5.1, the second component satisfies

$$0\le v(x,t)\le e^{\beta x}Q(t),\quad \beta=\frac{a}{D_v},$$

where Q solves

$$Q'(t)=\rho_v Q(t)\Big(1-\frac{Q(t)}{K_v}\Big),\quad Q(0)=\big\|e^{-\beta x}v_0(x)\big\|_{L^\infty(\Omega)}.$$

Since $(u_0,v_0)\in B$, it follows that $Q(0)\le M_{v,\beta}(B)$. Hence Lemma 6.1(iii) gives

$$0\le Q(t)\le K_v+\varepsilon\quad\text{for all } t\ge T^{\mathrm{F}}_{v,B}(\varepsilon).$$

Therefore

$$0\le v(x,t)\le e^{\beta x}(K_v+\varepsilon)\le e^{\beta L}(K_v+\varepsilon)\quad\text{for all } x\in\overline{\Omega},\ t\ge T^{\mathrm{F}}_{v,B}(\varepsilon).$$

Combining this with (6.1), we obtain the conclusion with

$$T^{\mathrm{F}}_B(\varepsilon):=\max\big\{T_{u,B}(\varepsilon),\,T^{\mathrm{F}}_{v,B}(\varepsilon)\big\}.$$

This completes the proof. □

Corollary 6.3. *For each fixed choice of boundary conditions in Assumption 1.1, the corresponding semiflow generated by* (1.8) *is point dissipative on* $\mathcal{X}_+$.

Proof. Let $(u_0,v_0)\in\mathcal{X}_+$. Applying Proposition 6.2 to the singleton set $B=\{(u_0,v_0)\}$, with $\varepsilon=1$, we find a bounded subset of $\mathcal{X}_+$ that eventually contains the orbit of (u_0,v_0).

More precisely, under (1.10) the orbit eventually enters $[0,K_u+1]\times[0,K_v+1]$, while under (1.11) it eventually enters $[0,K_u+1]\times[0,e^{\beta L}(K_v+1)]$. Thus every trajectory eventually enters a bounded subset of $\mathcal{X}_+$, and the semiflow is point dissipative. □

Proof of Theorem 1.6. Let $B\subset\mathcal{X}_+$ be bounded and let $\varepsilon>0$. Proposition 6.2 yields the eventual bounds stated in parts (i) and (ii) under the respective boundary conditions. More precisely, under (1.10) there exists a time $T^{\mathrm{N}}_B(\varepsilon)$ such that every trajectory issued from B enters the rectangle $[0,K_u+\varepsilon]\times[0,K_v+\varepsilon]$, while under (1.11) there exists a time $T^{\mathrm{F}}_B(\varepsilon)$ such that every trajectory issued from B enters the weighted region

$$0\le u(x,t)\le K_u+\varepsilon,\quad 0\le v(x,t)\le e^{\beta x}(K_v+\varepsilon)\le e^{\beta L}(K_v+\varepsilon).$$

The final statement of Theorem 1.6, namely point dissipativity of the semiflow on $\mathcal{X}_+$, is precisely Corollary 6.3. This completes the proof. □

6.2. **An absorbing set in higher regularity.** We now combine the absorbing L^∞-regions with the smoothing estimate from Lemma 5.6.

Lemma 6.4. *There exist $\tau_* > 0$, $\alpha \in (0,1)$, and a bounded set*

$$\mathscr{B}_* \subset C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$$

which is compact in $\mathcal{X}$ and absorbing for the semiflow in $\mathcal{X}_+$.

Proof. We distinguish the two boundary conditions, but the argument is the same in both cases.

Step 1. A bounded absorbing set in $\mathcal{X}$. Fix $\varepsilon = 1$. By Proposition 6.2, there exists a bounded absorbing set $\mathcal{R}_* \subset \mathcal{X}_+$ of the following form

$$\mathcal{R}_* = \big\{(u,v) \in \mathcal{X}_+ :\ 0 \le u \le K_u + 1,\ 0 \le v \le K_v + 1\big\}$$

under (1.10), and

$$\mathcal{R}_* = \big\{(u,v) \in \mathcal{X}_+ :\ 0 \le u \le K_u + 1,\ 0 \le v \le e^{\beta L}(K_v + 1)\big\}$$

under (1.11). In either case, $\mathcal{R}_*$ is a bounded absorbing subset of $\mathcal{X}_+$.

Step 2. Uniform tail regularity from the absorbing set. Choose $\tau_* := 1$. Since $\mathcal{R}_*$ is bounded in $\mathcal{X}_+$, Lemma 5.6 yields some $\alpha \in (0,1)$ and a constant $C_* > 0$ such that every global solution with initial datum in $\mathcal{R}_*$ satisfies

$$\sup_{t \ge \tau_*} \Big(\|u(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} \Big) \le C_*. \tag{6.2}$$

Step 3. Definition of the higher-regularity absorbing set. We define

$$\mathscr{B}_* := \big\{(u,v) \in \mathcal{X}_+ \cap \big(C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})\big) : \|u\|_{C^{1+\alpha}(\overline{\Omega})} + \|v\|_{C^{1+\alpha}(\overline{\Omega})} \le C_*\big\}.$$

By construction, $\mathscr{B}_*$ is bounded in $C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$.

Step 4. Compactness in $\mathcal{X}$. We claim that $\mathscr{B}_*$ is compact in $\mathcal{X}$. Let $\{(u_n,v_n)\} \subset \mathscr{B}_*$. Then

$$\|u_n\|_{C^{1+\alpha}(\overline{\Omega})} + \|v_n\|_{C^{1+\alpha}(\overline{\Omega})} \le C_* \quad \text{for all } n.$$

In particular, the families $\{u_n\}$, $\{v_n\}$, $\{u_n'\}$, and $\{v_n'\}$ are uniformly bounded and equicontinuous on $\overline{\Omega}$. By the Arzelà-Ascoli theorem, after passing to a subsequence we can assume that

$$u_n \to u, \quad v_n \to v, \quad u_n' \to p, \quad v_n' \to q$$

uniformly on $\overline{\Omega}$, for some continuous functions u, v, p, q.

Since $u_n \to u$ and $u_n' \to p$ uniformly, it follows that $u \in C^1(\overline{\Omega})$ and $u' = p$. Likewise, $v \in C^1(\overline{\Omega})$ and $v' = q$. Because each $(u_n, v_n) \in \mathcal{X}_+$, the uniform convergence also gives

$$u \ge 0, \quad v \ge 0 \quad \text{on } \overline{\Omega}.$$

Moreover, the uniform C^α-bounds on u_n' and v_n' pass to the limits. Indeed, for all $x, y \in \overline{\Omega}$,

$$|u'(x) - u'(y)| = \lim_{n\to\infty} |u_n'(x) - u_n'(y)| \le \big(\liminf_{n\to\infty} [u_n']_{C^\alpha(\overline{\Omega})}\big)|x-y|^\alpha,$$

and similarly

$$|v'(x) - v'(y)| \le \big(\liminf_{n\to\infty} [v_n']_{C^\alpha(\overline{\Omega})}\big)|x-y|^\alpha.$$

Hence

$$[u']_{C^\alpha(\overline{\Omega})} \le \liminf_{n\to\infty} [u_n']_{C^\alpha(\overline{\Omega})}, \quad [v']_{C^\alpha(\overline{\Omega})} \le \liminf_{n\to\infty} [v_n']_{C^\alpha(\overline{\Omega})}.$$

Combining these estimates with the uniform convergence of u_n, u_n', v_n, v_n', we obtain

$$\|u\|_{C^{1+\alpha}(\overline{\Omega})} + \|v\|_{C^{1+\alpha}(\overline{\Omega})} \le \liminf_{n\to\infty} \Big(\|u_n\|_{C^{1+\alpha}(\overline{\Omega})} + \|v_n\|_{C^{1+\alpha}(\overline{\Omega})} \Big) \le C_*.$$

Therefore $(u,v) \in \mathscr{B}_*$. Hence every sequence in $\mathscr{B}_*$ admits a subsequence converging in $\mathcal{X}$ to an element of $\mathscr{B}_*$, and $\mathscr{B}_*$ is compact in $\mathcal{X}$.

Step 5. Absorbing property. Let $B \subset \mathcal{X}_+$ be bounded. Since $\mathcal{R}_*$ is absorbing in $\mathcal{X}_+$, there exists $T_B \ge 0$ such that

$$S(t)B \subset \mathcal{R}_* \quad \text{for all } t \ge T_B.$$

We claim that
$$S(t)B \subset \mathscr{B}_* \quad \text{for all } t \ge T_B + \tau_*.$$
Indeed, fix $t \ge T_B + \tau_*$, and let $(u_0, v_0) \in B$. Set
$$(\tilde{u}_0, \tilde{v}_0) := S(T_B)(u_0, v_0) \in \mathcal{R}_*.$$
By the semigroup property,
$$S(t)(u_0, v_0) = S(t - T_B)(\tilde{u}_0, \tilde{v}_0).$$
Since $t - T_B \ge \tau_*$, estimate (6.2) applies to the trajectory starting from $(\tilde{u}_0, \tilde{v}_0) \in \mathcal{R}_*$. Therefore
$$\|u(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} + \|v(\cdot,t)\|_{C^{1+\alpha}(\overline{\Omega})} \le C_*,$$
which means precisely that
$$S(t)(u_0, v_0) \in \mathscr{B}_*.$$
Since $(u_0, v_0) \in B$ was arbitrary, this proves that $\mathscr{B}_*$ is absorbing. The proof is complete. □

Remark 6.5. Under (1.11), one could also formulate a sharper absorbing set adapted to the weighted barrier $e^{\beta x}(K_v + \varepsilon)$. For the dissipativity and attractor theory in the phase space $\mathcal{X} = C(\overline{\Omega}) \times C(\overline{\Omega})$, however, the simpler unweighted L^∞-bound
$$0 \le v(x,t) \le e^{\beta L}(K_v + \varepsilon)$$
is sufficient.

7. Semiflow and global attractor

Throughout this section, fix one of the two boundary conditions in Assumption 1.1. We show that the corresponding solution operator generated by (1.8) defines a continuous semiflow on $\mathcal{X}_+$, prove asymptotic compactness, and then construct its global attractor directly from a compact absorbing set.

7.1. **Semiflow structure.**

Proposition 7.1. *For each $t \ge 0$, we define*
$$S(t)(u_0, v_0) = (u(\cdot,t), v(\cdot,t)),$$
where (u, v) is the unique global mild solution of (1.8)-(1.9). *Then $\{S(t)\}_{t\ge0}$ defines a continuous semiflow on $\mathcal{X}_+$.*

Proof. We verify the semiflow properties one by one.

By Corollary 5.3, for every $(u_0, v_0) \in \mathcal{X}_+$, the corresponding mild solution exists globally on $[0, \infty)$. By Corollary 4.3, this solution remains in $\mathcal{X}_+$ for all times. Hence for each $t \ge 0$, the map $S(t) : \mathcal{X}_+ \to \mathcal{X}_+$ is well defined.

By definition of the solution,
$$S(0)(u_0, v_0) = (u_0, v_0) \quad \text{for all } (u_0, v_0) \in \mathcal{X}_+.$$
Thus $S(0) = I_{\mathcal{X}_+}$.

Fix $(u_0, v_0) \in \mathcal{X}_+$, and let (u, v) be the corresponding global mild solution. Let $s \ge 0$, and define
$$(\widetilde{u}(x,t), \widetilde{v}(x,t)) := \big(u(x, t+s), v(x, t+s)\big), \quad t \ge 0.$$
We claim that $(\widetilde{u}, \widetilde{v})$ is the mild solution of (1.8) with initial datum
$$(\widetilde{u}(\cdot,0), \widetilde{v}(\cdot,0)) = (u(\cdot,s), v(\cdot,s)) = S(s)(u_0, v_0).$$
Indeed, this is a standard consequence of the mild formulation and the autonomous nature of the equation. Equivalently, by uniqueness of mild solutions, the trajectory issued from $S(s)(u_0, v_0)$ must coincide with the time-shift of the original trajectory. Therefore, for every $t, s \ge 0$,
$$S(t)\big(S(s)(u_0, v_0)\big) = S(t+s)(u_0, v_0).$$
Since $(u_0, v_0) \in \mathcal{X}_+$ was arbitrary, we obtain
$$S(t+s) = S(t)S(s) \quad \text{for all } t, s \ge 0.$$

Fix $t \geq 0$. Let $(u_{0,n}, v_{0,n}) \to (u_0, v_0)$ in $\mathcal{X}$. By Lemma 3.6, the corresponding solutions converge uniformly on compact time intervals. In particular, on the compact interval $[0,t]$,

$$\sup_{0\leq\sigma\leq t} \|S(\sigma)(u_{0,n}, v_{0,n}) - S(\sigma)(u_0, v_0)\|_{\mathcal{X}} \to 0.$$

Evaluating at $\sigma = t$, we obtain

$$\|S(t)(u_{0,n}, v_{0,n}) - S(t)(u_0, v_0)\|_{\mathcal{X}} \to 0.$$

Hence each map $S(t) : \mathcal{X}_+ \to \mathcal{X}_+$ is continuous.

Fix $(u_0, v_0) \in \mathcal{X}_+$. Since the global mild solution belongs to $C([0,\infty); \mathcal{X})$, the map $t \mapsto S(t)(u_0, v_0)$ is continuous from $[0,\infty)$ into $\mathcal{X}$.

We now prove that

$$[0,\infty) \times \mathcal{X}_+ \ni (t, U_0) \mapsto S(t)U_0 \in \mathcal{X}_+$$

is jointly continuous. Let $t_n \to t$ in $[0,\infty)$, and let $U_{0,n} \to U_0$ in $\mathcal{X}_+$, where for brevity we write

$$U_{0,n} = (u_{0,n}, v_{0,n}), \quad U_0 = (u_0, v_0).$$

Choose $T > t$ so large that $t_n \in [0,T]$ for all sufficiently large n. Then

$$\|S(t_n)U_{0,n} - S(t)U_0\|_{\mathcal{X}} \leq \|S(t_n)U_{0,n} - S(t_n)U_0\|_{\mathcal{X}} + \|S(t_n)U_0 - S(t)U_0\|_{\mathcal{X}}.$$

The first term tends to zero by Lemma 3.6, uniformly for times in $[0,T]$. The second term tends to zero by continuity in time of the trajectory issued from U_0. Hence

$$S(t_n)U_{0,n} \to S(t)U_0 \quad \text{in } \mathcal{X}.$$

Therefore $\{S(t)\}_{t\geq 0}$ is a continuous semiflow on $\mathcal{X}_+$. This completes the proof. □

7.2. **Asymptotic compactness.**

Lemma 7.2. *Let $B \subset \mathcal{X}_+$ be bounded. Then for every sequence $\{(u_{0,n}, v_{0,n})\} \subset B$ and every sequence $t_n \to \infty$, the sequence $\{S(t_n)(u_{0,n}, v_{0,n})\}$ is precompact in $\mathcal{X}$.*

Proof. Let $B \subset \mathcal{X}_+$ be bounded, and let

$$U_{0,n} := (u_{0,n}, v_{0,n}) \in B, \quad t_n \to \infty.$$

By Lemma 6.4, there exist $\tau_* > 0$, $\alpha \in (0,1)$, and a compact absorbing set

$$\mathscr{B}_* \subset C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$$

which is compact in $\mathcal{X}$.

Since $\mathscr{B}_*$ is absorbing for the semiflow and B is bounded, there exists $T_B \geq 0$ such that

$$S(t)B \subset \mathscr{B}_* \quad \text{for all } t \geq T_B.$$

Because $t_n \to \infty$, there exists $N \in \mathbb{N}$ such that $t_n \geq T_B$ for all $n \geq N$. Hence

$$S(t_n)U_{0,n} \in \mathscr{B}_* \quad \text{for all } n \geq N.$$

Since the tail $\{S(t_n)U_{0,n}\}_{n\geq N}$ is contained in the compact set $\mathscr{B}_*$, it is precompact in $\mathcal{X}$. Adding finitely many initial terms preserves precompactness. Hence the full sequence is precompact in $\mathcal{X}$. This completes the proof. □

Proposition 7.3. *The semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact on $\mathcal{X}_+$.*

Proof. This is immediate from Lemma 7.2, since asymptotic compactness means precisely that for every bounded set $B \subset \mathcal{X}_+$, every sequence $U_{0,n} \in B$, and every sequence $t_n \to \infty$, the sequence $\{S(t_n)U_{0,n}\}$ has a convergent subsequence in $\mathcal{X}$. □

7.3. **Existence of the attractor.** For two nonempty subsets $E, F \subset \mathcal{X}$, we use the standard one-sided Hausdorff semidistance

$$\operatorname{dist}_{\mathcal{X}}(E,F) := \sup_{x\in E}\inf_{y\in F}\|x-y\|_{\mathcal{X}}.$$

Theorem 7.4. *The semiflow $\{S(t)\}_{t\geq 0}$ possesses a compact global attractor $\mathscr{A} \subset \mathcal{X}_+$. More precisely:*

(i) *$\mathscr{A}$ is compact in $\mathcal{X}$,*
(ii) *$\mathscr{A}$ is fully invariant, namely $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geq 0$,*
(iii) *for every bounded set $B \subset \mathcal{X}_+$,*
$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{X}}(S(t)B, \mathscr{A}) = 0,$$
(iv) *$\mathscr{A}$ is bounded in $C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$ for the exponent $\alpha \in (0,1)$ furnished by Lemma 6.4.*

Proof. The proof is divided into several steps.

Step 1. A compact set which is positively invariant after a fixed time. By Lemma 6.4, there exists a compact absorbing set $\mathscr{B}_* \subset \mathcal{X}_+$. Since $\mathscr{B}_*$ is bounded and absorbing for the semiflow, it absorbs itself. Therefore there exists $T_* > 0$ such that

$$S(t)\mathscr{B}_* \subset \mathscr{B}_* \quad \text{for all } t \geq T_*.$$

We define

$$\mathcal{K} := \overline{\cup_{t\geq T_*} S(t)\mathscr{B}_*}^{\mathcal{X}}.$$

Because $\cup_{t\geq T_*} S(t)\mathscr{B}_* \subset \mathscr{B}_*$, and $\mathscr{B}_*$ is compact in $\mathcal{X}$, it follows that $\mathcal{K}$ is a nonempty compact subset of $\mathcal{X}_+$.

We claim that $\mathcal{K}$ is positively invariant: $S(s)\mathcal{K} \subset \mathcal{K}$ or all $s \geq 0$. Indeed, let $y \in \cup_{t\geq T_*} S(t)\mathscr{B}_*$. Then $y = S(t)x$ for some $t \geq T_*$ and some $x \in \mathscr{B}_*$. Hence

$$S(s)y = S(s+t)x \in \cup_{r\geq T_*} S(r)\mathscr{B}_*,$$

since $s + t \geq T_*$. By continuity of $S(s)$, the same inclusion holds after taking the closure, so

$$S(s)\mathcal{K} \subset \mathcal{K}.$$

Step 2. Definition of the candidate attractor. Define the ω-limit set of $\mathcal{K}$ by

$$\mathscr{A} := \omega(\mathcal{K}) := \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau} S(t)\mathcal{K}}^{\mathcal{X}}.$$

Since $\mathcal{K}$ is compact and positively invariant, for every $\tau \geq 0$,

$$\overline{\cup_{t\geq\tau} S(t)\mathcal{K}}^{\mathcal{X}} \subset \mathcal{K}.$$

Therefore the family of sets

$$\mathcal{K}_\tau := \overline{\cup_{t\geq\tau} S(t)\mathcal{K}}^{\mathcal{X}}$$

is a decreasing family of nonempty compact subsets of $\mathcal{K}$. Hence their intersection $\mathscr{A}$ is nonempty and compact. Moreover, $\mathscr{A} \subset \mathcal{K} \subset \mathscr{B}_*$, so $\mathscr{A}$ is bounded in $C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$, where $\alpha \in (0,1)$ is the exponent furnished by Lemma 6.4. This proves (i) and (iv).

Step 3. Positive invariance of $\mathscr{A}$. Let $s \geq 0$. We first show that $S(s)\mathscr{A} \subset \mathscr{A}$. Let $x \in \mathscr{A} = \omega(\mathcal{K})$. Then there exist $x_n \in \mathcal{K}$ and $t_n \to \infty$ such that

$$S(t_n)x_n \to x \quad \text{in } \mathcal{X}.$$

By continuity of $S(s)$,

$$S(s)x = \lim_{n\to\infty} S(s)S(t_n)x_n = \lim_{n\to\infty} S(t_n+s)x_n.$$

Since $x_n \in \mathcal{K}$ and $t_n + s \to \infty$, the limit point $S(s)x$ belongs to $\omega(\mathcal{K}) = \mathscr{A}$. Therefore

$$S(s)\mathscr{A} \subset \mathscr{A} \quad \text{for all } s \geq 0.$$

Step 4. Reverse invariance. We now prove that

$$\mathscr{A} \subset S(s)\mathscr{A} \quad \text{for all } s \geq 0.$$

Fix $s \geq 0$, and let $x \in \mathscr{A}$. Since $x \in \omega(\mathcal{K})$, there exist $x_n \in \mathcal{K}$ and $t_n \to \infty$ such that

$$S(t_n)x_n \to x \quad \text{in } \mathcal{X}.$$

For n large enough, $t_n \geq s$. Consider the sequence $y_n := S(t_n - s)x_n$. Because $\mathcal{K}$ is compact and positively invariant, the set $\mathcal{K}$ is bounded, and $t_n - s \to \infty$. Hence Proposition 7.3 applied to the bounded set $\mathcal{K}$ yields a subsequence, not relabeled, and some $y \in \mathcal{X}$ such that

$$y_n \to y \quad \text{in } \mathcal{X}.$$

Since y is the limit of points of the form $S(t_n - s)x_n$, with $x_n \in \mathcal{K}$ and $t_n - s \to \infty$, we have $y \in \omega(\mathcal{K}) = \mathscr{A}$. By continuity of $S(s)$,

$$S(s)y = \lim_{n\to\infty} S(s)y_n = \lim_{n\to\infty} S(t_n)x_n = x.$$

Thus $x \in S(s)\mathscr{A}$. Since $x \in \mathscr{A}$ was arbitrary, $\mathscr{A} \subset S(s)\mathscr{A}$. Together with Step 3, this yields

$$S(s)\mathscr{A} = \mathscr{A} \quad \text{for all } s \geq 0.$$

Hence $\mathscr{A}$ is fully invariant, proving (ii).

Step 5. Attraction of $\mathcal{K}$. We claim that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{X}}(S(t)\mathcal{K}, \mathscr{A}) = 0.$$

Suppose, by contradiction, that this is false. Then there exist $\varepsilon_0 > 0$, a sequence $t_n \to \infty$, and points $x_n \in \mathcal{K}$ such that

$$\operatorname{dist}_{\mathcal{X}}(S(t_n)x_n, \mathscr{A}) \geq \varepsilon_0 \quad \text{for all } n.$$

Since $\mathcal{K}$ is bounded and $t_n \to \infty$, Proposition 7.3 yields a subsequence, not relabeled, and some $z \in \mathcal{X}$ such that $S(t_n)x_n \to z$ in $\mathcal{X}$. By definition of the ω-limit set, this implies $z \in \omega(\mathcal{K}) = \mathscr{A}$. Therefore

$$\operatorname{dist}_{\mathcal{X}}(S(t_n)x_n, \mathscr{A}) \leq \|S(t_n)x_n - z\|_{\mathcal{X}} \to 0,$$

contradicting $\operatorname{dist}_{\mathcal{X}}(S(t_n)x_n, \mathscr{A}) \geq \varepsilon_0$. Hence $S(t)\mathcal{K}$ is attracted by $\mathscr{A}$.

Step 6. Attraction of arbitrary bounded sets. Let $B \subset \mathcal{X}_+$ be bounded. Since $\mathscr{B}_*$ is absorbing, there exists $T_B \geq 0$ such that $S(t)B \subset \mathscr{B}_*$ for all $t \geq T_B$. Fix $t \geq T_B + T_*$. Then

$$S(t)B = S(t - T_B)\big(S(T_B)B\big).$$

Because $S(T_B)B \subset \mathscr{B}_*$ and $t - T_B \geq T_*$, we obtain

$$S(t)B \subset \cup_{r \geq T_*} S(r)\mathscr{B}_* \subset \mathcal{K}.$$

Thus $\mathcal{K}$ absorbs every bounded set $B \subset \mathcal{X}_+$.

We now prove that $\mathscr{A}$ attracts B. Let $\varepsilon > 0$. Since $\mathscr{A}$ attracts $\mathcal{K}$, there exists $T_\varepsilon \geq 0$ such that

$$\operatorname{dist}_{\mathcal{X}}(S(t)\mathcal{K}, \mathscr{A}) < \varepsilon \quad \text{for all } t \geq T_\varepsilon.$$

Hence for every $t \geq T_B + T_* + T_\varepsilon$, we have

$$S(t)B = S(t - T_B - T_*)\big(S(T_B + T_*)B\big),$$

and $S(T_B + T_*)B \subset \mathcal{K}$. Therefore

$$S(t)B \subset S(t - T_B - T_*)\mathcal{K},$$

which implies

$$\operatorname{dist}_{\mathcal{X}}(S(t)B, \mathscr{A}) \leq \operatorname{dist}_{\mathcal{X}}(S(t - T_B - T_*)\mathcal{K}, \mathscr{A}) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{X}}(S(t)B, \mathscr{A}) = 0.$$

This proves (iii) and completes the proof. $\square$

Proof of Theorem 1.7. Theorem 7.4 establishes the existence of a compact global attractor $\mathscr{A} \subset \mathcal{X}_+$, proves that it is fully invariant, and shows that it attracts every bounded subset of $\mathcal{X}_+$ with respect to the phase-space semidistance $\operatorname{dist}_{\mathcal{X}}$. In addition, part (iv) of Theorem 7.4 yields the bound

$$\mathscr{A} \subset \mathscr{B}_* \subset C^{1+\alpha}(\overline{\Omega}) \times C^{1+\alpha}(\overline{\Omega})$$

for the exponent $\alpha \in (0,1)$ supplied by Lemma 6.4. This is exactly the content of Theorem 1.7. $\square$

Concluding remarks

We have established global well-posedness, positivity, boundary-condition-dependent invariant and absorbing regions, and a compact global attractor for the one-dimensional RD-ARD glioblastoma system. Natural extensions include radially symmetric higher-dimensional geometries, spatially heterogeneous transport and growth coefficients, and numerical studies that use the comparison bounds proved here as analytical benchmarks.

Acknowledgments. This work is supported by National Natural Science Foundation of China (12301145, 12261107, 12561020) and Yunnan Fundamental Research Projects (202401AU070123, 202601AT070048). We want to express our gratitude to the anonymous referees for their meticulous review, and for the valuable feedback provided.

References

[1] H. Amann; *Linear and Quasilinear Parabolic Problems, Vol. I: Abstract Linear Theory*, Monographs in Mathematics, vol. 89, Birkhäuser, Basel, 1995.
[2] A. L. Baldock, S. Ahn, R. Rockne, S. Johnston, M. Neal, D. Corwin, K. Clark-Swanson, G. Sterin, A. D. Trister, H. Malone, V. Ebiana, A. M. Sonabend, M. Mrugala, J. K. Rockhill, D. L. Silbergeld, A. Lai, T. Cloughesy, G. M. McKhann II, J. N. Bruce, R. C. Rostomily, P. Canoll, K. R. Swanson; *Patient-specific metrics of invasiveness reveal significant prognostic benefit of resection in a predictable subset of gliomas*, PLoS One, **9** (2014), no. 10, e99057.
[3] M. Burger, P. Friele, J.-F. Pietschmann; *On a reaction-cross-diffusion system modeling the growth of glioblastoma*, SIAM J. Appl. Math., **80** (2020), no. 1, 160-182.
[4] H. Hatzikirou, D. Basanta, M. Simon, K. Schaller, A. Deutsch; *"Go or Grow": the key to the emergence of invasion in tumour progression?*, Math. Med. Biol., **29** (2012), no. 1, 49-65.
[5] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva; *Linear and Quasi-linear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, RI, 1968.
[6] G. M. Lieberman; *Second Order Parabolic Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[7] A. Lunardi; *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 16, Birkhäuser, Basel, 1995.
[8] A. A. Malik, K. C. Nguyen, J. T. Nardini, C. C. Krona, K. B. Flores, S. Nelander; *Mathematical modeling of multicellular tumor spheroids quantifies inter-patient and intra-tumor heterogeneity*, npj Syst. Biol. Appl., **11** (2025), Art. 20.
[9] C. V. Pao; *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.
[10] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.
[11] K. Pham, A. Chauviere, H. Hatzikirou, X. Li, H. M. Byrne, V. Cristini, J. Lowengrub; *Density-dependent quiescence in glioma invasion: instability in a simple reaction-diffusion model for the migration/proliferation dichotomy*, J. Biol. Dyn., **6** (2012), suppl. 1, 54-71.
[12] M. H. Protter, H. F. Weinberger; *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984.
[13] A. M. Stein, T. Demuth, D. Mobley, M. Berens, L. M. Sander; *A mathematical model of glioblastoma tumor spheroid invasion in a three-dimensional in vitro experiment*, Biophys. J., **92** (2007), no. 1, 356-365.
[14] T. L. Stepien, E. M. Rutter, Y. Kuang; *Traveling waves of a go-or-grow model of glioma growth*, SIAM J. Appl. Math., **78** (2018), no. 3, 1778-1801.
[15] K. R. Swanson, E. C. Alvord, Jr., J. D. Murray; *A quantitative model for differential motility of gliomas in grey and white matter*, Cell Prolif., **33** (2000), no. 5, 317-329.
[16] C. H. Wang, J. K. Rockhill, M. Mrugala, D. L. Peacock, A. Lai, K. Jusenius, J. M. Wardlaw, T. Cloughesy, A. M. Spence, R. Rockne, E. C. Alvord, Jr., K. R. Swanson; *Prognostic significance of growth kinetics in newly diagnosed glioblastomas revealed by combining serial imaging with a novel biomathematical model*, Cancer Res., **69** (2009), no. 23, 9133-9140.

Zhipeng Yang
Department of Mathematics, Yunnan Normal University, Kunming, China.
Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Kunming, China
Email address: yangzhipeng326@163.com

Zhenyu Zhan (corresponding author)
Department of Mathematics, Yunnan Normal University, Kunming, China.
Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Kunming, China
Email address: 20972850@qq.com