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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO HARDY-HÉNON BOUNDARY-VALUE PROBLEMS

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Abstract. We consider the elliptic equation

$$-\Delta u = d(x)^\alpha u^p, \quad x \in \Omega \quad (\text{or } x \in \mathbb{R}^N \setminus \overline{\Omega}),$$

where $\alpha, p \in \mathbb{R}$, $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. We establish estimates for the positive solutions when $1 < p < \frac{N+2}{N-2}$, and the nonexistence of positive solutions for exterior domains. For the corresponding Dirichlet problem we show the existence and uniqueness of positive solutions.

1. Introduction and main results

In this article we study the positive solutions to the elliptic equations

$$-\Delta u = d(x)^\alpha u^p, \quad x \in \Omega, \tag{1.1}$$

and

$$-\Delta u = d(x)^\alpha u^p, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{1.2}$$

where $\alpha, p \in \mathbb{R}$, $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. We will also consider the Dirichlet problem

$$\begin{gathered} -\Delta u = d(x)^\alpha u^p, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.3}$$

In the previous decades, there has been a lot of research on the elliptic equation

$$-\Delta u = a(x) u^p \quad \text{in } \Omega, \tag{1.4}$$

where $a(x)$ is a weight function and $\Omega \subseteq \mathbb{R}^N$ is a smooth domain. It is well-known that (1.4) is Lane-Emden equation if $a(x) \equiv 1$. There has been a great deal of work such as existence, nonexistence, symmetry, and uniqueness of positive solutions to the Lane-Emden equation. For example, Gidas [18] pointed out that the Lane-Emden equation has no positive solutions if $1 < p \leq \frac{N}{N-2}$ and $\Omega = \mathbb{R}^N$, and subsequently Gidas-Spruck [21, 22] ascertained that the Lane-Emden equation has no positive solutions if $1 < p < \frac{N+2}{N-2}$ and $\Omega = \mathbb{R}^N$ (or $\mathbb{R}^N_+$), Caffarelli-Gidas-Spruck [7], Chen-Li [9], and Gidas-Ni-Nirenberg [19, 20] established some interesting results related to the symmetry of positive solutions to the Lane-Emden equation with $p = \frac{N+2}{N-2}$ and $\Omega = \mathbb{R}^N$. If $a(x) = |x|^\alpha$ and $0 \in \Omega$, equation (1.4) is usually called Hardy-Hénon equation. Dancer-Du-Guo [12] proved that the Hardy-Hénon equation has no positive solutions in any domain Ω containing the origin if $\alpha \leq -2$ and $p > 1$; Phan-Souplet [31] showed that the Hardy-Hénon equation has no positive radial solutions in $\mathbb{R}^N$ if $\alpha > -2$ and $p < \frac{N+2+2\alpha}{N-2}$. In addition, Du-Guo [15] treated the existence and nonexistence of positive solutions as well as qualitative properties (such as stability, Morse index and isolated singularities) of the solutions to the Hardy-Hénon equation with $\alpha > -2$ and $p < 0$; Li-Zhang [25] claimed that Hénon-Lane-Emden elliptic system has no positive solutions when the exponent lies below the Hardy-Sobolev exponent, which covered

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the conjecture in Phan-Souplet [31] for $\mathbb{R}^3$; Cheng-Li-Zhang [10] established the Liouville-type theorems for generalized Hénon-Lane-Emden Schrödinger systems in $\mathbb{R}^2$ and $\mathbb{R}^3$. For (1.4) with $a(x) = -d(x)^\alpha$ and $-\Delta$ replaced by $-[\Delta + V(x)]$, researchers have been concerned with the existence, uniqueness and asymptotic behavior of boundary blow-up solution (or large solution), see [16, 26, 27] and references therein; and then Du [13] investigated the blow-up rate of any boundary blow-up solution to the singular p-Laplace equation. For the Hardy-Hénon equation subject to Dirichlet boundary condition, Cao-Peng-Yan [8] analyzed the profile of ground state solution and proved the existence and asymptotic behavior of multi-peaked solutions to equation (1.4) with $a(x) = |x|^\alpha$ and $\Omega = B_1(0)$. More recently, Cheng-Wei-Zhang [11] has presented the estimate, existence and nonexistence of positive solutions of Hardy-Hénon equations (1.4) with $a(x) = d(x)^\alpha$ and $\Omega = B_1(0)$.

In this article, we are interested in equations (1.1), (1.2) and (1.3), that is, equation (1.4) with $a(x) = d(x)^\alpha$ on a bounded domain Ω or an exterior domain $\mathbb{R}^N \backslash \overline{\Omega}$. Clearly, for the case $\alpha > 0$, $d(x)^\alpha$ converges to zero as $d(x) \to 0$, and for the case $\alpha < 0$, $d(x)^\alpha$ blows up as $d(x) \to 0$. Corresponding to the boundary Hardy potential $d(x)^{-2}$, the equation (1.1), (1.2) and (1.3) are called Hardy-Hénon boundary-value problems in this paper, in order to distinguish from the well-known Hardy-Hénon equations. For elliptic equations with the Hardy potential $d(x)^{-2}$, there have been many interesting problems and results. For instance, the well-known Hardy constant and Hardy inequality were established in [2, 28]. Bandle-Moroz-Reichel [5] dealt with

$$-\Delta u = \lambda \frac{u}{d(x)^2} - d(x)^\alpha u^p \quad \text{in } \Omega, \tag{1.5}$$

and presented some classification of positive solutions under conditions $p > 1$, $\alpha > -2$, and $\lambda \leq 1/4$. Du-Wei [17] obtained the uniqueness and asymptotic behavior of positive solutions to (1.5) for the case $p > 1$, $\alpha > -2$ and $\lambda > 1/4$. Bandle-Pozio [3] investigated the equation (1.5) with the sublinear term.

In view of the interesting work above, this paper is devoted to estimate the positive solutions to (1.1), the existence of positive solutions to (1.3) and the nonexistence of positive solutions to (1.2). The techniques described herein can also be applied to some other equations such as Camassa-Holm equation and mCH-Novikov equation [29, 32, 33, 34, 35].

Because of [31, Lemma 2.1], using a scaling argument and some analysis techniques we can obtain the following estimate of positive solutions to (1.1).

Theorem 1.1. *Suppose that $1 < p < \frac{N+2}{N-2}$ and $\alpha \in \mathbb{R}$. Then there exists $C = C(N, p, \alpha) > 0$ such that every nonnegative classical solution u to (1.1) satisfies*

$$u(x) \leq C d(x)^{-\frac{2+\alpha}{p-1}}, \quad x \in \Omega,$$
$$|\nabla u(x)| \leq C d(x)^{-\frac{p+1+\alpha}{p-1}}, \quad x \in \Omega.$$

As a direct consequence of Theorem 1.1 above, we have the following result.

Corollary 1.2. *Let $1 < p < \frac{N+2}{N-2}$ and $\alpha \leq -2$, then (1.1) has no boundary blow-up solutions. Specially, when $\alpha < -2$, any positive solution u to (1.1) satisfies $u(x) \to 0$ $(d(x) \to 0)$.*

Secondly, applying the "blow-up" a priori estimate method, Mountain Pass Lemma and the elliptic regularity we can derive the following existence of positive solutions to (1.3).

Theorem 1.3. *Assume that $1 < p < \frac{N+2}{N-2}$ and $\alpha \geq 0$. Then (1.3) has a positive solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.*

Theorem 1.4. *Suppose that $-2 < \alpha < 0$ and $1 < p < \frac{N+2+2\alpha}{N-2}$. Then (1.3) has a positive solution in $H_0^1(\Omega) \cap C^2(\Omega)$.*

Owing to [11, Lemma 3.2] and the similar proof to Theorem 1.1, it is easy to obtain the low estimate of nonnegative solutions to (1.1).

Theorem 1.5. *Assume that $0 < p < 1$ and $\alpha \in \mathbb{R}$. Then there exists $C = C(N, p, \alpha) > 0$ such that every nonnegative solution u to* (1.1) *satisfies*

$$u(x) \geq Cd(x)^{-\frac{2+\alpha}{p-1}}, \quad x \in \Omega.$$

Thirdly, by the method of supersolution and subsolution and the elliptic regularity, we can provide the existence of positive solutions to (1.3).

Theorem 1.6. *Let $\alpha > -2$ and $p < 1$, then* (1.3) *has a positive classical solution. Furthermore, if $p \leq 0$, the positive solution to* (1.3) *is unique.*

Finally, from the method of spherical averages, by indirect proof we can deduce the nonexistence of positive solutions to (1.2).

Theorem 1.7. (i) *Equation* (1.2) *has no positive solution satisfying*

$$u(x) \to +\infty \ (|x| \to +\infty);$$

(ii) *If $\alpha \geq -2$ and $p > 1$, then* (1.2) *has no positive solution such that*

$$u(x) \geq c > 0 \text{ when } |x| \text{ sufficiently large};$$

(iii) *Let $1 < p \leq \frac{N+\alpha}{N-2}$, then* (1.2) *has no positive solutions.*

The rest of this paper is organized as follows. In Section 2, we list some important lemmas which is useful in the proof of main results. In Section 3, we mainly establish the estimate of positive solutions to (1.1) and a priori estimate of positive solutions to (1.3) with perturbation. In Section 4, we prove the existence and uniqueness of positive solutions to (1.3) and the nonexistence of positive solutions to (1.2).

2. Preliminaries

In this section, we state several lemmas which will be used later.

Lemma 2.1 ([31, Lemma 2.1])**.** *Suppose $N \geq 3$, $1 < p < \frac{N+2}{N-2}$, $\mu \in (0, 1]$, and $a \in C^\mu(\overline{B}_1(0))$ satisfying*

$$\|a\|_{C^\mu(\overline{B}_1(0))} \leq C_1, \quad a(x) \geq C_2, \quad x \in \overline{B}_1(0),$$

where $C_1, C_2 > 0$. Then there exists $C := C(\mu, C_1, C_2, p, N) > 0$ such that any nonnegative solution u to

$$-\Delta u = a(x)u^p, \quad x \in B_1(0)$$

satisfies

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{\frac{p-1}{p+1}} \leq C\Big(1 + \frac{1}{1-|x|}\Big), \quad x \in B_1(0). \tag{2.1}$$

Lemma 2.2 ([11, Lemma 3.2])**.** *Suppose that $N \geq 3$, $p < 1$, $\mu \in (0, 1)$, and $a \in C^\mu(\overline{B}_1(0))$ satisfying*

$$a(x) \geq C, \ x \in \overline{B}_1(0),$$

where C is a positive constant. Then any positive classical solution u to

$$-\Delta u = a(x)u^p, \quad x \in B_1(0)$$

satisfies

$$|u(0)| \geq \Big(\frac{C}{\lambda_1(B_1(0))}\Big)^{\frac{1}{1-p}}, \tag{2.2}$$

where $\lambda_1(B_1(0))$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on $B_1(0)$.

Lemma 2.3 ([30, Theorem 3.43])**.** *Let ϕ be an arbitrary solution of*

$$\phi'' + p(t)f(\phi) \leq 0 \quad \text{in } [\alpha, \infty)$$

where $p(t) \geq 0$ in $[\alpha, \infty)$, $\int_{\pm 1}^{\pm\infty}(1/f(t))\,\mathrm{d}t < \infty$ and $tf(t) > 0$, $f'(t) > 0$ for all $t \neq 0$. If $\int^\infty tp(t)\,\mathrm{d}t = \infty$ then ϕ must have infinitely many positive zeros in $[\alpha, \infty)$.

3. Estimate of positive solutions

By the scaling method and Lemma 2.1, we obtain estimates for positive solutions to (1.1).

Proof of Theorem 1.1. Let x_0 be any given point in Ω, and define the function

$$U(x) = d(x_0)^{\frac{2+\alpha}{p-1}} u\Big(x_0 + \frac{d(x_0)}{2}x\Big), \quad x \in B_1(0).$$

Then $U(x)$ satisfies

$$-\Delta U = a(x; x_0)U^p, \quad x \in B_1(0),$$

where

$$a(x; x_0) = \frac{d(x_0 + \frac{d(x_0)}{2}x)^\alpha}{4d(x_0)^\alpha}.$$

For each $x \in B_1(0)$, denote $y = x_0 + \frac{d(x_0)}{2}x$. We have

$$d(y) \geq d(x_0) - d(x_0, y) \geq d(x_0) - \frac{d(x_0)}{2} = \frac{d(x_0)}{2}, \quad \text{when } |x| \leq 1,$$
$$d(y) \leq d(x_0) + d(x_0, y) \leq d(x_0) + \frac{d(x_0)}{2} = \frac{3d(x_0)}{2}, \quad \text{when } |x| \leq 1,$$

where $d(x_0, y) = \operatorname{dist}(x_0, y)$. So, we have

$$a(x; x_0) \geq \frac{1}{2^{\alpha+2}}, \quad \alpha \geq 0, a(x; x_0) \geq \frac{3^\alpha}{2^{\alpha+2}}, \quad \alpha < 0.$$

Next, we prove that

$$\|a(\cdot; x_0)\|_{C^1(\overline{B}_1(0))} \leq C, \tag{3.1}$$

where C only depends on α. In fact, for any $x \in B_1(0)$, when $\alpha \geq 0$, we have

$$4a(x; x_0) = \Big(\frac{d(x_0 + \frac{d(x_0)}{2}x)}{d(x_0)}\Big)^\alpha \leq \Big(\frac{\frac{3d(x_0)}{2}}{d(x_0)}\Big)^\alpha = \Big(\frac{3}{2}\Big)^\alpha.$$

When $\alpha < 0$, we have

$$4a(x; x_0) = \Big(\frac{d(x_0 + \frac{d(x_0)}{2}x)}{d(x_0)}\Big)^\alpha \leq \Big(\frac{\frac{d(x_0)}{2}}{d(x_0)}\Big)^\alpha = \Big(\frac{1}{2}\Big)^\alpha.$$

Since Ω is a smooth domain, the function $a(x; x_0)$ is also smooth. For any given $\tilde{x} \in B_1(0)$, there is a $\tilde{y} \in \partial\Omega$ such that

$$d\Big(x_0 + \frac{d(x_0)}{2}\tilde{x}\Big) = |x_0 + \frac{d(x_0)}{2}\tilde{x} - \tilde{y}|.$$

For a small $\delta > 0$, $a(x; x_0)$ is differentiable in $B_\delta(\tilde{x}) \subset B_1(0)$. According to the fastest growth direction of the function $d(x_0 + \frac{d(x_0)}{2}x)$ in $B_\delta(\tilde{x})$, we obtain

$$|\nabla d(x_0 + \frac{d(x_0)}{2}x)|_{x=\tilde{x}} = |\nabla |x_0 + \frac{d(x_0)}{2}x - \tilde{y}||_{x=\tilde{x}} = \frac{d(x_0)}{2}.$$

Hence, we see that

$$|D_i d(x_0 + \frac{d(x_0)}{2}x)|_{x=\tilde{x}} \leq |\nabla |x_0 + \frac{d(x_0)}{2}x - \tilde{y}||_{x=\tilde{x}} = \frac{d(x_0)}{2}.$$

Therefore,

$$D_i a(\tilde{x}; x_0) = \frac{\alpha}{4}\Big(\frac{d(x_0 + \frac{d(x_0)}{2}\tilde{x})}{d(x_0)}\Big)^{\alpha-1} \frac{1}{d(x_0)} \cdot D_i d(x_0 + \frac{d(x_0)}{2}x)|_{x=\tilde{x}}.$$

So, if $\alpha \geq 1$, then

$$|D_i a(\tilde{x}; x_0)| \leq \frac{\alpha}{8}\Big(\frac{3}{2}\Big)^{\alpha-1}, \quad \forall \tilde{x} \in \overline{B}_1(0).$$

If $\alpha < 1$, then

$$|D_i a(\tilde{x}; x_0)| \leq \frac{|\alpha|}{8}\Big(\frac{1}{2}\Big)^{\alpha-1}, \quad \forall \tilde{x} \in \overline{B}_1(0).$$

These imply
$$\|a(\cdot;x_0)\|_{C^1(\overline{B}_1(0))} \le C,$$
where C is independent of x_0. Hence, (3.1) holds. By Lemma 2.1, we obtain
$$U(0) + |\nabla U(0)| \le C.$$
This implies
$$u(x_0) \le Cd(x_0)^{-\frac{2+\alpha}{p-1}},$$
$$|\nabla u(x_0)| \le Cd(x_0)^{-\frac{p+1+\alpha}{p-1}}.$$
Using the arbitrariness of x_0, the proof is complete. □

Remark 3.1. Suppose that $1 < p < \frac{N+2}{N-2}$, $\alpha < 0$ and $\epsilon_0 > 0$. Then, for any $\rho > 0$ there exists $C := C(N, p, \alpha, \epsilon_0, \rho) > 0$ such that for any $\epsilon \in (0, \epsilon_0]$,
$$u_\epsilon(x) \le Cd(x)^{-\frac{2+\alpha}{p-1}}, \quad \text{as } d(x) > \rho,$$
$$|\nabla u_\epsilon(x)| \le Cd(x)^{-\frac{p+1+\alpha}{p-1}}, \quad \text{as } d(x) > \rho,$$
where u_ϵ is a nonnegative solution to
$$-\Delta u = (d(x) + \epsilon)^\alpha u^p, \quad x \in \Omega.$$

The proof is similar to that of Theorem 1.1, and hence we only give a simple description. Let ρ be a given small number, and x_0 be any given point in Ω_ρ, where $\Omega_\rho = \{x \in \Omega | d(x) > \rho\}$. We define
$$U(x) = d(x_0)^{\frac{2+\alpha}{p-1}} u_\epsilon\Big(x_0 + \frac{d(x_0)}{2}x\Big), \quad x \in B_1(0),$$
then we have
$$-\Delta U = a_\epsilon(x;x_0)U^p, \quad x \in B_1(0),$$
where
$$a_\epsilon(x;x_0) = \frac{[d(x_0 + \frac{d(x_0)}{2}x + \epsilon]^\alpha}{4d(x_0)^\alpha}.$$
Hence, it follows that
$$a_\epsilon(x;x_0) \ge \frac{(\frac{3}{2}d(x_0) + \epsilon_0)^\alpha}{4d(x_0)^\alpha} \ge \frac{1}{4}\Big(\frac{3}{2} + \frac{\epsilon_0}{\rho}\Big)^\alpha, \quad x \in B_1(0).$$
Now we show $\|a_\epsilon(\cdot;x_0)\|_{C^\mu(\overline{B}_1(0))} \le C$, where C is independent of ϵ and x_0. Firstly, we can see that
$$a_\epsilon(x;x_0) \le \frac{d(x_0 + \frac{d(x_0)}{2}x)^\alpha}{4d(x_0)^\alpha} \le 2^{-\alpha-2}, \quad x \in B_1(0),\ \epsilon \in (0, \epsilon_0].$$
In addition, we obtain
$$D_i a_\epsilon(x;x_0) = \frac{\alpha}{4}\Big(\frac{d(x_0 + \frac{d(x_0)}{2}x) + \epsilon}{d(x_0)}\Big)^{\alpha-1} \frac{1}{d(x_0)} D_i d(x_0 + \frac{d(x_0)}{2}x).$$
So, we have
$$|D_i a_\epsilon(x;x_0)| \le \frac{|\alpha|}{8}\Big(\frac{1}{2}\Big)^{\alpha-1}, \quad \forall x \in \overline{B}_1(0),\ \epsilon \in (0, \epsilon_0].$$
Hence, it follows that
$$\|a_\epsilon(\cdot;x_0)\|_{C^\mu(\overline{B}_1(0))} \le C.$$
By Lemma 2.1, we have $U(0) + |\nabla U(0)| \le C$. This implies
$$u_\epsilon(x_0) \le Cd(x_0)^{-\frac{2+\alpha}{p-1}},$$
$$|\nabla u_\epsilon(x_0)| \le Cd(x_0)^{-\frac{p+1+\alpha}{p-1}}.$$
By the arbitrariness of $x_0 \in \Omega_\rho$, the conclusion follows.

Next, by using the "blow-up" a priori estimate technique introduced in [22], we can obtain the existence of a priori bounds for positive solutions to (3.2).

Theorem 3.2. *Suppose that $1 < p < \frac{N+2}{N-2}$, $\alpha \geq 0$, $\epsilon_0 > 0, \epsilon \in (0, \epsilon_0]$ and $u_\epsilon \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a positive solution of*

$$\begin{gathered} -\Delta u = (d(x)+\epsilon)^\alpha u^p, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.2}$$

Then there exists $C = C(\alpha, p, \epsilon_0, N) > 0$ such that

$$\|u_\epsilon\|_{L^\infty(\Omega)} \leq C. \tag{3.3}$$

Proof. Suppose the conclusion is not true, then there are a sequence of numbers $\{\epsilon_k\} \subset (0, \epsilon_0]$, a corresponding positive solution sequence $\{u_k\}$ and a sequence of points $\{P_k\} \subset \Omega$ such that

$$M_k := \max_{x \in \overline{\Omega}} u_k(x) = u_k(P_k) \to +\infty \quad (k \to \infty).$$

Noticing that $\{\epsilon_k\}$ and $\{P_k\}$ are bounded, without loss of generality, assume that

$$\epsilon_k \to \tilde{\epsilon} \in [0, \epsilon_0], \quad P_k \to P \in \overline{\Omega} \quad \text{as } k \to \infty.$$

Case 1. $P \in \Omega$. We define

$$U_k(y) = \frac{1}{M_k} u_k(P_k + M_k^{-\frac{p-1}{2}} y).$$

We denote $2d_0 = d(P, \partial\Omega)$. Then

$$|M_k^{-\frac{p-1}{2}} y| \leq d_0, \quad \text{as } |y| \leq d_0 M_k^{\frac{p-1}{2}}.$$

So, when k is sufficiently large, we have

$$P_k + M_k^{-\frac{p-1}{2}} y \in B_{d_0}(P_k) \subset B_{2d_0}(P) \subset \Omega.$$

By calculations, U_k satisfies that $0 \leq U_k \leq 1$ and

$$\begin{gathered} -\Delta U_k = \Big[d(P_k + M_k^{-(p-1)/2} y) + \epsilon_k\Big]^\alpha {U_k}^p, \quad y \in B_{d_0 M_k^{\frac{p-1}{2}}}(0), \\ U_k(0) = 1. \end{gathered} \tag{3.4}$$

By the elliptic regularity [23, Theorem 6.14], we can extract a subsequence of $\{U_k\}$ such that it is convergent in $C^2_{\rm loc}(\mathbb{R}^N)$. Without loss of generality, assume that $\{U_k\}$ converges to U in $C^2_{\rm loc}(\mathbb{R}^N)$. For any given bounded domain $\widetilde{\Omega} \subset \mathbb{R}^N$, we obtain that

$$\lim_{k \to \infty} \Big[d(P_k + M_k^{-\frac{p-1}{2}} y) + \epsilon_k\Big]^\alpha = (2d_0 + \tilde{\epsilon})^\alpha, \quad \forall y \in \widetilde{\Omega}.$$

Letting $k \to \infty$ in (3.4), we have

$$\begin{gathered} -\Delta U = (2d_0 + \tilde{\epsilon})^\alpha U^p, \quad \text{in } \mathbb{R}^N, \\ U(0) = 1. \end{gathered} \tag{3.5}$$

This is a contradiction to Liouville Theorem [22, Theorem 1.2].

Case 2. $P \in \partial\Omega$. Since Ω is smooth, there exist an $r > 0$ small enough and a C^1 function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that—upon relabeling and reorienting the coordinates axes if necessary – we have

$$\Omega \cap B_r(P) = \{x \in B_r(P) |\ x_n > \psi(x'), x' = (x_1, x_2, \cdots, x_{n-1})\}.$$

Hence, we can use coordinate transformation to straighten out $\partial\Omega \cap B_r(P)$. Now define

$$\begin{gathered} y_i = x_i =: \Psi_i(x), \quad 1 \leq i < n, \\ y_n = x_n - \psi(x') =: \Psi_n(x), \end{gathered}$$

and write $y = \Psi(x)$. Similarly, we set

$$\begin{gathered} x_i = y_i =: \Phi_i(y), \quad 1 \leq i < n, \\ x_n = y_n + \psi(y') =: \Phi_n(y), \end{gathered}$$

and write $x = \Phi(y)$. Clearly, $\Psi = \Phi^{-1}$ with $\det\Psi = \det\Phi = 1$, and the mapping $x \mapsto \Psi(x) = y$ "straightens out" $\partial\Omega \cap B_r(P)$.

Let $v(y) = u(x)$ with $y = (y_1, y_2, \cdots, y_n)$ and $x = (x_1, x_2, \cdots, x_n)$, then there exists $\delta > 0$ small enough such that $v(y)$ satisfies

$$\begin{gathered} -\frac{\partial}{\partial y_i}(a^{ij}(y)\frac{\partial v}{\partial y_j}) = [d(\Phi(y)) + \epsilon]^{\alpha} v^p, \quad y \in B_{2\delta}(\Psi(P)) \cap \mathbb{R}^n_+, \\ v = 0, \quad y \in B_{2\delta}(\Psi(P)) \cap \{y_n = 0\} \end{gathered} \tag{3.6}$$

with $\mathbb{R}^n_+ = \{y = (y', y_n) : y' = (y_1, y_2, \cdots, y_{n-1}) \in \mathbb{R}^{n-1}, y_n > 0\}$. The coefficient matrix $[a^{ij}(y)]$ is symmetric and positive definite, in concrete,

$$a^{ij}(y) = \begin{cases} \delta_{ij}, \quad 1 \le i \le j < n, \\ -\psi'_{y_i}, \quad 1 \le i < j = n, \\ 1 + \sum_{i=1}^{n-1} (\psi'_{y_i})^2, \quad i = j = n. \end{cases}$$

where δ_{ij} is the Kronecker symbol.

Let $v_k(y) = u_k(x)$ and

$$V_k(z) := \frac{1}{M_k} v_k\Big(\Psi(P_k) + M_k^{-\frac{p-1}{2}} z\Big).$$

Then for sufficiently large k, $V_k(z)$ is well-defined in

$$D_k := B_{M_k^{\frac{p-1}{2}}\delta}(0) \cap \{z_n > -M_k^{\frac{p-1}{2}} d_k\},$$

where $d_k = \operatorname{dist}(\Psi(P_k), y_n = 0)$. Hence, V_k satisfies that $0 \le V_k \le 1$ and

$$\begin{gathered} -\frac{\partial}{\partial z_i}(a_k^{ij}(z)\frac{\partial V_k}{\partial z_j}) = \Big[d\Big(\Phi(\Psi(P_k) + M_k^{-\frac{p-1}{2}} z)\Big) + \epsilon_k\Big]^{\alpha} V_k^p, \quad z \in D_k, \\ V_k(0) = 1, \end{gathered} \tag{3.7}$$

where $a_k^{ij}(z) = a^{ij}(\Psi(P_k) + M_k^{-\frac{p-1}{2}} z)$.

If $\alpha \ge 0$, then $\big\{\big[d\big(\Phi(\Psi(P_k) + M_k^{-\frac{p-1}{2}} z)\big) + \epsilon_k\big]^{\alpha}\big\}$ is uniformly bounded with respect to k. By virtue of regularity of elliptic equations, $\{\nabla V_k\}$ is uniformly bounded, therefore

$$1 = |1 - 0| = \big|V_k(0) - V_k(0', -M_k^{\frac{p-1}{2}} d_k)\big| \le C M_k^{\frac{p-1}{2}} d_k,$$

which implies that

$$M_k^{\frac{p-1}{2}} d_k \ge C^{-1} > 0.$$

Now, we claim that $\{M_k^{\frac{p-1}{2}} d_k\}$ has upper bounds. In fact, if suppose that $\{M_k^{\frac{p-1}{2}} d_k\}$ has no upper bounds, then we can extract a subsequence relabeled as $\{M_k^{\frac{p-1}{2}} d_k\}$ such that

$$M_k^{\frac{p-1}{2}} d_k \to +\infty \quad \text{as } (k \to \infty.$$

It is clear to see that V_k is well-defined in $B_{M_k^{\frac{p-1}{2}} d_k}(0)$ and $V_k(0) = 1$. By the elliptic regularity, we can extract a subsequence relabeled as $\{V_k\}$ such that it converges to V in $C^2_{\rm loc}(\mathbb{R}^N)$. For any given bounded domain $\widetilde{\Omega} \subset \mathbb{R}^N$, we obtain that

$$\lim_{k\to\infty} \Big[d\Big(\Phi(\Psi(P_k) + M_k^{-\frac{p-1}{2}} z)\Big) + \epsilon_k\Big]^{\alpha} = \tilde{\epsilon}^{\ \alpha} := \begin{cases} \tilde{\epsilon}^{\alpha}, \quad \alpha > 0 \\ 1, \quad \alpha = 0, \end{cases}$$

for all $z \in \widetilde{\Omega}$. Letting $k \to \infty$ in (3.7), we have

$$\begin{gathered} -\frac{\partial}{\partial z_i}(a^{ij}(\Psi(P))\frac{\partial V}{\partial z_j}) = \tilde{\epsilon}^{\ \alpha} V^p, \quad \text{in } \mathbb{R}^N, \\ V(0) = 1. \end{gathered} \tag{3.8}$$

By an orthogonal transformation of coordinates (similar to the one used to go from [22, (2.8)] to [22, (2.9)]) and a stretching, we end up with Liouville Theorem [22, Theorem 1.2]. By that theorem, we conclude $V(z) \equiv 0$, which contradicts $V(0) = 1$.

Because of the boundedness of $\left\{M_k^{\frac{p-1}{2}}d_k\right\}$, without loss of generality assume that $M_k^{\frac{p-1}{2}}d_k \to s$, where $s > 0$. Repeating the compactness arguments above, we have that V satisfies

$$-\frac{\partial}{\partial z_i}(a^{ij}(\Psi(P))\frac{\partial V}{\partial z_j}) = \tilde{\epsilon}^{\ \alpha} V^p, \quad \text{in } \{z_n > -s\},$$
$$V(0) = 1.$$

By an orthogonal transformation and stretching of coordinates, we obtain that $V(z) \equiv 0$ due to Liouville Theorem [22, Theorem 1.3]. This contradicts $V(0) = 1$. □

Proof of Theorem 1.5. By Lemma 2.2 and the similar method used in the proof of Theorem 1.1, we can complete the proof, and hence omit the details here. □

4. Proof of existence, nonexistence, and uniqueness

This section is devoted to establishing Theorems 1.3-1.4 and Theorems 1.6-1.7.

Proof of Theorem 1.3. Firstly, we consider the perturbed problem

$$\begin{gathered} -\Delta u = (d(x) + \frac{1}{n})^\alpha u^p, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.1}$$

The corresponding energy functional is

$$F_n(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x) + \frac{1}{n}\Big)^\alpha |u|^{p+1}\mathrm{d}x, \quad u \in H_0^1(\Omega).$$

For given n, clearly F_n satisfies the mountain pass geometry conditions. Let $e_n \in H_0^1(\Omega)$ satisfy $F_n(e_n) < 0$. We denote

$$\Gamma_n := \{\gamma \in C([0,1], H_0^1(\Omega)) |\ \gamma(0) = 0,\ \gamma(1) = e_n\},$$
$$c_n := \inf_{\gamma\in\Gamma_n} \sup_{t\in[0,1]} F_n(\gamma(t)) > 0.$$

From $1 < p < \frac{N+2}{N-2}$, by standard arguments and Mountain Pass Lemma (see [1], [4, Theorem 4.3.1]), F_n has a nonnegative critical point $u_n \in H_0^1(\Omega)$, that is, u_n is the mountain pass solution to (4.1), and

$$c_n = F_n(u_n) > 0.$$

From regularity arguments it follows that $u_n \in C^2(\Omega) \cap C^1(\overline{\Omega})$. By the strong maximum principle, u_n is a positive solution of (4.1). By Theorem 3.2, there exists a positive constant C independent of n such that for all n,

$$\|u_n\|_{L^\infty(\Omega} \le C.$$

By elliptic regularity, $\{u_n\}$ has a convergent subsequence in $C^1(\overline{\Omega})$, and denote the limit function by w. Clearly, w is a nonnegative solution to (1.3) and $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Now, we will show that w is a positive solution. For any given n and $\tau > 0$, if u satisfies $\|u\| = \tau$, then

$$\begin{aligned} F_{n+1}(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x) + \frac{1}{n+1}\Big)^\alpha |u|^{p+1}\mathrm{d}x \\ &\ge \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x) + \frac{1}{n}\Big)^\alpha |u|^{p+1}\mathrm{d}x = F_n(u). \end{aligned}$$

This implies that for any pass $\gamma \in \Gamma_{n+1} (\subset \Gamma_n)$ we have

$$F_{n+1}(\gamma(t)) \ge F_n(\gamma(t)), \quad t \in [0,1].$$

Therefore we can obtain $c_{n+1} \ge c_n > 0$. Hence,

$$\begin{aligned} F(w) &= \frac{1}{2}\int_\Omega |\nabla w|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega d(x)^\alpha |w|^{p+1}\mathrm{d}x \\ &= \lim_{n\to\infty} F_n(u_n) = \lim_{n\to\infty} c_n > 0. \end{aligned}$$

So, w is a nontrivial nonnegative solution to (1.3). Furthermore, by the strong maximum principle, w is a positive solution to (1.3). $\square$

Proof of Theorem 1.4. We denote

$$F_n(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x)+\frac{1}{n}\Big)^\alpha |u|^{p+1}\mathrm{d}x, \quad u \in H_0^1(\Omega).$$

For any given n, clearly $(d(x)+\frac{1}{n})^\alpha$ is a positive bounded and continuous function on $\overline{\Omega}$. By $p>1$, F_n satisfies the mountain pass geometry, and take e_n such that $F_n(e_n)<0$. Denote

$$c_n := \inf_{\gamma\in\Gamma_n}\sup_{t\in[0,1]} F_n(\gamma(t)),$$

where

$$\Gamma_n := \{\gamma \in C([0,1], H_0^1(\Omega)) |\ \gamma(0)=0, \gamma(1)=e_n\}.$$

By the standard arguments and Mountain Pass Lemma, F_n has a positive critical point $u_n \in H_0^1(\Omega)$, and

$$c_n = F_n(u_n) > 0.$$

From elliptic regularity it follows that $u_n \in C^2(\Omega)\cap C^1(\bar{\Omega})$ is a positive solution to (4.1).

For u_n, we have

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x)+\frac{1}{n}\Big)^\alpha |u_n|^{p+1}\mathrm{d}x = c_n,$$
$$\int_\Omega |\nabla u_n|^2 \mathrm{d}x = \int_\Omega \Big(d(x)+\frac{1}{n}\Big)^\alpha |u_n|^{p+1}\mathrm{d}x.$$

So,

$$\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\int_\Omega |\nabla u_n|^2\mathrm{d}x = c_n. \tag{4.2}$$

For any $v \in H_0^1(\Omega)$, by $\alpha<0$ it follows that

$$\begin{aligned}
F_n(v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x)+\frac{1}{n}\Big)^\alpha |v|^{p+1}\mathrm{d}x \\
&\ge \frac{1}{2}\int_\Omega |\nabla v|^2 \mathrm{d}x - \frac{1}{p+1}\int_\Omega \Big(d(x)+\frac{1}{n+1}\Big)^\alpha |v|^{p+1}\mathrm{d}x \\
&= F_{n+1}(v).
\end{aligned}$$

This implies that for any $\gamma\in\Gamma_n$ $(\subset \Gamma_{n+1})$, we have

$$\sup_{t\in[0,1]} F_n(\gamma(t)) \ge \sup_{t\in[0,1]} F_{n+1}(\gamma(t)),$$

and then $c_n \ge c_{n+1}$. Hence, $\{c_n\}$ is a bounded sequence. By virtue of (4.2), $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Therefore, $\{u_n\}$ has a weakly convergent subsequence in $H_0^1(\Omega)$ (refer to [6, Chapter 3]). Without loss of generality, we assume

$$\begin{gathered}
u_n \rightharpoonup w \quad \text{in } H_0^1(\Omega), \text{ as } n\to\infty,\\
u_n \to w \quad \text{in } L^{p+1}(\Omega), \text{ as } n\to\infty.
\end{gathered}$$

For any given $\phi\in C_0^\infty(\Omega)$, noticing that u_n is a critical point of F_n, we obtain

$$\int_\Omega \nabla u_n \nabla\phi \mathrm{d}x = \int_\Omega \Big(d(x)+\frac{1}{n}\Big)^\alpha u_n^p \phi \mathrm{d}x.$$

Clearly, letting $n\to\infty$, we have

$$\int_\Omega \nabla w\nabla\phi\mathrm{d}x = \int_\Omega d(x)^\alpha w^p\phi\mathrm{d}x.$$

From the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ it follows that

$$\int_\Omega \nabla w\nabla\phi\mathrm{d}x = \int_\Omega d(x)^\alpha w^p\phi\mathrm{d}x, \quad \forall\phi\in H_0^1(\Omega), \tag{4.3}$$

this implies that w is a weak solution to (1.3). For the equality (4.3), in fact, for any $\phi \in H_0^1(\Omega)$, take a sequence of functions $\{\phi_n\} \subset C_0^\infty(\Omega)$ such that $\phi_n \to \phi$ $(n \to \infty)$ in $H_0^1(\Omega)$. Since

$$\int_\Omega \nabla w \nabla \phi_n \mathrm{d}x \to \int_\Omega \nabla w \nabla \phi \mathrm{d}x \quad \text{as } n \to \infty,$$

to show (4.3), it is sufficient to prove that

$$\int_\Omega d(x)^\alpha (\phi_n - \phi) w^p \mathrm{d}x \to 0, \quad \text{as } n \to \infty. \tag{4.4}$$

We can separate the proof into three cases. When $\alpha = -1$,

$$\begin{aligned}
\int_\Omega d(x)^{-1} |\phi_n - \phi| w^p \mathrm{d}x &\le \Big(\int_\Omega \frac{|\phi_n - \phi|^2}{d(x)^2} \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega w^{2p} \mathrm{d}x \Big)^{1/2} \\
&\le C \Big(\int_\Omega |\nabla \phi_n - \nabla \phi|^2 \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega w^{2p} \mathrm{d}x \Big)^{1/2}.
\end{aligned}$$

From $2p < 2\frac{N+2+2\alpha}{N-2} = \frac{2N}{N-2}$, we have

$$\Big(\int_\Omega |\nabla \phi_n - \nabla \phi|^2 \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega w^{2p} \mathrm{d}x \Big)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

So, we obtain (4.4). When $-1 < \alpha < 0$, clearly

$$\int_\Omega d(x)^\alpha |\phi_n - \phi| w^p \mathrm{d}x \le C \int_\Omega d(x)^{-1} |\phi_n - \phi| w^p \mathrm{d}x \to 0 \quad \text{as } n \to \infty,$$

which implies (4.4) is valid. When $-2 < \alpha < -1$, by $1 < p < \frac{N+2+2\alpha}{N-2}$, it follows that

$$\frac{1}{2} + \frac{-1-\alpha}{2} + \frac{(N-2)(p+1+\alpha)}{2N} < 1.$$

Therefore,

$$\begin{aligned}
&\int_\Omega d(x)^\alpha |\phi_n - \phi| w^p \mathrm{d}x \\
&= \int_\Omega \frac{|\phi_n - \phi|}{d(x)} \frac{w^{-1-\alpha}}{d(x)^{-1-\alpha}} w^{p+1+\alpha} \mathrm{d}x \\
&\le C \Big(\int_\Omega \frac{(\phi_n - \phi)^2}{d(x)^2} \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega \frac{w^2}{d(x)^2} \mathrm{d}x \Big)^{\frac{-1-\alpha}{2}} \Big(\int_\Omega w^{\frac{2N}{N-2}} \mathrm{d}x \Big)^{\frac{(N-2)(p+1+\alpha)}{2N}} \\
&\le C \Big(\int_\Omega |\nabla \phi_n - \nabla \phi|^2 \mathrm{d}x \Big)^{1/2} \Big(\int_\Omega |\nabla w|^2 \mathrm{d}x \Big)^{\frac{-1-\alpha}{2} + \frac{p+1+\alpha}{2}} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

and hence (4.4) holds.

We denote

$$F_\infty(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1} \int_\Omega d(x)^\alpha |u|^{p+1} \mathrm{d}x, \quad u \in H_0^1(\Omega).$$

Then

$$F_n(u) \ge F_\infty(u), \quad \forall u \in H_0^1(\Omega).$$

By $1 < p < \frac{N+2+2\alpha}{N-2}$ and $-2 < \alpha < 0$, we have

$$\frac{2(p+1+\alpha)}{2+\alpha} < \frac{2N}{N-2}.$$

Therefore,

$$\begin{aligned}
F_\infty(u) &= \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1} \int_\Omega (d(x)^\alpha |u|^{-\alpha}) |u|^{p+1+\alpha} \mathrm{d}x \\
&\ge \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1} \Big(\int_\Omega \frac{u^2}{d(x)^2} \Big)^{\frac{-\alpha}{2}} \Big(\int_\Omega |u|^{\frac{2(p+1+\alpha)}{2+\alpha}} \Big)^{\frac{2+\alpha}{2}} \\
&\ge \frac{1}{2} \|u\|^2 - C \|u\|^{-\alpha} \|u\|^{p+1+\alpha}
\end{aligned}$$

$$= \frac{1}{2}\|u\|^2 - C\|u\|^{p+1}.$$

Similar to the analysis of F_n and c_n, there is an $e_\infty \in H_0^1(\Omega)$ such that $F_\infty(e_\infty) < 0$. We denote

$$\Gamma_\infty := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e_\infty\},$$
$$c_\infty = \inf_{\gamma\in\Gamma_\infty} \sup_{t\in[0,1]} F_\infty(\gamma(t)).$$

Then $0 < c_\infty \le c_n$.

Next, we show that w a positive solution. Firstly, we prove that w is a nontrivial nonnegative solution. Suppose that $w \equiv 0$. By $\frac{2(p+1+\alpha)}{2+\alpha} < \frac{2N}{N-2}$, without loss of generality we assume

$$u_n \to 0 \quad \text{in } L^{\frac{2(p+1+\alpha)}{2+\alpha}}(\Omega), \quad \text{as } n \to \infty$$

So, we see

$$\begin{aligned}
c_n &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_\Omega \Big(d(x) + \frac{1}{n}\Big)^\alpha |u_n|^{p+1} \mathrm{d}x \\
&\le \Big(\frac{1}{2} - \frac{1}{p+1}\Big) \int_\Omega (d(x)^\alpha |u_n|^{-\alpha}) |u_n|^{p+1+\alpha} \mathrm{d}x \\
&\le \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\Big(\int_\Omega \frac{u_n^2}{d(x)^2}\Big)^{\frac{-\alpha}{2}} \Big(\int_\Omega |u_n|^{\frac{2(p+1+\alpha)}{2+\alpha}}\Big)^{\frac{2+\alpha}{2}} \\
&\le C\Big(\frac{1}{2} - \frac{1}{p+1}\Big)\Big(\int_\Omega |\nabla u_n|^2\Big)^{\frac{-\alpha}{2}} \Big(\int_\Omega |u_n|^{\frac{2(p+1+\alpha)}{2+\alpha}}\Big)^{\frac{2+\alpha}{2}} \\
&\le C\Big(\int_\Omega |u_n|^{\frac{2(p+1+\alpha)}{2+\alpha}}\Big)^{\frac{2+\alpha}{2}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

This is a contradiction to $c_n \ge c_\infty > 0$. Now, we show that w is a positive solution. By the regularity of elliptic equations, $w \in H_0^1(\Omega) \cap C^2(\Omega)$. Since w is nontrivial and nonnegative, there exists $x_0 \in \Omega$ such that $w(x_0) > 0$. For any $\Omega_0 \subset\subset \Omega$ containing x_0, clearly $w \in C^2(\overline{\Omega_0})$. By the strong maximum principle, it follows that $w(x) > 0$ for any $x \in \Omega_0$. By the arbitrariness of Ω_0, we have $w(x) > 0$ in Ω. □

Proof of Theorem 1.6. Firstly, we prove the existence of positive solutions to (1.3). Let $\phi_1 > 0$ and $\lambda_1(\Omega)$ be the first eigenfunction and eigenvalue of

$$\begin{gathered}
-\Delta\phi = \lambda\phi, \quad x \in \Omega, \\
\phi = 0, \quad x \in \partial\Omega.
\end{gathered}$$

We denote $\underline{u} = m\phi_1^\beta$, where $m > 0$, $\beta = \frac{2+\alpha}{1-p}$. Clearly, we can take $c_1, c_2 > 0$ such that

$$c_1\phi_1(x) \le d(x) \le c_2\phi_1(x), \quad x \in \Omega.$$

It is easy to see that

$$\begin{aligned}
-\Delta\underline{u} - d(x)^\alpha \underline{u}^p &= -m\beta\phi_1^{\beta-1}\Delta\phi_1 - m\beta(\beta-1)\phi_1^{\beta-2}|\nabla\phi_1|^2 - d(x)^\alpha m^p \phi_1^{p\beta} \\
&\le m\beta\lambda_1(\Omega)\phi_1^\beta - m\beta(\beta-1)\phi_1^{\beta-2}|\nabla\phi_1|^2 - c^\alpha m^p \phi_1^{p\beta+\alpha} \\
&= m\beta\phi_1^{\beta-2}\Big[\lambda_1(\Omega)\phi_1^2 - (\beta-1)|\nabla\phi_1|^2 - \frac{c^\alpha m^{p-1}}{\beta}\Big], \quad x \in \Omega,
\end{aligned}$$

for some constant $c > 0$. By $\beta > 0$ and $p - 1 < 0$, when $m > 0$ is sufficiently small, we have that

$$-\Delta\underline{u} - d(x)^\alpha \underline{u}^p \le 0, \quad x \in \Omega.$$

Denote $\bar{u} = K\phi_1^\rho$, where $K > 0$ and $0 < \rho < \min\{1, \beta\}$. Then

$$\begin{aligned}
&-\Delta\bar{u} - d(x)^\alpha \bar{u}^p \\
&\ge K\rho\lambda_1(\Omega)\phi_1^\rho - K\rho(\rho-1)\phi_1^{\rho-2}|\nabla\phi_1|^2 - C^\alpha K^p \phi_1^{p\rho+\alpha} \\
&= K\rho\phi_1^{\rho-2}\Big[\lambda_1(\Omega)\phi_1^2 - (\rho-1)|\nabla\phi_1|^2 - \frac{C^\alpha K^{p-1}\phi_1^{p\rho+\alpha-\rho+2}}{\rho}\Big], \quad x \in \Omega,
\end{aligned}$$

for some constant $C > 0$. Hence, for sufficiently large K, it follows that $\underline{u} \le \overline{u}$ and

$$-\Delta \bar{u} - d(x)^{\alpha} \bar{u}^{p} \ge 0, \quad x \in \Omega.$$

By the method of supersolution and subsolution (refer to [14, Chapter 4] or [24, Chapter 1]), and the elliptic regularity, (1.3) has a positive solution in $C^2(\Omega) \cap C(\overline{\Omega})$.

Now, we show that the positive solution to (1.3) is unique as $p \le 0$. By Lemma 2.2, there exists $c_* > 0$ such that for any positive solution u to (1.3), we have

$$u(x) \ge c_* \phi_1^{\frac{2+\alpha}{1-p}}, \forall x \in \Omega.$$

Take sufficiently small m satisfying $m < c_*$, then the exists a minimal positive solution v_* in $[m\phi_1^{\beta}, K\phi_1^{\rho}]$. Assume that v is an arbitrary positive solution to (1.3). Then $v(x) \ge m\phi_1^{\beta}$. Hence, $\min\{v, K\phi_1^{\rho}\}$ is a supersolution to (1.3), and satisfies

$$m\phi_1^{\beta} \le \min\{v, K\phi_1^{\rho}\}.$$

So, we have

$$v_* \le \min\{v, K\phi_1^{\rho}\},$$

Furthermore, we obtain $v_* \le v$. This implies that v_* is the minimal positive solution. In order to show the uniqueness, we only need to infer $v_* = v$. Suppose $v_* \ne v$, then there exists $x_0 \in \Omega$ such that

$$v_*(x_0) - v(x_0) = \min\{v_*(x) - v(x) : x \in \Omega\} < 0.$$

By $p < 0$ it follows that

$$0 \ge -\Delta(v_* - v)(x_0) = d(x)^{\alpha}(v_*(x_0)^p - v(x_0)^p) > 0,$$

which implies a contradiction. □

Proof of Theorem 1.7. Suppose that (1.2) has a positive solution $u \in C^2(\mathbb{R}^N \backslash \overline{\Omega})$. Take a $B_R := B_R(0)$ satisfying $\Omega \subset B_{R/2}$, then

$$C_1(\alpha)|x|^{\alpha} u^p \le -\Delta u \le C_2(\alpha)|x|^{\alpha} u^p, \quad x \in \mathbb{R}^N \setminus \overline{B}_R, \tag{4.5}$$

where

$$C_1(\alpha) = \begin{cases} (1/2)^{\alpha}, & \alpha \ge 0, \\ (3/2)^{\alpha}, & \alpha < 0, \end{cases} \quad C_2(\alpha) = \begin{cases} (3/2)^{\alpha}, & \alpha \ge 0, \\ (1/2)^{\alpha}, & \alpha < 0. \end{cases}$$

We denote

$$\bar{u}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta) \mathrm{d}\theta.$$

By the Jensen's inequality, we have

$$\bar{u}^p \le \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta)^p \mathrm{d}\theta.$$

From this and (4.5), we obtain

$$\bar{u}'' + \frac{N-1}{r} \bar{u}' \le -C_1(\alpha) r^{\alpha} \bar{u}^p, \quad r \in [R, \infty). \tag{4.6}$$

This implies

$$(r^{N-1} \bar{u}'(r))' < -C_1(\alpha) r^{N-1+\alpha} \bar{u}^p < 0, \quad r \in [R, +\infty). \tag{4.7}$$

Hence, $r^{N-1}\bar{u}'(r)$ is decreasing in $[R, +\infty)$. We denote $\beta := \lim_{r\to+\infty} r^{N-1}\bar{u}'(r)$, Then $\beta \in [-\infty, +\infty)$.

Now, we prove the conclusion (i). If $\beta \in [0, +\infty)$, then there exists a large $R_1 > R$ such that

$$R_1^{N-1} \bar{u}'(R_1) r^{1-N} = C r^{1-N} > \bar{u}'(r) > 0, \quad r \in [R_1, +\infty).$$

We denote $\gamma := \lim_{r\to+\infty} \bar{u}(r) \in (0, +\infty]$. For arbitrary $R_1 < r_1 < r_2 < +\infty$, integrate from r_1 to r_2, and we see that

$$C \int_{r_1}^{r_2} s^{1-N} \mathrm{d}s > \bar{u}(r_2) - \bar{u}(r_1),$$

which implies $\gamma < +\infty$. In fact,

$$\gamma = \lim_{r_2\to\infty} \bar{u}(r_2) \le \bar{u}(r_1) + C\int_{r_1}^{+\infty} s^{1-N}\,\mathrm{d}s = \bar{u}(r_1) + \frac{Cr_1^{2-N}}{N-2} < +\infty.$$

If $\beta < 0$, then there exists $R_2 > R$ satisfying

$$r^{N-1}\bar{u}'(r) < 0, \quad r \in [R_2, +\infty).$$

This implies $\bar{u}'(r) < 0$, $r \in [R_2, +\infty)$, and hence $\lim_{r\to+\infty} \bar{u}(r) < +\infty$. Therefore, it is impossible that $\lim_{|x|\to+\infty} u(x) = +\infty$.

(ii) Suppose that u satisfies $u(x) \ge c > 0$ for the sufficiently large $|x|$. Assume $\beta \in [0, +\infty)$. Clearly, when $r \ge R$, $r^{N-1}\bar{u}'(r) > 0$ holds. For (4.7), integrating in $[r_*, r] \subset [R, +\infty)$, we have

$$\int_{r_*}^{r} (s^{N-1}\bar{u}'(s))'\mathrm{d}s \le -C_1(\alpha)\int_{r_*}^{r} s^{N-1+\alpha}\bar{u}(s)^p\mathrm{d}s.$$

Hence, for $r \in (r_*, +\infty)$,

$$\int_{r_*}^{r} s^{N-1+\alpha}\bar{u}(s)^p\mathrm{d}s \le C_1(\alpha)^{-1}\left[r_*^{N-1}\bar{u}'(r_*) - r^{N-1}\bar{u}'(r)\right] < C_1(\alpha)^{-1}r_*^{N-1}\bar{u}'(r_*).$$

By $u'(r) > 0$, $u(r)$ is increasing in $[R, +\infty)$. So, it is easy to see that

$$\bar{u}(r_*)^p\int_{r_*}^{r} s^{N-1+\alpha}\mathrm{d}s < C_1(\alpha)^{-1}r_*^{N-1}\bar{u}'(r_*), \quad r \in (r_*, +\infty). \tag{4.8}$$

By $\alpha \ge -2$, clearly, $N + \alpha \ge 0$. Letting $r \to +\infty$ in (4.8), we obtain a contradiction.

Suppose $-\infty < \beta < 0$. Integrate (4.7) in $[r_*, r] \subset [R, +\infty)$, and we obtain

$$\int_{r_*}^{r} (s^{N-1}\bar{u}'(s))'\mathrm{d}s \le -C_1(\alpha)\int_{r_*}^{r} s^{N-1+\alpha}\bar{u}(s)^p\mathrm{d}s.$$

From $\beta < 0$, there exists $R_3 > R$ such that

$$r^{N-1}\bar{u}'(r) < 0, \quad r \in [R_3, +\infty).$$

So, $\bar{u}$ is decreasing in $[R_3, +\infty)$. Therefore, for $R_3 \le r_* < r$ we have

$$\begin{aligned}
\bar{u}(r)^p\int_{r_*}^{r} s^{N-1+\alpha}\mathrm{d}s &\le \int_{r_*}^{r} s^{N-1+\alpha}\bar{u}(s)^p\mathrm{d}s \\
&\le -C_1(\alpha)^{-1}\int_{r_*}^{r} (s^{N-1}\bar{u}'(s))'\mathrm{d}s \\
&= C_1(\alpha)^{-1}r_*^{N-1}\bar{u}'(r_*) - C_1(\alpha)^{-1}r^{N-1}\bar{u}'(r) \\
&\le -C_1(\alpha)^{-1}r^{N-1}\bar{u}'(r).
\end{aligned}$$

So, we obtain

$$c^p\int_{r_*}^{r} s^{N-1+\alpha}\mathrm{d}s \le -C_1(\alpha)^{-1}r^{N-1}\bar{u}'(r). \tag{4.9}$$

Letting $r \to +\infty$, we can derive a contradiction.

Suppose $\beta = -\infty$. By the condition there exists $R_4 > R$ such that for all $r \ge R_4$

$$(r^{N-1}\bar{u}'(r))' \le -C_1(\alpha)r^{N-1+\alpha}\bar{u}^p \le -c^pC_1(\alpha)r^{N-1+\alpha}.$$

By integrating we have

$$r^{N-1}\bar{u}'(r) - R_4^{N-1}\bar{u}'(R_4) \le \frac{c^pC_1(\alpha)}{N+\alpha}(R_4^{N+\alpha} - r^{N+\alpha})$$

By $\beta = -\infty$, there exists a sufficiently large $R_5 > R_4$ such that for any $r \ge R_5$

$$r^{N-1}\bar{u}'(r) \le \frac{c^pC_1(\alpha)}{N+\alpha}(R_4^{N+\alpha} - r^{N+\alpha}) \le -C(\alpha)r^{N+\alpha},$$

where

$$C(\alpha) = \frac{c^pC_1(\alpha)}{2(N+\alpha)} > 0.$$

Hence, we obtain

$$\bar{u}'(r) \le -C(\alpha) r^{1+\alpha}, \quad r \in [R_5, +\infty).$$

By integrating, we obtain

$$\bar{u}(r) - \bar{u}(R_5) \le -C(\alpha) \int_{R_5}^{r} s^{1+\alpha} \mathrm{d}s.$$

By $\alpha \ge -2$ and letting $r \to +\infty$, we can derive a contradiction.

(iii) From (4.6), we have

$$\bar{u}_{rr} + \frac{N-1}{r} \bar{u}_r \le -C_1(\alpha) r^{\alpha} \bar{u}^p, \quad r \in [R, \infty). \tag{4.10}$$

Let $s = r^{N-2}$ and $\phi(s) := s\bar{u}(r) = s\bar{u}(s^{\frac{1}{N-2}})$. By calculations, we obtain

$$\begin{aligned}
\phi_s &= \bar{u}(r) + \frac{1}{N-2} r \bar{u}_r, \\
\phi_{ss} &= \frac{1}{N-2} s^{\frac{3-N}{N-2}} \bar{u}_r + {\frac{1}{N-2}}^{2} s^{\frac{4-N}{N-2}} \bar{u}_{rr} + {\frac{1}{N-2}}^{2} s^{\frac{3-N}{N-2}} \bar{u}_r \\
&= {\frac{1}{N-2}}^{2} (N-1) s^{\frac{3-N}{N-2}} \bar{u}_r + {\frac{1}{N-2}}^{2} s^{\frac{4-N}{N-2}} \bar{u}_{rr} \\
&= {\frac{1}{N-2}}^{2} s^{\frac{4-N}{N-2}} (\bar{u}_{rr} + \frac{N-1}{r} \bar{u}_r).
\end{aligned}$$

From this and (4.10), we deduce

$$\begin{aligned}
\phi_{ss} &\le -{\frac{1}{N-2}}^{2} s^{\frac{4-N}{N-2}} C_1(\alpha) r^{\alpha} \bar{u}^p \\
&= -C_1(\alpha) {\frac{1}{N-2}}^{2} s^{\frac{4-N}{N-2}} s^{\frac{\alpha}{N-2}} s^{-p} \phi^p \\
&= -C_1(\alpha) \frac{1}{s^{1-\frac{2+\alpha}{N-2}+p}} \phi^p;
\end{aligned}$$

that is,

$$\phi_{ss} + C_1(\alpha) \frac{1}{s^{1-\frac{2+\alpha}{N-2}+p}} \phi^p \le 0.$$

From $1 < p \le \frac{N+\alpha}{N-2}$, it follows that

$$\int_R^{\infty} s \cdot \frac{1}{s^{1-\frac{2+\alpha}{N-2}+p}} \mathrm{d}s = \int_R^{\infty} s^{\frac{2+\alpha}{N-2}-p} \mathrm{d}s = \infty.$$

We denote $f(t) = t^p$, and hence

$$\phi_{ss} + \frac{C_1(\alpha)}{s^{1-\frac{2+\alpha}{N-2}+p}} f(\phi) \le 0.$$

From Lemma 2.3 we can derive a contradiction. □

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