

DYNAMICS OF PIECEWISE LINEAR LIÉNARD SYSTEMS NEAR INFINITY

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ABSTRACT. In this article, we study the dynamics near infinity for a class of piecewise linear Liénard systems $dx/dt = y - F(x)$, $dy/dt = -g(x)$, with n parallel switching lines, where $F(x)$ and $g(x)$ may be continuous or discontinuous. Using the qualitative theory of differential equations, we obtain a complete classification of the phase portraits near infinity in the Poincaré disc. More precisely, when equilibria exist on the equator of the Poincaré disc, we show that the dynamics near infinity are determined by the two outermost linear systems. When no equilibrium exists on the equator, a center-focus problem arises at infinity. In this case, we provide sufficient conditions for the dynamics near infinity to be a stable focus or an unstable focus, and derive a necessary condition for the dynamics near infinity to be a center. Furthermore, for the continuous case where the system has a unique finite equilibrium, we establish a global center criterion.

1. INTRODUCTION

Piecewise linear dynamical systems arise naturally in many models whose governing laws change according to thresholds, switching rules, or state dependent constraints. A typical example is provided by DC–DC buck converters in power electronics [1, 2]. Under each fixed switching configuration, a buck converter is governed by a linear circuit equation, whereas pulse width modulation or feedback control causes the overall dynamics to switch among different linear subsystems. Thus, from the viewpoint of state space modeling, buck converters naturally lead to switched linear, and hence piecewise linear, ordinary differential equations. These models are known to exhibit rich nonlinear phenomena, including periodic oscillations, subharmonic responses, bifurcations and chaotic dynamics, see [2, 13].

Piecewise linear models also play an important role in mathematical biology. A classical example is McKean’s piecewise linear caricature model [25, 26] of the FitzHugh–Nagumo model [15, 27], which was introduced as a tractable simplification of nerve impulse propagation models by replacing the smooth nonlinear reaction term with a piecewise linear function. This simplification preserves essential qualitative features such as excitability, oscillations and pulse propagation, while making the corresponding phase plane analysis more accessible. In particular, when formulated in the phase plane, such models naturally give rise to planar piecewise linear Liénard systems [12, 21, 29].

Piecewise linear differential systems also occur beyond the above two settings. They appear, for instance, in mechanical models with friction or impacts, relay and switching control systems, memristive circuits, neural networks, and genetic regulatory models, see [3, 10, 19, 28, 31]. In such problems, piecewise linear functions are often used to capture threshold effects, saturation, switching mechanisms or discontinuous feedback in a mathematically tractable form. This broad range of applications provides a natural motivation for the qualitative and global analysis of piecewise linear differential systems.

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We consider the general piecewise linear differential system with n parallel switching lines, which can be written as

$$\dot{X} = AX + B + \sum_{i=1}^n \left(C_i |L_i(X)| + D_i \operatorname{sgn}(L_i(X)) \right), \quad (1.1)$$

where $X = (x, y)^\top \in \mathbb{R}^2$, $A \in \mathbb{R}^{2 \times 2}$, $B, C_i, D_i \in \mathbb{R}^2$ and

$$L_i(X) = k_1 x + k_2 y + l_i, \quad i = 1, 2, \dots, n,$$

with $|k_1| + |k_2| > 0$. System (1.1) is characterized by $5n + 8$ independent parameters. By applying the coordinate transformations detailed in Appendix A, system (1.1) can be reduced, under the stated nondegeneracy assumption, to the following Liénard form

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -g(x), \quad (1.2)$$

where

$$F(x) = a_0 x + b_0 + \sum_{i=1}^n (a_i |x - x_i| + b_i \operatorname{sgn}(x - x_i)),$$

$$g(x) = c_0 x + d_0 + \sum_{i=1}^n (c_i |x - x_i| + d_i \operatorname{sgn}(x - x_i)),$$

and $a_0, b_0, c_0, d_0, a_i, b_i, c_i, d_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$. The reduced system (1.2) involves $5n + 4$ parameters. Its switching lines are $x = x_i$, where

$$-\infty < x_1 < x_2 < \dots < x_{k-1} < 0 < x_k < \dots < x_n < +\infty.$$

To simplify the notation, we partition the real line into $n + 1$ intervals

$$I_j = (x_{j-1}, x_j), \quad j = 1, 2, \dots, n + 1,$$

where $x_0 = -\infty$ and $x_{n+1} = +\infty$. On each interval I_j , the functions $F(x)$ and $g(x)$ are affine functions of x , namely

$$F(x) = A_j x + B_j, \quad g(x) = C_j x + D_j,$$

where

$$A_j = a_0 + \sum_{i=1}^{j-1} a_i - \sum_{i=j}^n a_i,$$

$$B_j = b_0 - \sum_{i=1}^{j-1} (a_i x_i - b_i) + \sum_{i=j}^n (a_i x_i - b_i),$$

$$C_j = c_0 + \sum_{i=1}^{j-1} c_i - \sum_{i=j}^n c_i,$$

$$D_j = d_0 - \sum_{i=1}^{j-1} (c_i x_i - d_i) + \sum_{i=j}^n (c_i x_i - d_i),$$

where $1 \leq i \leq n$ and $1 \leq j \leq n + 1$, and we adopt the convention that empty sums are taken to be zero.

From the viewpoint of the number of switching lines, most existing results focus on systems of the form (1.2) with a small number of switching lines, in particular, one or two parallel switching lines. For system (1.2) with a single switching line, both continuous and discontinuous piecewise linear Liénard systems have been extensively studied. In the continuous case, classical results have been obtained concerning global phase portraits, limit cycles, and bifurcations, see e.g., [20, 23]. In the discontinuous case, the corresponding dynamics have also been systematically investigated, see, e.g., [4, 16]. System (1.2) with two parallel switching lines represents another fundamental family of systems that has attracted considerable attention in recent years. In the continuous setting, extensive results on phase portrait classification and bifurcation analysis have been developed in

[5, 6, 7, 8, 9, 17, 18, 22, 23, 24]. These studies show that adding more switching lines can lead to significantly more complex global dynamics. However, the dynamics of general piecewise linear Liénard systems with an arbitrary number of switching lines remain largely unexplored, which motivates the present work.

When studying the global structure of planar differential systems, the behavior of orbits near infinity plays a crucial role. Indeed, the dynamics at infinity determines the structure of the vector field on the boundary of the Poincaré disc and describes the asymptotic directions of the orbits as $|x| + |y| \rightarrow \infty$. This information is essential for the study of global phase portraits and bifurcations, since it may determine possible separatrix connections, the existence or non-existence of closed orbits, and the way in which the finite part of the phase portrait is connected with infinity. Therefore, the analysis of singularities at infinity and their local phase portraits is a fundamental step in understanding the global dynamics of such systems.

When $F(x)$ and $g(x)$ in system (1.2) are arbitrary polynomial functions, system (1.2) becomes a generalized polynomial Liénard system. The dynamics near infinity of such systems were classically studied by Dumortier and Herssens in [11]. By using the Poincaré compactification and the Poincaré–Lyapunov compactification, they gave a complete classification of the behaviour near infinity. Their results show that, except for the so called center-focus problem at infinity, the necessary information is determined by the leading coefficients and degrees of the polynomial terms. In the exceptional case, the behavior near infinity also depends on the remaining coefficients of the system.

This naturally raises the question of what determines the dynamics near infinity of the piecewise linear Liénard system (1.2). In particular, when $F(x)$ and $g(x)$ are represented by absolute-value and sign functions, is the behaviour near infinity determined only by the linear systems in the outermost regions, or is it also affected by the switching lines and by the continuity or discontinuity across them? Answering these questions is an essential step toward understanding the global dynamics of system (1.2).

The rest of this paper is organized as follows. In Section 2, we state our main results, including the classification of the dynamics of system (1.2) near infinity, results on the center-focus problem at infinity, and the global center criterion. In Section 3, by means of the Poincaré compactification, we obtain the classification of the dynamics of system (1.2) near infinity. Section 4 is devoted to the study of the center-focus problem at infinity, where the Bendixson compactification is employed. In Section 5, we establish the global center criterion for the case where system (1.2) has a unique finite equilibrium.

2. MAIN RESULTS

For convenience, we introduce the following notation to represent the equilibria at infinity of system (1.2):

$$\begin{aligned}\tilde{I}_A : y &= \frac{A_{n+1}}{2}x, \text{ as } x \rightarrow +\infty, \\ \tilde{I}_B : y &= \frac{A_{n+1} - \sqrt{A_{n+1}^2 - 4C_{n+1}}}{2}x, \text{ as } x \rightarrow +\infty, \\ \tilde{I}_C : y &= \frac{A_{n+1} + \sqrt{A_{n+1}^2 - 4C_{n+1}}}{2}x, \text{ as } x \rightarrow +\infty, \\ \tilde{I}_D : y &= \frac{A_1}{2}x, \text{ as } x \rightarrow -\infty, \\ \tilde{I}_E : y &= \frac{A_1 - \sqrt{A_1^2 - 4C_1}}{2}x, \text{ as } x \rightarrow -\infty, \\ \tilde{I}_F : y &= \frac{A_1 + \sqrt{A_1^2 - 4C_1}}{2}x, \text{ as } x \rightarrow -\infty.\end{aligned}$$

The following theorem describes the dynamics at infinity of system (1.2) in the Poincaré disc.

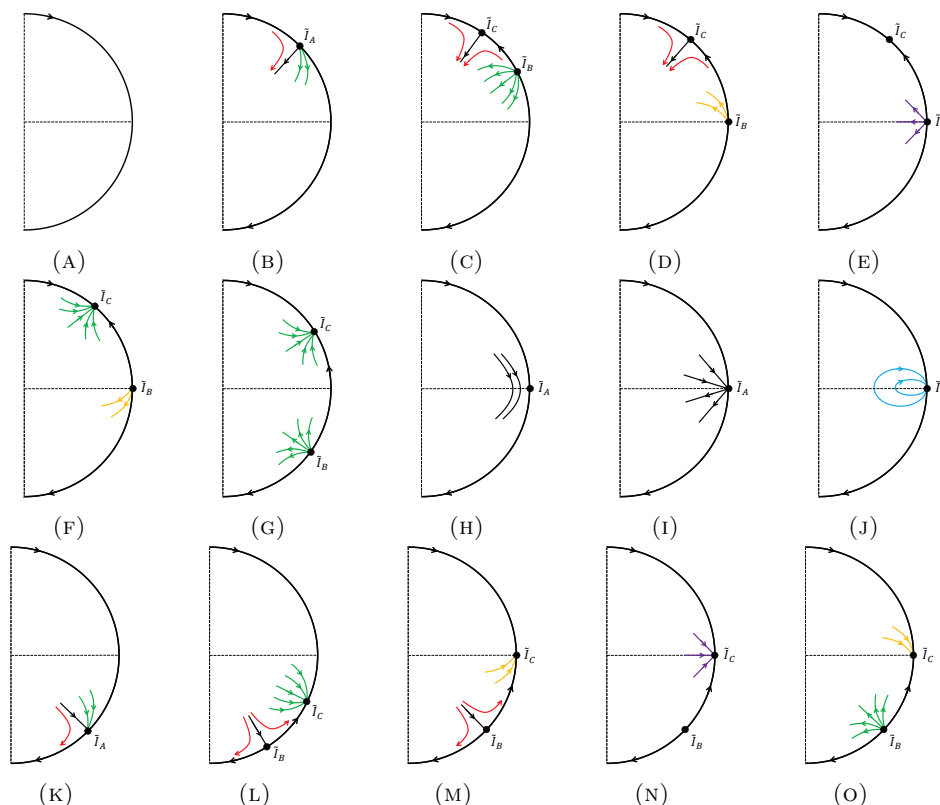


FIGURE 1. Dynamics near infinity in the Poincaré disc of system (1.2) as $x \rightarrow +\infty$

Theorem 2.1. *In the right (resp., left) half plane, the dynamics of system (1.2) at infinity are completely classified in Table 1 (resp., Table 2), yielding 15 distinct phase portraits for each case in the Poincaré disc, as shown in Figure 1 (resp., Figure 2). By combining the phase portraits on the right and left half discs of the Poincaré disc, system (1.2) has a total of 225 distinct phase portraits near infinity in the Poincaré disc.*

Remark 2.2. It is clear that the dynamics of system (1.2) at infinity in the right half plane are determined by the outer linear subsystem on $I_{n+1} = (x_n, +\infty)$, while the dynamics of system (1.2) at infinity in the left half plane are determined by the outer linear subsystem on $I_1 = (-\infty, x_1)$.

When system (1.2) has no equilibria at infinity, the dynamics near infinity gives rise to a center-focus problem. The following theorem provides a sufficient condition for the focus case, as well as a necessary condition for the center case.

Theorem 2.3. *Assume that $A_{n+1}^2 - 4C_{n+1} < 0$ and $A_1^2 - 4C_1 < 0$. If*

$$\frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}} + \frac{A_1\pi}{\sqrt{4C_1 - A_1^2}} < 0 \quad (\text{resp. } > 0),$$

then the dynamics of system (1.2) near infinity is a stable focus (resp., an unstable focus). If the dynamics of system (1.2) near infinity is a center, then

$$\frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}} + \frac{A_1\pi}{\sqrt{4C_1 - A_1^2}} = 0.$$

In particular, if system (1.2) is symmetric with respect to the y -axis, then the dynamics of system (1.2) near infinity is a center. These cases are illustrated in Figure 3.

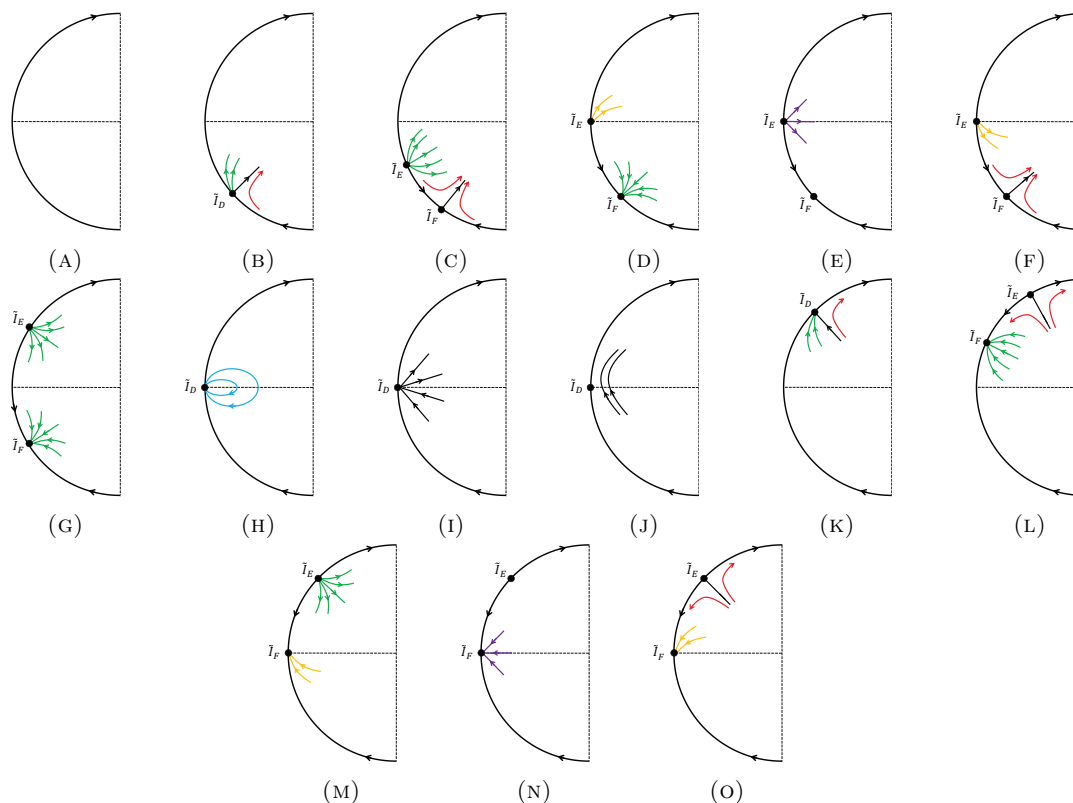
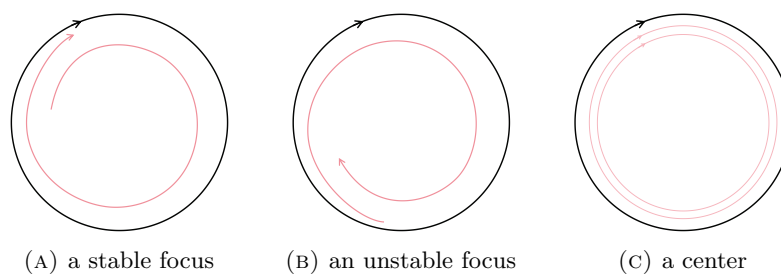
FIGURE 2. Dynamics near infinity in the Poincaré disc of system (1.2) as $x \rightarrow -\infty$ 

FIGURE 3. Center-focus problem for the dynamics near infinity

Remark 2.4. Unlike the focus case, if the dynamics of system (1.2) near infinity is a center, the two outermost regions alone are not sufficient to determine its behavior, since the intermediate regions also affect the dynamics at infinity.

In what follows, we only consider continuous piecewise linear Liénard systems, since discontinuities on switching lines may lead to nonunique orbits, sliding or escaping motions, and pseudo-equilibria. Without loss of generality, we assume that system (1.2) has an equilibrium at the origin. Let $I_k = (x_{k-1}, x_k)$ be the interval containing the origin. Then system (1.2) can be written as

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -g(x), \quad (2.1)$$

TABLE 1. Locations and topological structure of equilibria at infinity of system (1.2) as $x \rightarrow +\infty$

Parameters $(A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}) \in \mathbb{R}^4$			Equilibria and topological structure	Phase portraits
$A_{n+1} > 0$	$C_{n+1} > \frac{A_{n+1}^2}{4}$		None	Figure 1(a)
	$C_{n+1} = \frac{A_{n+1}^2}{4}$		$\tilde{I}_A$: consists of one HSR and one PSR	Figure 1(b)
	$0 < C_{n+1} < \frac{A_{n+1}^2}{4}$		$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of two HSRs	Figure 1(c)
	$C_{n+1} = 0$	$D_{n+1} > 0$	$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of two HSRs	Figure 1(d)
		$D_{n+1} = 0$	$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of the invariant lines	Figure 1(e)
		$D_{n+1} < 0$	$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of one PSR	Figure 1(f)
	$C_{n+1} < 0$		$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of one PSR	Figure 1(g)
$A_{n+1} = 0$	$C_{n+1} > 0$		None	Figure 1(a)
	$C_{n+1} = 0$	$D_{n+1} > 0$	$\tilde{I}_A$: consists of one HSR	Figure 1(h)
		$D_{n+1} = 0$	$\tilde{I}_A$: consists of the invariant lines	Figure 1(i)
		$D_{n+1} < 0$	$\tilde{I}_A$: consists of one ESR	Figure 1(j)
	$C_{n+1} < 0$		$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of one PSR	Figure 1(g)
$A_{n+1} < 0$	$C_{n+1} > \frac{A_{n+1}^2}{4}$		None	Figure 1(a)
	$C_{n+1} = \frac{A_{n+1}^2}{4}$		$\tilde{I}_A$: consists of one HSR and one PSR	Figure 1(k)
	$0 < C_{n+1} < \frac{A_{n+1}^2}{4}$		$\tilde{I}_B$: consists of two HSRs, $\tilde{I}_C$: consists of one PSR	Figure 1(l)
	$C_{n+1} = 0$	$D_{n+1} > 0$	$\tilde{I}_B$: consists of two HSRs, $\tilde{I}_C$: consists of one PSR	Figure 1(m)
		$D_{n+1} = 0$	$\tilde{I}_B$: consists of the invariant lines, $\tilde{I}_C$: consists of one PSR	Figure 1(n)
		$D_{n+1} < 0$	$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of one PSR	Figure 1(o)
	$C_{n+1} < 0$		$\tilde{I}_B$: consists of one PSR, $\tilde{I}_C$: consists of one PSR	Figure 1(g)
Denote HSR as hyperbolic sector region, PSR as parabolic sector region and ESR as elliptical sector region.				

where $F(x)$ and $g(x)$ are defined piecewise by

$$F(x) = \begin{cases} A_j x + \sum_{i=j}^{k-1} (A_{i+1} - A_i) x_i, & \text{if } x_{j-1} < x \leq x_j, \quad j < k, \\ A_k x, & \text{if } x_{k-1} < x \leq x_k, \\ A_j x + \sum_{i=k}^{j-1} (A_i - A_{i+1}) x_i, & \text{if } x_{j-1} < x \leq x_j, \quad j > k, \end{cases}$$

and

$$g(x) = \begin{cases} C_j x + \sum_{i=j}^{k-1} (C_{i+1} - C_i) x_i, & \text{if } x_{j-1} < x \leq x_j, \quad j < k, \\ C_k x, & \text{if } x_{k-1} < x \leq x_k, \\ C_j x + \sum_{i=k}^{j-1} (C_i - C_{i+1}) x_i, & \text{if } x_{j-1} < x \leq x_j, \quad j > k, \end{cases}$$

TABLE 2. Locations and topological structure of equilibria at infinity of system (1.2) as $x \rightarrow -\infty$

Parameters $(A_1, B_1, C_1, D_1) \in \mathbb{R}^4$			Equilibria and topological structure	Phase portraits
$A_1 > 0$	$C_1 > \frac{A_1^2}{4}$		None	Figure 2(a)
	$C_1 = \frac{A_1^2}{4}$		$\tilde{I}_D$: consists of one HSR and one PSR	Figure 2(b)
	$0 < C_1 < \frac{A_1^2}{4}$		$\tilde{I}_E$: consists of one PSR, two HSRs $\tilde{I}_F$: consists of	Figure 2(c)
	$C_1 = 0$	$D_1 > 0$	$\tilde{I}_E$: consists of one PSR, PSR $\tilde{I}_F$: consists of one	Figure 2(d)
		$D_1 = 0$	$\tilde{I}_E$: consists of one PSR, invariant lines $\tilde{I}_F$: consists of the	Figure 2(e)
		$D_1 < 0$	$\tilde{I}_E$: consists of one PSR, two HSRs $\tilde{I}_F$: consists of	Figure 2(f)
	$C_1 < 0$		$\tilde{I}_E$: consists of one PSR, PSR $\tilde{I}_F$: consists of one	Figure 2(g)
$A_1 = 0$	$C_1 > 0$		None	Figure 2(a)
	$C_1 = 0$	$D_1 > 0$	$\tilde{I}_D$: consists of one ESR	Figure 2(h)
		$D_1 = 0$	$\tilde{I}_D$: consists of the invariant lines	Figure 2(i)
		$D_1 < 0$	$\tilde{I}_D$: consists of one HSR	Figure 2(j)
	$C_1 < 0$		$\tilde{I}_E$: consists of one PSR, PSR $\tilde{I}_F$: consists of one	Figure 2(g)
$A_1 < 0$	$C_1 > \frac{A_1^2}{4}$		None	Figure 2(a)
	$C_1 = \frac{A_1^2}{4}$		$\tilde{I}_D$: consists of one HSR and one PSR	Figure 2(k)
	$0 < C_1 < \frac{A_1^2}{4}$		$\tilde{I}_E$: consists of two HSRs, one PSR $\tilde{I}_F$: consists of	Figure 2(l)
	$C_1 = 0$	$D_1 > 0$	$\tilde{I}_E$: consists of one PSR, PSR $\tilde{I}_F$: consists of one	Figure 2(m)
		$D_1 = 0$	$\tilde{I}_E$: consists of the invariant lines, consists of one PSR $\tilde{I}_F$:	Figure 2(n)
		$D_1 < 0$	$\tilde{I}_E$: consists of two HSRs, one PSR $\tilde{I}_F$: consists of	Figure 2(o)
	$C_1 < 0$		$\tilde{I}_E$: consists of one PSR, PSR $\tilde{I}_F$: consists of one	Figure 2(g)
Denote HSR as hyperbolic sector region, PSR as parabolic sector region and ESR as elliptical sector region.				

for $j = 1, 2, \dots, n+1$, with $1 < k < n+1$. We now state a global center criterion for system (2.1).

Theorem 2.5. *System (2.1) has a unique equilibrium, which is a global center if and only if the following conditions are satisfied:*

- (i) $A_k = 0$ and $C_k > 0$ on the interval $I_k = (x_{k-1}, x_k)$;
- (ii) $xg(x) > 0$ for all $x \neq 0$;
- (iii) $F(x_1) = F(x_2)$ when $G(x_1) = G(x_2)$ for all $x_2 < 0 < x_1$, where $G(x) = \int_0^x g(s)ds$;
- (iv) either $A_{n+1}^2 - 4C_{n+1} < 0$ and $A_1^2 - 4C_1 < 0$, or $A_{n+1} = C_{n+1} = A_1 = C_1 = 0$ and $D_1 < 0 < D_{n+1}$.

These two cases are illustrated in Figure 4.

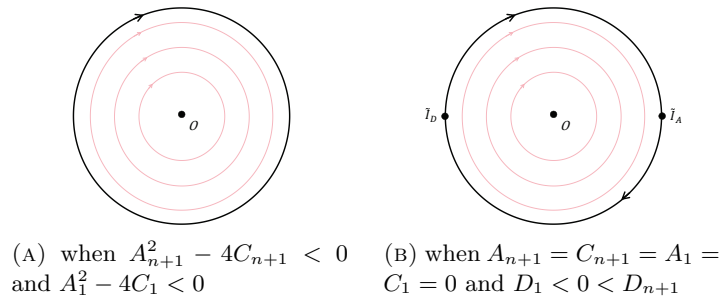


FIGURE 4. Global center of system (2.1)

3. POINCARÉ COMPACTIFICATION

To analyze the behavior of orbits of a planar differential system near infinity, we employ the Poincaré compactification, originally introduced by Poincaré [30] and systematically developed in [12, 29]. This approach transforms the non-compact phase plane into a compact manifold, allowing the study of the dynamics at infinity within a bounded domain. Geometrically, the phase plane is identified with a plane tangent to the unit sphere and mapped onto the Poincaré sphere by central projection, under which each direction at infinity corresponds to a point on the equator, so that infinity is resolved into a continuum of directions and the singularities at infinity are distributed along the equator.

For polynomial vector fields, the Poincaré compactification is particularly well suited, since after an appropriate time rescaling, the compactified vector field extends smoothly to the equator of the sphere; moreover, the dynamics at infinity is completely determined by the highest-degree homogeneous terms of the system.

Although system (1.2) is not a globally defined polynomial vector field because of the presence of the absolute value and sign functions, it is a piecewise affine system. More precisely, on each interval determined by the switching points $x_1, \dots, x_n$, the functions $F(x)$ and $g(x)$ reduce to affine functions of x . Hence, on each region of the phase plane, system (1.2) becomes an affine polynomial vector field of degree one. This allows the Poincaré compactification to be performed on each region, and thus the equilibria at infinity of system (1.2) can be investigated.

Proof of Theorem 2.1. To analyze the behavior of system (1.2) near infinity except along the y -axis, we apply the Poincaré transformation $x = 1/z$, $y = u/z$, system (1.2) becomes

$$\begin{aligned} \frac{du}{dt} &= -u^2 + z \left(uF\left(\frac{1}{z}\right) - g\left(\frac{1}{z}\right) \right), \\ \frac{dz}{dt} &= -uz + z^2 F\left(\frac{1}{z}\right). \end{aligned} \quad (3.1)$$

Substituting the expressions for F and g into (3.1), we obtain

$$\begin{aligned} \frac{du}{dt} &= -u^2 + a_0 u - c_0 + z(b_0 u - d_0) \\ &\quad + \sum_{i=1}^n \left[(a_i u - c_i) \operatorname{sgn}(z) |1 - x_i z| + (b_i u - d_i) z \operatorname{sgn}\left(\frac{1}{z} - x_i\right) \right], \\ \frac{dz}{dt} &= -uz + a_0 z + b_0 z^2 + \sum_{i=1}^n \left[a_i |z| |1 - x_i z| + b_i z^2 \operatorname{sgn}\left(\frac{1}{z} - x_i\right) \right]. \end{aligned} \quad (3.2)$$

For $z > 0$ sufficiently small, noting that $\operatorname{sgn}(z) = 1$, $\operatorname{sgn}(1/z - x_i) = 1$, and $|1 - x_i z| = 1 - x_i z$, system (3.2) reduces to

$$\begin{aligned}\frac{du}{dt} &= -u^2 + a_0 u - c_0 + z(b_0 u - d_0) + \sum_{i=1}^n \left[(a_i u - c_i)(1 - x_i z) + z(b_i u - d_i) \right] \\ &= -u^2 + A_{n+1} \cdot u + B_{n+1} \cdot uz - C_{n+1} - D_{n+1} \cdot z, \\ \frac{dz}{dt} &= -uz + a_0 z + b_0 z^2 + \sum_{i=1}^n \left[a_i z(1 - x_i z) + b_i z^2 \right] \\ &= -uz + A_{n+1} \cdot z + B_{n+1} \cdot z^2.\end{aligned}\tag{3.3}$$

We next analyze system (3.3). The number of equilibria of system (3.3) is determined by the number of real roots of the equation $u^2 - A_{n+1} \cdot u + C_{n+1} = 0$ on the u -axis. Let $\Delta_1 = A_{n+1}^2 - 4C_{n+1}$. Accordingly, system (3.3) has no equilibria when $\Delta_1 < 0$, a unique equilibrium $I_A(A_{n+1}/2, 0)$ when $\Delta_1 = 0$, and two equilibria $I_B((A_{n+1} - \sqrt{\Delta_1})/2, 0)$ and $I_C((A_{n+1} + \sqrt{\Delta_1})/2, 0)$ when $\Delta_1 > 0$. The Jacobian matrix of system (3.3) at (u, z) is given by

$$J(u, z) = \begin{pmatrix} -2u + A_{n+1} + B_{n+1} \cdot z & B_{n+1} \cdot u - D_{n+1} \\ -z & -u + A_{n+1} + 2B_{n+1} \cdot z \end{pmatrix}.$$

Case 1: Equilibrium I_A Substituting I_A into the Jacobian matrix yields

$$J_{I_A} = \begin{pmatrix} 0 & \frac{A_{n+1}B_{n+1}}{2} - D_{n+1} \\ 0 & \frac{A_{n+1}}{2} \end{pmatrix}\tag{3.4}$$

and we move the equilibrium I_A to the origin

$$\begin{aligned}\frac{du}{dt} &= -u^2 + B_{n+1} \cdot uz + \left(\frac{A_{n+1}B_{n+1}}{2} - D_{n+1} \right) z, \\ \frac{dz}{dt} &= \frac{A_{n+1}}{2} z - uz + B_{n+1} \cdot z^2\end{aligned}\tag{3.5}$$

by applying a translation $u \rightarrow u + A_{n+1}/2$. We consider two cases.

Case 1.1: $A_{n+1} \neq 0$. In this case, I_A is a degenerate equilibrium according to (3.4). By the transformation

$$(u, t) \rightarrow \left(u + \left(-B_{n+1} + \frac{2D_{n+1}}{A_{n+1}} \right) \cdot z, \frac{2}{A_{n+1}} \cdot \tau \right),$$

system (3.5) becomes

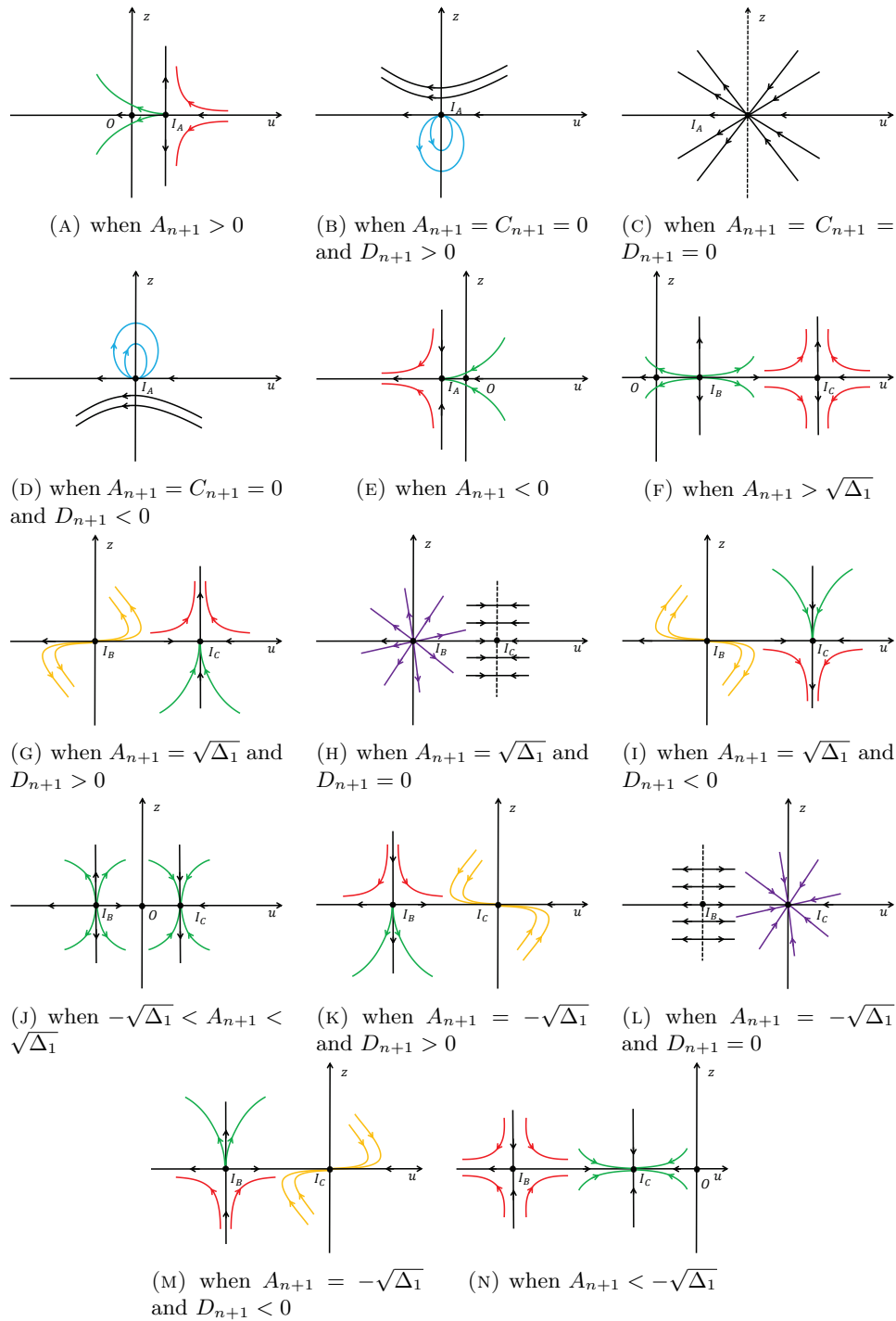
$$\begin{aligned}\frac{du}{d\tau} &= -\frac{2}{A_{n+1}} \cdot u^2 + \frac{4D_{n+1}}{A_{n+1}^2} \cdot uz = P(u, z), \\ \frac{dz}{d\tau} &= z - \frac{2}{A_{n+1}} \cdot uz + \frac{4D_{n+1}}{A_{n+1}^2} \cdot z^2 = z + Q(u, z).\end{aligned}\tag{3.6}$$

By solving $z + Q(u, z) = 0$, the implicit function theorem yields $z = 0$. Substituting $z = 0$ into $P(u, z)$ gives $P(u, 0) = -2u^2/A_{n+1}$. By [32, Theorem 7.1], the origin of system (3.6) is a saddle-node. Hence, I_A is also a saddle-node of system (3.3). Now, we determine the saddle and nodal parts of the saddle-node at I_A . On the one hand, from system (3.3), we have

$$\left. \frac{du}{dt} \right|_{z=0} = -\left(u - \frac{A_{n+1}}{2} \right)^2, \quad \left. \frac{dz}{dt} \right|_{z=0} = 0.$$

Hence, the line $z = 0$ is an invariant line, and the flow on this invariant line is oriented from right to left. Thus, the branch $u > A_{n+1}/2$ is attracted to I_A , while the branch $u < A_{n+1}/2$ is repelled from I_A . On the other hand, it is easy to verify that

$$u_1(z) = \left(B_{n+1} - \frac{2D_{n+1}}{A_{n+1}} \right) z + \frac{A_{n+1}}{2}$$

FIGURE 5. Qualitative properties of I_A , I_B and I_C of system (3.3)

is an invariant manifold of system (3.3). Substituting $u_1(z)$ into system (3.3), we obtain

$$\frac{dz}{dt} \Big|_{u=u_1} = \frac{A_{n+1}}{2}z + O(z^2).$$

Therefore, the direction of this invariant manifold is determined by the sign of A_{n+1} . More precisely, it is repelled from I_A if $A_{n+1} > 0$ and attracted to I_A if $A_{n+1} < 0$. Consequently, if

$A_{n+1} > 0$, the left side of the line $u = u_1(z)$ corresponds to the nodal part, whereas the right side corresponds to the saddle part. If $A_{n+1} < 0$, the positions of the two parts are interchanged, see Figures 5(a) and (e), respectively.

Case 1.2: $A_{n+1} = 0$. In this case, I_A is a completely degenerate equilibrium according to (3.4) and system (3.5) simplifies to

$$\begin{aligned}\frac{du}{dt} &= -u^2 + B_{n+1} \cdot uz - D_{n+1} \cdot z, \\ \frac{dz}{dt} &= -uz + B_{n+1} \cdot z^2.\end{aligned}\tag{3.7}$$

First, we consider $D_{n+1} \neq 0$. Applying the linear transformation $z \rightarrow -D_{n+1} \cdot z$ and then denoting the new variable $\tilde{z} = z - u^2 - B_{n+1} \cdot uz/D_{n+1}$, system (3.7) is transformed into

$$\frac{du}{dt} = \tilde{z}, \quad \frac{d\tilde{z}}{dt} = -u^3(1 + O(u)) - 3u\tilde{z}(1 + O(u)) + \tilde{z}^2 h(u, \tilde{z}).\tag{3.8}$$

By [32, Theorem 7.2], the origin of system (3.8) consists of one hyperbolic sector and one elliptic sector. The same holds for I_A of system (3.3). Now, we determine the positions of these two sector regions by analyzing the local directions of the vector field of system (3.7). On the one hand, by system (3.7), we calculate that

$$\left. \frac{du}{dt} \right|_{z=0} = -u^2, \quad \left. \frac{dz}{dt} \right|_{z=0} = 0.$$

Hence, the line $z = 0$ is an invariant line, and the flow on it is directed from right to left. More precisely, the positive branch $u > 0$ approaches the origin I_A along this invariant line, while the negative branch $u < 0$ moves away from the origin I_A along this invariant line. On the other hand, by system (3.7), we compute

$$\left. \frac{du}{dt} \right|_{u=0} = -D_{n+1} \cdot z.$$

This shows that the direction of the vector field on the line $u = 0$ is determined by the sign of D_{n+1} . If the orientation of the vector field on $u = 0$ is consistent with that on $z = 0$, the corresponding region is a hyperbolic sector, while the opposite region corresponds to an elliptic sector. Consequently, when $D_{n+1} > 0$, the upper region corresponds to a hyperbolic sector and the lower region corresponds to an elliptic sector. When $D_{n+1} < 0$, the positions of the hyperbolic and elliptic sectors are reversed. The same conclusion holds for I_A of system (3.3), see Figures 5(b) and (d), respectively.

Second, consider $D_{n+1} = 0$. System (3.7) reduces to

$$\dot{u} = -u(u + B_{n+1} \cdot z), \quad \dot{z} = -z(u + B_{n+1} \cdot z).\tag{3.9}$$

Clearly, all straight lines passing through the origin are invariant lines of system (3.9). So is the origin I_A of system (3.3), see Figure 5(c).

Case 2: Equilibria I_B and I_C Substituting I_B and I_C into the Jacobian matrix, we obtain

$$J_{I_B} = \begin{pmatrix} \sqrt{\Delta_1} & \frac{A_{n+1}-\sqrt{\Delta_1}}{2} B_{n+1} - D_{n+1} \\ 0 & \frac{A_{n+1}+\sqrt{\Delta_1}}{2} \end{pmatrix},$$

and

$$J_{I_C} = \begin{pmatrix} -\sqrt{\Delta_1} & \frac{A_{n+1}+\sqrt{\Delta_1}}{2} B_{n+1} - D_{n+1} \\ 0 & \frac{A_{n+1}-\sqrt{\Delta_1}}{2} \end{pmatrix}.$$

The corresponding determinants, traces, and discriminants are

$$\begin{aligned}\det J_{I_B} &= \frac{(A_{n+1} + \sqrt{\Delta_1})\sqrt{\Delta_1}}{2}, \quad \text{tr } J_{I_B} = \frac{A_{n+1} + 3\sqrt{\Delta_1}}{2}, \\ \Delta_{I_B} &= \frac{(A_{n+1} - \sqrt{\Delta_1})^2}{4} \geq 0, \quad \det J_{I_C} = \frac{-(A_{n+1} - \sqrt{\Delta_1})\sqrt{\Delta_1}}{2}, \\ \text{tr } J_{I_C} &= \frac{A_{n+1} - 3\sqrt{\Delta_1}}{2}, \quad \Delta_{I_C} = \frac{(A_{n+1} + \sqrt{\Delta_1})^2}{4} \geq 0.\end{aligned}$$

Depending on the values of A_{n+1} and $\sqrt{\Delta_1}$, we distinguish five cases.

Case 2.1: $A_{n+1} > \sqrt{\Delta_1}$. It is easy to verify that $\det J_{I_B} > 0$, $\operatorname{tr} J_{I_B} > 0$, $\Delta_{I_B} > 0$, and $\det J_{I_C} < 0$. Hence, I_B is an unstable node and I_C is a saddle, see Figure 5(f).

Case 2.2: $A_{n+1} = \sqrt{\Delta_1}$. First, consider $D_{n+1} \neq 0$. One can check that $\det J_{I_B} > 0$, $\operatorname{tr} J_{I_B} > 0$, $\Delta_{I_B} = 0$, and $\det J_{I_C} = 0$. Thus, the origin I_B is an unstable improper node and I_C is a degenerate equilibrium. By a translation

$$(u, t) \rightarrow (u + A_{n+1} + \frac{A_{n+1}B_{n+1} - D_{n+1}}{A_{n+1}} \cdot z, -\tau/A_{n+1}),$$

system (3.3) becomes

$$\begin{aligned} \frac{du}{d\tau} &= u + \frac{1}{A_{n+1}} \cdot u^2 - \frac{D_{n+1}}{A_{n+1}^2} \cdot uz = u + Q(u, z), \\ \frac{dz}{d\tau} &= \frac{1}{A_{n+1}} \cdot uz - \frac{D_{n+1}}{A_{n+1}^2} \cdot z^2 = P(u, z). \end{aligned}$$

Similar to the analysis of I_A , we solve $u + Q(u, z) = 0$ for u using the implicit function theorem, which yields $u = 0$. Substituting $u = 0$ into $P(u, z)$ gives $P(0, z) = -D_{n+1}/A_{n+1}^2 \cdot z^2$. It follows from [32, Theorem 7.1] that I_C is a saddle-node. Similarly, following the same argument as in Case 1.1, we obtain the Figures 5(g) and (i).

Second, consider $D_{n+1} = 0$. Since $C_{n+1} = 0$, system (3.3) becomes

$$\frac{du}{dt} = -u^2 + A_{n+1} \cdot u + B_{n+1} \cdot uz, \quad \frac{dz}{dt} = -uz + A_{n+1} \cdot z + B_{n+1} \cdot z^2.$$

The Jacobian matrix is

$$J_{I_B} = \begin{pmatrix} A_{n+1} & 0 \\ 0 & A_{n+1} \end{pmatrix},$$

so the origin I_B is an unstable proper node. Moreover, the equation $dz/dt = 0$ yields either $z = 0$ or $u = A_{n+1} + B_{n+1} \cdot z$. Substituting $u = A_{n+1} + B_{n+1}z$ into the first equation, we obtain $\frac{du}{dt} = 0$. Hence every point on the line $u = A_{n+1} + B_{n+1} \cdot z$ is an equilibrium. In particular, $I_C = (A_{n+1}, 0)$ is a non-isolated equilibrium, see Figure 5(h).

Case 2.3: $-\sqrt{\Delta_1} < A_{n+1} < \sqrt{\Delta_1}$. Evidently, $\det J_{I_B} > 0$, $\operatorname{tr} J_{I_B} > 0$, $\Delta_{I_B} > 0$, $\det J_{I_C} > 0$, $\operatorname{tr} J_{I_C} < 0$, and $\Delta_{I_C} > 0$. It follows that I_B is an unstable node and I_C is a stable node, see Figure 5(j).

Case 2.4: $A_{n+1} = -\sqrt{\Delta_1}$. Similar to the analysis of Case 2.2, when $D_{n+1} \neq 0$, we have $\det J_{I_B} = 0$, $\det J_{I_C} > 0$, $\operatorname{tr} J_{I_C} < 0$, and $\Delta_{I_C} = 0$. We can obtain that I_B is a saddle-node and the origin I_C is a stable improper node, see Figures 5(k) and (m). When $D_{n+1} = 0$, I_B is a non-isolated equilibrium and the origin I_C is a stable proper node, see Figure 5(l).

Case 2.5: $A_{n+1} < -\sqrt{\Delta_1}$. Trivially, $\det J_{I_B} < 0$, $\det J_{I_C} > 0$, $\operatorname{tr} J_{I_C} < 0$, and $\Delta_{I_C} > 0$. Therefore, I_B is a saddle and I_C is a stable node, see Figure 5(n).

For $z < 0$ sufficiently small, by $\operatorname{sgn}(z) = -1$, $\operatorname{sgn}(1/z - x_i) = -1$, and $|1 - x_i z| = 1 - x_i z$, system (3.2) is reduced to

$$\begin{aligned} \frac{du}{dt} &= -u^2 + a_0 u - c_0 + z(b_0 u - d_0) + \sum_{i=1}^n [(a_i u - c_i)(x_i z - 1) - z(b_i u - d_i)] \\ &= -u^2 + A_1 \cdot u + B_1 \cdot uz - C_1 - D_1 \cdot z, \\ \frac{dz}{dt} &= -uz + a_0 z + b_0 z^2 + \sum_{i=1}^n [a_i z(x_i z - 1) - b_i z^2] \\ &= -uz + A_1 \cdot z + B_1 \cdot z^2. \end{aligned} \tag{3.10}$$

It is important to note that system (3.10) is equivalent to system (3.3) under the parameter substitution

$$(A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}) \rightarrow (A_1, B_1, C_1, D_1).$$

Moreover,

$$A_{n+1} = A_1, \quad B_{n+1} = B_1, \quad C_{n+1} = C_1, \quad D_{n+1} = D_1$$

are equivalent to

$$\sum_{i=1}^n a_i = 0, \quad \sum_{i=1}^n b_i = \sum_{i=1}^n a_i x_i, \quad \sum_{i=1}^n c_i = 0, \quad \sum_{i=1}^n d_i = \sum_{i=1}^n c_i x_i.$$

Therefore, all the results obtained for system (3.3) apply directly to system (3.10). Letting $\Delta_2 = A_1^2 - 4C_1$, the number and types of equilibria of system (3.10) follow directly from those of system (3.3). More precisely, when $\Delta_2 < 0$, system (3.10) has no equilibria on the u -axis. If $\Delta_2 = 0$, system (3.10) has a unique equilibrium on the u -axis at $I_D(A_1/2, 0)$, which has the same qualitative properties as I_A of system (3.3). Finally, when $\Delta_2 > 0$, system (3.10) has two equilibria on the u -axis at $I_E((A_1 - \sqrt{\Delta_2})/2, 0)$ and $I_F((A_1 + \sqrt{\Delta_2})/2, 0)$, whose dynamical types correspond respectively to those of I_B and I_C of system (3.3).

Under the Poincaré transformation $x = v/z$, $y = 1/z$, system (1.2) is rewritten as

$$\frac{dv}{dt} = 1 - zF\left(\frac{v}{z}\right) + zv g\left(\frac{v}{z}\right), \quad \frac{dz}{dt} = z^2 g\left(\frac{v}{z}\right). \quad (3.11)$$

Clearly, the origin $(0, 0)$ is not an equilibrium of system (3.11). It follows that system (1.2) has no equilibria at infinity in the y -directions.

We now observe that the equilibrium I_A of system (3.3) corresponds, under the Poincaré compactification, to an equilibrium at infinity of system (1.2), denoted by $\tilde{I}_A$, lying on the positive x -direction of the equator of the Poincaré disc. Based on the above analysis, the local phase portrait of $\tilde{I}_A$ consists of one hyperbolic sector and one parabolic sector when $C_{n+1} = A_{n+1}^2/4$ and $A_{n+1} > 0$, as shown in Table 1. The same argument applies to $\tilde{I}_B$ and $\tilde{I}_C$. Consequently, in the right half of the Poincaré disc, we obtain 15 topologically distinct global phase portraits in the Poincaré disc, as illustrated in Figure 1. These portraits completely characterize the dynamics at infinity in the region $x \geq 0$.

It is observed that the equilibrium I_D of system (3.10) corresponds, under the Poincaré compactification, to an equilibrium at infinity of system (1.2), denoted by $\tilde{I}_D$, lying on the negative x -direction of the equator of the Poincaré disc. Based on the previous analysis, the local phase portrait of $\tilde{I}_D$ consists of one hyperbolic sector and one parabolic sector when $C_1 = A_1^2/4$ and $A_1 > 0$, as shown in Table 2. The same argument applies to $\tilde{I}_E$ and $\tilde{I}_F$. Consequently, in the left half of the Poincaré disc, we obtain 15 topologically distinct global phase portraits near infinity, as illustrated in Figure 2. These portraits completely characterize the dynamics in the region $x < 0$.

The dynamics in the right and left half discs of the Poincaré disc are governed independently by the corresponding outermost linear subsystems, each admitting exactly 15 distinct cases. Hence, each phase portrait of system (1.2) near infinity in the Poincaré disc is uniquely determined by an ordered pair of right and left half cases. It follows that system (1.2) admits exactly $15 \times 15 = 225$ topologically distinct phase portraits near infinity in the Poincaré disc. $\square$

4. CENTER-FOCUS PROBLEM

Proof of Theorem 2.3. From Theorem 2.1, it follows that system (1.2) has no equilibria at infinity if and only if $A_{n+1}^2 - 4C_{n+1} < 0$ and $A_1^2 - 4C_1 < 0$.

To study the center-focus problem, we apply the Bendixson reciprocal radius transformation [12]

$$x = \frac{\cos \theta}{r}, \quad y = \frac{\sin \theta}{r}, \quad (4.1)$$

which transforms a neighborhood of infinity in the (x, y) -plane into a neighborhood of $r = 0$ in the (r, θ) -coordinates. By the transformation (4.1), system (1.2) can be rewritten as

$$\begin{aligned} \frac{dr}{dt} &= -r \cos \theta \sin \theta + r^2 \cos(\theta) F\left(\frac{\cos \theta}{r}\right) + r^2 \sin(\theta) g\left(\frac{\cos \theta}{r}\right), \\ \frac{d\theta}{dt} &= -\sin^2 \theta + r \sin(\theta) F\left(\frac{\cos \theta}{r}\right) - r \cos(\theta) g\left(\frac{\cos \theta}{r}\right). \end{aligned} \quad (4.2)$$

Substituting $F(x)$ and $g(x)$ into (4.2), we obtain

$$\begin{aligned} \frac{dr}{dt} &= -r \cos \theta \sin \theta + r^2(b_0 \cos \theta + d_0 \sin \theta) + r(a_0 \cos^2 \theta + c_0 \sin \theta \cos \theta) \\ &\quad + \sum_{i=1}^n \left[r^2 \cos \theta \left(a_i \left| \frac{\cos \theta}{r} - x_i \right| + b_i \operatorname{sgn} \left(\frac{\cos \theta}{r} - x_i \right) \right) \right. \\ &\quad \left. + r^2 \sin \theta \left(c_i \left| \frac{\cos \theta}{r} - x_i \right| + d_i \operatorname{sgn} \left(\frac{\cos \theta}{r} - x_i \right) \right) \right], \\ \frac{d\theta}{dt} &= -\sin^2 \theta + a_0 \sin \theta \cos \theta - c_0 \cos^2 \theta + r(b_0 \sin \theta - d_0 \cos \theta) \\ &\quad + \sum_{i=1}^n \left[r \sin \theta \left(a_i \left| \frac{\cos \theta}{r} - x_i \right| + b_i \operatorname{sgn} \left(\frac{\cos \theta}{r} - x_i \right) \right) \right. \\ &\quad \left. - r \cos \theta \left(c_i \left| \frac{\cos \theta}{r} - x_i \right| + d_i \operatorname{sgn} \left(\frac{\cos \theta}{r} - x_i \right) \right) \right]. \end{aligned}$$

In a neighborhood of $r = 0$, the above system can be written as

$$\frac{dr}{dt} = \begin{cases} R_{n+1}(r, \theta), & \text{if } -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} - \varepsilon, \\ R_1(r, \theta), & \text{if } \frac{\pi}{2} + \varepsilon < \theta \leq \frac{3\pi}{2}, \end{cases}$$

and

$$\frac{d\theta}{dt} = \begin{cases} \Theta_{n+1}(r, \theta), & \text{if } -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} - \varepsilon, \\ \Theta_1(r, \theta), & \text{if } \frac{\pi}{2} + \varepsilon < \theta \leq \frac{3\pi}{2}, \end{cases}$$

where

$$\begin{aligned} R_i(r, \theta) &= r \cos \theta (A_i \cos \theta + (C_i - 1) \sin \theta) + r^2 (B_i \cos \theta + D_i \sin \theta), \\ \Theta_i(r, \theta) &= \cos \theta (A_i \sin \theta - C_i \cos \theta) - \sin^2 \theta + r (B_i \sin \theta - D_i \cos \theta) \end{aligned}$$

for $i = 1, n+1$, and $\varepsilon > 0$ is sufficiently small. Thus, when $\Theta_i(r, \theta) \neq 0$, we obtain

$$\frac{dr}{d\theta} = \begin{cases} \frac{R_{n+1}(r, \theta)}{\Theta_{n+1}(r, \theta)}, & \text{if } -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} - \varepsilon, \\ \frac{R_1(r, \theta)}{\Theta_1(r, \theta)}, & \text{if } \frac{\pi}{2} + \varepsilon < \theta \leq \frac{3\pi}{2}. \end{cases} \quad (4.3)$$

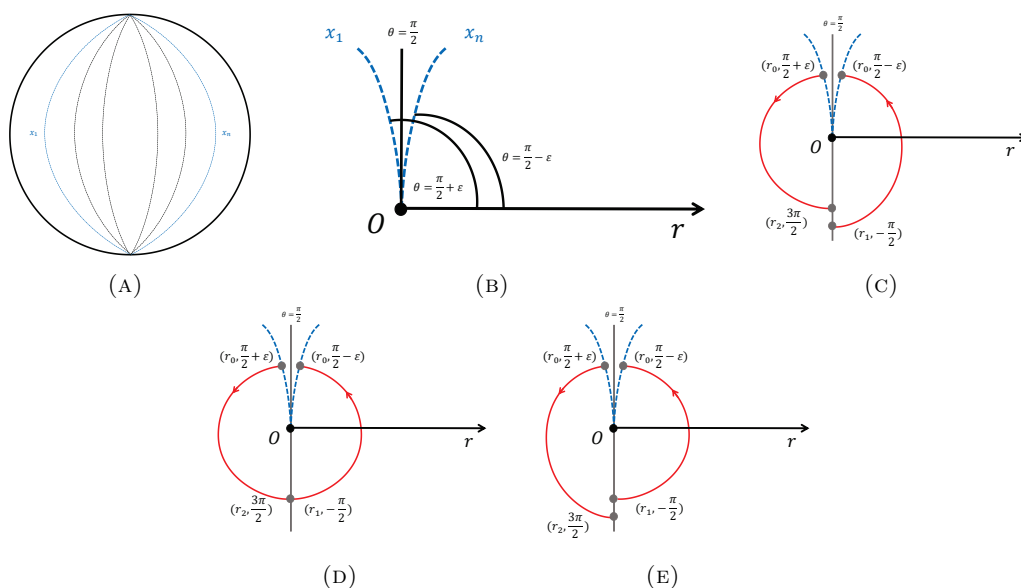


FIGURE 6. Discussion on the stability of the equator of the Poincaré sphere

Consider an initial point on the switching line $x = x_n$ (resp., $x = x_1$) with polar coordinates $(r_0, \pi/2 - \varepsilon)$ (resp., $(r_0, \pi/2 + \varepsilon)$), where $r_0 > 0$ and $\varepsilon > 0$ are sufficiently small, as shown in Figures 6(a) and (b). We now define the Poincaré maps associated with the lines $\theta = -\pi/2$ and $\theta = 3\pi/2$. Starting from the above initial point, we follow the orbit under the backward flow and denote by $(r_1, -\pi/2)$ the first hitting point on the line $\theta = -\pi/2$, while under the forward flow we denote by $(r_2, 3\pi/2)$ the first hitting point on the line $\theta = 3\pi/2$. Thus, we obtain the Poincaré return map defined on $\theta \in [-\pi/2, \pi/2 - \varepsilon)$, which can be computed as follows

$$\begin{aligned} \ln \left| \frac{r_0(\frac{\pi}{2} - \varepsilon)}{r_1(-\pi/2)} \right| &= \int_{-\pi/2}^{\frac{\pi}{2} - \varepsilon} \frac{A_{n+1} \cos^2 \theta + (C_{n+1} - 1) \sin \theta \cos \theta}{A_{n+1} \sin \theta \cos \theta - C_{n+1} \cos^2 \theta - \sin^2 \theta} d\theta \\ &= \int_{-\pi/2}^{\frac{\pi}{2} - \varepsilon} \frac{A_{n+1} + (C_{n+1} - 1) \tan \theta}{A_{n+1} \tan \theta - C_{n+1} - \tan^2 \theta} d\theta \\ &= \int_{-\infty}^{+\infty} \frac{t - A_{n+1}}{t^2 - A_{n+1}t + C_{n+1}} - \frac{t}{1 + t^2} dt \end{aligned} \quad (4.4)$$

as $\varepsilon \rightarrow 0$. Using the residue theorem in (4.4), we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} -\frac{t}{1 + t^2} + \frac{t}{t^2 - A_{n+1}t + C_{n+1}} - \frac{A_{n+1}}{t^2 - A_{n+1}t + C_{n+1}} dt \\ &= 0 + \frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}} - \frac{2A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}} \\ &= -\frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}}. \end{aligned} \quad (4.5)$$

Similarly, for the Poincaré return map defined on $\theta \in (\pi/2 + \varepsilon, 3\pi/2]$, we obtain

$$\ln \left| \frac{r_2(3\pi/2)}{r_0(\frac{\pi}{2} + \varepsilon)} \right| = \int_{\frac{\pi}{2} + \varepsilon}^{\frac{3\pi}{2}} \frac{A_1 \cos^2 \theta + (C_1 - 1) \sin \theta \cos \theta}{A_1 \sin \theta \cos \theta - C_1 \cos^2 \theta - \sin^2 \theta} d\theta = -\frac{A_1\pi}{\sqrt{4C_1 - A_1^2}} \quad (4.6)$$

when $\varepsilon \rightarrow 0$. By combining (4.5) and (4.6), we obtain

$$\ln \left| \frac{r_2(3\pi/2)}{r_1(-\pi/2)} \right| = -\frac{A_1\pi}{\sqrt{4C_1 - A_1^2}} - \frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}}. \quad (4.7)$$

By (4.7), we directly obtain that the dynamics of system (1.2) near infinity is a stable (resp., an unstable) focus, as shown in Figure 6(c) (resp., 6(e)), when

$$\frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}} + \frac{A_1\pi}{\sqrt{4C_1 - A_1^2}} < 0 \quad (\text{resp. } > 0).$$

When

$$\frac{A_{n+1}\pi}{\sqrt{4C_{n+1} - A_{n+1}^2}} + \frac{A_1\pi}{\sqrt{4C_1 - A_1^2}} = 0,$$

the dynamics of system (1.2) near infinity may be a center or a weak focus. In particular, if system (1.2) is symmetric with respect to the y -axis, then the dynamics of system (1.2) near infinity is a center, as shown in Figure 6(d). $\square$

5. GLOBAL CENTER

To prove Theorem 2.5, we present three preliminary lemmas.

Lemma 5.1. *System (2.1) has a unique equilibrium $O(0,0)$, which is a center if and only if $A_k = 0$, $C_k > 0$ on the interval $I_k = (x_{k-1}, x_k)$ and $xg(x) > 0$ for $x \neq 0$.*

Proof. It follows from the Jacobian matrix of system (2.1) at $O(0,0)$ that the equilibrium is a center if and only if $A_k = 0$ and $C_k > 0$.

Since $O(0,0)$ is the unique equilibrium, $g(x) \neq 0$ for $x \neq 0$, so that there are four possible cases:

- (C1) $g(x) > 0, x \neq 0$;
 (C2) $g(x) < 0, x \neq 0$;
 (C3) $xg(x) < 0, x \neq 0$; [item[(C4)] $xg(x) > 0, x \neq 0$.

As above, the index of the origin coincides with that of the vector field $(y, -g(x))$. In Case (C1), this vector field is homotopic to $(y, -|x|)$, whose index at the origin is 0. In Case (C2), it is homotopic to $(y, |x|)$, which also has index 0. In Case (C3), it is homotopic to (y, x) , whose index is -1 . Only in Case (C4) is it homotopic to $(y, -x)$, whose index is 1. Since the center has index 1, the first three cases are excluded. Therefore, only Case (C4) can occur, namely, $xg(x) > 0$, for $x \neq 0$. Conversely, if $xg(x) > 0$ for $x \neq 0$, then the vector field is homotopic to $(y, -x)$, whose index at the origin is 1. Hence, the equilibrium is an anti-saddle, and therefore a center. $\square$

Lemma 5.2. Assume that $xg(x) > 0$ for $x \neq 0$. Then all orbit segments of system (2.1) surrounding the origin are closed if and only if $F(x_1) = F(x_2)$ when $G(x_1) = G(x_2)$ for all $x_2 < 0 < x_1$, where $G(x) = \int_0^x g(s) ds$.

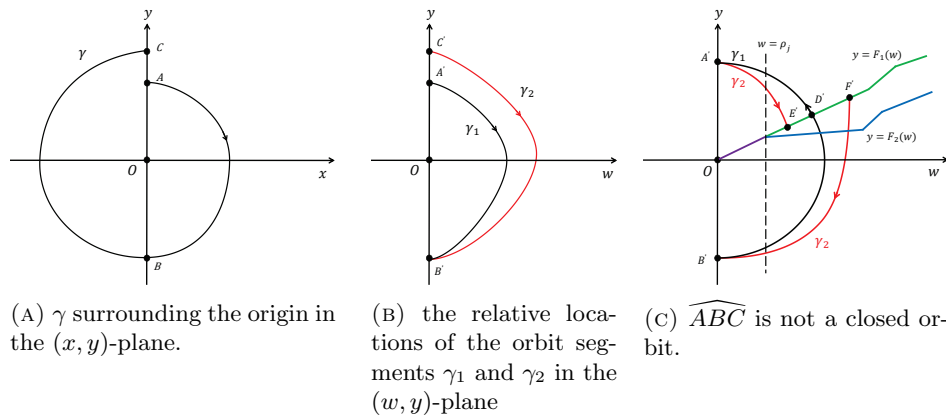


FIGURE 7. Discussion of closed orbits of system (2.1) under the Filippov transformation.

Proof. To compare the dynamics in the two half planes $x > 0$ and $x < 0$ under the same scaling, we introduce the Filippov transformation $w = G(x) = \int_0^x g(s) ds$ [14]. Since $xg(x) > 0$ for $x \neq 0$, $G(x)$ is strictly decreasing for $x < 0$ and strictly increasing for $x > 0$, with $G(0) = 0$. Thus, for any $w > 0$, the equation $w = G(x)$ has two inverse branches, denoted by $x_2(w) < 0 < x_1(w)$. Under this Filippov transformation, system (2.1) reduces to

$$\frac{dw}{dy} = F(x_1(w)) - y = F_1(w) - y, \quad w > 0, \quad (5.1)$$

and

$$\frac{dw}{dy} = F(x_2(w)) - y = F_2(w) - y, \quad w > 0, \quad (5.2)$$

respectively.

Let $\widehat{ABC}$ be an orbit of system (2.1) surrounding the origin in the xy -plane. It intersects the positive y -axis at A and C , and the negative y -axis at B , see Figure 7(a). Under this Filippov transformation, the branch $\widehat{AB}$ for $x > 0$ and the branch $\widehat{BC}$ for $x < 0$ are mapped to the integral curves γ_1 of (5.1) and γ_2 of (5.2) in the wy -plane, respectively. Denoting the images of A , B , and C by A' , B' , and C' in the wy -plane, γ_1 is the integral curve connecting A' to B' , while γ_2 is the integral curve connecting C' to B' , see Figure 7(b). Let D' denote the intersection point of γ_1 with the curve $y = F_1(w)$ in the wy -plane, see Figure 7(c).

Sufficiency: Assume that $F(x_1) = F(x_2)$ when $G(x_1) = G(x_2)$ for $x_2 < 0 < x_1$. By the definition of the Filippov transformation $w = G(x)$, this assumption is equivalent to $F_1(w) \equiv F_2(w)$ for $w > 0$. Thus, the two integral curves γ_1 of (5.1) and γ_2 of (5.2), respectively, satisfy the same

differential equation and pass through the same point B' . By the uniqueness of solutions, it follows that γ_1 and γ_2 coincide in the wy -plane. Thus, their intersections with the y -axis also coincide, that is, $A' = C'$. This implies that the corresponding orbit $\widehat{ABC}$ of system (2.1) is closed. Since this argument holds for every such orbit, all orbits surrounding the origin are closed.

Necessity: Since system (2.1) is continuous and piecewise linear, the switching lines $x = x_i$ ($i = 1, 2, \dots, n$) in the xy -plane are mapped to a set of new switching lines $w = \rho_j$ ($j = 1, 2, \dots, m$) in the wy -plane under the Filippov transformation $w = G(x)$. Let their coordinates on the w -axis be denoted by

$$0 = \rho_0 < \rho_1 < \dots < \rho_{m-1} < \rho_m = +\infty.$$

The functions $F_1(w)$ and $F_2(w)$ are still continuous and piecewise linear.

When all orbits surrounding the origin of system (2.1) are closed in the xy -plane, we claim that $F_1(w) \equiv F_2(w)$ for all $w > 0$. Otherwise, there exists a $w \in (\rho_j, \rho_{j+1})$ such that $F_1(w) \neq F_2(w)$. Without loss of generality, we assume that $F_1(w) \equiv F_2(w)$ for $w \in (0, \rho_j)$ and $F_1(w) > F_2(w)$ for $w \in (\rho_j, \rho_{j+1})$. Then, we obtain $F_1(w) - y > F_2(w) - y$ for $w \in (\rho_j, \rho_{j+1})$. By the comparison theorem [12], the integral curve of (5.2) passing through A' intersects the curve $y = F_1(w)$ at a point E' lying to the left of D' , while the integral curve of (5.2) passing through B' intersects the curve $y = F_1(w)$ at a point F' lying to the right of D' , see Figure 7(c). This implies that the corresponding orbit $\widehat{ABC}$ of system (2.1) is not closed. This contradicts the fact that all orbits surrounding the origin are closed. Then, the assertion is proven. By $F_1(w) \equiv F_2(w)$ for $w > 0$, we directly conclude that $G(x_1) = G(x_2) = w$ for every pair (x_1, x_2) satisfying $x_2 < 0 < x_1$. $\square$

Lemma 5.3. *There is no orbit of system (2.1) connecting equilibria at infinity in the Poincaré disc if and only if one of the following conditions holds:*

- (i) $A_{n+1}^2 - 4C_{n+1} < 0$ and $A_1^2 - 4C_1 < 0$;
- (ii) $A_{n+1} = C_{n+1} = A_1 = C_1 = 0$ and $D_1 < 0 < D_{n+1}$.

The proof of the above lemma follows directly from Theorem 2.1, so we omit it.

Proof of Theorem 2.5. Sufficiency: Assume that conditions (i)–(iv) hold. By Lemma 5.1, conditions (i) and (ii) imply that the origin $O(0,0)$ is the unique finite equilibrium of system (2.1) and is a center. By Lemma 5.2, condition (iii) implies that all orbit segments surrounding the origin $O(0,0)$ are closed. By Lemma 5.3, condition (iv) implies that there is no orbit connecting equilibria at infinity in the Poincaré disc. Combining Lemmas 5.1–5.3, the period annulus of the origin $O(0,0)$ extends to the whole finite plane. In other words, the origin $O(0,0)$ is a global center.

Necessity: Conversely, assume that system (2.1) has a global center at the origin $O(0,0)$. Since Lemmas 5.1, 5.2, and 5.3 show necessary and sufficient conditions, conditions (i)–(iv) follow immediately. This proves the necessity. $\square$

6. APPENDIX A

We present a rigorous derivation of the reduction of system (1.1) to the Liénard system (1.2).

Step 1: Linear normalization of switching direction. Consider the invertible linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \frac{1}{k_1^2 + k_2^2} \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad k_1^2 + k_2^2 > 0.$$

Under this transformation, the switching function becomes

$$L_i(X) = k_1x + k_2y + l_i = x + l_i$$

and each switching manifold is mapped to a vertical line $x = -l_i$.

Step 2: System pushforward. Applying the above transformation, we obtain an equivalent system (1.1) of the form

$$\begin{aligned}\frac{dx}{dt} &= m_1x + n_1y + p_1 + \sum_{i=1}^n (\alpha_i|x + l_i| + \beta_i \operatorname{sgn}(x + l_i)), \\ \frac{dy}{dt} &= m_2x + n_2y + p_2 + \sum_{i=1}^n (\gamma_i|x + l_i| + \delta_i \operatorname{sgn}(x + l_i)),\end{aligned}\quad (6.1)$$

where $m_1, n_1, p_1, m_2, n_2, p_2, \alpha_i, \beta_i, \gamma_i$ and δ_i are real coefficients determined by the original parameters.

Step 3: Reduction to the Liénard form. Assume $n_1 \neq 0$. Consider the transformation

$$(x, y, t) \rightarrow \left(x, -y + \frac{n_2}{n_1}x, -\frac{t}{n_1}\right).$$

Then system (6.1) is equivalent to

$$\frac{dx}{dt} = y - F(x), \quad \frac{dy}{dt} = -g(x),$$

where

$$\begin{aligned}F(x) &= \frac{m_1 + n_2}{n_1}x + \frac{p_1}{n_1} + \sum_{i=1}^n \left(\frac{\alpha_i}{n_1}|x + l_i| + \frac{\beta_i}{n_1} \operatorname{sgn}(x + l_i)\right), \\ g(x) &= \frac{n_1m_2 - n_2m_1}{n_1^2}x + \frac{n_1p_2 - n_2p_1}{n_1^2} + \sum_{i=1}^n \left(\frac{n_1\gamma_i - n_2\alpha_i}{n_1^2}|x + l_i| + \frac{n_1\delta_i - n_2\beta_i}{n_1^2} \operatorname{sgn}(x + l_i)\right).\end{aligned}$$

We define the parameters in system (1.2) via the mapping induced by the transformation

$$\begin{aligned}a_0 &= \frac{m_1 + n_2}{n_1}, \quad b_0 = \frac{p_1}{n_1}, \quad a_i = \frac{\alpha_i}{n_1}, \quad b_i = \frac{\beta_i}{n_1}, \quad c_0 = \frac{n_2m_1 - n_1m_2}{n_1^2}, \\ d_0 &= \frac{n_2p_1 - n_1p_2}{n_1^2}, \quad c_i = \frac{n_2\alpha_i - n_1\gamma_i}{n_1^2}, \quad d_i = \frac{n_2\beta_i - n_1\delta_i}{n_1^2}, \quad x_i = -l_i,\end{aligned}$$

for $i = 1, \dots, n$. This defines an explicit parameter correspondence between systems (1.1) and (1.2).

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