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DYNAMICS OF WEAK SOLUTIONS FOR TWO-DIMENSIONAL MICROPOLAR EQUATIONS WITH NONLINEAR DAMPING

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Abstract. In this article, we consider 2D micropolar fluid equations with a nonlinear damping term $\sigma|u|^{\beta-1}u$ with $1 \leq \beta < 7/3$. We establish the existence and uniqueness of weak solutions. Based on the method of ℓ-trajectory proposed by Málek and Pražák, we analyze the long-time behavior of the system in the trajectory space X_ℓ, deduce the existence of the global attractor of the semigroup in the initial phase space $H \times L^2(\Omega)$, and establish that this global attractor has finite fractal dimension.

1. Introduction

This article considers the long-time behavior of weak solutions for the two-dimensional micropolar equations with nonlinear damping,

$$
\begin{gathered}
u_t + (u\cdot\nabla)u - (\nu+k)\Delta u + \sigma|u|^{\beta-1}u + \nabla p = 2k\nabla\times w + f_1, \quad (x,t)\in\Omega\times\mathbb{R}^+,\\
w_t + (u\cdot\nabla)w + 4kw - \gamma\Delta w = 2k\nabla\times u + f_2, \quad (x,t)\in\Omega\times\mathbb{R}^+,\\
\nabla\cdot u = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+,\\
u(x,0)=u_0(x), w(x,0)=w_0(x), \quad x\in\Omega,\\
u(x,t)|_{\partial\Omega}=0, w(x,t)|_{\partial\Omega}=0, \quad t\geq 0,
\end{gathered}
\tag{1.1}
$$

where $\Omega \subseteq \mathbb{R}^d$ $(d = 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $u = (u_1, u_2)$ is the velocity of the fluid, $w = w(x,t)$ is the angular velocity of micro-rotation, $p(x,t)$ is the fluid pressure, $f_1 = f_1(x)$ and $f_2 = f_2(x)$ represent the time-independent external force terms, the positive constant σ is the damping coefficient, ν, k and γ are all positive constants, γ represents the angular viscosity. Moreover,

$$
\nabla\times u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \nabla\times w = \Big(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1}\Big).
$$

Damping phenomena are extremely common in nature. Originating from fluid motion, they can describe flows in porous media, resistance or friction effects, as well as certain dissipative mechanisms(such as [2, 3, 11, 12]). Real fluid motion is generally accompanied by dissipation, and the damping term is the core physical term that characterizes such dissipative mechanisms.

When $w = 0$, system (1.1) reduces to the two-dimensional Navier-Stokes equations with damping. For the Navier-Stokes equations with damping, a fairly systematic theoretical framework has been established for the study of their long-time behavior (such as [7, 13, 21]). However, the classical Navier-Stokes equations neglect the microscopic rotational degrees of freedom of fluid elements, so the micropolar fluid theory needs to be introduced. The micropolar flow model was first proposed by Eringen in the 1960s [10]. It describes the dynamic behavior of fluids with non-uniform stress tensors, including animal blood, polymer suspensions, liquid crystals, and dilute aqueous polymer solutions, and holds significant practical value.

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Numerous scholars (such as [4, 8, 9, 15, 23]) have investigated the mathematical theory of the two-dimensional micropolar fluid model, yet the existing results on the two-dimensional micropolar fluid equations with damping remain are relatively limited. Therefore, in this paper, we will discuss the existence of solutions and attractors for the two-dimensional micropolar flow equations with nonlinear damping terms. For $1 \le \beta < 7/3$, we rigorously establish the existence and uniqueness of weak solutions to system (1.1) via the Galerkin approximation method [6]. Nevertheless, due to the lack of sufficient regularity of weak solutions, traditional methods fail to directly prove the existence of a global attractor for the system, which constitutes a major challenge in this field. Ai and Tan [1] addressed this type of problem by proving the existence of global and exponential attractors for the two-dimensional non-Newtonian micropolar fluid equations via the ℓ-trajectory method [18], which offers a crucial insight. Motivated by [1], to establish the existence of a global attractor for system (1.1), we derive a priori estimates for weak solutions, construct the trajectory space X_ℓ, and equate the limit behavior of solutions in the original phase space with that in the trajectory space. This approach overcomes the regularity barrier of weak solutions, and we finally prove the existence of a global attractor for the system in the phase space.

The structure of this paper is as follows. In Section 2, we present the notations used in this paper, as well as the relevant definitions and theorems involved in the research process. In Section 3, we establish the existence of a global attractor $\mathcal{A}_\ell$ with finite fractal dimension in the trajectory space by estimating the solutions of the system. In Section 4, we prove the existence of a global attractor $\mathcal{A}$ in the original phase space, which also has a finite fractal dimension.

2. Preliminaries

Throughout this paper, $\mathbf{W}^{m,p} = (W^{m,p}(\Omega))^2$, $L^q = (L^q(\Omega))^2$, respectively, stand for the Sobolev and Lebesgue spaces on Ω with $m \in \mathbf{N}$ and $1 \le q \le \infty$. In particular, $\mathbf{H}^m(\Omega) = \mathbf{W}^{m,2}(\Omega)$ is a Hilbert space. The Lebesgue spaces $L^q(0,\ell;X)$ are composed of all functions u that map into X for almost every $s \in (0,\ell)$ and satisfy $\big(\int_0^\ell \|u(s)\|_X^q \mathrm{d}s\big)^{1/q} < +\infty$. Moreover, let C be the generic positive constants and $C(\cdot)$, C_0, C_1, … be the positive constants that need special emphasis.

To study problem (1.1), we also introduce the following function spaces:

$$
\begin{aligned}
&\mathcal{V} = \left\{u \in (C_0^\infty(\Omega))^2 : \operatorname{div} u = 0\right\},\\
&H = \text{the closure of } \mathcal{V} \text{ in } (L^2(\Omega))^2,\\
&V = \text{the closure of } \mathcal{V} \text{ in } (H_0^1(\Omega))^2.
\end{aligned}
$$

In addition, let V^{-1} be the dual spaces of V. And $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^{-1}.

The inner products of H and $L^2(\Omega)$ are defined by

$$
(u,v) := \int_\Omega u \cdot v \,\mathrm{d}x, \quad \text{for } u,v \in H \text{ or } u,v \in L^2(\Omega),
$$

and the inner products of V and $H_0^1(\Omega)$ are defined by

$$
(u,v) := \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i \,\mathrm{d}x, \quad \text{for } u,v \in V \text{ or } u,v \in H_0^1(\Omega).
$$

Moreover, the L^q-norm is defined as $\|\cdot\|_{L^q(\Omega)}$ and is abbreviated as $\|\cdot\|_{L^q}$.

Next, we will introduce some operators used in this paper. Let the bilinear form $a_0 : V \times V \to \mathbb{R}$ satisfy

$$
a_0(u,v) = \sum_{i=1,j=1}^{2} \int_\Omega \partial_{x_j} u_i \cdot \partial_{x_j} v_i \,\mathrm{d}x, \quad \text{for } u,v \in V,
$$

the bilinear form $a_1 : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ satisfy

$$a_1(u,v) = \sum_{i=1,j=1}^{2} \int_\Omega \partial_{x_j} u_i \cdot \partial_{x_j} v_i \,\mathrm{d}x, \quad \text{for } u,v \in H_0^1(\Omega).$$

Then let P be the Helmholtz-Leray orthogonal projection operator from $L^2(\Omega)$ onto H. We define the operator $A : V \to V^{-1}$ satisfying $Au = -P\Delta u$, which is the Stokes operator with domain $D(A) = \mathbf{H}^2(\Omega) \cap V$, endowed with inner product

$$\langle Au, v\rangle = \sum_{i=1}^{2} \int_\Omega \nabla u_i \cdot \nabla v_i \,\mathrm{d}x, \quad \text{for } u,v \in V \text{ or } u,v \in H_0^1(\Omega).$$

The mapping $B_0 : V \times V \to V^{-1}$ is defined by

$$B_0(u,v) = P((u\cdot\nabla)v), \quad \text{for } u,v \in V,$$

the mapping $B_1 : V \times H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by

$$B_1(u,w) = (u\cdot\nabla)w, \quad \text{for } u \in V, w \in H_0^1(\Omega),$$

which satisfy

$$b(u,v,w) = \langle B(u,v), w\rangle = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x,$$
$$b(u,v,v) = 0, \quad b(u,v,w) = -b(u,w,v),$$

where $B(\cdot,\cdot)$ stands for $B_0(\cdot,\cdot)$ or $B_1(\cdot,\cdot)$.

Let us apply the orthogonal projection operator P to problem (1.1), then (1.1) is rewritten as the equivalent functional differential equation,

$$\begin{gathered} u_t + B_0(u,u) + (\nu+k)Au + G(u) = 2kP(\nabla\times w) + Pf_1, \\ w_t + B_1(u,w) + 4kw - \gamma\Delta w = 2k\nabla\times u + f_2, \\ \nabla\cdot u = 0, \\ u(x,t)\big|_{t=0} = u_0(x), \quad w(x,t)\big|_{t=0} = w_0(x), \end{gathered} \tag{2.1}$$

where we let $G(u) = P(\sigma|u|^{\beta-1}u)$.

Poincaré's inequality [20] gives

$$\|u\|_{L^2} \le \lambda_{1,u}^{-1/2}\|\nabla u\|_{L^2}, \quad \|w\|_{L^2} \le \lambda_{1,w}^{-1/2}\|\nabla w\|_{L^2}, \quad \text{for } (u,w) \in V \times H_0^1(\Omega),$$

where $\lambda_{1,u}, \lambda_{1,w}$ are the first eigenvalues for Au and Aw, respectively.

Now we give the definition of solutions to the two-dimensional micropolar fluid equations (1.1) with nonlinear damping.

Definition 2.1. Assume $(u_0, w_0) \in H \times L^2(\Omega)$ and $(f_1, f_2) \in H \times L^2(\Omega)$. For any fixed $T > 0$, the function $(u(t), w(t))$ is called a weak solution of equations (1.1) with initial condition (u_0, w_0) on $(0,T)$, if

$$\begin{gathered} u(t) \in \mathcal{C}_w(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)), \\ w(t) \in \mathcal{C}_w(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \\ (u_t, w_t) \in L^2(0,T;V^{-1} \times H^{-1}(\Omega)) \end{gathered}$$

for all $(\varphi,\psi) \in V \times H_0^1(\Omega)$, satisfy

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(u(t),\varphi) + (B_0(u(t),u(t)),\varphi) + (\nu+k)a_0(u(t),\varphi) + (G(u(t)),\varphi) \\ &= 2k(\nabla\times w(t),\varphi) + (f_1(x),\varphi) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(w(t),\psi) + (B_1(u(t),w(t)),\psi) + 4k(w(t),\psi) + \gamma a_1(w(t),\psi) \\ &= 2k(\nabla\times u(t),\psi) + (f_2(x),\psi) \end{aligned} \tag{2.3}$$

hold on $(0,\infty)$ in the sense of distributions.

Subsequently, we present the proof of well-posedness for weak solutions of the two-dimensional nonlinear damped micropolar fluid equations (1.1).

Theorem 2.2. *Assume that $(f_1, f_2) \in H \times L^2(\Omega)$. Then for any $(u_0, w_0) \in H \times L^2(\Omega)$ with $1 \le \beta < \frac{7}{3}$, there exists a weak solution $(u(t), w(t))$ of problem (1.1).*

Proof. By the proof of [6, Theorem 8], we consider $\{\varphi_m\}_{m\in\mathbf{N}} \subset V$ and $\{\psi_m\}_{m\in\mathbf{N}} \subset H_0^1(\Omega)$. We denote by $V_m = \operatorname{span}\{\varphi_1,\dots,\varphi_m\} \subset V$ the linear subspace spanned by $\{\varphi_1,\dots,\varphi_m\}$, and by $W_m = \operatorname{span}\{\psi_1,\dots,\psi_m\} \subset H_0^1(\Omega)$ the linear subspace spanned by $\{\psi_1,\dots,\psi_m\}$. Additionally, we consider the pair (u_m, w_m) satisfying

$$u_m(x,t) = \sum_{r=1}^{m} b_{rm}(t)\varphi_r(x), \quad w_m(x,t) = \sum_{r=1}^{m} d_{rm}(t)\psi_r(x)$$

are the solutions of the approximate problem

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}\big(u_m(t),\varphi_r\big) + \big(B_0(u_m(t),u_m(t)),\varphi_r\big) + (\nu+k)a_0(u_m(t),\varphi_r) + \big(G(u_m(t)),\varphi_r\big) \\
= 2k\big(\nabla\times w_m(t),\varphi_r\big) + \big(f_1(x),\varphi_r\big), \quad r=1,2,\dots,m, \\
\frac{\mathrm{d}}{\mathrm{d}t}\big(w_m(t),\psi_r\big) + \big(B_1(u_m(t),w_m(t)),\psi_r\big) + 4k\big(w_m(t),\psi_r\big) + \gamma a_1(w_m(t),\psi_r) \\
= 2k\big(\nabla\times u_m(t),\psi_r\big) + \big(f_2(x),\psi_r\big), \quad r=1,2,\dots,m, \\
u_m(0) = u_{0m} \to u_0, \quad \text{strongly in } H, \\
w_m(0) = w_{0m} \to w_0, \quad \text{strongly in } L^2(\Omega).
\end{gathered} \tag{2.4}$$

Since a local solution to (2.4) exists on the interval $[0,t_m]$ with $0 < t_m < T$, we extend this solution to the entire interval $[0,T]$ by the first estimates in the following.

First estimates. Multiplying both sides of $(2.4)_1$ by b_{rm} and $(2.4)_2$ by d_{rm}, followed by summing over $r=1$ to m, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{L^2}^2 + (\nu+k)\|\nabla u_m(t)\|_{L^2}^2 + \sigma\|u_m(t)\|_{L^{\beta+1}}^{\beta+1} \\
&= 2k(\nabla\times w_m(t),u_m(t)) + (f_1(x),u_m(t)) \\
&= 2k(\nabla\times u_m(t),w_m(t)) + (f_1(x),u_m(t)) \\
&\le 2k\|w_m(t)\|_{L^2}\|\nabla u_m(t)\|_{L^2} + \|f_1(x)\|_{V^{-1}}\|\nabla u_m(t)\|_{L^2}
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(t)\|_{L^2}^2 + 4k\|w_m(t)\|_{L^2}^2 + \gamma\|\nabla w_m(t)\|_{L^2}^2 \\
&= 2k(\nabla\times u_m(t),w_m(t)) + (f_2(x),w_m(t)) \\
&\le 2k\|\nabla u_m(t)\|_{L^2}\|w_m(t)\|_{L^2} + \|f_2(x)\|_{H^{-1}(\Omega)}\|\nabla w_m(t)\|_{L^2}.
\end{aligned} \tag{2.6}$$

Applying Young's inequality, we derive the following inequalities from (2.5) and (2.6) separately,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{L^2}^2 + (\nu+k)\|\nabla u_m(t)\|_{L^2}^2 + \sigma\|u_m(t)\|_{L^{\beta+1}}^{\beta+1} \\
&\le \frac{\nu+k}{6}\|\nabla u_m(t)\|_{L^2}^2 + \frac{6k^2}{\nu+k}\|w_m(t)\|_{L^2}^2 + \frac{\nu+k}{6}\|\nabla u_m(t)\|_{L^2}^2 + \frac{3}{2(\nu+k)}\|f_1(x)\|_{V^{-1}}^2
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(t)\|_{L^2}^2 + 4k\|w_m(t)\|_{L^2}^2 + \gamma\|\nabla w_m(t)\|_{L^2}^2 \\
&\le \frac{\nu+k}{6}\|\nabla u_m(t)\|_{L^2}^2 + \frac{6k^2}{\nu+k}\|w_m(t)\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla w_m(t)\|_{L^2}^2 + \frac{1}{2\gamma}\|f_2(x)\|_{H^{-1}(\Omega)}^2.
\end{aligned} \tag{2.8}$$

By adding (2.7) and (2.8) and integrating the result from 0 to t with $0 \le t \le T$, we deduce that

$$\begin{aligned}&\big(\|u_m(t)\|_{L^2}^2 + \|w_m(t)\|_{L^2}^2\big) + (\nu + k)\int_0^t \|\nabla u_m(s)\|_{L^2}^2 \,\mathrm{d}s\\&+ 2\sigma \int_0^t \|u_m(s)\|_{L^{\beta+1}}^{\beta+1} \,\mathrm{d}s + \gamma \int_0^t \|\nabla w_m(s)\|_{L^2}^2 \,\mathrm{d}s\\&\le C + C\int_0^t \big(\|u_m(s)\|_{L^2}^2 + \|w_m(s)\|_{L^2}^2\big) \,\mathrm{d}s.\end{aligned} \tag{2.9}$$

Using Gronwall's inequality, we obtain

$$\|u_m(t)\|_{L^2}^2 + \|w_m(t)\|_{L^2}^2 \le C. \tag{2.10}$$

Consequently, using estimate (2.9), we conclude that

$$\{u_m\} \text{ is bounded in } L^\infty(I;H), \tag{2.11}$$
$$\{u_m\} \text{ is bounded in } L^{\beta+1}(I;L^{\beta+1}), \tag{2.12}$$
$$\{u_m\} \text{ is bounded in } L^2(I;V), \tag{2.13}$$
$$\{w_m\} \text{ is bounded in } L^\infty(I;L^2(\Omega)), \tag{2.14}$$
$$\{w_m\} \text{ is bounded in } L^2(I;H_0^1(\Omega)), \tag{2.15}$$

where $I = (0,T)$ is a time interval.

Second estimates. The mapping $P_m : V \to V_m$ is defined as the orthogonal projection from V to V_m,

$$P_m u = \sum_{j=1}^m (u,\varphi_j)\varphi_j, \quad \forall u(t) \in V.$$

We further consider the adjoint operator of P_m, denoted by $P_m^* : V^{-1} \to V^{-1}$. We note that $P_m^* u_m' = u_m'$. Through the choice of the special basis $\{\varphi_m\}$, we obtain

$$\|P_m\|_{\mathfrak{L}(V,V)} \le 1, \quad \|P_m^*\|_{\mathfrak{L}(V^{-1},V^{-1})} \le 1. \tag{2.16}$$

From $(2.4)_1$, we obtain

$$u_m' = -P_m^* B_0 u_m - (\nu + k)P_m^* A u_m - P_m^* G(u_m) + 2kP_m^* \nabla \times w_m + P_m^* f_1. \tag{2.17}$$

Using the Gagliardo-Nirenberg's inequality, we derive that

$$\begin{aligned}|\langle B_0(u_m,u_m), v\rangle| &= |b(u_m,u_m,v)|\\&= |b(u_m,v,u_m)|\\&\le \|u_m\|_{L^4}\|\nabla v\|_{L^2}\|u_m\|_{L^4}\\&\le C\|u_m\|_{L^2}^{1/2}\|\nabla u_m\|_{L^2}^{1/2}\|\nabla v\|_{L^2}\|u_m\|_{L^2}^{1/2}\|\nabla u_m\|_{L^2}^{1/2}\\&\le C\|u_m\|_{L^2}\|\nabla u_m\|_{L^2}\|\nabla v\|_{L^2}\end{aligned}$$

for any $u_m(t), v(t) \in V$. Therefore, from (2.11) and (2.13),

$$\{B_0(u_m,u_m)\} \text{ is bounded in } L^2(I;V^{-1}). \tag{2.18}$$

Since $|\langle Au_m, v\rangle| \le \|\nabla u_m\|_{L^2}\|\nabla v\|_{L^2}$, for all $u_m(t), v(t) \in V$. From (2.13), we take

$$\{Au_m\} \text{ is bounded in } L^2(I;V^{-1}). \tag{2.19}$$

Let $u_m(t), v(t) \in V$, using Hölder's inequality, we have

$$|\langle G(u_m), v\rangle| \le |\langle |u_m|^{\beta-1}u_m, v\rangle| \le \||u_m|^{\beta-1}u_m\|_{L^{6/5}}\|v\|_{L^6} \le C\|u_m\|_{L^{6\beta/5}}\|\nabla v\|_{L^2}.$$

Then we deduce that when $1 \le \beta \le 5/3$

$$|\langle G(u_m), v\rangle| \le C\|u_m\|_{L^2}^{\beta}\|\nabla v\|_{L^2}.$$

So

$$\|G(u_m)\|_{V^{-1}} \le C\|u_m\|_{L^2}^{\beta},$$

and from this we have

$$\int_0^T \|G(u_m)\|_{V^{-1}}^2 \mathrm{d}s \le C \int_0^T \|u_m\|_{L^2}^{2\beta} \mathrm{d}s \le C\|u_m\|_{L^\infty(I;H)}^{2\beta} T. \tag{2.20}$$

For $5/3 < \beta < 7/3$, using the Gagliardo-Nirenberg's inequality, we obtain

$$|\langle G(u_m), v\rangle| \le C\|u_m\|_{L^{\beta+1}}^{\frac{(\beta+1)(3\beta-5)}{3(\beta-1)}} \|u_m\|_{L^2}^{\frac{5-\beta}{3(\beta-1)}} \|\nabla v\|_{L^2}.$$

So

$$\|G(u_m)\|_{V^{-1}} \le C\|u_m\|_{L^{\beta+1}}^{\frac{(\beta+1)(3\beta-5)}{3(\beta-1)}} \|u_m\|_{L^2}^{\frac{5-\beta}{3(\beta-1)}},$$

and from the above inequality we obtain

$$\begin{aligned} \int_0^T \|G(u_m)\|_{V^{-1}}^2 \mathrm{d}s &\le C \int_0^T \|u_m\|_{L^{\beta+1}}^{\frac{2(\beta+1)(3\beta-5)}{3(\beta-1)}} \|u_m\|_{L^2}^{\frac{2(5-\beta)}{3(\beta-1)}} \mathrm{d}s \\ &\le C\|u_m\|_{L^\infty(I;H)}^{\frac{2(5-\beta)}{3(\beta-1)}} \int_0^T \|u_m\|_{L^{\beta+1}}^{\frac{2(\beta+1)(3\beta-5)}{3(\beta-1)}} \mathrm{d}s \\ &\le C\|u_m\|_{L^\infty(I;H)}^{\frac{2(5-\beta)}{3(\beta-1)}} \Big(\int_0^T \|u_m\|_{L^{\beta+1}}^{\beta+1} \mathrm{d}s\Big)^{\frac{2(3\beta-5)}{3(\beta-1)}} T^{\frac{7-3\beta}{3(\beta-1)}}. \end{aligned} \tag{2.21}$$

Therefore, from (2.20), (2.21), (2.11) and (2.12), we have

$$\{G(u_m)\} \text{ is bounded in } L^2(I;V^{-1}). \tag{2.22}$$

Let $w_m(t) \in H_0^1(\Omega)$ and $v(t) \in V$,

$$|\langle \nabla \times w_m, v\rangle| = |\langle w_m, \nabla \times v\rangle| \le \|w_m\|_{L^2}\|\nabla v\|_{L^2},$$

it follows from (2.14) that

$$\{\nabla \times w_m\} \text{ is bounded in } L^2(I;V^{-1}). \tag{2.23}$$

From (2.18), (2.19), (2.22), (2.23), (2.16) and the assumptions on f_1, we obtain

$$\{u_m'\} \text{ is bounded in } L^2(I;V^{-1}). \tag{2.24}$$

Similarly, let $R_m : H_0^1(\Omega) \to W_m$ be the orthogonal projection,

$$R_m w = \sum_{j=1}^m (w, \psi_j)\psi_j, \ \forall w \in H_0^1(\Omega).$$

We further consider the adjoint operator of R_m, denoted by $R_m^* : H^{-1}(\Omega) \to H^{-1}(\Omega)$. We note that $R_m^* w_m' = w_m'$. Through the choice of the special basis $\{\psi_m\}$, we obtain

$$\|R_m\|_{\mathfrak{L}(H_0^1(\Omega),H_0^1(\Omega))} \le 1, \quad \|R_m^*\|_{\mathfrak{L}(H^{-1}(\Omega),H^{-1}(\Omega))} \le 1. \tag{2.25}$$

From $(2.4)_2$, we obtain

$$w_m' = -R_m^* B_1(u_m, w_m) - 4kR_m^* w_m - \gamma R_m^* A w_m + 2kR_m^* \nabla \times u_m + R_m^* f_2. \tag{2.26}$$

Using the Gagliardo-Nirenberg's inequality, we derive that

$$\begin{aligned} |\langle B_1(u_m, w_m), v\rangle| &= |b(u_m, w_m, v)| \\ &= |b(u_m, v, w_m)| \\ &\le \|u_m\|_{L^4}\|\nabla v\|_{L^2}\|w_m\|_{L^4} \\ &\le C\|u_m\|_{L^2}^{1/2}\|\nabla u_m\|_{L^2}^{1/2}\|\nabla v\|_{L^2}\|w_m\|_{L^2}^{1/2}\|\nabla w_m\|_{L^2}^{1/2} \\ &\le C\|u_m\|_{L^2}\|\nabla u_m\|_{L^2}\|\nabla v\|_{L^2} + \|w_m\|_{L^2}\|\nabla w_m\|_{L^2}\|\nabla v\|_{L^2} \end{aligned}$$

for all $u_m(t) \in V$, $w_m(t), v(t) \in H_0^1(\Omega)$. Therefore, from (2.11), (2.13)-(2.15)

$$\{B_1(u_m, w_m)\} \text{ is bounded in } L^2(I;H^{-1}(\Omega)). \tag{2.27}$$

From (2.15), we obtain

$$\{w_m\} \text{ is bounded in } L^2(I;H^{-1}(\Omega)). \tag{2.28}$$

Analogously, we have $|\langle Aw_m, v\rangle| \le \|\nabla w_m\|_{L^2}\|\nabla v\|_{L^2}$, for all $w_m(t), v(t) \in H_0^1(\Omega)$. From (2.15), we take

$$\{Aw_m\} \text{ is bounded in } L^2(I; H^{-1}(\Omega)). \tag{2.29}$$

Let $u_m(t) \in V$ and $v(t) \in H_0^1(\Omega)$

$$|\langle \nabla \times u_m, v\rangle| = |\langle u_m, \nabla \times v\rangle| \le \|u_m\|_{L^2}\|\nabla v\|_{L^2},$$

it follows from (2.11) that

$$\{\nabla \times u_m\} \text{ is bounded in } L^2(I; H^{-1}(\Omega)). \tag{2.30}$$

From (2.27)-(2.30), (2.25) and the assumptions on f_2, we obtain

$$\{w_m'\} \text{ is bounded in } L^2(I; H^{-1}(\Omega)). \tag{2.31}$$

From (2.11)-(2.15), (2.24), (2.31) and the Aubin-Lions lemma, we conclude that there exist subsequences of $\{u_m\}$ and $\{w_m\}$, still denoted by $\{u_m\}$ and $\{w_m\}$ respectively, such that

$$u_m \to u \quad \text{strongly in } L^2(I; H), \tag{2.32}$$
$$u_m \to u \quad \text{strongly in } L^{\beta+1}(I; L^{\beta+1}), \tag{2.33}$$
$$u_m \to u \quad \text{weak star in } L^\infty(I; H), \tag{2.34}$$
$$u_m \to u \quad \text{weakly in } L^2(I; V), \tag{2.35}$$
$$u_m' \to u' \quad \text{weakly in } L^2(I; V^{-1}), \tag{2.36}$$
$$w_m \to w \quad \text{strongly in } L^2(I; L^2(\Omega)), \tag{2.37}$$
$$w_m \to w \quad \text{weak star in } L^\infty(I; L^2(\Omega)), \tag{2.38}$$
$$w_m \to w \quad \text{weakly in } L^2(I; H_0^1(\Omega)), \tag{2.39}$$
$$w_m' \to w' \quad \text{weakly in } L^2(I; H^{-1}(\Omega)). \tag{2.40}$$

We note that (2.13) and (2.24) imply that $u \in C^0(I; H)$. Similarly, (2.15) and (2.31) imply that $w \in C^0(I; L^2(\Omega))$. Therefore, it makes sense to consider $u(0) = u_0$ and $w(0) = w_0$.

From (2.32) and (2.37) and the proof of [6, Theorem 8], we deduce that

$$\begin{gathered}\int_0^T b(u_m, u_m, \varphi) \to \int_0^T b(u, u, \varphi),\\ \int_0^T b(u_m, w_m, \psi) \to \int_0^T b(u, w, \psi).\end{gathered} \tag{2.41}$$

From (2.33), we deduce that

$$\int_0^T \langle |u_m|^{\beta-1}u_m, \varphi\rangle \,\mathrm{d}t \to \int_0^T \langle |u|^{\beta-1}u, \varphi\rangle \,\mathrm{d}t. \tag{2.42}$$

The convergences (2.32)-(2.42) allow us to pass to the limit in the system (2.4), where φ_r and ψ_r are fixed to obtain

$$u' + B_0(u, u) + (\nu + k)Au + G(u) = 2k\nabla \times w + f_1, \quad \text{in } L^2(I; V^{-1}), \tag{2.43}$$
$$w' + B_1(u, w) + 4kw + \gamma Aw = 2k\nabla \times u + f_2, \quad \text{in } L^2(I; H^{-1}(\Omega)). \tag{2.44}$$

□

Theorem 2.3. *Assume that $(f_1, f_2) \in H \times L^2(\Omega)$. Then for any $(u_0, w_0) \in H \times L^2(\Omega)$ with $1 \le \beta < 7/3$, the weak solution $(u(t), w(t))$ to problem (1.1) in the sense of Theorem 2.2 is unique.*

Proof. Let (u_1, w_1) and (u_2, w_2) be weak solutions to problem (1.1) in the sense of Theorem 2.2, and let $(\bar{u}, \bar{w}) = (u_1 - u_2, w_1 - w_2)$, then $(\bar{u}, \bar{w})$ satisfies the equations

$$\begin{gathered}\frac{\partial \bar{u}}{\partial t} + B_0(u_1, u_1) - B_0(u_2, u_2) + (\nu + k)A\bar{u} + G(u_1) - G(u_2) = 2kP(\nabla \times \bar{w}),\\ \frac{\partial \bar{w}}{\partial t} + B_1(u_1, w_1) - B_1(u_2, w_2) + 4k\bar{w} + \gamma A\bar{w} = 2k(\nabla \times \bar{u}),\\ \bar{u}(0) = \bar{w}(0) = 0.\end{gathered} \tag{2.45}$$

Multiplying $(2.45)_1$ and $(2.45)_2$ by $\bar{u}$ and $\bar{w}$ respectively, and integrating by parts, we find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}\|_{L^2}^2 + (\nu + k)\|\nabla \bar{u}\|_{L^2}^2 + \int_\Omega (G(u_1) - G(u_2)) \cdot \bar{u}\,\mathrm{d}x + b(\bar{u}, u_1, \bar{u})\\ &= 2k\int_\Omega (\nabla \times \bar{w}) \cdot \bar{u}\,\mathrm{d}x,\end{aligned} \tag{2.46}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{w}\|_{L^2}^2 + 4k\|\bar{w}\|_{L^2}^2 + \gamma\|\nabla \bar{w}\|_{L^2}^2 + b(\bar{u}, w_2, \bar{w}) = 2k\int_\Omega (\nabla \times \bar{u}) \cdot \bar{w}\,\mathrm{d}x. \tag{2.47}$$

By [22, Lemma 2.2] and the proof of [6, Theorem 9], combining (2.46) with (2.47), we infer that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar{u}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2\right) + (\nu + k)\|\nabla \bar{u}\|_{L^2}^2 + \gamma\|\nabla \bar{w}\|_{L^2}^2 + C(\beta)\|\bar{u}\|_{L^{\beta+1}}^{\beta+1} + 4k\|\bar{w}\|_{L^2}^2\\ &\le |b(\bar{u}, \bar{u}, u_1)| + |b(\bar{u}, \bar{w}, w_2)| + 4k\int_\Omega (\nabla \times \bar{u}) \cdot \bar{w}\mathrm{d}x\\ &\le |b(\bar{u}, u_1, \bar{u})| + |b(\bar{u}, w_2, \bar{w})| + 4k\|\nabla \bar{u}\|_{L^2}\|\bar{w}\|_{L^2}\\ &\le C\|\nabla u_1\|_{L^2}\|\bar{u}\|_{L^2}\|\nabla \bar{u}\|_{L^2} + C\|\nabla w_2\|_{L^2}\|\bar{u}\|_{L^2}^{1/2}\|\nabla \bar{u}\|_{L^2}^{1/2}\|\bar{w}\|_{L^2}^{1/2}\|\nabla \bar{w}\|_{L^2}^{1/2} + 4k\|\nabla \bar{u}\|_{L^2}\|\bar{w}\|_{L^2}\\ &\le \frac{\nu + k}{6}\|\nabla \bar{u}\|_{L^2}^2 + C\|\nabla u_1\|_{L^2}^2\|\bar{u}\|_{L^2}^2 + \sqrt{\frac{\nu + k}{3}}\|\nabla \bar{u}\|_{L^2}\sqrt{\gamma}\|\nabla \bar{w}\|_{L^2}\\ &\quad + C\|\bar{u}\|_{L^2}\|\bar{w}\|_{L^2}\|\nabla w_2\|_{L^2}^2 + 4k\|\nabla \bar{u}\|_{L^2}\|\bar{w}\|_{L^2}\\ &\le \frac{\nu + k}{6}\|\nabla \bar{u}\|_{L^2}^2 + C\|\nabla u_1\|_{L^2}^2\|\bar{u}\|_{L^2}^2 + \frac{\nu + k}{6}\|\nabla \bar{u}\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla \bar{w}\|_{L^2}^2 + C\|\nabla w_2\|_{L^2}^2\|\bar{u}\|_{L^2}^2\\ &\quad + C\|\nabla w_2\|_{L^2}^2\|\bar{w}\|_{L^2}^2 + 4k\|\nabla \bar{u}\|_{L^2}\|\bar{w}\|_{L^2}\\ &\le \frac{\nu + k}{6}\|\nabla \bar{u}\|_{L^2}^2 + C\|\nabla u_1\|_{L^2}^2\|\bar{u}\|_{L^2}^2 + \frac{\nu + k}{6}\|\nabla \bar{u}\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla \bar{w}\|_{L^2}^2 + C\|\nabla w_2\|_{L^2}^2\|\bar{u}\|_{L^2}^2\\ &\quad + C\|\nabla w_2\|_{L^2}^2\|\bar{w}\|_{L^2}^2 + \frac{\nu + k}{6}\|\nabla \bar{u}\|_{L^2}^2 + C\|\bar{w}\|_{L^2}^2,\end{aligned}$$

which implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar{u}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2\right) \le C(1 + \|\nabla u_1\|_{L^2}^2 + \|\nabla w_2\|_{L^2}^2)(\|\bar{u}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2). \tag{2.48}$$

Integrating (2.48) from 0 to t, we obtain

$$\|\bar{u}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2 \le C\int_0^t (1 + \|\nabla u_1\|_{L^2}^2 + \|\nabla w_2\|_{L^2}^2)(\|\bar{u}\|_{L^2}^2 + \|\bar{w}\|_{L^2}^2)\,\mathrm{d}s. \tag{2.49}$$

Applying Gronwall's inequality to (2.49) and combining with (2.13) and (2.15), we deduce that

$$u_1(t) = u_2(t), \quad w_1(t) = w_2(t), \quad \forall t \in [0, T]. \tag{2.50}$$

□

Based on the well-posedness of weak solutions to the system (1.1) established in Theorem 2.2 and Theorem 2.3, and by referring to [1, Corollary 2.1], we obtain the following results.

Corollary 2.4. *Suppose that $(f_1, f_2) \in H \times L^2(\Omega)$, $(u_{0m}, w_{0m}) \rightharpoonup (u_0, w_0) \in H \times L^2(\Omega)$, let $\{(u_m(t), w_m(t))\}_{m=1}^\infty$ be a sequence of weak solutions of problem* (1.1) *such that $(u_m(0), w_m(0)) = (u_{0m}, w_{0m})$. For any $T > 0$, if there exists a subsequence of $\{(u_m(t), w_m(t))\}_{m=1}^\infty$ that converging $(*-)$ weakly to a certain function $(u(t), w(t))$ in the space $\{(u(t), w(t)) \in (L^\infty(0, T; H) \cap$*

$L^2(0,T;V)\cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega)))\times(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))) : (u_t,w_t)\in L^2(0,T;V^{-1}\times H^{-1}(\Omega))\}$, then $(u(t),w(t))$ is a weak solution on $[0,T]$ with $(u(0),w(0))=(u_0,w_0)$.

Now we present some definitions and results for general dynamical systems [18]. Considering a nonlinear system of differential equations that can be formulated as an abstract evolutionary problem,

$$\begin{gathered}\frac{\mathrm{d}u(t)}{\mathrm{d}t}=F(u(t)),\quad \text{in } X\ (t>0),\\ u(0)=u_0,\end{gathered}\tag{2.51}$$

where X is an Banach space, $F:X\to X$ is a nonlinear operator, and $u_0\in X$. Assume that for each $u_0\in X$, the system (2.49) admits a unique solution $u(t)\in X$ on $t\geq 0$. Then the solutions of system (2.49) define a solution semigroup $\{S(t)\}_{t\geq 0}$ such that $S(t)u_0$ is the set of all solutions with initial value u_0, that is, $S(t)u_0=\{u(t;u_0)\}$, and $S(t)u_0\subset X$ for all $u_0\in X$ and $t\geq 0$. Furthermore, no matter how complex the initial states of the system are, after evolving for a sufficiently long time, all solutions will fall into a specific set, which is the global attractor $\mathcal{A}$ possessed by $\{S(t)\}_{t\geq 0}$, that is to say, $\mathcal{A}$ is the long-term convergence set of all solutions of the original system.

Lemma 2.5 ([5, 17, 18, 19])**.** *Let $p_1\in(1,\infty]$, $p_2\in[1,\infty)$. Let X be a Banach space, Y and Z be separable and reflexive Banach spaces satisfying $Y\hookrightarrow\hookrightarrow X\hookrightarrow Z$. Then for any $\ell\in(0,\infty)$,*

$$\{u\in L^{p_1}(0,\ell;Y);\ u'\in L^{p_2}(0,\ell;Z)\}\hookrightarrow\hookrightarrow L^{p_1}(0,\ell;X).$$

Definition 2.6 ([16])**.** Let X be a Banach space, and $\{S(t)\}_{t\geq 0}$ be a family of operators on X. We call that $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous semigroup on X, if the following conditions hold

(I) $S(0)=\mathrm{Id}$ (the identity);
(II) $S(t)S(s)=S(t+s)$, for any $t,s\geq 0$;
(III) $S(t_n)x_n\rightharpoonup S(t)x$, if $t_n\to t$ and $x_n\to x$ in X.

Definition 2.7 ([18, 20])**.** Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Banach space X. A set $\mathcal{A}\subset X$ is called a global attractor of the dynamical system, if $\mathcal{A}$ satisfies that

(I) $\mathcal{A}$ is compact in X;
(II) $\mathcal{A}$ is strictly invariant, i.e., $S(t)\mathcal{A}=\mathcal{A}$, for all $t\geq 0$;
(III) $\mathcal{A}$ attracts all bounded sets in X, i.e., for any bounded set $B\subset X$, $\operatorname{dist}_X(S(t)B,\mathcal{A})\to 0$ as $t\to\infty$, where dist_X denotes the Hausdorff semi-distance in X, which is defined as

$$\operatorname{dist}_X(B,A)=\sup_{b\in B}\inf_{a\in A}\|b-a\|_X,\quad \text{for any } A,B\subset X.$$

Lemma 2.8 ([18, 20])**.** *Let $(S(t),X)$ be a dynamical system and X be a Banach space. Assume that there exists a compact set $X_0\subset X$, which is uniformly absorbing and positively invariant with respect to $S(t)$. Moreover, let $S(t)$ be continuous on X_0. Then $(S(t),X)$ has a global attractor.*

Definition 2.9 ([18])**.** For any non-empty compact subset $X_0\subset X$, the fractal dimension of X_0 is the number

$$d_f^X(X_0)=\limsup_{\epsilon\to 0^+}\frac{\log N_\epsilon^X(X_0)}{\log\left(\frac{1}{\epsilon}\right)},$$

where $N_\epsilon^X(X_0)$ is the minimum number of open balls in X with radius $\epsilon>0$ that are necessary to cover X_0.

Lemma 2.10 ([18])**.** *Let X,K be normed spaces such that $K\hookrightarrow\hookrightarrow X$ and $\mathcal{A}\subset X$ is bounded. Assume that there exists a mapping L such that $L(\mathcal{A})\subset\mathcal{A}$ and $L:X\to K$ is Lipschitz continuous on $\mathcal{A}$. Then $d_f^X(\mathcal{A})$ is finite.*

Lemma 2.11 ([18])**.** *Let X and Y be metric spaces and $F:X\to Y$ be α-Hölder continuous on the subset $\mathcal{A}\subset X$. Then*

$$d_f^Y(F(\mathcal{A}))\leq\frac{1}{\alpha}d_f^X(\mathcal{A}).$$

In particular, if the mapping F is Lipschitz continuous, then the fractal dimension of its image is finite.

3. Existence of a global attractor in X_ℓ

In this section, we transfer the phase space $H \times L^2(\Omega)$ of the solutions of problem (1.1) to the trajectory space X_ℓ for consideration.

It follows from Theorem 2.2 and Theorem 2.3, we can infer that problem (1.1) admits a unique weak solution $(u(t), w(t))$, and this weak solution satisfies $(u(x,0), w(x,0)) = (u_0, w_0)$ in $H \times L^2(\Omega)$. Thus, we define a $(H \times L^2(\Omega), H \times L^2(\Omega))$-continuous semigroup $\{S(t)\}_{t \geq 0}$ governed by problem (1.1) in the space $H \times L^2(\Omega)$, which satisfies

$$S(t)(u_0, w_0) = (u(t), w(t)) = (u(t; (u_0, w_0)), w(t; (u_0, w_0)))$$

for all $t \geq 0$. Since for any $(u_0, w_0) \in H \times L^2(\Omega)$ and $\ell > 0$, there exists at most one solution $(u(t), w(t)) \in \mathcal{C}_w([0, \ell]; H \times L^2(\Omega))$ satisfying the initial condition $(u(0), w(0)) = (u_0, w_0) \in H \times L^2(\Omega)$. To this end, we introduce the space X_ℓ, which is the set of all solution trajectories $\chi(\tau, (u_0, w_0)) = (u, w)\big|_{[0,\ell]}$ in $L^2(0, \ell; H \times L^2(\Omega))$ in the sense of the norm topology. Since $X_\ell \subset \mathcal{C}_w([0, \ell]; H \times L^2(\Omega))$, it makes it meaningful to discuss the point values of the trajectories. Moreover, since it has not been proved whether the trajectory space X_ℓ is closed in $L^2(0, \ell; H \times L^2(\Omega))$, in general, X_ℓ is not necessarily a complete metric space.

For any $\chi \in X_\ell$, we define the mapping $e_\epsilon : X_\ell \to H \times L^2(\Omega)$ by

$$e_\epsilon(\chi) = \chi(\epsilon \ell)$$

for any $\epsilon \in [0, 1]$. The mapping $b : H \times L^2(\Omega) \to X_\ell$ satisfies

$$b((u_0, w_0)) = (u, w)(\tau; (u_0, w_0)) = S(\tau)(u_0, w_0), \quad \tau \in [0, \ell]$$

for any $(u_0, w_0) \in H \times L^2(\Omega)$.

The operators $L_t : X_\ell \to X_\ell$ are given by the relation

$$L_t(\chi(\tau, (u_0, w_0))) = (u, w)(t + \tau, (u_0, w_0)), \ \tau \in [0, \ell]$$

for any $(u_0, w_0)) \in H \times L^2(\Omega)$.

First, we obtain the existence of a positively invariant subset of the solution operator in $H \times L^2(\Omega)$ by establishing some a priori estimates for the solutions of problem (1.1).

Theorem 3.1. *Assume that $(f_1, f_2) \in H \times L^2(\Omega)$, $1 \leq \beta < 7/3$. Then there exist positive constants ρ_1 and ρ_2 such that for any bounded subset $B \subset H \times L^2(\Omega)$, there exists a time $\tau_1 = \tau_1(B) > 0$ depending on B such that for any weak solution $(u(t), w(t))$ of problem (1.1) with initial value $(u_0, w_0) \in B$, we have*

$$\begin{gathered} \|u(t)\|_H^2 + \|w(t)\|_{L^2}^2 \leq \rho_1, \\ \int_0^\ell \Big(\|u(t+r)\|_{L^2}^2 + \|w(t+r)\|_{L^2}^2 \Big) \mathrm{d}r \leq \rho_2, \end{gathered}$$

for any $t \geq \tau_1$, where

$$\rho_1 := 1 + \frac{\alpha_2}{\alpha_1 \lambda^2} \left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 \right), \quad \rho_2 := \frac{1}{\alpha_1 \lambda} + \left(\frac{\alpha_2}{\alpha_1^2 \lambda^3} + \frac{\alpha_2 \ell}{\alpha_1 \lambda^2} \right) \left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 \right).$$

Proof. Taking $\varphi = u$ and $\psi = w$ in equations (2.2) and (2.3) respectively, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2}^2 + (\nu + k) \|\nabla u\|_{L^2}^2 + \sigma \|u\|_{L^{\beta+1}}^{\beta+1} \\ &= 2k(\nabla \times w, u) + (f_1, u) \\ &= 2k(\nabla \times u, w) + (f_1, u) \\ &\leq 2k\|w\|_{L^2}^2 + \frac{k}{2} \|\nabla u\|_{L^2}^2 + \frac{\nu}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2\nu \lambda_{1,u}} \|f_1\|_{L^2}^2 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + 4k\|w\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 &= 2k(\nabla\times u, w) + (f_2, w)\\ &\le \frac{k}{2}\|\nabla u\|_{L^2}^2 + 2k\|w\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla w\|_{L^2}^2 + \frac{1}{2\gamma\lambda_{1,w}}\|f_2\|_{L^2}^2,\end{aligned} \tag{3.2}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + (\nu+k)\|\nabla u\|_{L^2}^2 + 2\sigma\|u\|_{L^{\beta+1}}^{\beta+1} \le 4k\|w\|_{L^2}^2 + \frac{1}{\nu\lambda_{1,u}}\|f_1\|_{L^2}^2, \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + 4k\|w\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 \le k\|\nabla u\|_{L^2}^2 + \frac{1}{\gamma\lambda_{1,w}}\|f_2\|_{L^2}^2. \tag{3.4}$$

Combining (3.3) with (3.4), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \gamma\|\nabla w\|_{L^2}^2 + 2\sigma\|u\|_{L^{\beta+1}}^{\beta+1}\\ &\le \frac{1}{\nu\lambda_{1,u}}\|f_1\|_{L^2}^2 + \frac{1}{\gamma\lambda_{1,w}}\|f_2\|_{L^2}^2.\end{aligned} \tag{3.5}$$

From (3.5), by using the Poincaré's inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2\right) + \alpha_1\lambda\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2\right) \le \frac{\alpha_2}{\lambda}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right), \tag{3.6}$$

where $\alpha_1 = t\{\nu,\gamma\}$, $\alpha_2 = \max\{\frac{1}{\nu},\frac{1}{\gamma}\}$ and $\lambda = \min\{\lambda_{1,u},\lambda_{1,w}\}$.

Multiplying (3.6) by $e^{\alpha_1\lambda t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\alpha_1\lambda t}\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2\right)\right] \le \frac{\alpha_2}{\lambda}e^{\alpha_1\lambda t}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right). \tag{3.7}$$

By integrating (3.7) from 0 to t, we obtain

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \le e^{-\alpha_1\lambda t}\left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\right) + \frac{\alpha_2}{\alpha_1\lambda^2}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right). \tag{3.8}$$

From (3.8), we can deduce that for any bounded subset B of $H\times L^2(\Omega)$, there exists a time $\tau_0 = \tau_0(B) = \alpha_1^{-1}\lambda^{-1}\ln\left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\right) > 0$ such that for any $(u_0,w_0)\in B$, we have

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \le 1 + \frac{\alpha_2}{\alpha_1\lambda^2}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right) \tag{3.9}$$

for any $t\ge\tau_0$. Furthermore, by integrating (3.7) from t to $t+r$, we obtain

$$\|u(t+r)\|_{L^2}^2 + \|w(t+r)\|_{L^2}^2 \le e^{-\alpha_1\lambda r}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\right) + \frac{\alpha_2}{\alpha_1\lambda^2}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right). \tag{3.10}$$

Combining with (3.9), by integrating (3.10) with respect to r over $(0,\ell)$, we obtain

$$\begin{aligned}&\int_0^\ell\left(\|u(t+r)\|_{L^2}^2 + \|w(t+r)\|_{L^2}^2\right)\mathrm{d}r\\ &\le \frac{1}{\alpha_1\lambda}\left(1 - e^{-\alpha_1\lambda\ell}\right)\left[1 + \frac{\alpha_2}{\alpha_1\lambda^2}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right)\right] + \frac{\alpha_2\ell}{\alpha_1\lambda^2}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right)\\ &\le \frac{1}{\alpha_1\lambda} + \left(\frac{\alpha_2}{\alpha_1^2\lambda^3} + \frac{\alpha_2\ell}{\alpha_1\lambda^2}\right)\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right).\end{aligned} \tag{3.11}$$

Therefore, for any bounded subset $B\subset H\times L^2(\Omega)$, there exists $\tau_1 = 2\alpha_1^{-1}\lambda^{-1}\ln\|B\|_{H\times L^2(\Omega)} \ge \tau_0$ such that for any $(u_0,w_0)\in B$ and $t\ge\tau_1$, we have

$$\int_0^\ell\left(\|u(t+r)\|_{L^2}^2 + \|w(t+r)\|_{L^2}^2\right)\mathrm{d}r \le \frac{1}{\alpha_1\lambda} + \left(\frac{\alpha_2}{\alpha_1^2\lambda^3} + \frac{\alpha_2\ell}{\alpha_1\lambda^2}\right)\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right), \tag{3.12}$$

where $\|B\|_{H\times L^2(\Omega)} := \sup_{(u_0,w_0)\in B}\left(\|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2\right)$. $\square$

Let

$$B_0 = \{(u,w) \in H \times L^2(\Omega) : \|u(t)\|_H^2 + \|w(t)\|_{L^2}^2 \le \rho_1\}$$

be a bounded subset of $H \times L^2(\Omega)$. By Theorem 3.1, for any weak solution of problem (1.1) with initial value $(u_0, w_0) \in B_0$, there exists a time $t_0 = t_0(B) \ge 0$ such that $(u(t), w(t)) \in B_0$, where $(u(t), w(t))$ is the solution of problem (1.1) with initial data $(u_0, w_0) \in B_0$, for any $t \ge t_0$. Then, we obtain

$$S(t)B_0 \subset B_0$$

for any $t \ge t_0$.

We define

$$B_1 = \cup_{t\in[0,t_0]} S(t)B_0, \quad B_2 = \overline{B_1}^{H\times L^2(\Omega)},$$
$$B_0^\ell = \{\chi \in X_\ell : e_0(\chi) \in B_2\},$$

from the proof of the bounded absorbing set in Theorem 3.1, we have

$$S(t)B_2 \subset B_2, \quad \text{and} \quad L_t B_0^\ell \subset B_0^\ell$$

for any $t \ge 0$, and B_2 is a bounded subset of $H \times L^2(\Omega)$. Moreover,we obtain the following conclusion by Theorem 3.1.

Corollary 3.2. *Suppose $(f_1, f_2) \in H \times L^2(\Omega), 1 \le \beta < \frac{7}{3}$. Then for any bounded subset $B_0^\ell \subset X_\ell$, there exists a time $\tau_2 = \tau_2(B_0^\ell) > 0$ such that for any weak solution $(u(t), w(t))$ of problem (1.1) with ℓ-trajectory $\chi \in B_0^\ell \subset X_\ell$, it satisfies*

$$\|u(t)\|_H^2 + \|w(t)\|_{L^2}^2 \le \rho_1,$$
$$\int_0^\ell \left(\|u(t+r)\|_{L^2}^2 + \|w(t+r)\|_{L^2}^2\right) \mathrm{d}r \le \rho_2$$

for all $t \ge \tau_2$.

Next, we establish the existence of a positively invariant compact absorbing set in X_ℓ of the semigroup $\{L_t\}_{t\ge0}$.

Theorem 3.3. *Suppose $(f_1, f_2) \in H \times L^2(\Omega), 1 \le \beta < \frac{7}{3}$. Then for any bounded subset $B_0^\ell \subset X_\ell$, there exist positive constants ρ_3, ρ_4 and a time $\tau_3 = \tau_3(B_0^\ell) > 0$ such that for the ℓ-trajectory $\chi \in B_0^\ell \subset X_\ell$ corresponding to any weak solution of problem (1.1), we have*

$$\int_0^\ell \|\nabla u(t+\tau)\|_{L^2}^2 \mathrm{d}\tau + \int_0^\ell \|\nabla w(t+\tau)\|_{L^2}^2 \mathrm{d}\tau + \int_0^\ell \|u(t+\tau)\|_{L^{\beta+1}}^{\beta+1} \mathrm{d}\tau \le \rho_3,$$
$$\int_0^\ell \|u_t(t+\tau)\|_{V^{-1}}^2 \mathrm{d}\tau + \int_0^\ell \|w_t(t+\tau)\|_{H^{-1}(\Omega)}^2 \mathrm{d}\tau \le \rho_4$$

for any $t \ge \tau_3$.

Proof. By Theorem 3.1 and Corollary 3.2, we obtain that there exists a time $\tau_2 = \tau_2(B_0^\ell)$ such that

$$\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \le 1 + \frac{\alpha_2}{\alpha_1\lambda^2}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right) \tag{3.13}$$

for all $t \ge \tau_2$.

For each $t \ge \tau_2 + \frac{\ell}{2}$, combining (3.13) and integrating (3.5) from $t-s$ to $t+\ell$, where $s \in \left(0, \frac{\ell}{2}\right)$, we obtain

$$\begin{aligned}
&\|u(t+\ell)\|_{L^2}^2 + \|w(t+\ell)\|_{L^2}^2 + \nu\int_{t-s}^{t+\ell} \|\nabla u\|_{L^2}^2 \mathrm{d}r \\
&+ \gamma\int_{t-s}^{t+\ell} \|\nabla w\|_{L^2}^2 \mathrm{d}r + 2\sigma\int_{t-s}^{t+\ell} \|u\|_{L^{\beta+1}}^{\beta+1} \mathrm{d}r \\
&\le \|u(t-s)\|_{L^2}^2 + \|w(t-s)\|_{L^2}^2 + \frac{3\alpha_2\ell}{2\lambda}\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right) \\
&\le 1 + \left(\frac{\alpha_2}{\alpha_1\lambda^2} + \frac{3\alpha_2\ell}{2\lambda}\right)\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right).
\end{aligned} \tag{3.14}$$

We denote $\tau_3 = \tau_2 + \frac{\ell}{2}$. From (3.14), we have

$$\begin{aligned}&\int_{t-s}^{t+\ell} \|\nabla u\|_{L^2}^2 \mathrm{d}r + \int_{t-s}^{t+\ell} \|\nabla w\|_{L^2}^2 \mathrm{d}r + \int_{t-s}^{t+\ell} \|u\|_{L^{\beta+1}}^{\beta+1} \mathrm{d}r \\ &\le \frac{1}{\delta} + \Big(\frac{\alpha_2}{\alpha_1\delta\lambda^2} + \frac{3\alpha_2\ell}{2\delta\lambda}\Big)\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right)\end{aligned} \tag{3.15}$$

for each $t \ge \tau_3$, where $\delta = \min\{\nu, \gamma, 2\sigma\}$. Thus, we obtain

$$\int_0^\ell \|\nabla u(t+\tau)\|_{L^2}^2 \mathrm{d}\tau + \int_0^\ell \|\nabla w(t+\tau)\|_{L^2}^2 \mathrm{d}\tau + \int_0^\ell \|u(t+\tau)\|_{L^{\beta+1}}^{\beta+1} \mathrm{d}\tau \le \rho_3,$$

where

$$\rho_3 = \frac{1}{\delta} + \Big(\frac{\alpha_2}{\alpha_1\delta\lambda^2} + \frac{3\alpha_2\ell}{2\delta\lambda}\Big)\left(\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2\right).$$

By (2.1), we have

$$u_t = -B_0(u,u) - (\nu+k)Au - G(u) + 2kP(\nabla\times w) + Pf_1, \tag{3.16}$$

$$w_t = -B_1(u,w) - 4kw - \gamma Aw + 2k(\nabla\times u) + f_2. \tag{3.17}$$

For (3.16), by the proof of [6, Theorem 8], using Hölder's inequality, we deduce that when $1 \le \beta \le 5/3$

$$\begin{aligned}\|u_t\|_{V^{-1}} &\le C_0\|u\|_{L^2}\|\nabla u\|_{L^2} + (\nu+k)\|\nabla u\|_{L^2} + C_1\|u\|_{L^{\frac{6}{5}\beta}}^{\beta} + 2k\|\nabla w\|_{L^2} + \|f_1\|_{L^2} \\ &\le C_0\|u\|_{L^2}\|\nabla u\|_{L^2} + (\nu+k)\|\nabla u\|_{L^2} + C_1\|u\|_{L^2}^{\beta} + 2k\|\nabla w\|_{L^2} + \|f_1\|_{L^2}.\end{aligned} \tag{3.18}$$

For (3.18), we obtain

$$\begin{aligned}\|u_t\|_{L^2(t,t+\ell;V^{-1})} &\le C_0\|u\|_{L^\infty(t,t+\ell;H)}\|u\|_{L^2(t,t+\ell;V)} + (\nu+k)\|u\|_{L^2(t,t+\ell;V)} \\ &\quad + C_1\ell^{1/2}\|u\|_{L^\infty(t,t+\ell;H)}^{\beta} + 2k\|w\|_{L^2(t,t+\ell;H_0^1(\Omega))} + \ell^{1/2}\|f_1\|_{L^2(\Omega)}.\end{aligned} \tag{3.19}$$

As $5/3 < \beta < 7/3$, using the Gagliardo-Nirenberg's inequality and Young's inequality, we have

$$\begin{aligned}\|u_t\|_{V^{-1}} &\le C_0\|u\|_{L^2}\|\nabla u\|_{L^2} + (\nu+k)\|\nabla u\|_{L^2} + C_1\|u\|_{L^{\frac{6}{5}\beta}}^{\beta} + 2k\|\nabla w\|_{L^2} + \|f_1\|_{L^2} \\ &\le C_0\|u\|_{L^2}\|\nabla u\|_{L^2} + (\nu+k)\|\nabla u\|_{L^2} + C\|u\|_{L^{\beta+1}}^{\frac{(\beta+1)(3\beta-5)}{3(\beta-1)}}\|u\|_{L^2}^{\frac{5-\beta}{3(\beta-1)}} \\ &\quad + 2k\|\nabla w\|_{L^2} + \|f_1\|_{L^2} \\ &\le C_0\|u\|_{L^2}\|\nabla u\|_{L^2} + (\nu+k)\|\nabla u\|_{L^2} + C\|u\|_{L^{\beta+1}}^{\frac{\beta+1}{2}} + C\|u\|_{L^2}^{\frac{5-\beta}{7-3\beta}} \\ &\quad + 2k\|\nabla w\|_{L^2} + \|f_1\|_{L^2},\end{aligned} \tag{3.20}$$

noting that

$$\|u\|_{L^{\frac{6}{5}\beta}} \le C\|u\|_{L^{\beta+1}}^{\frac{(\beta+1)(3\beta-5)}{3\beta(\beta-1)}}\|u\|_{L^2}^{\frac{5-\beta}{3\beta(\beta-1)}},$$

where $\frac{(\beta+1)(3\beta-5)}{3\beta(\beta-1)} \to 0^+$ for $\beta \to \frac{5}{3}^+$, and $\frac{(\beta+1)(3\beta-5)}{3\beta(\beta-1)} \to \frac{5}{7}^-$ for $\beta \to \frac{7}{3}^-$. Therefore, we always have $\frac{(\beta+1)(3\beta-5)}{3\beta(\beta-1)} \in (0,1)$ as $\beta \in (\frac{5}{3}, \frac{7}{3})$.

For (3.20), we obtain

$$\begin{aligned}\|u_t\|_{L^2(t,t+\ell;V^{-1})} &\le C_0\|u\|_{L^\infty(t,t+\ell;H)}\|u\|_{L^2(t,t+\ell;V)} + (\nu+k)\|u\|_{L^2(t,t+\ell;V)} \\ &\quad + C\|u\|_{L^{\beta+1}(t,t+\ell;L^{\beta+1}(\Omega))}^{\frac{\beta+1}{2}} + C\ell^{1/2}\|u\|_{L^\infty(t,t+\ell;H)}^{\frac{2(5-\beta)}{7-3\beta}} \\ &\quad + 2k\|w\|_{L^2(t,t+\ell;H_0^1(\Omega))} + \ell^{1/2}\|f_1\|_{L^2(\Omega)}.\end{aligned} \tag{3.21}$$

For (3.17), by the proof of [6, Theorem 8], using Hölder's inequality, we deduce that

$$\begin{aligned}&\|w_t\|_{H^{-1}(\Omega)} \\ &\le C_2\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|w\|_{L^2}^{1/2}\|\nabla w\|_{L^2}^{1/2} + 4k\|w\|_{L^2} + \gamma\|\nabla w\|_{L^2} + 2k\|\nabla u\|_{L^2} + \|f_2\|_{L^2} \\ &\le C\|u\|_{L^2}\|\nabla u\|_{L^2} + C\|w\|_{L^2}\|\nabla w\|_{L^2} + 4k\|w\|_{L^2} + \gamma\|\nabla w\|_{L^2} + 2k\|\nabla u\|_{L^2} + \|f_2\|_{L^2}.\end{aligned} \tag{3.22}$$

For (3.22), we obtain

$$
\begin{aligned}
&\|w_t\|_{L^2(t,t+\ell;H^{-1}(\Omega))} \\
&\le C\|u\|_{L^\infty(t,t+\ell;H)}\|u\|_{L^2(t,t+\ell;V)} + C\|w\|_{L^\infty(t,t+\ell;L^2(\Omega))}\|w\|_{L^2(t,t+\ell;H_0^1(\Omega))} \\
&\quad + 4k\ell^{1/2}\|w\|_{L^\infty(t,t+\ell;L^2(\Omega))} + \gamma\|w\|_{L^2(t,t+\ell;H_0^1(\Omega))} + 2k\|u\|_{L^2(t,t+\ell;V)} + \ell^{1/2}\|f_2\|_{L^2(\Omega)}.
\end{aligned} \tag{3.23}
$$

Combining (3.19), (3.21) with (3.23), we conclude that there exists a positive constant ρ_4 such that

$$
\int_0^\ell \|u_t(t+\tau)\|_{V^{-1}}^2 \mathrm{d}\tau + \int_0^\ell \|w_t(t+\tau)\|_{H^{-1}(\Omega)}^2 \mathrm{d}\tau \le \rho_4 \tag{3.24}
$$

for any $t \ge \tau_3$. □

Let

$$
Y = \{\chi \in X_\ell : \chi \in L^2(0,\ell;V\times H_0^1(\Omega)),\ \chi_t \in L^2(0,\ell;V^{-1}\times H^{-1}(\Omega))\},
$$

endowed with the norm

$$
\|\chi\|_Y = \big\{\|\chi\|_{L^2(0,\ell;V\times H_0^1(\Omega))} + \|\chi_t\|_{L^2(0,\ell;V^{-1}\times H^{-1}(\Omega))}\big\}.
$$

From Corollary 3.2 and Theorem 3.3, we obtain $L_tB_0^\ell \subset B_0^\ell$ for any $t \ge 0$, and $L_tB_0^\ell \subset Y$ for any $t \ge \tau_3$.

We define

$$
B_1^\ell := \overline{L_{\tau_3}B_0^\ell}^{L^2(0,\ell;H\times L^2(\Omega))},
$$

where $\tau_3 = \tau_2 + \frac{\ell}{2}$.

Lemma 3.4. *Suppose $(f_1,f_2) \in H\times L^2(\Omega), 1 \le \beta < \frac{7}{3}$. Then $\overline{L_tB_0^\ell}^{L^2(0,\ell;H\times L^2(\Omega))} \subset B_0^\ell$, for any $t \ge 0$.*

The proof method for the above lemma is similar to that of [14, Lemma 3.5], so we omit it.

Lemma 3.5. *Assume that $(f_1,f_2) \in H\times L^2(\Omega), 1 \le \beta < \frac{7}{3}$. Then the mapping $L_t : X_\ell \to X_\ell$ is Lipschitz continuous on B_0^ℓ for all $t \ge 0$.*

Proof. For any fixed $t > 0$ and any $\chi^1,\chi^2 \in B_0^\ell$, let $L_t\chi^1 = (u_1(t+\tau), w_1(t+\tau))$, $L_t\chi^2 = (u_2(t+\tau), w_2(t+\tau))$ and let $(\bar u,\bar w) = (u_1-u_2, w_1-w_2)$, then $(\bar u,\bar w)$ satisfies (2.45) in the proof of Theorem 2.3. And the following inequality is valid:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar u\|_{L^2}^2 + \|\bar w\|_{L^2}^2\right) + (\nu+k)\|\nabla\bar u\|_{L^2}^2 + \gamma\|\nabla\bar w\|_{L^2}^2 + C(\beta)\|\bar u\|_{L^{\beta+1}}^{\beta+1} \le \eta(t)(\|\bar u\|_{L^2}^2 + \|\bar w\|_{L^2}^2), \tag{3.25}
$$

where $\eta(t) = C(1+\|\nabla u_1\|_{L^2}^2 + \|\nabla w_2\|_{L^2}^2)$. Integrating (3.25) from s to $t+s$, we deduce that for any $s \in (0,\ell)$ and any $t \ge 0$,

$$
\|\bar u(t+s)\|_{L^2}^2 + \|\bar w(t+s)\|_{L^2}^2 \le \int_s^{t+s} \eta(\tau)\left(\|\bar u(\tau)\|_{L^2}^2 + \|\bar w(\tau)\|_{L^2}^2\right)\mathrm{d}\tau + \|\bar u(s)\|_{L^2}^2 + \|\bar w(s)\|_{L^2}^2. \tag{3.26}
$$

By using Gronwall's lemma, we obtain

$$
\begin{aligned}
\|\bar u(t+s)\|_{L^2}^2 + \|\bar w(t+s)\|_{L^2}^2 &\le \left(\|\bar u(s)\|_{L^2}^2 + \|\bar w(s)\|_{L^2}^2\right)\exp\Big(\int_s^{t+s}\eta(\tau)\mathrm{d}\tau\Big) \\
&\le N_\ell(t)\left(\|\bar u(s)\|_{L^2}^2 + \|\bar w(s)\|_{L^2}^2\right),
\end{aligned} \tag{3.27}
$$

where $N_\ell(t) = \exp\big(\int_0^{t+\ell}\eta(\tau)\mathrm{d}\tau\big)$.

Since $e_0(\chi)$ is uniformly bounded in $H\times L^2(\Omega)$ for any $\chi \in B_0^\ell$, we conclude that $N_\ell(t) = \exp\Big(\int_0^{t+\ell}\eta(\tau)\mathrm{d}\tau\Big)$ is a finite number depending on $e_0(\chi) \in B_2$, for any $\ell \ge 0$.

Integrating (3.27) with respect to s from 0 to ℓ, we have

$$
\int_0^\ell \left(\|\bar u(t+s)\|_{L^2}^2 + \|\bar w(t+s)\|_{L^2}^2\right)\mathrm{d}s \le N_\ell(t)\int_0^\ell \left(\|\bar u(s)\|_{L^2}^2 + \|\bar w(s)\|_{L^2}^2\right)\mathrm{d}s. \tag{3.28}
$$

From (3.28), we conclude that

$$\|L_t\chi^1 - L_t\chi^2\|_{L^2(0,\ell;H\times L^2(\Omega))} \le N_\ell(t)\|\chi^1-\chi^2\|_{L^2(0,\ell;H\times L^2(\Omega))}. \tag{3.29}$$

Therefore, the mapping $L_t : X_\ell \to X_\ell$ is Lipschitz continuous on B_0^ℓ for all $t\ge 0$. □

By Corollary 3.2, Lemma 3.4, Lemma 3.5, and the invariance of B_2 under the action of $\{L_t\}_{t\ge0}$, we can deduce that $W = B_1^\ell \subset B_0^\ell$ is a compact, positively invariant, and uniformly bounded subset in X_ℓ. According to Lemma 2.8, we can get the existence of a global attractor in X_ℓ.

Theorem 3.6. *Assume that $(f_1, f_2) \in H \times L^2(\Omega), 1 \le \beta < \frac{7}{3}$. Then the semigroup $\{L_t\}_{t\ge0}$ generated by problem (1.1) possesses a global attractor $\mathcal{A}_\ell$ in X_ℓ and $e_\epsilon(\mathcal{A}_\ell)$ is uniformly bounded in $H\times L^2(\Omega)$ with respect to $\epsilon\in[0,1]$, where*

$$e_\epsilon(\mathcal{A}_\ell) = \{e_\epsilon(\chi) : \chi \in \mathcal{A}_\ell\}$$

for any $\epsilon \in [0,1]$.

Next, we prove the smooth property of the semigroup $\{L_t\}_{t\ge0}$ and thereby estimate the finite fractal dimension of the global attractor $\mathcal{A}_\ell$.

Lemma 3.7. *Assume that $(f_1, f_2) \in H\times L^2(\Omega)$, $1 \le \beta < 7/3$. Then there exist a positive constant ρ_5 such that for any bounded subset $B \subset H \times L^2(\Omega)$, there exists a time $\tau_4 = \tau_4(B) > 0$ depending on B such that for any weak solution $(u(t), w(t))$ of problem (1.1) with initial value $(u_0, w_0) \in B$, we have*

$$\|u\|_{L^\infty(t,t+\ell;V)} + \|w\|_{L^\infty(t,t+\ell;H_0^1(\Omega))} \le \rho_5$$

for any $t \ge \tau_4$.

Proof. Taking the inner product of $(1.1)_1$ and $(1.1)_2$ with Au and Aw in $H\times L^2(\Omega)$, respectively, and by Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) + 4k\|\nabla w\|_{L^2}^2 + (\nu+k)\|Au\|_{L^2}^2 + \gamma\|Aw\|_{L^2}^2\\
&+ \sigma\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4\sigma(\beta-1)}{(\beta+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2\\
&= -\int_\Omega (u\cdot\nabla u)\cdot Au\mathrm{d}x - \int_\Omega (u\cdot\nabla w)\cdot Aw\mathrm{d}x + 4k\int_\Omega(\nabla\times w)\cdot Au\mathrm{d}x\\
&\quad + \int_\Omega f_1\cdot Au\mathrm{d}x + \int_\Omega f_2\cdot Aw\mathrm{d}x\\
&\le \frac{1}{\nu}\int_\Omega |u|^2|\nabla u|^2\mathrm{d}x + \frac{\nu}{4}\|Au\|_{L^2}^2 + \frac{1}{2\gamma}\int_\Omega |u|^2|\nabla w|^2\mathrm{d}x + \frac{\gamma}{2}\|Aw\|_{L^2}^2 + 4k\|\nabla w\|_{L^2}^2\\
&+ k\|Au\|_{L^2}^2 + \frac{1}{\nu}\|f_1\|_{L^2}^2 + \frac{\nu}{4}\|Au\|_{L^2}^2 + \frac{1}{2\gamma}\|f_2\|_{L^2}^2 + \frac{\gamma}{2}\|Aw\|_{L^2}^2\\
&\le \frac{1}{\nu}\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \frac{1}{2\gamma}\|u\|_{L^\infty}^2\|\nabla w\|_{L^2}^2 + 4k\|\nabla w\|_{L^2}^2 + \left(\frac{\nu}{2}+k\right)\|Au\|_{L^2}^2 + \gamma\|Aw\|_{L^2}^2\\
&\quad + \frac{1}{\nu}\|f_1\|_{L^2}^2 + \frac{1}{2\gamma}\|f_2\|_{L^2}^2\\
&\le \frac{1}{\nu}\|u\|_{L^2}\|Au\|_{L^2}\|\nabla u\|_{L^2}^2 + \frac{1}{2\gamma}\|u\|_{L^2}\|Au\|_{L^2}\|\nabla w\|_{L^2}^2 + 4k\|\nabla w\|_{L^2}^2 + \left(\frac{\nu}{2}+k\right)\|Au\|_{L^2}^2\\
&\quad + \gamma\|Aw\|_{L^2}^2 + \frac{1}{\nu}\|f_1\|_{L^2}^2 + \frac{1}{2\gamma}\|f_2\|_{L^2}^2\\
&\le \frac{1}{\nu^3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4 + \frac{\nu}{4}\|Au\|_{L^2}^2 + \frac{1}{4\nu\gamma^2}\|u\|_{L^2}^2\|\nabla w\|_{L^2}^4 + \frac{\nu}{4}\|Au\|_{L^2}^2 + 4k\|\nabla w\|_{L^2}^2\\
&\quad + \left(\frac{\nu}{2}+k\right)\|Au\|_{L^2}^2 + \gamma\|Aw\|_{L^2}^2 + \frac{1}{\nu}\|f_1\|_{L^2}^2 + \frac{1}{2\gamma}\|f_2\|_{L^2}^2\\
&\le \frac{C}{\nu^3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4 + \frac{C}{\gamma^3}\|u\|_{L^2}^2\|\nabla w\|_{L^2}^4 + 4k\|\nabla w\|_{L^2}^2 + (\nu+k)\|Au\|_{L^2}^2 + \gamma\|Aw\|_{L^2}^2
\end{aligned}$$

$$+\frac{1}{\nu}\|f_1\|_{L^2}^2+\frac{1}{2\gamma}\|f_2\|_{L^2}^2,$$

which implies that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2\right)\\
&\le \frac{C}{\nu^3}\|u\|_{L^2}^2\|\nabla u\|_{L^2}^4+\frac{C}{\gamma^3}\|u\|_{L^2}^2\|\nabla w\|_{L^2}^4+\alpha_3(\|f_1\|_{L^2}^2+\|f_2\|_{L^2}^2)\\
&\le C(\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2+\|u\|_{L^2}^2\|\nabla w\|_{L^2}^2)(\|\nabla u\|_{L^2}^2+\|\nabla w\|_{L^2}^2)+\alpha_3(\|f_1\|_{L^2}^2+\|f_2\|_{L^2}^2)
\end{aligned}\tag{3.30}$$

where $\alpha_3=\max\{2/\nu,1/\gamma\}$.

With the aid of the uniform Gronwall's inequality, it can be inferred from the proofs of Theorem 3.1 and Theorem 3.3 that there exists a positive constant ρ_5 such that

$$\|u\|_{L^\infty(t,t+\ell;V)}+\|w\|_{L^\infty(t,t+\ell;H_0^1(\Omega))}\le\rho_5 \tag{3.31}$$

for any $t\ge\tau_4$. □

Theorem 3.8. *Assume that $(f_1,f_2)\in H\times L^2(\Omega)$, $1\le\beta<\frac{7}{3}$, and let χ^1, χ^2 be two ℓ-trajectories in $\mathcal{A}_\ell$. Then there exists a compact subset Y in X_ℓ such that for any $t\ge\ell$, the mapping $L_t: X_\ell\to Y$ is Lipschitz continuous on $\mathcal{A}_\ell$.*

Proof. By Lemma 2.5, we deduce that $Y\hookrightarrow\hookrightarrow X_\ell$. For any fixed $t>0$ and any $\chi^1,\chi^2\in\mathcal{A}_\ell$, let $L_t\chi^1=(u_1(t+\tau),w_1(t+\tau))$, $L_t\chi^2=(u_2(t+\tau),w_2(t+\tau))$ and let $(\bar u,\bar w)=(u_1-u_2,w_1-w_2)$. Since $e_\epsilon(\chi^1)$ and $e_\epsilon(\chi^2)$ are uniformly bounded in $H\times L^2(\Omega)$ with respect to $\epsilon\in[0,1]$ for any $\chi^1,\chi^2\in\mathcal{A}_\ell$, from the proof of Theorem 2.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\bar u\|_{L^2}^2+\|\bar w\|_{L^2}^2\right)+(\nu+k)\|\nabla\bar u\|_{L^2}^2+\gamma\|\nabla\bar w\|_{L^2}^2\le\eta(t)\left(\|\bar u\|_{L^2}^2+\|\bar w\|_{L^2}^2\right), \tag{3.32}$$

where $\eta(t)=C(1+\|\nabla u_1\|_{L^2}^2+\|\nabla w_2\|_{L^2}^2)$.

It follows from Gronwall's lemma that

$$\|\bar u(t+\ell)\|_{L^2}^2+\|\bar w(t+\ell)\|_{L^2}^2\le\left(\|\bar u(t-s)\|_{L^2}^2+\|\bar w(t-s)\|_{L^2}^2\right)\exp\Big(\int_{t-s}^{t+\ell}\eta(\tau)\mathrm{d}\tau\Big). \tag{3.33}$$

For any $s\in\left[0,\frac{\ell}{2}\right]$, integrating (3.32) from $t-s$ to $t+\ell$ with $t\ge\ell$, we conclude that

$$\begin{aligned}
&\|\bar u(t+\ell)\|_{L^2}^2+\|\bar w(t+\ell)\|_{L^2}^2+\kappa\int_{t-s}^{t+\ell}\left(\|\nabla\bar u(\tau)\|_{L^2}^2+\|\nabla\bar w(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\
&\le\int_{t-s}^{t+\ell}\eta(\tau)\left(\|\bar u(\tau)\|_{L^2}^2+\|\bar w(\tau)\|_{L^2}^2\right)\mathrm{d}\tau+\|\bar u(t-s)\|_{L^2}^2+\|\bar w(t-s)\|_{L^2}^2,
\end{aligned}\tag{3.34}$$

where $\kappa=\min\{\nu+k,\gamma\}>0$.

Considering (3.33) and (3.34), we obtain

$$\begin{aligned}
&\kappa\int_{t-s}^{t+\ell}\left(\|\nabla\bar u(\tau)\|_{L^2}^2+\|\nabla\bar w(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\
&\le\int_{t-s}^{t+\ell}\eta(\tau)\Big[\left(\|\bar u(t-s)\|_{L^2}^2+\|\bar w(t-s)\|_{L^2}^2\right)\exp\Big(\int_{t-s}^{\tau}\eta(\xi)\mathrm{d}\xi\Big)\Big]\mathrm{d}\tau\\
&\quad+\|\bar u(t-s)\|_{L^2}^2+\|\bar w(t-s)\|_{L^2}^2\\
&\le\Big[\left(\|\bar u(t-s)\|_{L^2}^2+\|\bar w(t-s)\|_{L^2}^2\right)\exp\Big(\int_{t-s}^{t+\ell}\eta(\xi)\mathrm{d}\xi\Big)\Big]\int_{\frac{\ell}{2}}^{t+\ell}\eta(\tau)\mathrm{d}\tau\\
&\quad+\|\bar u(t-s)\|_{L^2}^2+\|\bar w(t-s)\|_{L^2}^2.
\end{aligned}\tag{3.35}$$

Based on the above estimates, we obtain

$$\begin{aligned}&\|\bar{u}(t+\ell)\|_{L^2}^2+\|\bar{w}(t+\ell)\|_{L^2}^2+\kappa\int_{t-s}^{t+\ell}\left(\|\nabla\bar{u}(\tau)\|_{L^2}^2+\|\nabla\bar{w}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\&\le W_\ell(t)\left(\|\bar{u}(t-s)\|_{L^2}^2+\|\bar{w}(t-s)\|_{L^2}^2\right)\exp\Big(\int_{t-s}^{t+\ell}\eta(\tau)\mathrm{d}\tau\Big)\\&\quad+\|\bar{u}(t-s)\|_{L^2}^2+\|\bar{w}(t-s)\|_{L^2}^2,\end{aligned}\tag{3.36}$$

where $W_\ell(t)=\int_{\frac{\ell}{2}}^{t+\ell}\eta(\tau)\mathrm{d}\tau+1$.

Integrating (3.32) from s to $t-s$, for any $t\ge\ell$ and any $s\in[0,\ell/2]$, we deduce

$$\|\bar{u}(t-s)\|_{L^2}^2+\|\bar{w}(t-s)\|_{L^2}^2\le\int_s^{t-s}\eta(\tau)\left(\|\bar{u}(\tau)\|_{L^2}^2+\|\bar{w}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau+\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2.\tag{3.37}$$

Using Gronwall's lemma, we have

$$\begin{aligned}\|\bar{u}(t-s)\|_{L^2}^2+\|\bar{w}(t-s)\|_{L^2}^2&\le\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right)\exp\Big(\int_s^{t-s}\eta(\tau)\mathrm{d}\tau\Big)\\&\le\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right)\exp\Big(\int_0^{t-s}\eta(\tau)\mathrm{d}\tau\Big).\end{aligned}\tag{3.38}$$

Combining (3.36) with (3.38), we obtain

$$\begin{aligned}&\kappa\int_0^\ell\left(\|\nabla\bar{u}(t+\tau)\|_{L^2}^2+\|\nabla\bar{w}(t+\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\&\le W_\ell(t)\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right)\exp\Big(\int_0^{t+\ell}\eta(\tau)\mathrm{d}\tau\Big)\\&\quad+\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right)\exp\Big(\int_0^{t-s}\eta(\tau)\mathrm{d}\tau\Big)\\&\le 2W_\ell(t)N_\ell(t)\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right).\end{aligned}\tag{3.39}$$

Integrating (3.39) over $(0,\frac{\ell}{2})$ with respect to s, we obtain

$$\begin{aligned}&\int_0^\ell\left(\|\nabla\bar{u}(t+\tau)\|_{L^2}^2+\|\nabla\bar{w}(t+\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\&\le\frac{4W_\ell(t)N_\ell(t)}{\kappa\ell}\int_0^{\frac{\ell}{2}}\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right)\mathrm{d}s.\end{aligned}\tag{3.40}$$

Since $W_\ell(t)$ and $N_\ell(t)$ are bounded for any fixed $t\in[\ell,+\infty)$. Thus, we can infer that

$$\begin{aligned}&\int_0^\ell\left(\|\nabla\bar{u}(t+\tau)\|_{L^2}^2+\|\nabla\bar{w}(t+\tau)\|_{L^2}^2\right)\mathrm{d}\tau\\&\le\frac{4W_\ell(t)N_\ell(t)}{\kappa\ell}\int_0^{\ell}\left(\|\bar{u}(s)\|_{L^2}^2+\|\bar{w}(s)\|_{L^2}^2\right)\mathrm{d}s.\end{aligned}$$

From the above estimates, we obtain

$$\|L_t\chi^1-L_t\chi^2\|_{L^2(0,\ell;V\times H_0^1(\Omega))}\le\kappa_0(t,\ell)\|\chi^1-\chi^2\|_{L^2(0,\ell;H\times L^2(\Omega))}\tag{3.41}$$

for any $\chi^1,\chi^2\in\mathcal{A}_\ell$ and $t\ge\ell$.

Suppose (u_1,w_1) and (u_2,w_2) are two solutions of (2.1) corresponding to different initial values, substitute them into (2.1) respectively and take the difference to obtain

$$\bar{u}_t=-\left(B_0(u_1,u_1)-B_0(u_2,u_2)\right)-(\nu+k)A\bar{u}-\left(G(u_1)-G(u_2)\right)+2kP(\nabla\times\bar{w}),\tag{3.42}$$

$$\bar{w}_t=-\left(B_1(u_1,w_1)-B_1(u_2,w_2)\right)-4k\bar{w}-\gamma A\bar{w}+2\kappa(\nabla\times\bar{u}).\tag{3.43}$$

We deduce from the proof of [6, Theorem 9] that when $\beta=1$,

$$\|\bar{u}_t\|_{V^{-1}}\le\|\bar{u}\|_{L^4}\|u_1\|_{L^4}+\|u_2\|_{L^4}\|\bar{u}\|_{L^4}+(\nu+\kappa)\|\nabla\bar{u}\|_{L^2}+C\|\bar{u}\|_{L^2}+2k\|\nabla\bar{w}\|_{L^2}.\tag{3.44}$$

For (3.44), we infer from Lemma 3.7 that

$$\begin{aligned} \|\bar{u}_t\|_{L^2(t,t+\ell;V^{-1})} \le\ & C\|u_1\|_{L^\infty(t,t+\ell;V)}\|\bar{u}\|_{L^2(t,t+\ell;V)} + C\|u_2\|_{L^\infty(t,t+\ell;V)}\|\bar{u}\|_{L^2(t,t+\ell;V)} \\ & + C(\nu+k)\|\bar{u}\|_{L^2(t,t+\ell;V)} + C\|\bar{u}\|_{L^2(t,t+\ell;V)} + Ck\|\bar{w}\|_{L^2(t,t+\ell;H_0^1(\Omega))}. \end{aligned} \tag{3.45}$$

As $1 < \beta < 7/3$, using Hölder's inequality, we deduce from [22, Lemma 2.2] that

$$\begin{aligned} \|\bar{u}_t\|_{V^{-1}} \le\ & \|\bar{u}\|_{L^4}\|u_1\|_{L^4} + \|u_2\|_{L^4}\|\bar{u}\|_{L^4} + (\nu+k)\|\nabla\bar{u}\|_{L^2} + C\|(|u_1|+|u_2|)^{\beta-1}\|_{L^{\frac{3}{2}}}\|\bar{u}\|_{L^6} \\ & + 2k\|\nabla\bar{w}\|_{L^2} \\ \le\ & \|\bar{u}\|_{L^4}\|u_1\|_{L^4} + \|u_2\|_{L^4}\|\bar{u}\|_{L^4} + (\nu+k)\|\nabla\bar{u}\|_{L^2} + C\||u_1|+|u_2|\|^{\beta-1}_{L^{\frac{3}{2}(\beta-1)}}\|\nabla\bar{u}\|_{L^2} \\ & + 2k\|\nabla\bar{w}\|_{L^2} \\ \le\ & \|\bar{u}\|_{L^4}\|u_1\|_{L^4} + \|u_2\|_{L^4}\|\bar{u}\|_{L^4} + (\nu+k)\|\nabla\bar{u}\|_{L^2} + C\||u_1|\|^{\beta-1}_{L^{\frac{3}{2}(\beta-1)}}\|\nabla\bar{u}\|_{L^2} \\ & + C\||u_2|\|^{\beta-1}_{L^{\frac{3}{2}(\beta-1)}}\|\nabla\bar{u}\|_{L^2} + 2k\|\nabla\bar{w}\|_{L^2} \\ \le\ & \|\bar{u}\|_{L^4}\|u_1\|_{L^4} + \|u_2\|_{L^4}\|\bar{u}\|_{L^4} + (\nu+k)\|\nabla\bar{u}\|_{L^2} + C\|u_1\|^{\beta-1}_{L^2}\|\nabla\bar{u}\|_{L^2} \\ & + C\|u_2\|^{\beta-1}_{L^2}\|\nabla\bar{u}\|_{L^2} + 2k\|\nabla\bar{w}\|_{L^2}. \end{aligned} \tag{3.46}$$

For (3.46), we infer from Lemma 3.7 that

$$\begin{aligned} \|\bar{u}_t\|_{L^2(t,t+\ell;V^{-1})} \le\ & C\|u_1\|_{L^\infty(t,t+\ell;V)}\|\bar{u}\|_{L^2(t,t+\ell;V)} + C\|u_2\|_{L^\infty(t,t+\ell;V)}\|\bar{u}\|_{L^2(t,t+\ell;V)} \\ & + C(\nu+k)\|\bar{u}\|_{L^2(t,t+\ell;V)} + C\|u_1\|^{\beta-1}_{L^\infty(t,t+\ell;H)}\|\bar{u}\|_{L^2(t,t+\ell;V)} \\ & + C\|u_2\|^{\beta-1}_{L^\infty(t,t+\ell;H)}\|\bar{u}\|_{L^2(t,t+\ell;V)} + Ck\|\bar{w}\|_{L^2(t,t+\ell;H_0^1(\Omega))}. \end{aligned} \tag{3.47}$$

Similarly, we deduce that

$$\|\bar{w}_t\|_{H^{-1}(\Omega)} \le \|\bar{u}\|_{L^4}\|w_1\|_{L^4} + \|u_2\|_{L^4}\|\bar{w}\|_{L^4} + 4k\|\bar{w}\|_{L^2} + \gamma\|\nabla\bar{w}\|_{L^2} + 2k\|\nabla\bar{u}\|_{L^2}. \tag{3.48}$$

For (3.48), we conclude from Lemma 3.7 that

$$\begin{aligned} & \|\bar{w}_t\|_{L^2(t,t+\ell;H^{-1}(\Omega))} \\ & \le C\|w_1\|_{L^\infty(t,t+\ell;H_0^1(\Omega))}\|\bar{u}\|_{L^2(t,t+\ell;V)} + C\|u_2\|_{L^\infty(t,t+\ell;V)}\|\bar{w}\|_{L^2(t,t+\ell;H_0^1(\Omega))} \\ & \quad + 4kC\|\bar{w}\|_{L^2(t,t+\ell;H_0^1(\Omega))} + \gamma\|\bar{w}\|_{L^2(t,t+\ell;H_0^1(\Omega))} + Ck\|\bar{u}\|_{L^2(t,t+\ell;V)}. \end{aligned} \tag{3.49}$$

Combining (3.45), (3.47) with (3.49), we conclude that there exists a positive constant $\kappa_1(t,\ell)$ such that

$$\int_0^\ell \Big(\|\bar{u}_t(t+\tau)\|^2_{V^{-1}} + \|\bar{w}_t(t+\tau)\|^2_{H^{-1}(\Omega)}\Big)\,\mathrm{d}\tau \le \kappa_1(t,\ell)\int_0^\ell \Big(\|\bar{u}(s)\|^2_V + \|\bar{w}(s)\|^2_{H_0^1(\Omega)}\Big)\,\mathrm{d}s. \tag{3.50}$$

Combining (3.41) with (3.50), we have

$$\|(L_t\chi^1 - L_t\chi^2)'\|_{L^2(0,\ell;V^{-1}\times H^{-1}(\Omega))} \le \kappa_2(t,\ell)\|\chi^1-\chi^2\|_{L^2(0,\ell;H\times L^2(\Omega))} \tag{3.51}$$

for all $\chi^1,\chi^2 \in \mathcal{A}_\ell$ and $t \ge \ell$, where $\kappa_2(t,\ell) = \kappa_0(t,\ell)\cdot\kappa_1(t,\ell)$. □

Combining with Lemma 2.10, Theorem 3.6 and Theorem 3.8, we obtain the following result.

Theorem 3.9. *Assume that $(f_1,f_2) \in H \times L^2(\Omega)$, $1 \le \beta < \frac{7}{3}$. Then the fractal dimension of the global attractor $\mathcal{A}_\ell$ in X_ℓ for the semigroup $\{L_t\}_{t\ge0}$ generated by problem* (1.1) *established in Theorem 3.6 is finite.*

4. Existence of a global attractor in $H \times L^2(\Omega)$

In this section, we prove that the semigroup $\{S(t)\}_{t\ge0}$ generated by problem (1.1) admits a global attractor $\mathcal{A}$ with finite fractal dimension in the initial phase space $H \times L^2(\Omega)$.

By Theorems 2.2 and 2.3, we deduce that for a each initial condition $(u_0,w_0) \in B_2 \subset H\times L^2(\Omega)$, problem (1.1) admits a unique solution. Thus, the solution operator $S(t)$ restricted to B_2 is a semigroup. Moreover, B_2 is positively invariant with respect to $S(t)$.

Theorem 4.1. *Suppose $(f_1, f_2) \in H \times L^2(\Omega), 1 \leq \beta < \frac{7}{3}$, then the map $e_1 : \mathcal{A}_\ell \to \mathcal{A} = e_1(\mathcal{A}_\ell)$ is Lipschitz continuous. That is, for any two ℓ-trajectories $\chi^1, \chi^2 \in \mathcal{A}_\ell$, there exists a positive constant θ depending on ℓ such that*

$$\|e_1(\chi^1) - e_1(\chi^2)\|^2_{H\times L^2(\Omega)} \leq \theta \int_0^\ell \|\chi^1(\tau) - \chi^2(\tau)\|^2_{H\times L^2(\Omega)} \mathrm{d}\tau.$$

Proof. For each fixed $t \geq 0$ and any $\chi^1, \chi^2 \in \mathcal{A}_\ell$, let $L_t\chi^1 = (u_1(t + \tau), w_1(t + \tau))$, $L_t\chi^2 = (u_2(t + \tau), w_2(t + \tau))$ and let $(\bar{u}, \bar{w}) = (u_1 - u_2, w_1 - w_2)$. Since $e_0(\chi^1)$ and $e_0(\chi^2)$ are uniformly bounded in $H \times L^2(\Omega)$ for any $\chi^1, \chi^2 \in \mathcal{A}_\ell$, from the proof of Theorem 2.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{u}\|^2_{L^2} + \|\bar{w}\|^2_{L^2}) + (\nu + k)\|\nabla\bar{u}\|^2_{L^2} + \gamma\|\nabla\bar{w}\|^2_{L^2} \leq \eta(t)(\|\bar{u}\|^2_{L^2} + \|\bar{w}\|^2_{L^2}), \tag{4.1}$$

where $\eta(t) = C(1 + \|\nabla u_1\|^2_{L^2} + \|\nabla w_2\|^2_{L^2})$.

For $s \in (0, \ell)$, we infer from Gronwall's lemma that

$$\begin{aligned} \|\bar{u}(\ell)\|^2_{L^2} + \|\bar{w}(\ell)\|^2_{L^2} &\leq \big(\|\bar{u}(s)\|^2_{L^2} + \|\bar{w}(s)\|^2_{L^2}\big) \exp\Big(\int_s^\ell \eta(\tau)\mathrm{d}\tau\Big) \\ &\leq \big(\|\bar{u}(s)\|^2_{L^2} + \|\bar{w}(s)\|^2_{L^2}\big) \exp\Big(\int_0^\ell \eta(\tau)\mathrm{d}\tau\Big). \end{aligned} \tag{4.2}$$

Integrating (4.2) with respect to s over $(0, \ell)$, we obtain

$$\|\bar{u}(\ell)\|^2_{L^2} + \|\bar{w}(\ell)\|^2_{L^2} \leq \frac{1}{\ell} N_\ell(0) \int_0^\ell \big(\|\bar{u}(s)\|^2_{L^2} + \|\bar{w}(s)\|^2_{L^2}\big)\,\mathrm{d}s, \tag{4.3}$$

where $N_\ell(0) := \exp\big(\int_0^\ell \eta(\tau)\mathrm{d}\tau\big)$ is a finite number, which depends on (u_{1_0}, w_{1_0}) and (u_{2_0}, w_{2_0}), for any fixed $\ell \geq 0$. Based on the above results, for $\theta = \frac{1}{\ell}N_\ell(0)$, then we have

$$\|e_1(\chi^1) - e_2(\chi^2)\|^2_{H\times L^2(\Omega)} \leq \theta \int_0^\ell \|\chi^1(\tau) - \chi^2(\tau)\|^2_{H\times L^2(\Omega)} \mathrm{d}\tau. \tag{4.4}$$

From (4.4), we conclude that the map $e_1 : \mathcal{A}_l \to \mathcal{A}$ is Lipschitz continuous. □

Theorem 4.2. *Suppose $(f_1, f_2) \in H \times L^2(\Omega), 1 \leq \beta < \frac{7}{3}$ and $(u_0, w_0) \in H \times L^2(\Omega)$, then the semigroup $(S(t), B_2)$ generated by (1.1) possesses a global attractor $\mathcal{A} = e_1(\mathcal{A}_\ell)$ in $H \times L^2(\Omega)$. Moreover, the fractal dimension of the global attractor $\mathcal{A}$ is finite.*

Proof. From Lemma 2.11, Theorem 3.9 and Theorem 4.1, we can conclude that $\mathcal{A}$ is compact in $H \times L^2(\Omega)$ and the fractal dimension of $\mathcal{A}$ is finite. Since $L_t\mathcal{A}_\ell = \mathcal{A}_\ell$, we have

$$S(t)\mathcal{A} = S(t)e_1(\mathcal{A}_\ell) = e_1(L_t\mathcal{A}_\ell) = e_1(\mathcal{A}_\ell) = \mathcal{A}$$

for any $t \geq 0$. From the definition of B_2, we can infer that for any bounded subset B in $H \times L^2(\Omega)$, there exists a time $t' = t'(B)$ such that for any $t \geq t'$, it follows that

$$S(t)B \subset B_2.$$

Therefore, we need to prove that

$$\lim_{t\to+\infty} \mathrm{dist}_{H\times L^2(\Omega)}(S(t)B_2, \mathcal{A}) = 0.$$

Suppose the above formula does not hold, then there exists a positive constant ζ_0, a sequence $\{(u_n, w_n)\}_{n=1}^\infty \subset B_2$, and some time ${t_n}'$ such that

$$\mathrm{dist}_{H\times L^2(\Omega)}(S(t_n)(u_n, w_n), \mathcal{A}) \geq \zeta_0 \tag{4.5}$$

for any $t_n \geq {t_n}'$.

From the definition of B_2, we can infer that there exists a sequence $\{\chi_n\}_{n=1}^\infty \subset B_0^l$ such that

$$(u_n, w_n) = e_0(\chi_n).$$

Since $\{\chi_n\}_{n=1}^{\infty}$ is bounded in X_ℓ and $\mathcal{A}_\ell$ is the global attractor in X_ℓ for the semigroup $\{L_t\}_{t\geq 0}$ generated by problem (1.1), then there exist a subsequence $\{\chi_{n_j}\}_{n_j=1}^{\infty}$ of $\{\chi_n\}_{n=1}^{\infty}$ and a subsequence $\{t_{n_j}\}_{n_j=1}^{\infty}$ of $\{t_n\}_{n=1}^{\infty}$ such that

$$L_{t_{n_j}-\ell}\chi_{n_j} \to \chi \in \mathcal{A}_\ell \quad \text{in } X_\ell \text{ as } j \to +\infty.$$

By the continuity of e_1, we have

$$S(t_{n_j})(u_{n_j}, w_{n_j}) = e_1(L_{t_{n_j}-\ell}\chi_{n_j}) \to e_1(\chi) \in \mathcal{A} \quad \text{in } H \times L^2(\Omega) \text{ as } j \to +\infty,$$

which contradicts (4.5). □

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