Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 73, pp. 1–22.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.73

STABILIZATION OF A DEGENERATE BEAM ON AN ELASTIC FOUNDATION WITH AXIAL FORCE UNDER FRACTIONAL DERIVATIVE-TYPE DELAYED BOUNDARY FEEDBACK

LANZHI CAO, SHENG-JIE LI, SHUGEN CHAI

Abstract. In this article, we investigate the exponential stabilization of a degenerate Euler-Bernoulli beam on an elastic foundation with axial force. The control input at the boundary features a fractional derivative-type delay. We construct a tailored Lyapunov functional, specifically designed to offset the effects of time delay and capture the intrinsic internal dynamics of the degenerate beam, thereby rigorously establishing the exponential stability of the system.

1. Introduction

This article investigates a degenerate Euler-Bernoulli beam equation fo the form

$$\begin{gathered} u_{tt}(x,t)+(au_{xx})_{xx}(x,t)-(bu_x)_x(x,t)+cu(x,t)=0, \quad (x,t)\in Q, \\ Bu(0,t)=0, \quad u_{xx}(1,t)=0, \quad t>0, \\ (au_{xx})_x(1,t)-(bu_x)(1,t)=r_1u_t(1,t)+r_2\partial_t^{\alpha,\eta}u(1,t-s), \quad t>0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in[0,1], \\ u_t(1,t-s)=f_0(t-s), \quad t\in(0,s), \end{gathered} \tag{1.1}$$

where $Q:=(0,1)\times(0,+\infty)$, u_0, u_1 and f_0 are the initial data given in suitable function spaces. $a\in C^0[0,1]$, with $a>0$ on (0,1] and $a(0)=0$, the constant $s>0$ represents the time delay. r_1 and r_2 are positive real numbers such that

$$\eta^{\alpha-1}r_2<r_1<b_0\Big(\max\Big\{\frac{1}{b_0},2\max\Big\{\frac{1}{b_0},\frac{1}{a(1)(2-K_a)}\Big\}\Big\}\max\Big\{\frac{3}{b_0},\frac{2}{a(1)(2-K_a)}\Big\}\Big)^{-1}. \tag{1.2}$$

In particular,

$$Bu:=\begin{cases} u(0,t)=0, \\ u_x(0,t)=0, & \text{if } a \text{ is weakly degenerate (WD)}, \\ (au_{xx})(0,t)=0 & \text{if } a \text{ is strongly degenerate (SD)}, \end{cases} \tag{1.3}$$

The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $\alpha>0$, with respect to the time variable. It is defined as follows

$$\partial_t^{\alpha,\eta}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}e^{-\eta(t-\tau)}\frac{d}{d\tau}u(\tau)\,d\tau, \quad \eta\geq 0. \tag{1.4}$$

Frist, we provide a brief explanation of the definition of the Caputo's fractional derivative $\partial_t^{\alpha,\eta}u(t-s)$. Let us define $v(t)=u(t-s)$. Then, $\frac{d}{dt}v(t)=\frac{d}{dt}u(t-s)$. Thus, the form of the

---

2020 *Mathematics Subject Classification.* P58J35, 53E20.

Key words and phrases. Degenerate beam equation; elastic foundation; axial force; fractional derivative; time delay; Lyapunov functional.

Submitted February 12, 2026. Published September 22, 2026.

Caputo's fractional derivative can be written as

$$\begin{aligned}\partial_t^{\alpha,\eta} v(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} \frac{d}{d\tau} v(\tau)\, d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} \frac{d}{d\tau} u(\tau - s)\, d\tau,\end{aligned} \tag{1.5}$$

i.e.,

$$\partial_t^{\alpha,\eta} u(t-s) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} \frac{d}{d\tau} u(\tau - s)\, d\tau. \tag{1.6}$$

The degeneracy of (1.1) at $x = 0$ is measured by the parameter

$$K_a := \sup_{x\in(0,1]} \frac{xa'(x)}{a(x)}. \tag{1.7}$$

And, we say that

$$\begin{gathered} a \text{ is weakly degenerate (WD) if } a \in C^0[0,1] \cap C^1(0,1] \text{ and } K_a \in [0,1), \\ a \text{ is Strongly degenerate (SD) if } a \in C^1(0,1] \text{ and } K_a \in [1,2). \end{gathered} \tag{1.8}$$

The condition $K_a < 2$ is essential for the ensuing analysis. The axial force disturbance function b and foundation's stiffness coefficient c ($c > 0$) satisfy the following conditions

$$\begin{gathered} b, c \in W^{1,\infty}(0,1), \\ 0 < b_0 \le b(x) \le b_1, \quad 0 < c_0 \le c(x) \le c_1, \quad \forall x \in [0,1], \\ |b'(x)| \le b_2, \quad |c'(x)| \le c_2, \quad \forall x \in [0,1], \end{gathered} \tag{1.9}$$

where $b_0, b_1, b_2, c_0, c_1, c_2$ are positive constants satisfying

$$\frac{4c_0c_1 + 3c_1c_2}{2c_0b_0} < 2 - K_{a,b,c}, \tag{1.10}$$

where the constant $K_{a,b,c}$ is given by (4.25) below.

The control of flexible structures - typified by Euler-Bernoulli beams (e.g., [18]) - stands as a foundational component of modern engineering, with pivotal applications spanning aerospace structures [28], robotic manipulators [13, 17], long-span bridges [4], and precision instruments [27]. In these application domains, the structure interacts with its environment being mounted on an elastic foundation or subjected to significant axial loads [19]. This interaction is typically modeled by two key terms: a distributed restoring term, cu, representing the elastic foundation, and a geometric term, $(bu_x)_x$, representing the axial force. The cu term represents an elastic restoring force (elastic foundation) proportional to the transverse displacement u, where $c > 0$ is the foundation's stiffness coefficient. The $(bu_x)_x$ term captures the geometric stiffness induced by the axial load. Collectively, these terms critically influence the system's vibrational dynamics and stability.

Consequently, a particularly challenging scenario arises when the beam exhibits material degradation or geometric tailoring, leading to a loss of stiffness at a boundary. This degeneracy, mathematically characterized by a flexural rigidity coefficient $a(x)$ that vanishes (*e.g.*, $a(0) = 0$), renders classical analysis tools ineffective and demands a formulation in weighted Sobolev spaces, see [7, 8]. In [7], the authors considered the following viscoelastic beam

$$\begin{gathered} u_{tt}(x,t) + (au_{xx})_{xx}(x,t) = 0, \quad (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, \quad t > 0, \\ u_x(0,t) = 0, \quad \text{if } a \text{ is (WD)}, \quad t \in (0,T) \\ (au_{xx})(0,t) = 0, \quad \text{if } a \text{ is (SD)}, \quad t \in (0,T), \\ \beta u(1,t) - \big(au_{xx}\big)_x(1,t) + u_t(1,t) = 0, \quad t \in (0,T), \\ \gamma u_x(1,t) + \big(au_{xx}\big)(1,t) + u_{tx}(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,1), \end{gathered}$$

where $\beta, \gamma \geq 0$. Within the framework of weighted spaces, they examined the boundary stability of above system.

While the stabilization of non-degenerate beams is well-established (see for example [9, 10, 24, 25]), research on controlling degenerate systems, particularly under the combined effects of an elastic foundation and axial force, remains considerably limited. A further layer of practical complexity is introduced by time delays, which are invariably present in sensor and actuator signals and can destabilize an otherwise stable system [11]. Consequently, the stabilization of time-delay systems has garnered significant attention and remains an active area of research in both mathematical control theory and engineering applications (*e.g.*, [15, 16, 21, 26]). Notably, the fractional derivative-type delayed feedback (see for example [3, 1]), which incorporates the system's state history through fractional calculus, provides a powerful framework for modeling complex memory effects and hereditary properties in both materials and control systems. The preceding analysis substantiates the necessity of this investigation.

This paper is organized as follows: In section 2, we introduce the functional framework and reframe the system (1.1) into an augmented system, followed by relevant preliminary lemmas and key preparatory work. In Section 3, we establish the well-posedness of the augmented system by employing the semigroup theory. Finally, we derive the exponential stability result via the method proposed by Komornik [14], and this stability result is closely associated with the order of the fractional derivative in Section 4.

2. Preliminary and augmented model

This section is devoted to defining the functional spaces employed throughout the paper and reformulating the problem (1.1) into an augmented system. We start with the following lemmas.

Lemma 2.1 ([20]). *Let*

$$\mu(\xi) := |\xi|^{\frac{(2\alpha-1)}{2}}, \quad \xi \in \mathbb{R}, \quad 0 < \alpha < 1.$$

Then the relationship between the "input" U and "output" O of the system

$$\begin{gathered} \phi_t(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad \xi \in \mathbb{R}, \quad t > 0, \quad \beta \geq 0, \\ \phi(\xi,0) = 0, \\ O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty} \phi(\xi,t)\mu(\xi)d\xi \end{gathered} \tag{2.1}$$

is given by

$$O := I^{1-\alpha,\eta}U,$$

where

$$I^{\alpha,\eta}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}u(\tau)\,d\tau.$$

Lemma 2.2 ([1]). *If $\lambda \in D = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \neq 0\}$, then*

$$D_\mu := \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}d\xi = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\eta)^{\alpha-1}.$$

From now on, we restrict D_μ so that $D_\mu < 2$, which is crucial for the proof of our conclusion.

Lemma 2.3 ([2]). *Let $\alpha \in (0,1)$, $\eta \geq 0$, then the following integrals*

$$\mathsf{S}_0(\eta,\alpha) = \kappa(\alpha)\int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1+\xi^2+\eta}d\xi, \quad \mathsf{S}_1(\eta,\alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1+\xi^2+\eta)^2}d\xi, \mathsf{S}_2(\eta,\alpha) = \int_0^{+\infty} \frac{\xi^{2\alpha+1}}{(1+\xi^2+\eta)^2}d\xi$$

are well defined.

As in [22], we introduce the new variable

$$z(x,\rho,t) = u_t(1,t-s\rho), \quad x \in (0,1),\ \rho \in (0,1),\ t > 0. \tag{2.2}$$

Then, we deduce that

$$sz_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in (0,1),\ \rho \in (0,1),\ t > 0. \tag{2.3}$$

Therefore, by (2.2), (2.3), Lemma 2.1 and (1.2), system (1.1) is recast into the augmented model

$$\begin{gathered}
u_{tt}(x,t)+(au_{xx})_{xx}(x,t)-(bu_x)_x(x,t)+cu(x,t)=0, \quad (x,t)\in Q,\\
\phi_t(x,\xi,t)+(\xi^2+\eta)\phi(x,\xi,t)-z(x,1,t)\mu(\xi)=0, \quad (x,t)\in Q,\ \xi\in\mathbb{R},\\
sz_t(x,\rho,t)+z_\rho(x,\rho,t)=0, \quad (x,t)\in Q,\ \rho\in(0,1),\\
Bu(0,t)=0, \quad t>0,\\
u_{xx}(1,t)=0, \quad t>0,\\
(au_{xx})_x(1,t)-(bu_x)(1,t)=r_1z(x,0,t)+\gamma\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi, \quad t>0, \quad \xi\in\mathbb{R},\\
\phi(x,\xi,0)=0, \quad x\in(0,1),\ \xi\in\mathbb{R},\\
z(x,0,t)=u_t(1,t) \quad (x,t)\in Q,\\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in[0,1],\\
z(x,\rho,0)=f_0(x,-\rho s), x\in(0,1), \quad \rho\in(0,1),
\end{gathered} \tag{2.4}$$

where

$$\gamma:=(\pi)^{-1}\sin(\alpha\pi)r_2. \tag{2.5}$$

Here, we claim that

$$\gamma D_\mu=r_2\eta^{\alpha-1}. \tag{2.6}$$

We define the energy associated with the solution of the system (2.4) as

$$E(t):=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\sqrt{a}u_{xx}\|_2^2+\frac{1}{2}\|\sqrt{b}u_x\|_2^2+\frac{1}{2}\|\sqrt{c}u\|_2^2+\frac{\gamma}{2}\int_{-\infty}^{+\infty}|\phi(\xi)|^2d\xi+\frac{\delta}{2}\int_0^1|z(\rho)|^2d\rho, \tag{2.7}$$

where δ is a positive constant satisfying

$$s\gamma D_\mu<\delta<s[2r_1-\gamma D_\mu] \tag{2.8}$$

and D_μ is defined in Lemma 2.2.

Lemma 2.4. *Let (u,ϕ,z) be a regular solution of* (2.4)*. Then, the energy functional defined by* (2.7) *satisfies*

$$\frac{dE(t)}{dt}\le -C_e[z^2(x,0,t)+z^2(x,1,t)] \tag{2.9}$$

where $C_e:=\min\Big\{\big(\frac{\delta}{2s}-\frac{\gamma}{2}D_\mu\big),\big(r_1-\frac{\gamma}{2}D_\mu-\frac{\delta}{2s}\big)\Big\}$.

Proof. Multiplying the first equation in (2.4) by u_t, integrating by parts over $(0,1)$, we obtain

$$\frac{d}{dt}\Big\{\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\sqrt{a}u_{xx}\|_2^2+\frac{1}{2}\|\sqrt{b}u_x\|_2^2+\frac{1}{2}\|\sqrt{c}u\|_2^2\Big\}=-r_1u_t^2(1)-\gamma\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi u_t(1). \tag{2.10}$$

Multiplying the second equation in (2.4) by $\gamma\phi$, andintegrating by parts over $(-\infty,+\infty)$, we obtain

$$\frac{d}{dt}\Big\{\frac{\gamma}{2}\int_{-\infty}^{+\infty}|\phi(\xi)|^2d\xi\Big\}+\gamma\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2d\xi-\gamma z(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi=0. \tag{2.11}$$

Multiplying the third equation in (2.4) by $\frac{\delta}{s}z$, and integrating by parts over $(0,1)$, we have

$$\frac{d}{dt}\Big\{\frac{\delta}{2}\int_0^1|z(\rho)|^2d\rho\Big\}=-\frac{\delta}{2s}\int_0^1\frac{\partial}{\partial\rho}z^2(\rho)d\rho=-\frac{\delta}{2s}[z^2(x,1,t)-z^2(x,0,t)]. \tag{2.12}$$

From (2.10), (2.11), (2.12) and using $z(x,0,t)=u_t(1,t)$, we obtain

$$\begin{aligned}
\frac{dE(t)}{dt}&=-r_1z^2(x,0,t)-\gamma\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi u_t(1)-\gamma\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2d\xi\\
&\quad+\gamma z(x,1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi-\frac{\delta}{2s}[z^2(x,1,t)-z^2(x,0,t)].
\end{aligned} \tag{2.13}$$

By using Cauchy-Schwarz inequality, Young inequality, (2.8) and Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{dE(t)}{dt}\\
&\le -r_1 z^2(x,0,t) + \frac{\gamma}{2} D_\mu z^2(x,0,t) + \frac{\gamma}{2}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2 d\xi + \frac{\gamma}{2} D_\mu z^2(x,1,t)\\
&\quad + \frac{\gamma}{2}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2 d\xi - \gamma\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi)|^2 d\xi - \frac{\delta}{2s}[z^2(x,1,t) - z^2(x,0,t)]\\
&= -\Big(\frac{\delta}{2s} - \frac{\gamma}{2}D_\mu\Big) z^2(x,1,t) - \Big(r_1 - \frac{\delta}{2s} - \frac{\gamma}{2}D_\mu\Big) z^2(x,0,t)\\
&\le -C_e[z^2(x,1,t) + z^2(x,0,t)].
\end{aligned} \tag{2.14}$$

The proof is complete. □

As in [7], we introduce the following weighted Hilbert spaces:

$$\begin{gathered}
V_a^2(0,1) := \{u \in H^1(0,1) : u_x \text{ is absolutely continuous in } [0,1], \sqrt{a}u_{xx} \in L^2(0,1) \text{ if } a \text{ is (WD)}\},\\
V_a^2(0,1) := \{u \in H^1(0,1) : u_x \text{ is locally absolutely cont. in } (0,1], \sqrt{a}u_{xx} \in L^2(0,1) \text{ if } a \text{ is (SD)}\},\\
\langle u,v\rangle_a := \int_0^1 uv dx + \int_0^1 u_x v_x dx + \int_0^1 a u_{xx} v_{xx} dx, \text{ for every } u,v \in V_a^2.
\end{gathered} \tag{2.15}$$

The previous inner products induces the related norm

$$\|u\|_{2,a}^2 := \|u\|_{L^2(0,1)}^2 + \|u_x\|_{L^2(0,1)}^2 + \|\sqrt{a}u_{xx}\|_{L^2(0,1)}^2, \quad \forall\, u \in V_a^2(0,1). \tag{2.16}$$

Also, we define the weighted space

$$H_a^2(0,1) := \{u \in V_a^2(0,1) : u(0) = 0\}, \tag{2.17}$$

endowed with the inner product defined by (2.15).

Proposition 2.5 ([7])**.** *Assume a is weakly or strongly degenrate. For all $u \in H_a^2$, we have*

$$\|u\|_{L^2(0,1)}^2 \le \|u_x\|_{L^2(0,1)}^2 \le \frac{1}{b_0}\|\sqrt{b}u_x\|_{L^2(0,1)}^2, \tag{2.18}$$

$$|u_x(1)|^2 \le 2\Big(\frac{1}{b_0}\|\sqrt{b}u_x\|_{L^2(0,1)}^2 + \frac{\|\sqrt{a}u_{xx}\|_{L^2(0,1)}^2}{a(1)(2-K_a)}\Big). \tag{2.19}$$

We can find the proofs of (2.18) and (2.19) in [7]. We set

$$K(0,1) := \{u \in H_a^2(0,1) : au_{xx} \in H^2(0,1)\}. \tag{2.20}$$

Next, we introduce the functional spaces

$$\begin{gathered}
H_{a,0}^2(0,1) := \{u \in H_a^2(0,1) : u_x(0) = 0 \text{ if } a \text{ is } (WD)\},\\
H_{a,0}^2(0,1) := \{H_{a,0}^2(0,1) \text{ if } a \text{ is } (SD)\},\\
K_0(0,1) := \{u \in K(0,1) : u_x(0) = 0 \text{ if } a \text{ is } (WD)\},\\
K_0(0,1) := \{u \in K(0,1) : (au_{xx})(0) = 0 \text{ if } a \text{ is } (SD)\}.
\end{gathered} \tag{2.21}$$

3. Well-posedness

In this section, we aim to establish the well-posedness of system (2.4) through semigroup theory. We consider $v(x,t) = u_t(x,t)$. Let $U = (u,v,\phi,z)^\top$, then (2.4) can be written as

$$\begin{gathered}
U_t - \mathcal{A}U = 0,\\
U_0 = (u_0, u_1, 0, f_0(-\rho s))^\top,
\end{gathered} \tag{3.1}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U := \begin{pmatrix} v \\ -(au_{xx})_{xx} + (bu_x)_x - cu \\ -(\xi^2+\eta)\phi(\xi) + z(1)\mu(\xi) \\ -\frac{1}{s} z_\rho(\rho) \end{pmatrix} \tag{3.2}$$

with domain $D(\mathcal{A})$ as follows:

$$\begin{gathered} U \in \mathcal{H} : u \in K_0(0,1), v \in H^2_{a,0}(0,1), \\ z_\rho \in L^2(0,1), v = z(\cdot,0), \\ \phi, |\xi|\phi \in L^2(-\infty,+\infty), \\ (\xi^2+\eta)\phi(\xi) - z(1)\mu(\xi) \in L^2(-\infty,+\infty), \\ (au_{xx})_x(1) - bu_x(1) - r_1 u_t(1) - \gamma \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)d\xi = 0, \end{gathered} \tag{3.3}$$

where the Hilbert space $\mathcal{H}$ is

$$\mathcal{H} := H^2_{a,0} \times L^2(0,1) \times L^2(-\infty,+\infty) \times L^2(0,1) \tag{3.4}$$

equipped with the inner product

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} := \int_0^1 v\tilde{v}dx + \int_0^1 au_{xx}\tilde{u}_{xx}dx + \int_0^1 bu_x\tilde{u}_x dx + \int_0^1 cu\tilde{u}dx + \delta\int_0^1 z\tilde{z}dx + \gamma\int_{-\infty}^{+\infty}\phi^2 d\xi, \tag{3.5}$$

where r_1, r_2 satisfy (1.2), and δ satisfies (2.8).

Proposition 3.1. *Assume a is weakly or strongly degenrate. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is nonpositive with dense domain and generates a contraction C_0-semigroup $(T(t))_{t\geq 0}$.*

Proof. We proceed in the following two steps.

Step 1. We prove that $\mathcal{A}$ is dissipative: let $U = u, v, \phi, z \in D(\mathcal{A})$. We have

$$\Re(\langle \mathcal{A}U, U\rangle_{\mathcal{H}}) \leq -C_e[z^2(x,1,t) + z^2(x,0,t)] \leq 0. \tag{3.6}$$

Then, $\mathcal{A}$ is dissipative.

Step 2. We show that $I - \mathcal{A}$ is surjective: given $F = (f,g,h,l)^\top \in \mathcal{H}$. We need to prove the existence of $U = u, v, \phi, z \in D(\mathcal{A})$ unique solution of the equation

$$(I - \mathcal{A})U = F, \tag{3.7}$$

which is equivalent to the system

$$\begin{gathered} u - v = f, \\ v + (au_{xx})_{xx} - (bu_x)_x + cu = g, \\ \phi + (\xi^2+\eta)\phi - z(1)\mu(\xi) = h, \\ z + \frac{1}{s} z_\rho = l. \end{gathered} \tag{3.8}$$

From $(3.8)_3$, we have

$$\phi = \frac{h}{\xi^2+\eta+1} + \frac{\mu(\xi)z(1)}{\xi^2+\eta+1}. \tag{3.9}$$

Combining $(3.8)_1$ and $(3.8)_4$, we obtain

$$\begin{gathered} z = (u(1) - f(1))e^{-s\rho} + s\int_0^\rho e^{s(r-\rho)} l(r)dr, \quad \rho \in (0,1), \\ v = u - f, \end{gathered} \tag{3.10}$$

which satisfies $z(x,0,t) = u(1,t) - f(1,t)$. Then, from $(3.8)_2$, it can be deduced that

$$(au_{xx})_{xx} - (bu_x)_x + cu + u = f + g. \tag{3.11}$$

Let $\psi \in H^2_{a,0}(0,1)$. Multiplying (3.11) by ψ and integrating over (0,1), we obtain

$$\begin{aligned}
&\int_0^1 (f+g)\psi\, dx \\
&= \int_0^1 [(au_{xx})_{xx} - (bu_x)_x + (c+1)u]\psi dx \\
&= \int_0^1 [au_{xx}\psi_{xx} + bu_x\psi_x + (c+1)u\psi]dx + r_1 v \\
&\quad + \gamma \int_{-\infty}^{+\infty} \mu(\xi)\frac{h}{\xi^2+\eta+1} d\xi \psi(1) + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi z(x,1)\psi(1) \\
&= \int_0^1 [au_{xx}\psi_{xx} + bu_x\psi_x + (c+1)u\psi]dx + r_1 v \\
&\quad - r_1 f(1) + \gamma \int_{-\infty}^{+\infty} \mu(\xi)\frac{h}{\xi^2+\eta+1} d\xi \psi(1) + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi u(1)e^{-s\rho}\psi(1) \\
&\quad - \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi f(1) e^{-s\rho}\psi(1) + \gamma s \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi \int_0^\rho e^{s(r-\rho)} l(r) dr \psi(1).
\end{aligned} \tag{3.12}$$

Then,

$$\begin{aligned}
&\int_0^1 [(au_{xx})_{xx} - (bu_x)_x + (c+1)u]\psi dx + \Big(r_1 + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi e^{-s\rho}\Big) u(1)\psi(1) \\
&= \int_0^1 (f+g)\psi(1) dx + \Big[r_1 f(1) + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi f(1) e^{-s\rho} \\
&\quad - \gamma s \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi \int_0^\rho e^{s(r-\rho)} l(r) dr - \gamma \int_{-\infty}^{+\infty} \mu(\xi)\frac{h}{\xi^2+\eta+1} d\xi\Big]\psi(1).
\end{aligned} \tag{3.13}$$

So, the preceding equation can be written as

$$\mathcal{L}(u,\psi) = \mathcal{M}(\psi), \quad \forall \psi \in H^2_{a,0}(0,1), \tag{3.14}$$

where the bilinear form $\mathcal{L}(u,\psi) : H^2_{a,0}(0,1) \times H^2_{a,0}(0,1) \to \mathbb{R}$ is defined by

$$\mathcal{L}(u,\psi) := \int_0^1 [(au_{xx})_{xx} - (bu_x)_x + (c+1)u]\psi dx + A_1 u(1)\psi(1), \tag{3.15}$$

and the linear form $\mathcal{M}(\psi) : H^2_{a,0}(0,1) \to \mathbb{R}$ by

$$\mathcal{M}(\psi) := \int_0^1 (f+g)\psi(1) dx + A_2 \psi(1), \tag{3.16}$$

where

$$\begin{gathered}
A_1 := r_1 + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi e^{-s\rho}, \\
A_2 := A_1 f(1) - \gamma s \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi \int_0^\rho e^{s(r-\rho)} l(r) dr - \gamma \int_{-\infty}^{+\infty} \mu(\xi)\frac{h}{\xi^2+\eta+1} d\xi \psi(1).
\end{gathered} \tag{3.17}$$

It is clear that $\mathcal{L}$ is a continuous and coercive form on $H^2_{a,0}(0,1) \times H^2_{a,0}(0,1)$, and $\mathcal{M}$ is continuous form on $H^2_{a,0}(0,1)$. Hence, according to the Lax-Milgram theorem, the variational problem (3.14) has a unique solution $u \in H^2_{a,0}(0,1)$. In particular, for any $\psi \in \mathcal{D}(0,1)$ ($\mathcal{D}(0,1)$ is dense in $L^2(0,1)$) in (3.14), we obtain

$$\int_0^1 [(au_{xx})_{xx} - (bu_x)_x + (c+1)u]\bar{\psi} dx = \int_0^1 (f+g)\bar{\psi}(1) dx \text{ in } \mathcal{D}'(0,1). \tag{3.18}$$

As $(f+g) \in L^2(0,1)$, from (3.14), we deduce that

$$(au_{xx})_{xx} - (bu_x)_x + (c+1)u = f + g \quad \text{in} \quad L^2(0,1). \tag{3.19}$$

B ecuase $u \in H^2_{a,0}(0,1)$, and $au_{xx} \in H^2(0,1)$. Thus, $u \in K_0(0,1)$ and solves (3.14).

Setting $v := u - f$, we obtain $v \in H^2_{a,0}(0,1)$. It remains to prove that $U \in D(\mathcal{A})$ and solves (3.7). To this end, we need the following arguments. We define

$$\phi = \frac{h}{\xi^2+\eta+1} + \frac{\mu(\xi)z(1)}{\xi^2+\eta+1}. \tag{3.20}$$

We need to prove $\phi, |\xi|\phi \in L^2(-\infty,+\infty)$. From (3.9), using Lemma 2.3 and the fact that $h \in L^2(-\infty,+\infty)$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} |\phi|^2 d\xi &\le 2\int_{-\infty}^{+\infty} \frac{h^2}{(\xi^2+\eta+1)^2} d\xi + \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)z^2(1)}{(\xi^2+\eta+1)^2} d\xi \\ &\le 2\int_{-\infty}^{+\infty} \frac{h^2}{(\xi^2+\eta+1)^2} d\xi + 2\mathsf{S}_1 z^2(1) < \infty \end{aligned} \tag{3.21}$$

and

$$\int_{-\infty}^{+\infty} |\xi\phi|^2 d\xi \le 2\int_{-\infty}^{+\infty} \frac{\xi^2 h^2}{(\xi^2+\eta+1)^2} d\xi + \int_{-\infty}^{+\infty} \frac{\xi^{2\alpha+1} z^2(1)}{(\xi^2+\eta+1)^2} d\xi < \infty. \tag{3.22}$$

Then, we deduce that $\phi, |\xi|\phi \in L^2(-\infty,+\infty)$. From (3.20), we have

$$-(\xi^2+\eta)\phi + \mu(\xi)z(1) = \phi - h \in L^2(-\infty,+\infty). \tag{3.23}$$

Also, (3.14) holds for every $\psi \in H^2_{a,0}(0,1)$ and integrating by parts, we obtain that

$$\begin{aligned} &-(au_{xx})_x(1) + (bu_x)(1) + r_1 u(1) + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi e^{-s\rho} u(1) \\ &= r_1 f(1) + \gamma \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi e^{-s\rho} f(1) - \gamma \int_{-\infty}^{+\infty} \mu(\xi) \frac{h}{\xi^2+\eta+1} d\xi \\ &\quad + \gamma s \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+1} d\xi \int_0^\rho e^{s(r-\rho)} l(r) dr \end{aligned} \tag{3.24}$$

Combining (3.20), we deduce that

$$-(au_{xx})_x(1) + (bu_x)(1) + r_1 v(1) + \gamma \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) = 0. \tag{3.25}$$

Therefore,

$$U \in D(\mathcal{A}). \tag{3.26}$$

And the operator $I - \mathcal{A}$ is surjective. Then, $\mathcal{A}$ is densely defined in $\mathcal{H}$. According to Lümer-Phillips theorem, $\mathcal{A}$ generates a C_0-semigroup of contractions (see, e.g. [12]). $\square$

Building upon Proposition 3.1 and following the approach in [5], we first establish the existence theorem of the solution. To further achieve higher regularity of the solution in the time domain, we apply the Hille-Yosida theorem [23], with the specific conclusions detailed as follows:

Theorem 3.2. *Assume a is weakly or strongly degenrate. If $U_0 \in \mathcal{H}$, then problem (2.4) admits a unique weak solution satisfying*

$$U(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, (2.4) admits a unique strong solution U satisfying

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

4. Stability result

In this section, we consider the exponential stability of problem (2.4), we will present this result based on the Lyapunov functional method. At the beginning of this section, we study the degenerate elliptic equation associated to problem (2.4), which is crucial to obtain the stability result of problem (2.4).

Proposition 4.1. *Assume a is weakly or strongly degenrate. Define*

$$|||y|||^2 := \int_0^1 a(y_{xx})^2 dx + \int_0^1 b(y_x)^2 dx + \int_0^1 cy^2 dx \tag{4.1}$$

for all $y \in K_0(0,1)$. Then the norm $\cdot$ and $\|\cdot\|_{2,a}$ are equivalent on $K_0(0,1)$. In addition, for all $\lambda_1, \lambda_2 \in \mathbb{R}$, the variational problem

$$\int_0^1 ay_{xx}\varphi_{xx} dx + \int_0^1 by_x\varphi_x dx + \int_0^1 cy\varphi dx = \lambda_1\varphi(1) + \lambda_2\varphi_x(1), \quad \forall\varphi \in K_0(0,1) \tag{4.2}$$

admits a unique solution $y \in K_0(0,1)$, which satisfies the following estimates:

$$|||y|||^2 \le C^2_{\lambda,\mu,b_0,K_a,a} \text{ and } \|y\|^2_{L^2(0,1)} \le \frac{1}{b_0} C^2_{\lambda,\mu,b_0,K_a,a}, \tag{4.3}$$

with the constant

$$C_{\lambda,\mu,b_0,K_a,a} := |\lambda|\sqrt{\frac{1}{b_0}} + |\mu|\sqrt{2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}}. \tag{4.4}$$

Moreover $y \in D(\mathcal{A}_a) := K_0(0,1)$ satisfies

$$\begin{gathered} \mathcal{A}_a y := (ay_{xx})_{xx} - (by_x)_x + cy = 0, \\ (by_x)(1) - (ay_{xx})_x(1) = \lambda_1, \\ a(1)y_{xx}(1) = \lambda_2. \end{gathered} \tag{4.5}$$

Proof. From (2.16) and (4.1), we directly obtain the estimates

$$\begin{aligned} \int_0^1 a(y_{xx})^2 dx + \int_0^1 b(y_x)^2 dx + \int_0^1 cy^2 dx &\le \int_0^1 a(y_{xx})^2 dx + b_1 \int_0^1 (y_x)^2 dx + c_1 \int_0^1 y^2 dx \\ &\le \max\{c_1, b_1\}\|y\|^2_{L^2(0,1)} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \int_0^1 a(y_{xx})^2 dx + \int_0^1 b(y_x)^2 dx + \int_0^1 cy^2 dx &\ge \int_0^1 a(y_{xx})^2 dx + b_0 \int_0^1 (y_x)^2 dx + c_0 \int_0^1 y^2 dx \\ &\ge \min\{c_0, b_0\}\|y\|^2_{L^2(0,1)}. \end{aligned} \tag{4.7}$$

Thus, the first part of Proposition 4.1 holds in $K_0(0,1)$.

Now, we consider the bilinear and symmetric form $\Lambda : K_0(0,1) \times K_0(0,1) \to \mathbb{R}$ such that

$$\Lambda(y,\varphi) := \int_0^1 ay_{xx}\varphi_{xx} dx + \int_0^1 by_x\varphi_x dx + \int_0^1 cy\varphi dx. \tag{4.8}$$

It is straight forward to prove that Λ is coercive and continuous. Now, consider the linear functional

$$\mathfrak{L}(\varphi) := \lambda_1\varphi(1) + \lambda_2\varphi_x(1), \tag{4.9}$$

with $\varphi \in K_0(0,1)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. The functional $\mathfrak{L}$ is continuous and linear. Thus, by the Lax-Milgram theorem, there exists a unique solution $y \in K_0(0,1)$ of

$$\Lambda(y,\varphi) = \mathfrak{L}(\varphi) \tag{4.10}$$

for all $\varphi \in K_0(0,1)$. Then, we obtain

$$|y(1)|^2 \le \tfrac{1}{b_0}\|\sqrt{b}y_x\|^2_{L^2(0,1)} \le \tfrac{1}{b_0}|||y|||^2, \tag{4.11}$$

and

$$|y_x(1)|^2 \quad \le 2\Big(\tfrac{1}{b_0}\|\sqrt{b}y_x\|^2_{L^2(0,1)} + \tfrac{\|\sqrt{a}y_{xx}\|^2_{L^2(0,1)}}{a(1)(2-K_a)}\Big) \le 2\max\Big\{\tfrac{1}{b_0}, \tfrac{1}{a(1)(2-K_a)}\Big\}\|y\|^2. \tag{4.12}$$

Combining the above results, we have

$$|||y|||^2 = \lambda_1\varphi(1) + \lambda_2\varphi_x(1) \le \Big(|\lambda|\sqrt{\frac{1}{b_0}} + |\mu|\sqrt{2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}}\Big)|||y|||. \tag{4.13}$$

Thus

$$|||y||| \le |\lambda|\sqrt{\frac{1}{b_0}} + |\mu|\sqrt{2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}}. \tag{4.14}$$

Moreover, by (2.18), we have

$$\|y\|^2_{L^2(0,1)} \le \frac{1}{b_0}|||y|||^2 \le \frac{1}{b_0}C^2_{\lambda,\mu,b_0,K_a,a}. \tag{4.15}$$

Now, we prove that y belongs to $D(\mathcal{A}_a)$ and solves (4.5). To this aim, consider (4.10) again. It is clear that (4.10) holds for every $\varphi \in C_c^\infty(0,1)$, so that

$$\int_0^1 [(ay_{xx})_{xx} - (by_x)_x + cy]\varphi dx = 0, \tag{4.16}$$

then $(ay_{xx})_{xx} - (by_x)_x + cy = 0$ *a.e.* in $(0,1)$, in particular, for $y \in K_0(0,1)$, we have $\mathcal{A}_a := (ay_{xx})_{xx} - (by_x)_x + cy \in L^2(0,1)$, which implies that $ay_{xx} \in H^2(0,1)$, then we deduce that $y \in K_0(0,1)$. Moreover, let $\varphi \in K_0(0,1)$, combining (4.5) and (4.16), we deduce that

$$\int_0^1 ay_{xx}\varphi_{xx}dx + \int_0^1 by_x\varphi_x dx + \int_0^1 cy\varphi dx = \lambda_1\varphi(1) + \lambda_2\varphi_x(1)$$

if and only if

$$((by_x)_x(1) - (ay_{xx})_x(1) - \lambda_1)\varphi(1) + (a(1)y_{xx}(1) - \lambda_2)\varphi_x(1) = 0,$$

and $y_x(0) = 0$ in the weakly degenerate case and

$$\int_0^1 ay_{xx}\varphi_{xx}dx + \int_0^1 by_x\varphi_x dx + \int_0^1 cy\varphi dx = \lambda_1\varphi(1) + \lambda_2\varphi_x(1)$$

if and only if

$$((by_x)_x(1) - (ay_{xx})_x(1) - \lambda_1)\varphi(1) + (a(1)y_{xx}(1) - \lambda_2)\varphi_x(1) - (ay_{xx})(0)\varphi_x(0) = 0$$

in the strongly degenerate case. Thus,

$$(by_x)_x(1) - (ay_{xx})_x(1) = \lambda_1, \quad a(1)y_{xx}(1) = \lambda_2 \tag{4.17}$$

and $(ay_{xx})(0) = 0$ in the strongly degenerate case. To sum up, the proof is complete. □

From now and on, We focus on the stability result of the system. The main result of this paper is based on the following theorem presented in Kormonik' work in [14].

Theorem 4.2. *Suppose that $E : [0,+\infty)$ is a non-increasing function and that there exists $N > 0$ such that*

$$\int_t^\infty E(s)ds \le NE(t), \qquad \forall t \in [0,+\infty). \tag{4.18}$$

Then we have

$$E(t) \le e^{1-\frac{t}{N}}E(0), \qquad \forall t \in [N,+\infty). \tag{4.19}$$

We consider the Lyapunov functional

$$L(t) := ME(t) + I_1 + M_1I_2 + I_3. \tag{4.20}$$

Here,

$$I_1 := \int_0^1 u_t\Big(2xu_x + \frac{K_{a,b,c}}{2}u\Big)dx, \tag{4.21}$$

$$I_2 := \frac{\delta}{2}\int_0^1 e^{-2s\rho}z^2(\rho)d\rho, \tag{4.22}$$

$$I_3 := \frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau)+F\Big]^2 d\xi + \frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(x,\xi,t)d\xi, \tag{4.23}$$

where

$$F := -\frac{s\mu(\xi)}{\xi^2+\eta}\int_0^1 f_0(x,-\rho s)d\rho + \frac{\mu(\xi)u_0(1)}{\xi^2+\eta}, \tag{4.24}$$

$$K_{a,b,c} := \max\Big\{K_a, \frac{b_2}{b_0}, \frac{c_2}{c_0}\Big\} < 2, \tag{4.25}$$

and M, M_1 are positive real numbers to be chosen appropriately later.

Lemma 4.3. *Assume a is weakly or strongly degenerate, and (1.2) and (2.8) hold. Let (u,ϕ,z) be a regular solution of the problem (2.4). Then, the functional I_1 in (4.21) satisfies*

$$\begin{aligned}
\frac{dI_1}{dt} &\le -2(au_{xx})_x(1)u_x(1) + b(1)u_x^2(1) - c(1)u^2(1) + \frac{K_{a,b,c}}{2}(bu_x)(1)u(1)\\
&\quad - \frac{K_{a,b,c}}{2}(au_{xx})_x(1)u(1) + u_t^2(1)\\
&\quad - \Big(3-\frac{K_{a,b,c}}{2}\Big)\int_0^1 au_{xx}^2dx - \Big(1-\frac{K_{a,b,c}}{2} - \frac{4c_0b_0+3c_1c_2}{2c_0b_0}\Big)\int_0^1 bu_x^2dx\\
&\quad - \Big(-\frac{c_2}{4c_0}+\frac{K_{a,b,c}}{2}\Big)\int_0^1 cu^2dx - \Big(1-\frac{K_{a,b,c}}{2}\Big)\int_0^1 u_t^2dx
\end{aligned} \tag{4.26}$$

where $K_{a,b,c}$ is defined by (4.25).

Proof. By taking the time derivative of (4.21), we obtain

$$\begin{aligned}
\frac{dI_1}{dt} &= 2\int_0^1 xu_{tt}u_xdx + \frac{K_{a,b,c}}{2}\int_0^1 u_{tt}udx + 2\int_0^1 xu_tu_{xt}dx + \frac{K_{a,b,c}}{2}\int_0^1 u_t^2dx\\
&= \underbrace{2\int_0^1 \Big[-(au_{xx})_{xx} + (bu_x)_x - cu\Big]\cdot(xu_x)dx}_{I_{1-1}}\\
&\quad + \underbrace{\frac{K_{a,b,c}}{2}\Big[-(au_{xx})_{xx} + (bu_x)_x - cu\Big]\cdot udx}_{I_{1-2}}\\
&\quad + \underbrace{x(u_t^2)_xdx + \frac{K_{a,b,c}}{2}\int_0^1 u_t^2dx}_{I_{1-3}}.
\end{aligned} \tag{4.27}$$

Then, we have

$$\begin{aligned}
I_{1-1} &= \Big[-2(au_{xx})_x\cdot(xu_x)\Big]_0^1 + 2\int_0^1 (au_{xx})_x(u_x+xu_{xx})dx + \Big[2(bu_x)\cdot(xu_x)\Big]_0^1\\
&\quad - 2\int_0^1 (bu_x)(u_x+xu_{xx})dx - \Big[xcu^2\Big]_0^1 + \int_0^1 (c+xc')u^2dx\\
&= -2(au_{xx})_x(1)u_x(1) + b(1)u_x^2(1) - c(1)u^2(1) + \int_0^1 (-3a+xa')u_{xx}^2dx\\
&\quad + \int_0^1 (-b+xb')u_x^2dx + \int_0^1 (c+xc')u^2dx,
\end{aligned} \tag{4.28}$$

$$
\begin{aligned}
I_{1-2} &= \frac{K_{a,b,c}}{2}\Big[-(au_{xx})_x u\Big]_0^1 + \frac{K_{a,b,c}}{2}\int_0^1 (au_{xx})_x u_x dx \\
&\quad + \Big[\frac{K_{a,b,c}}{2} bu_x u\Big]_0^1 - \frac{K_{a,b,c}}{2}\int_0^1 bu_x^2 dx - \frac{K_{a,b,c}}{2}\int_0^1 cu^2 dx \\
&= -\frac{K_{a,b,c}}{2}\int_0^1 au_{xx}^2 dx - \frac{K_{a,b,c}}{2}\int_0^1 bu_x^2 dx - \frac{K_{a,b,c}}{2}\int_0^1 cu^2 dx \\
&\quad + \frac{K_{a,b,c}}{2}(bu_x)(1)u(1) - \frac{K_{a,b,c}}{2}(au_{xx})_x(1)u(1),
\end{aligned} \tag{4.29}
$$

and

$$
I_{1-3} = \Big[xu_t^2\Big]_0^1 - \int_0^1 u_t^2 dx + \frac{K_{a,b,c}}{2}\int_0^1 u_t^2 dx = \Big(\frac{K_{a,b,c}}{2} - 1\Big)\int_0^1 u_t^2 dx + u_t^2(1). \tag{4.30}
$$

Substituting (4.28), (4.29) and (4.30) into (4.27), we obtain

$$
\begin{aligned}
\frac{dI_1}{dt} &= -2(au_{xx})_x(1)u_x(1) + b(1)u_x^2(1) - c(1)u^2(1) + \frac{K_{a,b,c}}{2}(bu_x)(1)u(1) \\
&\quad - \frac{K_{a,b,c}}{2}(au_{xx})_x(1)u(1) + u_t^2(1) + \int_0^1 \Big(-3a + xa' - \frac{K_{a,b,c}}{2}a\Big)u_{xx}^2 dx \\
&\quad + \int_0^1 \Big(-b + xb' - \frac{K_{a,b,c}}{2}b\Big)u_x^2 dx + \int_0^1 \Big(c + xb' - \frac{K_{a,b,c}}{2}c\Big)u^2 dx \\
&\quad + \int_0^1 \Big(\frac{K_{a,b,c}}{2} - 1\Big)u_t^2 dx.
\end{aligned} \tag{4.31}
$$

Using (2.18), (1.10) and (4.25), we deduce that

$$
\begin{aligned}
\frac{dI_1}{dt} &\le -2(au_{xx})_x(1)u_x(1) + b(1)u_x^2(1) - c(1)u^2(1) + \frac{K_{a,b,c}}{2}(bu_x)(1)u(1) \\
&\quad - \frac{K_{a,b,c}}{2}(au_{xx})_x(1)u(1) + u_t^2(1) - \Big(3 - K_{a,b,c} + \frac{K_{a,b,c}}{2}\Big)\int_0^1 au_{xx}^2 dx \\
&\quad - \Big(1 - K_{a,b,c} + \frac{K_{a,b,c}}{2} - \frac{4c_0b_0 + 3c_1c_2}{2c_0b_0}\Big)\int_0^1 bu_x^2 dx \\
&\quad - \Big(-\frac{c_2}{4c_0} + \frac{K_{a,b,c}}{2}\Big)\int_0^1 cu^2 dx - \Big(1 - \frac{K_{a,b,c}}{2}\Big)\int_0^1 u_t^2 dx
\end{aligned} \tag{4.32}
$$

where

$$
\begin{gathered}
\frac{c_2}{4c_0} - \frac{K_{a,b,c}}{2} \le \frac{3K_{a,b,c}}{4} \le 0, \quad -2 + \frac{K_{a,b,c}}{2} \le 0, \\
-1 + \frac{K_{a,b,c}}{2} + \frac{4c_0b_0 + 3c_1c_2}{2c_0b_0} \le 0, \quad -1 + \frac{K_{a,b,c}}{2} \le 0.
\end{gathered}
$$

□

Lemma 4.4. *Assume a is weakly or strongly degenrate. Let (u, ϕ, z) be a regular solution of the problem (2.4), then, the functional I_2 in (4.22) satisfies*

$$
\frac{dI_2}{dt} \le -I_2 - \frac{C_{s,\delta}}{2s}z^2(1) + \frac{\delta}{2s}z^2(0) \tag{4.33}
$$

where $C_{s,\delta}$ is defined by (4.35).

Proof. Differentiating (4.22) with respect to t and using the third equation in (2.4), we obtain

$$\begin{aligned} \frac{dI_2}{dt} &= \frac{\delta}{2}\Big[-\frac{2}{s}\cdot\frac{\partial}{\partial t}\Big(\int_0^1 e^{-2s\rho} z z_\rho d\rho\Big)\Big] \\ &= \frac{\delta}{2}\Big[-\frac{1}{s}\Big(\int_0^1 e^{-2s\rho}\frac{\partial}{\partial t}(z^2) d\rho\Big)\Big] \\ &= \frac{\delta}{2}\Big[-\frac{1}{s}e^{-2s}z^2(1)+\frac{1}{s}z^2(0)+2\int_0^1 e^{-2s\rho}(z^2)d\rho\Big] \\ &\le -I_2-\frac{C_{s,\delta}}{2s}z^2(1)+\frac{\delta}{2s}z^2(0) \end{aligned} \tag{4.34}$$

where

$$C_{s,\delta} := \frac{\delta e^{-2s}}{2s}. \tag{4.35}$$

□

Lemma 4.5. *Assume a is weakly or strongly degenrate. Let (u,ϕ,z) be a regular solution of the problem (2.4). Then, the functional I_3 in (4.23) satisfies*

$$\begin{aligned} \frac{dI_3}{dt} &\le \frac{s^2 r_2}{4\varepsilon_1}\int_0^1 z^2(\rho)d\rho + \frac{r_2 D_\mu}{4\varepsilon_1}u^2(1)+\frac{r_2 D_\mu}{4\varepsilon_1}z^2(1)-\frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi \\ &\quad -\Big[\gamma-\frac{2r_2\eta^{\alpha-1}\sin(\pi\alpha)\varepsilon_1}{4D_\mu\pi}-\frac{\sin(\pi\alpha)\varepsilon_1 r_2\eta^{\alpha-1}}{4\pi}\Big]\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi, \end{aligned} \tag{4.36}$$

where ε_1 is a suitable positive constant.

Proof. By taking the time derivative of (4.23), we have

$$\frac{dI_3}{dt} = \frac{\gamma}{2}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big(\int_0^t \phi(x,\xi,\tau)\,d\tau + F\Big)\phi(x,\xi,t)d\xi + \gamma\int_{-\infty}^{+\infty}\phi(\xi)\phi_t(\xi)d\xi. \tag{4.37}$$

Using the second equation in (2.4), we have

$$\int_0^t \phi_t(x,\xi,\tau)\,d\tau + (\xi^2+\eta)\int_0^t \phi(x,\xi,\tau)\,d\tau = \mu(\xi)\int_0^t z(1)\,d\tau. \tag{4.38}$$

Then, we obtain

$$\int_0^t \phi(x,\xi,\tau)\,d\tau = \frac{\mu(\xi)}{\xi^2+\eta}\int_0^t z(1)\,d\tau - \frac{\phi(x,\xi,t)}{\xi^2+\eta}. \tag{4.39}$$

Using the third equation in (2.4), we obtain

$$s\frac{\partial}{\partial t}\int_0^1 z(\rho)d\rho + z(x,1,t)-z(x,0,t)=0, \tag{4.40}$$

$$s\int_0^t\frac{\partial}{\partial t}\int_0^1 z(\rho)d\rho\,d\tau = s\Big[\int_0^1 z d\rho\Big]_0^t = s\int_0^1 z(x,\rho,t)d\rho - s\int_0^1 z(x,\rho,0)d\rho. \tag{4.41}$$

Observe that

$$\int_0^t z(1)\,d\tau = s\int_0^1 z d\rho + s\int_0^1 f_0(x,-s\rho)d\rho + u(1,t)-u(1,0). \tag{4.42}$$

Therefore,

$$\begin{aligned} &\int_0^t \phi(x,\xi,\tau)\,d\tau \\ &= -\frac{s\mu(\xi)}{\xi^2+\eta}\int_0^1 z(\rho)d\rho + \frac{s\mu(\xi)}{\xi^2+\eta}\int_0^1 f_0(x,-s\rho)d\rho + \frac{s\mu(\xi)u(1)}{\xi^2+\eta}+\frac{s\mu(\xi)u_0(1)}{\xi^2+\eta}-\frac{\phi(x,\xi,t)}{\xi^2+\eta}. \end{aligned} \tag{4.43}$$

Substituting (4.43) into (4.37), we obtain

$$\begin{aligned}\frac{dI_3}{dt} &= -\frac{\gamma}{2}\int_{-\infty}^{+\infty} s\mu(\xi)\int_0^1 z(\rho)\phi(\xi)d\rho d\xi + \frac{\gamma}{2}\int_{-\infty}^{+\infty}\mu(\xi)u(1)\phi(\xi)d\xi \\ &\quad + \gamma\int_{-\infty}^{+\infty} z(1)\mu(\xi)\phi(\xi)d\xi - \gamma\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi - \frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi.\end{aligned} \tag{4.44}$$

Now, we will estimate the term on the right side of (4.44). Using the Cauchy-Schwarz inequality, Young's inequality, (1.2) and (2.6), the first three items on the right can be written as follows:

$$\begin{aligned}&\frac{\gamma}{2}\int_{-\infty}^{+\infty} s\mu(\xi)\int_0^1 z(\rho)\phi(\xi)d\rho d\xi \\ &\le \frac{s^2r_2}{4\varepsilon_1}\int_0^1 z^2(\rho)d\rho + \frac{\gamma^2 D_\mu\varepsilon_1}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi \\ &= \frac{s^2r_2}{4\varepsilon_1}\int_0^1 z^2(\rho)d\rho + \frac{\sin(\pi\alpha)\varepsilon_1 r_2\eta^{\alpha-1}}{4\pi}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi,\end{aligned} \tag{4.45}$$

$$\begin{aligned}\frac{\gamma}{2}\int_{-\infty}^{+\infty}\mu(\xi)u(1)\phi(\xi)d\xi &\le \frac{r_2D_\mu}{4\varepsilon_1}u^2(1) + \frac{\gamma^2\varepsilon_1}{4r_2}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi \\ &\le \frac{r_2D_\mu}{4\varepsilon_1}u^2(1) + \frac{r_2\eta^{\alpha-1}\sin(\pi\alpha)\varepsilon_1}{4D_\mu\pi}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi,\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}\frac{\gamma}{2}\int_{-\infty}^{+\infty}\mu(\xi)z(1)\phi(\xi)d\xi &\le \frac{r_2D_\mu}{4\varepsilon_1}z^2(1) + \frac{\gamma^2\varepsilon_1}{4r_2}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi \\ &\le \frac{r_2D_\mu}{4\varepsilon_1}z^2(1) + \frac{r_2\eta^{\alpha-1}\sin(\pi\alpha)\varepsilon_1}{4D_\mu\pi}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi\end{aligned} \tag{4.47}$$

where γ and D_μ are defined in (2.5) and Lemma 2.2. Then, we obtain

$$\begin{aligned}\frac{dI_3}{dt} &\le \frac{s^2r_2}{4\varepsilon_1}\int_0^1 z^2(\rho)d\rho + \frac{r_2D_\mu}{4\varepsilon_1}u^2(1) + \frac{r_2D_\mu}{4\varepsilon_1}z^2(1) - \frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi \\ &\quad - \Big[\gamma - \frac{2r_2\eta^{\alpha-1}\sin(\pi\alpha)\varepsilon_1}{4D_\mu\pi} - \frac{\sin(\pi\alpha)\varepsilon_1 r_2\eta^{\alpha-1}}{4\pi}\Big]\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi,\end{aligned} \tag{4.48}$$

where $\varepsilon_1 \in \big(0, \frac{4\pi}{\sin(\alpha\pi)(2-D_\mu)}\big)$. $\square$

Next, we estimate $L(t)$ and $L'(t)$. Before estimating $L(t)$, we present the following lemma.

Lemma 4.6. *Assume a is weakly or strongly degenrate. Let (u,ϕ,z) be a regular solution of the problem (2.4). Then*

$$\begin{aligned}&\Big|\frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau) + F\Big]^2 d\xi\Big| \\ &\le \frac{3\gamma}{4\eta}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi + \frac{3\gamma}{4}D_\mu u^2(1,t) + \Big(\frac{\gamma}{4} + \frac{\gamma}{2}s^2D_\mu\Big)\int_0^1 z^2(\rho)d\rho.\end{aligned} \tag{4.49}$$

Proof. Using the second equation in (2.4), as in (4.43), we have

$$\begin{aligned}&\Big|\frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau) + F\Big]d\xi\Big| \\ &\le |\frac{\gamma}{4}\phi(x,\xi,t) + u(1,t)\mu(\xi) - s\mu(\xi)\int_0^1 z(\rho)d\rho.\end{aligned} \tag{4.50}$$

Therefore,

$$
\begin{aligned}
&\Big|\frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau)+F\Big]^2 d\xi\Big|\\
&\le \frac{\gamma}{4}\int_{-\infty}^{+\infty}\frac{\phi^2(\xi)}{\xi^2+\eta}d\xi+\frac{\gamma}{4}D_\mu u^2(1,t)+\frac{\gamma}{4}s^2D_\mu\int_0^1 z^2(\rho)d\rho\\
&\quad+\frac{\gamma}{2}\int_{-\infty}^{+\infty}\frac{\phi(\xi)\mu(\xi)u(1,t)}{\xi^2+\eta}\int_0^1 z(\rho)d\rho d\xi\\
&\quad+\frac{\gamma}{2}\int_{-\infty}^{+\infty}\frac{\phi(\xi)\mu(\xi)u(1,t)}{\xi^2+\eta}d\xi+\frac{\gamma}{2}\int_{-\infty}^{+\infty}s\mu(\xi)\phi(\xi)\frac{\int_0^1 z(\rho)d\rho}{\xi^2+\eta}d\xi.
\end{aligned}
\tag{4.51}
$$

Using Cauchy-Schwarz, Young's, Cauchy-Schwarz, Höder's inequality and the fact that $\frac{1}{\xi^2+\eta}\le\frac{1}{\eta}$, we have

$$
\begin{aligned}
&\Big|\frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau)+F\Big]^2 d\xi\Big|\\
&\le \frac{\gamma}{4\eta}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi+\frac{\gamma}{4}D_\mu u^2(1,t)+\frac{\gamma}{4}s^2D_\mu\int_0^1 z^2(\rho)d\rho+\frac{\gamma}{4}D_\mu u^2(1,t)+\frac{\gamma}{4}\int_0^1 z^2(\rho)d\rho\\
&\quad+\frac{\gamma}{4}D_\mu u^2(1,t)+\frac{\gamma}{4\eta}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi+\frac{\gamma s^2 D_\mu}{4}\int_0^1 z^2(\rho)d\rho+\frac{\gamma}{4\eta}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi.
\end{aligned}
\tag{4.52}
$$

By organizing the above inequalities, we obtain (4.49). $\square$

Lemma 4.7. *Assume that a is weakly or strongly degenerate. For $M>0$ large enough, there exist two positive constants l_1 and l_2 such that*

$$
l_1E(t)\le L(t)\le l_2E(t),\quad \forall t>0,
\tag{4.53}
$$

where

$$
l_1:=M-C_{a,b,c},\qquad l_2:=M+C_{a,b,c}.
\tag{4.54}
$$

Proof. Recalling the definition (4.20) of $L(t)$ and setting

$$
G(t):=I_1+M_1I_2+I_3,
\tag{4.55}
$$

where I_1,I_2,I_3 are given in (4.21)-(4.23), and M_1 is a suitable constant defined in (4.67). From (4.21), (4.22) and (4.23), using Young's inequality and Poincaré inequality, we have

$$
\begin{aligned}
|G(t)|&\le \Big|\int_0^1 u_t\Big(2xu_x+\frac{K_{a,b,c}}{2}u\Big)dx\Big|+\Big|\frac{\delta}{2}\int_0^1 e^{-2s\rho}z^2(\rho)d\rho\Big|\\
&\quad+\Big|\frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau)+F\Big]^2 d\xi+\frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(x,\xi,t)d\xi\Big|\\
&\le\Big(1+\frac{K_{a,b,c}}{4}\Big)\|u_t\|^2_{L^2(0,1)}+\frac{1}{b_0}\|\sqrt{b}u_x\|^2_{L^2(0,1)}+\frac{K_{a,b,c}}{4c_0}\|\sqrt{c}u\|^2_{L^2(0,1)}+\frac{\delta}{2}\int_0^1 z^2(\rho)d\rho\\
&\quad+\Big|\frac{\gamma}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\Big[\int_0^t \phi(x,\xi,\tau)+F\Big]^2 d\xi+\frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(x,\xi,t)d\xi\Big|.
\end{aligned}
\tag{4.56}
$$

Then, by using Lemma 4.6, we obtain

$$
\begin{aligned}
&|G(t)|\\
&\le\Big(1+\frac{K_{a,b,c}}{4}\Big)\|u_t\|^2_{L^2(0,1)}+\frac{1}{b_0}\|\sqrt{b}u_x\|^2_{L^2(0,1)}+\frac{K_{a,b,c}}{4c_0}\|\sqrt{c}u\|^2_{L^2(0,1)}+\frac{\delta}{2}\int_0^1 z^2(\rho)d\rho\\
&\quad+\frac{3\gamma}{4\eta}\int_{-\infty}^{+\infty}\phi^2(\xi)d\xi+\frac{3\gamma}{4}D_\mu u^2(1,t)+\Big(\frac{\gamma}{4}+\frac{\gamma}{2}s^2D_\mu\Big)\int_0^1 z^2(\rho)d\rho+\frac{\gamma}{2}\int_{-\infty}^{+\infty}\phi^2(x,\xi,t)d\xi\\
&\le C_{a,b,c}E(t)
\end{aligned}
\tag{4.57}
$$

where $C_{a,b,c} := \max\{2+\frac{K_{a,b,c}}{2}, \frac{2}{b_0}, \frac{K_{a,b,c}}{2c_0}, 1+\frac{3\gamma}{2\eta}, 1+\frac{\gamma(1+2s^2D_\mu)}{2\delta}\}$. Therefore,

$$|L(t) - ME(t)| \le C_{a,b,c}E(t) \tag{4.58}$$

So, we can choose M large enough such that $l_1 := M - C_{a,b,c} > 0$ and $l_2 := M + C_{a,b,c} > 0$. Then, we deduce the desired result. □

Based on the previous lemma, we give the following proposition, which is crucial for our main result.

Proposition 4.8. *Assume that a is wealy or strongly degenerate. Let (u, ϕ, z) be a regular solution of problem (2.4), for $M > 0$ large enough. Then*

$$\frac{dL(t)}{dt} \le -2C_1E(t) + C_2(u^2(1) + u_x^2(1)) \tag{4.59}$$

where C_1, C_2 are defined in (4.63), (4.64), and M satisfies (4.68).

Proof. Combining Lemmas 4.3, 4.4 and 4.5, we have

$$\begin{aligned}
&-(au_{xx})_x(1)u_x(1)\\
&= \Big[-(bu_x)(1) - r_1u_t(1) - \gamma\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi\Big]u_x(1)\\
&\le -\frac{1}{2}b(1)u_x^2(1) + \frac{r_1^2}{2b(1)}u_t^2(1) + \frac{r_2D_\mu}{\varepsilon_2} + \frac{r_2\eta^{\alpha-1}\sin(\alpha\pi)\varepsilon_2}{4D_\mu\pi}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi
\end{aligned}\tag{4.60}$$

and

$$\begin{aligned}
&[-(au_{xx})_x(1) + (bu_x)(1)]\Big(\frac{K_{a,b,c}}{2}u(1) + u_x(1)\Big)\\
&= -\gamma\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)d\xi\Big(\frac{K_{a,b,c}}{2}u(1) + u_x(1)\Big)\\
&\le \frac{1}{2}b(1)u_x^2(1) + \frac{r_1^2}{2b(1)}u_t^2(1) + \frac{b(1)K_{a,b,c}}{2\varepsilon_2}u^2(1) + \frac{r_1^2\varepsilon_2}{2b(1)}u_t^2(1) + \frac{K_{a,b,c}D_\mu}{4\varepsilon_2}u^2(1)\\
&\quad + \frac{K_{a,b,c}\gamma^2\varepsilon_2}{4}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi + \frac{1}{\varepsilon_2}b(1)u_x^2(1) + \frac{D_\mu\gamma^2\varepsilon_2}{4b(1)}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi\\
&= \Big(\frac{1}{2} + \frac{1}{\varepsilon_2}\Big)b(1)u_x^2(1) + \frac{b(1)K_{a,b,c}}{2\varepsilon_2}u^2(1) + \frac{r_1^2\varepsilon_2}{2b(1)}u_t^2(1)\\
&\quad + \frac{K_{a,b,c}D_\mu}{4\varepsilon_2}u^2(1) + \Big(\frac{K_{a,b,c}r_2\eta^{\alpha-1}\gamma\varepsilon_2}{4} + \frac{r_2\eta^{\alpha-1}\gamma\varepsilon_2}{4b(1)}\Big)\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi.
\end{aligned}\tag{4.61}$$

Then

$$\begin{aligned}
\frac{dL(t)}{dt} &\le -2C_1E(t) + C_2(u^2(1) + u_x^2(1)) + \Big(\frac{\delta M_1}{2s} + \frac{r_1^2(1+\varepsilon_1)}{2b(1)}\Big)z^2(0)\\
&\quad + \Big(\frac{C_{s,\delta}}{2s} + \frac{r_2D_\mu}{4\varepsilon_1}\Big)z^2(1)\\
&\quad - \Big[\gamma - \frac{\eta^{\alpha-1}r_2\sin(\alpha\pi)(2+K_{a,b,c})(\varepsilon_1+\varepsilon_2)}{4\max\{1, D_\mu, b(1)\}\pi}\Big]\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi^2(\xi)d\xi\\
&\le -2C_1E(t) + C_2(u^2(1) + u_x^2(1)) - \Big(MC_e - \frac{\delta M_1}{2s} - \frac{r_1^2(1+\varepsilon_1)}{2b(1)}\Big)z^2(0)\\
&\quad - \Big(MC_e - \frac{C_{s,\delta}}{2s} - \frac{r_2D_\mu}{4\varepsilon_1}\Big)z^2(1),
\end{aligned}\tag{4.62}$$

where

$$C_1 := \min\Big\{3 - \frac{K_{a,b,c}}{2}, 1 - \frac{K_{a,b,c}}{2} - \frac{4c_0b_0 + 3c_2c_1}{4c_0b_0}, -\frac{c_2}{4c_0} + \frac{K_{a,b,c}}{2}, 1 - \frac{K_{a,b,c}}{2}, 1, M_1 - \frac{s^2r_2}{2\varepsilon_1\delta}\Big\}, \tag{4.63}$$

$$C_2 := \max\Big\{\frac{r_2 D_\mu}{4\varepsilon_1} + \frac{b(1)K_{a,b,c}}{2\varepsilon_2} + \frac{K_{a,b,c}D_\mu}{4\varepsilon_2}, \frac{b(1)}{\varepsilon_2}\Big\}, \tag{4.64}$$

$$\begin{aligned} &\max\Big\{\frac{s^2 r_2}{2\delta M_1}, \frac{r_2 D_\mu}{4(MC_e - \frac{C_{s,\delta}}{2s})}\Big\} \\ &\le \varepsilon_1 \le \min\Big\{\frac{4\pi}{\sin(\alpha\pi)(2-D_\mu)}, \frac{4\max\{1, D_\mu, b(1)\}}{\eta^{\alpha-1}(2+K_{a,b,c})} - \varepsilon_2, \frac{(MC_e - \frac{C_{s,\delta}}{2s})2b(1) - r_1^2}{r_1^2}\Big\}, \end{aligned} \tag{4.65}$$

$$\varepsilon_2 \le \frac{4\max\{1, D_\mu, b(1)\}}{\eta^{\alpha-1}(2+K_{a,b,c})}, \tag{4.66}$$

$$\frac{s^2 r_2}{2\varepsilon_1\delta} \le M_1 \le \frac{2sMC_e}{\delta}. \tag{4.67}$$

where $C_e, \varepsilon_1, \varepsilon_2$ and M_1 are given in Lemma 2.4, (4.65), (4.66) and (4.67). We choose M large enough such that

$$MC_e - \frac{\delta M_1}{2s} > 0, \quad MC_e - \frac{C_{s,\delta}}{2s} > 0. \tag{4.68}$$

Then, we obtain (4.59). □

Based on Proposition 4.8, we have the following lemma.

Lemma 4.9. *Assume that a is weakly or strongly degenerate. Let (u, ϕ, z) be a regular solution of the problem* (2.4)*, for any $r \in (0, T)$, then*

$$2C_1 \int_r^T E(t)dt \le L(r) - L(T) + C_2 \int_r^T (u^2(1) + u_x^2(1))dt \tag{4.69}$$

where C_1 and C_2 are defined in (4.63) *and* (4.64)*, respectively.*

Proof. Using Proposition 4.8, and integrating (4.59) over (r, T), we obtain the estimate (4.69). □

Next, we estimate $u^2(1) + u_x^2(1)$.

Lemma 4.10. *Assume a weakly or strongly degenerate. Let (u, ϕ, z) be a regular solution of the problem* (2.4)*. Then*

$$\int_r^T (u^2(1) + u_x^2(1))dt \le C_3\Big[\varepsilon_4 \int_r^T E(t)dt + C_{b_0}(E(r) + E(T)) + C_{\delta,b_0}(E(r) - E(T))\Big], \tag{4.70}$$

where C_{b_0}, C_{δ,b_0} and C_3 are defined by (4.81)*,* (4.82) *and* (4.83)*.*

Proof. Multiplying the first equation in (2.4) by y and the first equation in (4.5) by u, then integrating over $(r, T) \times (0, 1)$, we obtain

$$\begin{aligned} 0 &= \int_r^T \int_0^1 [u_{tt} + (au_{xx})_{xx} - (bu_x)_x + cu]y\,dx\,dt \\ &= \Big[\int_0^1 u_t y dx\Big]_r^T + \int_r^T [(au_{xx})_x(1) - (bu_x)(1)]y(1)dt - \int_r^T \int_0^1 u_t y_t\,dx\,dt \\ &\quad + \int_r^T \int_0^1 au_{xx}y_{xx}\,dx\,dt + \int_r^T \int_0^1 bu_x y_x\,dx\,dt + \int_r^T \int_0^1 cuy\,dx\,dt \end{aligned} \tag{4.71}$$

and

$$\begin{aligned} 0 &= \int_r^T \int_0^1 [(ay_{xx})_{xx} - (by_x)_x + cy]u\,dx\,dt \\ &= \int_r^T \int_0^1 ay_{xx}u_{xx}\,dx\,dt + \int_r^T \int_0^1 by_x u_x\,dx\,dt + \int_r^T \int_0^1 cuy\,dx\,dt \\ &\quad + \int_r^T \Big((by_x)(1) - (ay_{xx})_x(1)\Big)u(1)dt - \int_r^T (ay_{xx})(1)u_x(1)dt. \end{aligned} \tag{4.72}$$

Let $\lambda_1 = u(1), \lambda_2 = u_x(1)$, and using the boundary condition $(2.4)_{4-6}$ and $(4.5)_{2-3}$, we have

$$\begin{aligned} &\int_r^T \int_0^1 u_t y_t dt - \Big[\int_0^1 u_t y dx\Big]_r^T \\ &= \int_r^T \Big(r_1 u_t(1,t) + \gamma \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)d\xi\Big) y(1)dt + \int_r^T (u^2(1) + u_x^2(1))dt. \end{aligned} \tag{4.73}$$

Integrating (2.13)) over (r,T), we obtain

$$\begin{aligned} E(T) - E(r) = &\int_r^T -r_1 u_t^2(1)dt \\ &- \int_r^T \gamma \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)d\xi u_t(1)dt - \int_r^T \gamma \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi\, dt \\ &+ \int_r^T \gamma z(1) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi\, dt - \int_r^T \frac{\delta}{2s}[z^2(1) + z^2(0)]dt, \end{aligned} \tag{4.74}$$

where $z(x,0,t) = u_t(1,t)$. Using Cauchy-Schwarz and Young's inequality, combining (1.2), (2.6), (2.8) and (4.74), we have

$$\begin{aligned} &\int_r^T (u^2(1) + u_x^2(2))dt \le \int_r^T \int_0^1 u_t y_t\, dx\, dt - \Big[\int_0^1 u_t y dx\Big]_r^T + \frac{r_1 D_\mu}{\varepsilon_3} \int_r^T u_t^2(1)dt + \frac{r_1\varepsilon_3}{4} \int_r^T y^2(1)dt \\ &+ \frac{\gamma}{2} \int_r^T \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi^2(\xi)\, d\xi\, dt + \frac{\gamma D_\mu}{2} \int_r^T y^2(1)dt \\ &= E(r) - E(T) - \int_r^T \gamma \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)d\xi u_t(1)dt + \int_r^T \gamma z(x,1,t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi\, dt \\ &\quad - \int_r^T \frac{\delta}{2s}[z^2(1) + z^2(0)]dt + \frac{r_1\varepsilon_3}{4} \int_r^T y^2(1)dt + \frac{\gamma D_\mu}{2} \int_r^T y^2(1)dt \\ &\quad - \frac{\gamma}{2} \int_r^T \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi^2(\xi)\, d\xi\, dt + (\frac{r_1 D_\mu}{\varepsilon_3} - r_1) \int_r^T z^2(0)dt + \int_r^T \int_0^1 u_t y_t\, dx\, dt - \Big[\int_0^1 u_t y dx\Big]_r^T \\ &\le \int_r^T \int_0^1 u_t y_t\, dx\, dt - \Big[\int_0^1 u_t y dx\Big]_r^T + (\frac{r_1 D_\mu}{\varepsilon_3} - r_1) \int_r^T z^2(0)dt + \Big(\frac{r_1\varepsilon_3}{4} + \frac{\gamma D_\mu}{2}\Big) \int_r^T y^2(1)dt \\ &\quad + E(r) - E(T) + \gamma D_\mu \int_r^T z^2(0)dt + \gamma D_\mu \int_r^T z^2(1)dt - \int_r^T \frac{\delta}{2s}[z^2(1) + z^2(0)]dt \\ &\le \int_r^T \int_0^1 u_t y_t\, dx\, dt - \Big[\int_0^1 u_t y dx\Big]_r^T + (\frac{r_1 D_\mu}{\varepsilon_3} - r_1) \int_r^T z^2(0)dt + \Big(\frac{r_1\varepsilon_3}{4} + \frac{r_1}{2}\Big) \int_r^T y^2(1)dt \\ &\quad + E(r) - E(T) + r_1 \int_r^T z^2(0)dt + r_1 \int_r^T z^2(1)dt - \int_r^T \frac{\delta}{2s}[z^2(1) + z^2(0)]dt \\ &\le \int_r^T \int_0^1 u_t y_t\, dx\, dt - \Big[\int_0^1 u_t y dx\Big]_r^T + E(r) - E(T) + \Big(\frac{r_1\varepsilon_3}{4} + \frac{r_1}{2}\Big) \int_r^T y^2(1)dt \\ &\quad + \Big(-\frac{\delta}{2s} + r_1\Big) \int_r^T z^2(1)dt + \Big(-\frac{\delta}{2s} + \frac{r_1 D_\mu}{\varepsilon_3}\Big) \int_r^T z^2(0)dt \\ &\le \int_r^T \int_0^1 u_t y_t\, dx\, dt - \Big[\int_0^1 u_t y dx\Big]_r^T + E(r) - E(T) + \Big(\frac{r_1\varepsilon_3}{4} + \frac{r_1}{2}\Big) \int_r^T y^2(1)dt \\ &\quad + \max\Big\{-\frac{\delta}{2s} + r_1, -\frac{\delta}{2s} + \frac{r_1 D_\mu}{\varepsilon_3}\Big\} \int_r^T (z^2(1) + z^2(0))dt, \end{aligned}$$

where $\varepsilon_3 \le \frac{2sr_1D_\mu}{\delta}$. Using Lemma 2.4, we obtain

$$\begin{aligned}
&\int_r^T (u^2(1)+u_x^2(2))dt \\
&\le \int_r^T\int_0^1 u_t y_t\,dx\,dt - \Big[\int_0^1 u_t y dx\Big]_r^T + E(r) - E(T) + \Big(\frac{r_1\varepsilon_3}{4}+\frac{r_1}{2}\Big)\int_r^T y^2(1)dt \\
&\quad + \max\Big\{-\frac{\delta}{2s}+r_1, -\frac{\delta}{2s}+\frac{r_1D_\mu}{\varepsilon_3}\Big\}\int_r^T -\frac{1}{C_e}\frac{dE(T)}{dt}dt \\
&\le \int_r^T\int_0^1 u_t y_t dt - \Big[\int_0^1 u_t y dx\Big]_r^T + E(r) - E(T) + \Big(\frac{r_1\varepsilon_3}{4}+\frac{r_1}{2}\Big)\int_r^T y^2(1)dt \\
&\quad + \frac{\max\Big\{-\frac{\delta}{2s}+r_1, -\frac{\delta}{2s}+\frac{r_1D_\mu}{\varepsilon_3}\Big\}}{C_e}(E(r)-E(T)).
\end{aligned} \tag{4.75}$$

Now, we estimate the term $\int_r^T\int_0^1 u_t y_t\,dx\,dt$. Differentiating (4.5) with respect to t yields

$$\begin{gathered}
\mathcal{A}_a y := (a(y_t)_{xx})_{xx} - (b(y_t)_x)_x + cy_t = 0, \\
b(y_t)_x)(1) - (a(y_t)_{xx})_x(1) = \lambda_1, \\
a(1)(y_t)_{xx}(1) = \lambda_2.
\end{gathered} \tag{4.76}$$

Then, y_t satisfies the conditions of Proposition 4.1, so

$$\|y_t\|^2_{L^2(0,1)} \le \frac{1}{b_0}\|\!|y|\!\|^2 \le \frac{2}{b_0}\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}\int_r^T (u^2(1)+u_x^2(2))dt. \tag{4.77}$$

where the norm $\cdot$ is given in (4.1). Thus, for $\varepsilon_4>0$, we have

$$\begin{aligned}
\int_r^T\int_0^1 u_t y_t\,dx\,dt &\le \frac{\varepsilon_4}{2}\int_r^T\int_0^1 u_t^2\,dx\,dt + \frac{1}{2\varepsilon_4}\int_r^T\int_0^1 y_t^2\,dx\,dt \\
&\le \varepsilon_4\int_r^T E(t)dt + \frac{\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}}{b_0\varepsilon_4}\int_r^T (u^2(1)+u_x^2(2))dt \\
&\le \varepsilon_4\int_r^T E(t)dt + \frac{\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}}{C_e b_0\varepsilon_4}(E(r)-E(T)).
\end{aligned} \tag{4.78}$$

Combining Proposition 2.5 and (4.77), we obtain

$$\begin{aligned}
\Big[\int_0^1 u_t y dx\Big]_r^T &\le \max\Big\{1, \frac{2}{b_0}\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\} \\
&\quad \times \max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\Big\}(E(r)+E(T)).
\end{aligned} \tag{4.79}$$

By a similar process, we obtain

$$\begin{aligned}
&\Big(\frac{r_1\varepsilon_3}{4}+\frac{r_1}{2}\Big)\int_r^T y^2(1)dt \\
&\le \frac{2}{b_0}\Big(\frac{r_1\varepsilon_3}{4}+\frac{r_1}{2}\Big)\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\} \\
&\quad \times \max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\int_r^T (u^2(1)+u_x^2(1))dt.
\end{aligned}$$

Thus, (4.75) can be written as

$$\begin{aligned}
&\int_r^T (u^2(1)+u_x^2(2))dt \\
&\le \varepsilon_4\int_r^T E(t)dt + \max\Big\{1, \frac{2}{b_0}\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\times \max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\Big\}(E(r)+E(T))\\
&+ \frac{2}{b_0}\Big(\frac{r_1\varepsilon_3}{4}+\frac{r_1}{2}\Big)\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}\\
&\times \max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\int_r^T (u^2(1)+u_x^2(1))dt\\
&+ \Big(1+\frac{\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}}{C_e b_0 \varepsilon_4} + \frac{\max\Big\{-\frac{\delta}{2s}+r_1, \frac{\delta}{2s}-\varepsilon_3 D_\mu\Big\}}{C_e}\Big)(E(r)-E(T)).
\end{aligned}
$$

Recalling (1.2) and choosing

$$
\begin{aligned}
\varepsilon_3 \le \min\Big\{&\frac{2sr_1D_\mu}{\delta}, \frac{2b_0}{r_1}\Big(\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}\\
&\times \max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\Big)^{-1} - 2\Big\},
\end{aligned} \tag{4.80}
$$

$$
C_{b_0} := \max\Big\{1, \frac{2}{b_0}\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}\max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\Big\}, \tag{4.81}
$$

$$
C_{\delta,b_0} := \Big(1+\frac{\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}}{C_e b_0\varepsilon_4} + \frac{\max\Big\{-\frac{\delta}{2s}+r_1, \frac{\delta}{2s}-\varepsilon_3 D_\mu\Big\}}{C_e}\Big), \tag{4.82}
$$

$$
C_3 := \Big[1-\frac{2}{b_0}\Big(\frac{r_1\varepsilon_3}{4}+\frac{r_1}{2}\Big)\max\Big\{\frac{1}{b_0}, 2\max\Big\{\frac{1}{b_0}, \frac{1}{a(1)(2-K_a)}\Big\}\Big\}\max\Big\{\frac{3}{b_0}, \frac{2}{a(1)(2-K_a)}\Big\}\Big]^{-1}, \tag{4.83}
$$

we obtain the desired estimate (4.70). □

Theorem 4.11. *Assume a is weakly or strongly degenerate. Let (u, ϕ, z) be a regular solution of the problem* (2.4)*, we have*

$$
E(t) \le e^{1-\frac{t}{N}}E(0), \quad \forall t \in [N, +\infty), \tag{4.84}
$$

where $N > 0$ is given by (4.88)*.*

Proof. Using Proposition 4.8, Lemma 4.9 and Lemma 4.10, we have

$$
(2C_1-\varepsilon_4C_3)\int_r^T E(t)dt \le L(r)-L(T)+C_2C_3C_{b_0}(E(r)+E(T))+C_2C_3C_{\delta,b_0}(E(r)-E(T)) \tag{4.85}
$$

By choosing $\varepsilon_4 \in (0, \frac{2C_1}{C_3})$, we deduce that

$$
\begin{aligned}
\int_r^T E(t)dt \le (2C_1-\varepsilon_4C_3)^{-1}[&L(r)-L(T)+C_2C_3C_{b_0}(E(r)+E(T))\\
&+C_2C_3C_{\delta,b_0}(E(r)-E(T))].
\end{aligned} \tag{4.86}
$$

Combining this with (4.53), we have $l_1E(r)-l_2E(T) \le L(r)-L(T) \le l_2E(r)-l_1E(T)$. Then

$$
\int_r^T E(t)dt \le (2C_1-\varepsilon_4C_3)^{-1}(l_2+C_2C_3C_{b_0}+2C_2C_3C_{\delta,b_0})E(r), \tag{4.87}
$$

where

$$
N := (2C_1-\varepsilon_4C_3)^{-1}(l_2+C_2C_3C_{b_0}+2C_2C_3C_{\delta,b_0}), \tag{4.88}
$$

and N is related to M_1, α, a, b, c. □

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 12271316). Sheng-Jie Li was supported by the National Funded Postdoctoral Researcher Program (Tier C) (No. GZC20261655), by the China Postdoctoral Science Foundation (No. 2025M783086), by the National Key R&D Program of China (No. 2024YFA1012802), and by the Science & Technology Commission of Shanghai Municipality (No. 23JC1400800).

References

[1] Z. Achouri, N. E. Amroun, A. Benaissa; *The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type*. Math. Methods Appl. Sci., 40(11)(2017),3837-3854.

[2] M. Akil, I. Issa, A. Wehbe; *Energy Decay of some boundary coupled systems involving wave \ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping*. arXiv preprint arXiv:2102.12732.

[3] A. Benaissa, S. Benazzouz; *Well-posedness and asymptotic behavior of Timoshenko beam system with dynamic boundary dissipative feedback of fractional derivative type*. Z. Angew. Math. Phys., 68(4) (2017), 94.

[4] V. Benci, D. Fortunato, F. Gazzola; *Existence of torsional solitons in a beam model of suspension bridge*. Arch. Ration. Mech. Anal., 226(2) (2017), 559-585.

[5] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter; *Representation and control of infinite dimensional systems*. Boston, MA: Birkhäuser Boston, 2007.

[6] H. Brezis; *Analyse fonctionnelle*. Théorie et applications, 1983.

[7] A. Camasta, G. Fragnelli; *New results on controllability and stability for degenerate Euler-Bernoulli type equations*. arXiv preprint arXiv:2306.11851, 2023.

[8] A. Camasta, G. Fragnelli; *A stability result for a degenerate beam equation*. SIAM J. Control Optim., 62(1)(2024), 630-649.

[9] G. Chen, M. C. Delfour, M. A. Krall, G. Payre; *Stabilization and control of serially connected beam*. Proceedings of the 23rd IEEE Conference on Decision and Control, Las Vegas. 1984.

[10] G. Chen, S. G. Krantz, D. W. Ma, C. E. Wayne, H. H. West; *The Euler-Bernoulli beam equation with boundary energy dissipation*. Operator methods for optimal control problems (New Orleans, La., 1986), 67-96. Lecture Notes in Pure and Appl. Math., 108.

[11] R. Datko; *Two questions concerning the boundary control of certain elastic systems*. J. Differ. Equ., 92(1)(1991), 27-44.

[12] K. J. Engel, R. Nagel; *One-parameter semigroups for linear evolution equations*. GTM 194, Springer New York, 2000.

[13] P. C. Han, Y. F. Li, G. Q. Xu, D. H. liu; *The exponential stability result of an Euler-Bernoulli beam equation with interior delays and boundary damping*. J. Differ. Equ., 2016(1) (2016), 3732176.

[14] V. Komornik; *Exact controllability and stabilization: the multiplier method*. Elsevier Masson, 1994.

[15] S.-J. Li, S. G. Chai; *Stabilization of a semilinear wave equation with time-dependent variable coefficients and a time varying delay on the viscoelastic boundary*, Math. Methods Appl. Sci., 48(5)(2025), 6110-6130.

[16] S.-J. Li, S. G. Chai, I. Lasiecka; *Stabilization of a weak viscoelastic wave equation in Riemannian geometric setting with an interior delay under nonlinear boundary dissipation*, Nonlinear Anal-Real., 84(2025), 104252.

[17] Z. H. Luo; *Direct strain feedback control of flexible robot arms: new theoretical and experimental results*. IEEE Trans. Autom. Control., 38(11)(1993), 1610-1622.

[18] Z. H. Luo, B. Z. Guo,Ö. Morgül; *Stability and stabilization of infinite dimensional systems with applications*. Springer Science & Business Media, 2012.

[19] B. G. J. Marc, T. K. Augustin, Y. Gozo; *A study of exponential stability for a flexible Euler-Bernoulli beam with variable coefficients under a force control in rotation and velocity rotation*. J. Pure and Appl. Math., 13(10)(2017), 6991-7008.

[20] B. Mbodje; *Wave energy decay under fractional derivative controls*. IMA J. Math. Control Inf., 23(2)(2006), 237-257.

[21] S. Nicaise, C. Pignotti; *Exponential stability of abstract evolution equations with time delay*. J. Evol. Equ., 15(1)(2015), 107-129.

[22] S. Nicaise, C. Pignotti; *Stabilization of the wave equation with boundary or internal distributed delay*. 2008.

[23] A. Pazzy; *Semigroups of linear operators and applications to partial differential equations*. Appl. Math. Sci., 44(1983).

[24] S. H. Sandilo, W. T. V. Horssen; *On boundary damping for an axially moving tensioned beam*. 2012.

[25] M. A. Shubov; *Spectral analysis of Euler-Bernoulli beam model with distributed damping and fully non-conservative boundary feedback matrix*. Asymptotic Anal., 129(1)(2022), 75-112.

[26] B. B. J. Siriki, A. Coulibaly; *Exponential Stability of a Degenerate Euler-Bernoulli Beam with Axial Force and Delayed Boundary Control*. arXiv preprint arXiv:2510.25484, 2025.

[27] J. A. Turner, J. S. Wiehn; *Sensitivity of flexural and torsional vibrationmodes of atomic force microscope cantilevers to surfacestiffness variations*. Nanotechnology, 12(3)(2001), 322.

[28] K. G. Vinod, S. Gopalakrishnan, R. Ganguli; *Free vibration and wave propagation analysis of uniform and tapered rotating beams using spectrally formulated finite elements*. Int. J. Solids Struct., 44(18-19)(2007), 5875-5893.

Lanzhi Cao

School of Mathematics and Statistics, Shanxi University, Taiyuan 030006, China

Email address: caolanzhi1114@163.com

Sheng-Jie Li
School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
Systems Research Institute, Polish Academy of Sciences, Newelska 6, 01-447 Warsaw, Poland
Email address: lishengjiebo@163.com, li_sj@fudan.edu.cn

Shugen Chai
School of Mathematics and Statistics, Shanxi University, Taiyuan, 030006, China
Email address: sgchai@sxu.edu.cn