

WELL-POSEDNESS, STABILITY, AND BIFURCATION OF A FRACTIONAL-ORDER PREDATOR-PREY MODEL WITH FEAR AND PREDATOR INTERFERENCE

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ABSTRACT. We study a fractional-order predator-prey model with fear in predator interference and a non-monotonic functional response, formulated via the Caputo derivative. The fractional-order setting incorporates memory effects in the population dynamics and extends the corresponding integer-order system. We first prove well-posedness by establishing the existence and uniqueness of solutions, together with positivity and boundedness. We then analyze the equilibria and derive conditions for local stability in terms of the Jacobian matrix and the fractional order. Sufficient criteria for global asymptotic stability are obtained by using an appropriate Lyapunov function. In addition, we show that the system can undergo a Hopf bifurcation about an interior equilibrium as the fractional order varies. Numerical simulations are provided to illustrate the theoretical results on stability and bifurcation.

1. INTRODUCTION

Predator-prey models are a cornerstone of population dynamics and have long been used to understand how trophic interactions shape species persistence, oscillations, and bifurcations. Classical and modern variants incorporate mechanisms such as carrying capacity, functional responses, and additional ecological effects to better represent observed dynamics in real ecosystems (see, e.g., [22] and the references therein). From a mathematical viewpoint, predator-prey systems are known to exhibit rich dynamical structures, including global behaviors and bifurcation phenomena; see, for instance, the global analysis in [6], Hopf-bifurcation and stability studies in [13, 27], and the rigorous multiplicity and bifurcation analysis of positive solutions in diffusive predator-prey models [8, 9].

A key modeling ingredient is the *functional response*, which describes how prey consumption depends on prey density, and which strongly influences stability and long-term behavior [14, 15]. In many systems, predator–predator interactions also play a central role: conspecific predators may interfere with each other while searching for prey, reducing search efficiency (mutual interference), or alternatively may benefit from aggregation or cooperation, thereby enhancing predation success. To capture such predator-density dependence in a unified way, we introduce

$$g(y) = y^q, \quad q > 0,$$

where $y(t)$ denotes predator density and q quantifies interference or aggregation. In the predation term of (1.1), total predation scales like y^q , hence the per-capita predation rate scales like y^{q-1} . The regimes $0 < q < 1$ (interference), $q = 1$ (no interference), and $q > 1$ (cooperation/aggregation) are summarized in Table 1; see also [2, 10, 11, 31].

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TABLE 1. Interpretation of the exponent q in the predator density function $g(y) = y^q$.

q	Interpretation
$0 < q < 1$	Predation increases sublinearly with predator density due to predator interference, resulting in reduced per capita predation at higher predator densities.
$q = 1$	Predation depends linearly on predator density, corresponding to independent predator foraging without interference.
$q > 1$	Predation increases superlinearly with predator density, representing cooperative or aggregated predator behavior.

Motivated by substrate-inhibition mechanisms observed at the microbial level, we employ a *non-monotonic* functional response of Monod-Haldane (Holling type IV) type. This class was originally proposed in [3] and has been widely used to represent inhibition, group defense, and complex bifurcation structures in predator-prey dynamics; see, e.g., [4, 24, 25, 30, 32]. Specifically, we take

$$f(x) = \frac{px}{x^2 + \alpha\beta x + d},$$

so that the total predation rate is $f(x)g(y)$. To clarify the role of this modeling choice, we compare the prey-dependent functional response used in the present model with the classical Holling type II and Holling type III responses. In a unified notation, these responses may be written as

$$f_{II}(x) = \frac{px}{m_1 + x}, \quad f_{III}(x) = \frac{px^2}{m_2 + x^2}, \quad f_{IV}(x) = \frac{px}{x^2 + \alpha\beta x + d},$$

where $m_1, m_2 > 0$ are saturation constants. The Holling type II response is monotone increasing and saturating, while the Holling type III response is also monotone increasing but sigmoidal, reflecting low predation efficiency at small prey densities. In contrast, the Holling type IV response adopted here is non-monotone. Indeed,

$$f'_{IV}(x) = \frac{p(d - x^2)}{(x^2 + \alpha\beta x + d)^2}.$$

Thus the predation rate increases for $x < \sqrt{d}$, reaches its maximum at $x = \sqrt{d}$, and decreases for $x > \sqrt{d}$. This decreasing part can describe inhibition, prey group defense, or predator confusion at high prey density. Therefore, unlike models with Holling type II or Holling type III functional responses, the present model can capture the suppression of predation at large prey density, which is a key dynamical feature of the Monod-Haldane/Holling type IV response.

In addition to consumptive effects, predators can induce *fear effects* that reduce prey reproduction and alter demographic outcomes. A common modeling approach modifies the prey birth term through a decreasing function of predator density, for example

$$h(y) = \frac{1}{1 + ky},$$

as in [29]. To incorporate predator aggregation/interference into the fear mechanism, we use

$$h(y) = \frac{1}{1 + k\beta y},$$

in line with aggregation-induced fear modeling [19, 26].

Combining the above mechanisms yields the integer-order predator-prey system

$$\begin{aligned} \frac{dx}{dt} &= \frac{ax}{1 + k\beta y} - bx - cx^2 - \frac{pxy^q}{x^2 + \alpha\beta x + d}, & x(0) &\geq 0, \\ \frac{dy}{dt} &= -ey + \frac{\mu pxy^q}{x^2 + \alpha\beta x + d}, & y(0) &\geq 0, \end{aligned} \tag{1.1}$$

TABLE 2. Description of model parameters and their biological meanings.

Parameter	Biological meaning
a	intrinsic growth rate of the prey population
b	natural death rate of the prey population
c	intraspecific competition rate among prey
d	half-saturation/regularization constant in the functional response denominator
e	intrinsic growth rate of predator
k	strength of fear
p	intensity of predation
q	predator interference parameter
α	predator tolerance limit
β	predator aggregate sensitivity
μ	biomass conversion efficiency from prey to predator

where $x(t)$ and $y(t)$ denote prey and predator densities, respectively. All parameters in (1.1) are positive and described in Table 2. Throughout, we assume $0 < \mu \leq 1$ and $a > b$ to ensure biologically meaningful conversion efficiency and positive net prey growth in the absence of predation. When $k = 0$ (no fear effect), (1.1) reduces to a group-defense type predator-prey model closely related to [12, 30]. The case $q = 1$ was studied in [26], while mutual-interference effects ($0 < q < 1$) have been investigated in related predator-prey settings; see [10, 11, 19].

Real ecological systems often exhibit history dependence and gradual responses that are difficult to capture using integer-order dynamics alone. Fractional derivatives provide a standard framework to incorporate memory effects and non-local temporal interactions [5, 23, 18, 7]. Accordingly, we study the Caputo fractional-order extension of (1.1)

$$\begin{aligned} {}^C D_t^\gamma x &= \frac{ax}{1+k\beta y} - bx - cx^2 - \frac{pxy^q}{x^2 + \alpha\beta x + d}, \quad x(0) \geq 0, \\ {}^C D_t^\gamma y &= -ey + \frac{\mu pxy^q}{x^2 + \alpha\beta x + d}, \quad y(0) \geq 0, \end{aligned} \quad (1.2)$$

where ${}^C D_t^\gamma$ denotes the Caputo derivative of order $\gamma \in (0, 1)$, introduced by Caputo [5]. The *Caputo (fractional) derivative* of order γ for a function $f(t)$ is defined as

$${}^C D_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{f'(s)}{(t-s)^\gamma} ds,$$

where Γ is the Gamma function, defined as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$. We refer to [23, 18, 7] for background and properties. Fractional predator-prey systems of related type have also been considered in the literature; see, e.g., [17].

The main contributions of this paper are as follows. First, we establish a well-posedness theory (existence, uniqueness, positivity, and boundedness) for (1.2) for all $q > 0$, with special attention to the biologically relevant regime $0 < q < 1$ where y^q may fail to be locally Lipschitz at $y = 0$. Second, we characterize equilibrium points and analyze local stability via linearization together with the fractional-order stability criterion. Third, we provide sufficient conditions for global asymptotic stability via Lyapunov-type arguments. We also investigate Hopf-type bifurcation (or order-induced stability switching) triggered by variations in the fractional order, revealing its role in the emergence of oscillatory behavior. Finally, numerical simulations illustrate the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we prove existence and uniqueness of solutions together with positivity and boundedness. Section 3 provides equilibrium and stability analysis, and Section 4 presents numerical simulations and bifurcation illustrations.

2. WELL-POSEDNESS AND BASIC PROPERTIES

In this section, we establish the well-posedness and basic qualitative properties of model (1.2), including positivity, boundedness, existence, and uniqueness of solutions. Note that the system admits boundary solutions: if $x(0) = 0$, then $x(t) \equiv 0$ is a solution of the first equation, and if $y(0) = 0$, then $y(t) \equiv 0$ is a solution of the second equation. Since our main interest is the biologically meaningful coexistence dynamics, we assume

$$x(0) > 0 \quad \text{and} \quad y(0) > 0$$

throughout this section. In particular, when $0 < q < 1$, the assumption $y(0) > 0$ is essential for uniqueness, because the map $y \mapsto y^q$ is not locally Lipschitz at $y = 0$.

We recall the definition of the Mittag-Leffler functions, which will be used throughout this section. We refer to [18, 7, 23] for further properties of Mittag-Leffler functions.

Definition 2.1. The one-parameter Mittag-Leffler function is defined by

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)}, \quad \text{where } \gamma > 0 \text{ and } z \in \mathbb{C}.$$

The two-parameter Mittag-Leffler function is defined by

$$E_{\gamma_1, \gamma_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma_1 k + \gamma_2)}, \quad \text{where } \gamma_1 > 0, \gamma_2 > 0, \text{ and } z \in \mathbb{C}.$$

2.1. Positivity.

Lemma 2.2. Let $(x(t), y(t))$ be a solution of the model (1.2) with $x(0) > 0$ and $y(0) > 0$. Then

$$x(t) > 0 \quad \text{and} \quad y(t) > 0 \quad \text{for all } t \geq 0.$$

Proof. Assume by contradiction that at least one component reaches 0. Define the first hitting time

$$\tau := \inf \{t > 0 : x(t) = 0 \text{ or } y(t) = 0\}.$$

By continuity, we have $x(t) > 0$ and $y(t) > 0$ for all $t \in [0, \tau)$, and $x(\tau) = 0$ or $y(\tau) = 0$.

Since x and y are continuous on $[0, \tau]$, they are bounded above. We set

$$M_x := \max_{t \in [0, \tau]} x(t) \quad \text{and} \quad M_y := \max_{t \in [0, \tau]} y(t).$$

Step 1: $y(t)$ cannot hit 0. On $[0, \tau]$, we have $x(t) \geq 0$ and $y(t) \geq 0$, hence the second equation of (1.2) can be rewritten as

$${}^C D_t^\gamma y(t) + e y(t) = \frac{\mu p x(t) y(t)^q}{x(t)^2 + \alpha \beta x(t) + d} =: g(t) \geq 0.$$

Using the integral form for Caputo fractional ODE (see, e.g., [7, 18, 23]), we obtain for $t \in [0, \tau]$,

$$y(t) = y(0) E_\gamma(-e t^\gamma) + \int_0^t (t-s)^{\gamma-1} E_{\gamma, \gamma}(-e(t-s)^\gamma) g(s) ds.$$

The kernel $(t-s)^{\gamma-1} E_{\gamma, \gamma}(-e(t-s)^\gamma)$ is nonnegative and $g(s) \geq 0$, hence

$$y(t) \geq y(0) E_\gamma(-e t^\gamma) > 0 \quad \text{for all } t \in [0, \tau].$$

In particular, $y(\tau) > 0$, so it is impossible that $y(\tau) = 0$.

Step 2: $x(t)$ cannot hit 0. On $[0, \tau]$, using $0 \leq y(t) \leq M_y$, $0 \leq x(t) \leq M_x$, and $x^2 + \alpha \beta x + d \geq d$, we estimate

$$\begin{aligned} {}^C D_t^\gamma x(t) &= \frac{a x(t)}{1 + k \beta y(t)} - b x(t) - c x(t)^2 - \frac{p x(t) y(t)^q}{x(t)^2 + \alpha \beta x(t) + d} \\ &\geq -b x(t) - c x(t)^2 - \frac{p}{d} x(t) y(t)^q \\ &\geq -\left(b + c M_x + \frac{p}{d} M_y^q\right) x(t) =: -L x(t) \end{aligned}$$

for all $t \in [0, \tau]$, where $L := b + cM_x + \frac{p}{d}M_y^q > 0$. Let $z(t) := x(0)E_\gamma(-Lt^\gamma)$, which satisfies ${}^C D_t^\gamma z(t) = -Lz(t)$. Set $w(t) := x(t) - z(t)$. Then

$${}^C D_t^\gamma w(t) + Lw(t) = ({}^C D_t^\gamma x(t) + Lx(t)) - ({}^C D_t^\gamma z(t) + Lz(t)) \geq 0, \quad w(0) = 0.$$

Applying again the formula to the Caputo fractional ODE ${}^C D_t^\gamma w + Lw = h(t)$ with $h(t) \geq 0$, we get $w(t) \geq 0$ on $[0, \tau]$. Therefore,

$$x(t) \geq z(t) = x(0)E_\gamma(-Lt^\gamma) > 0, \quad t \in [0, \tau],$$

and in particular $x(\tau) > 0$, contradicting $x(\tau) = 0$.

Thus $\tau = \infty$ and both $x(t)$ and $y(t)$ remain strictly positive for all $t \geq 0$. $\square$

2.2. Boundedness.

Lemma 2.3. *Assume $0 < \mu \leq 1$. Then all solutions $(x(t), y(t))$ of (1.2) with $x(0) > 0$ and $y(0) > 0$ are bounded on $[0, \infty)$.*

Proof. Let $\Theta(t) = x(t) + y(t)$. Adding the two equations in (1.2) yields

$${}^C D_t^\gamma \Theta(t) = \frac{ax(t)}{1 + k\beta y(t)} - bx(t) - cx(t)^2 + \frac{p(\mu - 1)x(t)y(t)^q}{x(t)^2 + \alpha\beta x(t) + d} - ey(t).$$

Since $0 < \mu \leq 1$, the term involving $(\mu - 1)$ is nonpositive. Also, $\frac{ax}{1+k\beta y} \leq ax$ for $x, y \geq 0$. Hence,

$${}^C D_t^\gamma \Theta(t) \leq (a - b)x(t) - cx(t)^2 - ey(t).$$

Fix any $\eta \in (0, e]$. Then

$$\begin{aligned} {}^C D_t^\gamma \Theta(t) + \eta\Theta(t) &\leq (a - b + \eta)x(t) - cx(t)^2 - (e - \eta)y(t) \\ &\leq (a - b + \eta)x(t) - cx(t)^2 \leq \frac{(a - b + \eta)^2}{4c} =: C. \end{aligned}$$

By the standard estimate for the Caputo inequality ${}^C D_t^\gamma u + \eta u \leq C$ (see [7, Chapter 7]), we have that for all $t \geq 0$,

$$\Theta(t) \leq \Theta(0)E_\gamma(-\eta t^\gamma) + C \int_0^t s^{\gamma-1} E_{\gamma,\gamma}(-\eta s^\gamma) ds.$$

Using $0 < E_\gamma(-\eta t^\gamma) \leq 1$ and $\int_0^\infty s^{\gamma-1} E_{\gamma,\gamma}(-\eta s^\gamma) ds = \frac{1}{\eta}$ (see [7, Chapter 7]), we obtain that for all $t \geq 0$,

$$\Theta(t) \leq \Theta(0) + \frac{C}{\eta} = \Theta(0) + \frac{(a - b + \eta)^2}{4c\eta}.$$

Therefore $x(t)$ and $y(t)$ are bounded on $[0, \infty)$. $\square$

2.3. Lower boundedness on finite time intervals (needed for $0 < q < 1$). When $0 < q < 1$, the map $y \mapsto y^q$ is not locally Lipschitz at $y = 0$. To apply the existence and uniqueness theorem for Caputo fractional differential systems (see Section 2.4 below), it is crucial to guarantee that $y(t)$ stays away from 0 on any finite time interval.

Lemma 2.4. *Let $(x(t), y(t))$ be a solution of the model (1.2) with $y(0) > 0$. Then $y(t) > 0$ for all $t \geq 0$. Moreover, for every $T > 0$,*

$$y(t) \geq y(0) E_\gamma(-eT^\gamma) > 0 \quad \text{for all } t \in [0, T].$$

Proof. From the argument in the proof of Lemma 2.2, we have the estimate

$$y(t) \geq y(0)E_\gamma(-et^\gamma) \quad \text{for all } t \in [0, T].$$

Since the function $t \mapsto E_\gamma(-et^\gamma)$ is monotonically decreasing on $[0, \infty)$, we obtain

$$y(t) \geq y(0)E_\gamma(-eT^\gamma) > 0 \quad \text{for all } t \in [0, T],$$

which completes the proof. $\square$

2.4. Well-posedness.

Theorem 2.5 (Existence and uniqueness). *Assume that all parameters in the model (1.2) are positive and $0 < \mu \leq 1$, and let $0 < \gamma < 1$.*

- (i) *If $q \geq 1$, then for any initial condition $(x(0), y(0)) \in \mathbb{R}_{\geq 0}^2$, then model (1.2) admits a unique solution $(x(t), y(t))$ for all $t \geq 0$.*
- (ii) *If $0 < q < 1$, then for any strictly positive initial condition $(x(0), y(0)) \in \mathbb{R}_{> 0}^2$, the model (1.2) admits a unique solution $(x(t), y(t))$ for all $t \geq 0$.*

Proof. Let $\mathbf{u}(t) = (x(t), y(t))$ and rewrite (1.2) as the Caputo fractional differential system

$${}^C D_t^\gamma \mathbf{u}(t) = F(\mathbf{u}(t)), \quad 0 < \gamma < 1,$$

where $F = (F_1, F_2)$ with

$$\begin{aligned} F_1(x, y) &= \frac{ax}{1 + k\beta y} - bx - cx^2 - \frac{pxy^q}{x^2 + \alpha\beta x + d}, \\ F_2(x, y) &= -ey + \frac{\mu pxy^q}{x^2 + \alpha\beta x + d}. \end{aligned}$$

Note that $1 + k\beta y \geq 1$ for $y \geq 0$ and $x^2 + \alpha\beta x + d \geq d > 0$ for $x \geq 0$.

Step 1: Volterra integral formulation. By the equivalence between Caputo initial value problems and Volterra integral equations [7, Lemma 6.2] (and its extension to systems, [7, Remark 6.1]), $\mathbf{u}$ is a solution on $[0, T]$ if and only if it satisfies

$$\mathbf{u}(t) = \mathbf{u}_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} F(\mathbf{u}(s)) ds, \quad t \in [0, T], \quad (2.1)$$

where $\mathbf{u}_0 = (x(0), y(0))$.

Step 2: Local existence and uniqueness via Banach fixed point.

Step 2.1: Choice of a compact set where F is Lipschitz.

Case (i) $q \geq 1$, $(x(0), y(0)) \in \mathbb{R}_{\geq 0}^2$. Fix $R > 0$ and define the compact rectangle

$$K := [0, x(0) + R] \times [0, y(0) + R] \subset \mathbb{R}_{\geq 0}^2.$$

Since $q \geq 1$, the map $y \mapsto y^q$ is C^1 on $[0, \infty)$ and hence Lipschitz on $[0, y(0) + R]$. All remaining terms of the function F are smooth on $\mathbb{R}_{\geq 0}^2$ because the denominators are bounded away from 0. Therefore F is continuous on K and Lipschitz on K . Set

$$M_K := \sup_{\mathbf{u} \in K} \|F(\mathbf{u})\| < \infty, \quad L_K := \sup_{\mathbf{u} \in K} \|DF(\mathbf{u})\| < \infty.$$

Then we have

$$\|F(\mathbf{u}) - F(\mathbf{v})\| \leq L_K \|\mathbf{u} - \mathbf{v}\| \quad \text{for all } \mathbf{u}, \mathbf{v} \in K.$$

Case (ii) $0 < q < 1$, $(x(0), y(0)) \in \mathbb{R}_{> 0}^2$. Choose $R \in (0, \min\{\frac{x(0)}{2}, \frac{y(0)}{2}\})$ and define

$$K := [x(0) - R, x(0) + R] \times [y(0) - R, y(0) + R].$$

Then $y \geq y(0) - R \geq \frac{y(0)}{2} > 0$ on K , hence $y \mapsto y^q$ is C^1 and Lipschitz on $[y(0) - R, y(0) + R]$. As above, the remaining terms of F are smooth. Thus F is Lipschitz on K and we define M_K, L_K exactly as in case (i).

Step 2.2: Fixed point setting. Let $X := C([0, T]; \mathbb{R}^2)$ with $\|\mathbf{u}\|_\infty := \sup_{t \in [0, T]} \|\mathbf{u}(t)\|$. Define the closed subset

$$\mathcal{B} := \{\mathbf{u} \in X : \mathbf{u}(0) = \mathbf{u}_0, \mathbf{u}(t) \in K \text{ for all } t \in [0, T]\}.$$

Define the Volterra operator $\mathcal{T} : \mathcal{B} \rightarrow X$ by

$$(\mathcal{T}\mathbf{u})(t) := \mathbf{u}_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} F(\mathbf{u}(s)) ds.$$

Then fixed points of $\mathcal{T}$ are exactly solutions of (2.1).

Step 2.3: $\mathcal{T}$ maps $\mathcal{B}$ into itself. For any $\mathbf{u} \in \mathcal{B}$ and $t \in [0, T]$,

$$\begin{aligned} \|(\mathcal{T}\mathbf{u})(t) - \mathbf{u}_0\| &\leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|F(\mathbf{u}(s))\| ds \\ &\leq \frac{M_K}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} ds = \frac{M_K}{\Gamma(\gamma+1)} t^\gamma. \end{aligned}$$

Choose $T > 0$ such that

$$\frac{M_K}{\Gamma(\gamma+1)} T^\gamma \leq R. \quad (2.2)$$

Then $(\mathcal{T}\mathbf{u})(t) \in K$ for all $t \in [0, T]$, so $\mathcal{T}\mathbf{u} \in \mathcal{B}$.

Step 2.4: $\mathcal{T}$ is a contraction on $\mathcal{B}$. For $\mathbf{u}, \mathbf{v} \in \mathcal{B}$ and $t \in [0, T]$,

$$\begin{aligned} \|(\mathcal{T}\mathbf{u})(t) - (\mathcal{T}\mathbf{v})(t)\| &\leq \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|F(\mathbf{u}(s)) - F(\mathbf{v}(s))\| ds \\ &\leq \frac{L_K}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|\mathbf{u} - \mathbf{v}\|_\infty ds, \end{aligned}$$

so

$$\|(\mathcal{T}\mathbf{u})(t) - (\mathcal{T}\mathbf{v})(t)\| \leq \frac{L_K}{\Gamma(\gamma+1)} T^\gamma \|\mathbf{u} - \mathbf{v}\|_\infty.$$

Taking the supremum in t yields

$$\|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\|_\infty \leq \frac{L_K}{\Gamma(\gamma+1)} T^\gamma \|\mathbf{u} - \mathbf{v}\|_\infty.$$

Choose $T > 0$ (possibly smaller, in addition to (2.2)) so that

$$\frac{L_K}{\Gamma(\gamma+1)} T^\gamma < 1.$$

Then $\mathcal{T}$ is a contraction on $\mathcal{B}$.

Step 2.5: Local solution. By Banach's fixed point theorem, $\mathcal{T}$ has a unique fixed point in $\mathcal{B}$. By Step 1, (1.2) admits a unique solution on $[0, T]$.

Step 3: Maximal existence time. Let $T_{\max} \in (0, \infty]$ be the supremum of all T such that a solution exists on $[0, T]$. By Step 2, $T_{\max}$ is well-defined and $T_{\max} > 0$.

Step 4: Global existence ($T_{\max} = \infty$) using boundedness (and lower bound if $0 < q < 1$).

Step 4.1: Uniform bound on $[0, T_{\max})$. By Lemma 2.3, there exists a constant $C > 0$ such that

$$0 \leq x(t) \leq C \text{ and } 0 \leq y(t) \leq C \text{ for all } t \in [0, T_{\max}).$$

If $0 < q < 1$ and $y(0) > 0$, then Lemma 2.4 gives

$$y(t) \geq y(0)E_\gamma(-et^\gamma) > 0 \text{ for all } t \in [0, T_{\max}),$$

so in particular $y(t)$ is bounded away from 0 on any neighborhood of any finite time.

Step 4.2: Existence of the left-sided limit $\mathbf{u}(T_{\max})$ when $T_{\max} < \infty$. Assume $T_{\max} < \infty$ and let $0 \leq r < t < T_{\max}$. Using the integral form (2.1) and the bound $\|F(\mathbf{u}(s))\| \leq M_*$ on $[0, T_{\max})$, we estimate

$$\|\mathbf{u}(t) - \mathbf{u}(r)\| \leq \frac{M_*}{\Gamma(\gamma)} \left(\int_0^r |(t-s)^{\gamma-1} - (r-s)^{\gamma-1}| ds + \int_r^t (t-s)^{\gamma-1} ds \right).$$

A direct computation gives

$$\|\mathbf{u}(t) - \mathbf{u}(r)\| \leq \frac{2M_*}{\Gamma(\gamma+1)} (t-r)^\gamma.$$

Hence $\mathbf{u}$ is uniformly continuous on $[0, T_{\max})$ and admits a limit

$$\mathbf{u}^* := \lim_{t \nearrow T_{\max}} \mathbf{u}(t).$$

Step 4.3: Extension beyond $T_{\max}$ (contradiction). Define for $t \in [T_{\max}, T_{\max} + \delta]$ the known function

$$\Phi(t) := \mathbf{u}_0 + \frac{1}{\Gamma(\gamma)} \int_0^{T_{\max}} (t-s)^{\gamma-1} F(\mathbf{u}(s)) ds.$$

By the dominated convergence, Φ is continuous and $\Phi(T_{\max}) = \mathbf{u}^*$. Consider the Volterra integral equation on $[T_{\max}, T_{\max} + \delta]$:

$$\mathbf{v}(t) = \Phi(t) + \frac{1}{\Gamma(\gamma)} \int_{T_{\max}}^t (t-s)^{\gamma-1} F(\mathbf{v}(s)) ds. \quad (2.3)$$

Then we can solve (2.3) by a contraction argument exactly as in Step 2. The Volterra operator associated with (2.3) has a unique fixed point $\mathbf{v}$. Defining

$$\tilde{\mathbf{u}}(t) = \begin{cases} \mathbf{u}(t), & t \in [0, T_{\max}), \\ \mathbf{v}(t), & t \in [T_{\max}, T_{\max} + \delta], \end{cases}$$

we obtain an extension of the solution beyond $T_{\max}$, contradicting the definition of $T_{\max}$ in Step 3. Therefore $T_{\max} = \infty$.

Step 5: Uniqueness. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions with the same initial data. Fix any finite $T > 0$. By Lemma 2.3 (and Lemma 2.4 if $0 < q < 1$), there exists a compact set K_T such that $\mathbf{u}_1(t), \mathbf{u}_2(t) \in K_T$ for all $t \in [0, T]$, and F is Lipschitz on K_T with some constant L_T . Using the integral formulation (2.1) for both solutions and subtracting, we obtain

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \leq \frac{L_T}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\| ds, \quad t \in [0, T].$$

Applying the fractional Grönwall-type inequality [7, Lemma 6.19] with $\varepsilon_1 = 0$ yields $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ for all $t \in [0, T]$. Since $T > 0$ is arbitrary, we conclude that $\mathbf{u}_1(t) \equiv \mathbf{u}_2(t)$ for all $t \geq 0$. This completes the proof. $\square$

Remark 2.6. (i) If $x(0) = 0$ and $y(0) = 0$, then $(x(t), y(t)) \equiv (0, 0)$ is a trivial solution.
(ii) If $x(0) > 0$ and $y(0) = 0$, then $(x(t), y(t)) = (\bar{x}(t), 0)$ is a solution, where $\bar{x}(t)$ solves the fractional logistic equation ${}^C D_t^\gamma \bar{x} = (a-b)\bar{x} - c\bar{x}^2$. In this case, the solution $(x(t), y(t))$ is referred to as the predator-extinction solution.
(iii) If $x(0) = 0$ and $y(0) > 0$, then $(x(t), y(t)) = (0, y(0)E_\gamma(-et^\gamma))$ is a solution. In this case, the solution $(x(t), y(t))$ is referred to as the prey-extinction solution.

3. DYNAMICAL ANALYSIS

In this section, we investigate the behavior of the fractional-order model (1.2) from both local and global perspectives.

We first provide the definition of an equilibrium point. Let $\mathbf{u}(t) = (x(t), y(t))$ and rewrite (1.2) as the Caputo fractional differential system

$${}^C D_t^\gamma \mathbf{u}(t) = F(\mathbf{u}(t)), \quad 0 < \gamma < 1,$$

where $F = (F_1, F_2)$ is given by

$$\begin{aligned} F_1(x, y) &= \frac{ax}{1+k\beta y} - bx - cx^2 - \frac{pxy^q}{x^2 + \alpha\beta x + d}, \\ F_2(x, y) &= -ey + \frac{\mu pxy^q}{x^2 + \alpha\beta x + d}. \end{aligned} \quad (3.1)$$

Definition 3.1. We say that $\mathbf{u}^* = (x^*, y^*)$ is an equilibrium point of the model (1.2) if and only if $F_1(\mathbf{u}^*) = F_2(\mathbf{u}^*) = 0$.

It is clear that for $F_1(x, y) = F_2(x, y) = 0$, the model (1.2) always contains the trivial and axial equilibria at $E_0 = (0, 0)$ and $E_1 = (\frac{a-b}{c}, 0)$ respectively. For the coexistence state, we recall the following existence result established for the closely related integer-order model in [19].

Theorem 3.2. *There exists at least one positive interior equilibrium point of (1.2).*

Remark 3.3. We clarify the relationship between the present fractional-order model and the integer-order model studied in [19]. The two models share the same ecological mechanisms, namely predator aggregation-induced fear, predator interference, and a non-monotonic functional response. They also have the same steady-state equations $F_1(x, y) = 0$ and $F_2(x, y) = 0$. However, the two models differ in several important ways. In the fractional-order setting, local stability is not determined solely by the signs of the real parts of the eigenvalues. Instead, it is governed by the fractional-order stability condition $|\arg(\lambda_i)| > \frac{\gamma\pi}{2}$, which explicitly depends on the fractional order γ , see Section 3.1. Moreover, the present work establishes well-posedness for the fractional-order system, with special attention to the case $0 < q < 1$, where the term y^q is not locally Lipschitz at $y = 0$, see Section 2. We also investigate Hopf bifurcation obtained by varying the fractional order γ , see Section 3.3. These fractional-order stability, well-posedness, and bifurcation features are specific to the present model and do not appear in the same way in the integer-order model of [19].

To describe the stability properties of the equilibria of (1.2), we recall the standard notions of stability for general fractional-order systems.

Definition 3.4. Consider Caputo fractional differential systems

$${}^C D_t^\gamma \mathbf{u} = \mathbf{f}(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^n,$$

where $\mathbf{f}$ is continuously differentiable. Let $\mathbf{u}^*$ be an equilibrium point, that is, $\mathbf{f}(\mathbf{u}^*) = 0$. Then we have the following definitions.

- (i) $\mathbf{u}^*$ is *stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\mathbf{u}(0) - \mathbf{u}^*\| < \delta$, then for all $t \geq 0$,

$$\|\mathbf{u}(t) - \mathbf{u}^*\| < \varepsilon.$$

- (ii) $\mathbf{u}^*$ is *locally asymptotically stable* if it is stable and there exists $\delta > 0$ such that if $\|\mathbf{u}(0) - \mathbf{u}^*\| < \delta$, then

$$\lim_{t \rightarrow \infty} \mathbf{u}(t) = \mathbf{u}^*.$$

That is, nearby trajectories converge to $\mathbf{u}^*$ over time.

- (iii) $\mathbf{u}^*$ is *globally asymptotically stable* if it is stable and

$$\lim_{t \rightarrow \infty} \mathbf{u}(t) = \mathbf{u}^* \quad \text{for all initial conditions } \mathbf{u}(0) \in \mathbb{R}^n.$$

That is, all solutions, regardless of where they start, eventually converge to $\mathbf{u}^*$.

- (iv) $\mathbf{u}^*$ is *unstable* if it is not stable; i.e., there exists a neighborhood of $\mathbf{u}^*$ such that some trajectories starting arbitrarily close to $\mathbf{u}^*$ move away from it.
- (v) $\mathbf{u}^*$ is a *saddle point* if the linearized system has both positive and negative eigenvalues. Note that a saddle point is always unstable.

3.1. Local stability. We investigate the stability and instability properties of the equilibrium points of the model (1.2). Consider the Caputo fractional differential system ${}^C D_t^\gamma \mathbf{u} = \mathbf{f}(\mathbf{u})$ and let $\mathbf{u}^*$ be an equilibrium point. Since $\mathbf{f}$ is differentiable, in a sufficiently small neighborhood of $\mathbf{u}^*$ the vector field can be decomposed into its linear part, determined by the Jacobian matrix $J(\mathbf{u}^*) = D\mathbf{f}(\mathbf{u}^*)$, and a higher-order nonlinear remainder. As $\mathbf{u} \rightarrow \mathbf{u}^*$, this remainder becomes negligible compared with the linear term. Consequently, the local dynamics near $\mathbf{u}^*$ are well approximated by the linearized system, and the stability of $\mathbf{u}^*$ can be determined from the eigenvalues of the Jacobian matrix, in conjunction with the fractional-order stability criterion.

To establish the stability properties of the equilibria, we recall the following results from [7].

Lemma 3.5. Let $J(\mathbf{u}^*)$ be the Jacobian matrix of the system (1.2) at an equilibrium point $\mathbf{u}^*$, and let λ_i ($i = 1, 2$) be the eigenvalues of $J(\mathbf{u}^*)$. If some eigenvalue λ_1 satisfies $|\arg(\lambda_1)| > \frac{\gamma\pi}{2}$ and the other eigenvalue λ_2 satisfies $|\arg(\lambda_2)| < \frac{\gamma\pi}{2}$, then $\mathbf{u}^*$ is a saddle point.

Lemma 3.6. Let $\mathbf{u}^*$ and $J(\mathbf{u}^*)$ be an equilibrium point and the Jacobian matrix of (1.2), respectively, and let λ_i be the eigenvalues of $J(\mathbf{u}^*)$. Then

- (i) $\mathbf{u}^*$ is *locally asymptotically stable* if and only if $|\arg(\lambda_i)| > \frac{\gamma\pi}{2}$ for all λ_i ($i = 1, 2$).

- (ii) $\mathbf{u}^*$ is stable if and only if $|\arg(\lambda_i)| \geq \frac{\gamma\pi}{2}$ for all λ_i ($i = 1, 2$), and the eigenvalues with $|\arg(\lambda_i)| = \frac{\gamma\pi}{2}$ have the same geometric multiplicity and algebraic multiplicity.

We focus on the local stabilities of an interior equilibrium point $E_2 = (x^*, y^*)$ of the model (1.2). Then, the Jacobian matrix of (1.2) at E_2 is

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}, \quad (3.2)$$

where F_1 and F_2 are given by (3.1). We directly compute from $F_1(x^*, y^*) = F_2(x^*, y^*) = 0$ that

$$\begin{aligned} J_{11} &= x^* \left(\frac{p(y^*)^q(2x^* + \alpha\beta)}{((x^*)^2 + \alpha\beta x^* + d)^2} - c \right), \\ J_{12} &= -x^* \left(\frac{ak\beta}{(1 + k\beta y^*)^2} + \frac{pq(y^*)^{q-1}}{(x^*)^2 + \alpha\beta x^* + d} \right) < 0, \\ J_{21} &= \frac{ey^*}{x^*} \frac{d - (x^*)^2}{(x^*)^2 + \alpha\beta x^* + d}, \\ J_{22} &= e(q - 1). \end{aligned}$$

We define

$$\begin{aligned} R &:= (x^*)^2 + \alpha\beta x^* + d > 0, \quad P := \frac{ak\beta}{(1 + k\beta y^*)^2} + \frac{pq(y^*)^{q-1}}{R} > 0, \\ Q &:= \frac{p(y^*)^q(2x^* + \alpha\beta)}{R^2} - c. \end{aligned}$$

Then the Jacobian matrix $J = (J_{ij})$ can be written as

$$J_{11} = x^*Q, \quad J_{12} = -x^*P < 0, \quad J_{21} = \frac{ey^*}{x^*} \frac{d - (x^*)^2}{R}, \quad J_{22} = e(q - 1).$$

With the notation above, we can derive the trace, determinant, and discriminant of the Jacobian matrix $J = (J_{ij})$ as follows:

$$\begin{aligned} \text{tr}(J) &= J_{11} + J_{22} = x^*Q + e(q - 1), \\ \det(J) &= J_{11}J_{22} - J_{12}J_{21} = e \left(x^*(q - 1)Q + y^*P \frac{d - (x^*)^2}{R} \right), \end{aligned} \quad (3.3)$$

and the discriminant

$$\begin{aligned} \Delta &:= \text{tr}(J)^2 - 4\det(J) = (J_{11} - J_{22})^2 + 4J_{12}J_{21} \\ &= (x^*Q - e(q - 1))^2 - \frac{4ey^*P}{R} (d - (x^*)^2). \end{aligned} \quad (3.4)$$

Before classifying the sign of Δ , we note how the non-monotonic Holling type IV response enters the local stability analysis. The factor $d - (x^*)^2$ in (3.4) comes directly from the derivative of the prey-dependent functional response

$$f_{IV}(x) = \frac{px}{x^2 + \alpha\beta x + d}, \quad f'_{IV}(x) = \frac{p(d - x^2)}{(x^2 + \alpha\beta x + d)^2}.$$

Thus $f'_{IV}(x)$ changes sign at $x = \sqrt{d}$. This is different from the Holling type II and Holling type III responses, whose prey-dependent parts are monotone increasing and therefore do not exhibit the same sign change. In the present model, the sign of $d - (x^*)^2$ determines the sign of J_{21} and directly affects the discriminant Δ . This leads naturally to the three cases $x^* < \sqrt{d}$, $x^* = \sqrt{d}$, and $x^* > \sqrt{d}$ considered in the following lemma.

Lemma 3.7. *Let Δ be given by (3.4).*

- (i) *If $x^* > \sqrt{d}$, then $d - (x^*)^2 < 0$, hence $4J_{12}J_{21} > 0$ and therefore $\Delta > 0$. In particular, both eigenvalues of J are real.*
(ii) *If $x^* = \sqrt{d}$, then $d - (x^*)^2 = 0$ and $\Delta = (J_{11} - J_{22})^2$; thus*

$$\Delta > 0 \iff x^*Q \neq e(q - 1), \quad \Delta = 0 \iff x^*Q = e(q - 1).$$

(iii) If $x^* < \sqrt{d}$, then $d - (x^*)^2 > 0$ and $4J_{12}J_{21} < 0$, hence

$$\Delta < 0 \iff (x^*Q - e(q-1))^2 < \frac{4ey^*P}{R} (d - (x^*)^2).$$

In this case, J has a complex conjugate pair.

We now classify the local stability of an interior equilibrium point $E_2 = (x^*, y^*)$ of (1.2).

Theorem 3.8 (Local stability). *Let $\mathbf{u} = (x, y)$ and consider the linearized system of (1.2) at an equilibrium E_2 :*

$${}^C D_t^\gamma \mathbf{u} = J \mathbf{u}, \quad 0 < \gamma < 1,$$

where $J = J(E_2)$ is the Jacobian matrix. Let $\lambda_{1,2} = \frac{\text{tr}(J) \pm \sqrt{\Delta}}{2}$ be the eigenvalues of J . Then E_2 is locally asymptotically stable if and only if

$$|\arg(\lambda_i)| > \frac{\gamma\pi}{2} \quad \text{for all } \lambda_i \ (i = 1, 2).$$

Furthermore, the following classification holds.

- (i) The case $\Delta > 0$ (real eigenvalues). In this case, $\lambda_{1,2} \in \mathbb{R}$ and the stability depends on the sign pattern of the real eigenvalues.
 - (i-a) If $\Delta > \text{tr}(J)^2$ (equivalently $\det(J) < 0$), then $\lambda_1 \lambda_2 < 0$ and E_2 is a saddle, hence unstable for all $0 < \gamma < 1$.
 - (i-b) If $0 < \Delta < \text{tr}(J)^2$ (equivalently $\det(J) > 0$), then λ_1 and λ_2 have the same sign. Moreover, E_2 is locally asymptotically stable for all $0 < \gamma < 1$ if $\text{tr}(J) < 0$ (both eigenvalues negative), and unstable if $\text{tr}(J) > 0$ (both eigenvalues positive).
 - (i-c) If $\Delta = \text{tr}(J)^2$ (equivalently $\det(J) = 0$), then one eigenvalue is zero, so E_2 is indeterminate.
- (ii) The case $\Delta = 0$ (repeated real eigenvalues). In this case, $\lambda_1 = \lambda_2 = \frac{\text{tr}(J)}{2} \in \mathbb{R}$.
 - (ii-a) If $\text{tr}(J) < 0$, then E_2 is locally asymptotically stable for all $0 < \gamma < 1$.
 - (ii-b) If $\text{tr}(J) > 0$, then E_2 is unstable.
 - (ii-c) If $\text{tr}(J) = 0$, then $\lambda_1 = \lambda_2 = 0$ and E_2 is indeterminate.
- (iii) The case $\Delta < 0$ (complex conjugate eigenvalues). In this case, $\lambda_{1,2} = \frac{\text{tr}(J)}{2} \pm \frac{\sqrt{-\Delta}}{2}i$.
 - (iii-a) If $\text{tr}(J) \leq 0$, then E_2 is locally asymptotically stable for all $0 < \gamma < 1$.
 - (iii-b) If $\text{tr}(J) > 0$, then E_2 is locally stable if and only if

$$\arctan\left(\frac{\sqrt{-\Delta}}{\text{tr}(J)}\right) \geq \frac{\gamma\pi}{2}, \quad \text{equivalently, } -\Delta \geq \text{tr}(J)^2 \tan^2\left(\frac{\gamma\pi}{2}\right). \quad (3.5)$$

In particular, E_2 is locally asymptotically stable if the strict inequality holds in (3.5).

Based on Theorem 3.8, the local stability classification is summarized in Table 3.

TABLE 3. Stability classification of a Caputo fractional differential system ($0 < \gamma < 1$) using the discriminant, determinant, and trace. The symbol * indicates that the corresponding quantity does not affect stability.

Δ	$\det J$	$\text{tr } J$	Stability ($0 < \gamma < 1$)
> 0	< 0	*	Unstable (saddle)
> 0	> 0	< 0	Asymptotically stable
> 0	> 0	> 0	Unstable
> 0	$= 0$	*	Indeterminate
$= 0$	*	< 0	Asymptotically stable
$= 0$	*	> 0	Unstable
$= 0$	*	$= 0$	Indeterminate
< 0	*	≤ 0	Asymptotically stable
< 0	*	> 0	Asymptotically stable if $\arctan\left(\frac{\sqrt{-\Delta}}{\text{tr}(J)}\right) > \frac{\gamma\pi}{2}$

3.2. Global stability. We prove the global stability of an interior equilibrium solution E_2 by using the Lyapunov function and [16, Lemma 4.6] (see also [21, 1]).

Proposition 3.9 (Generalized LaSalle Invariance Principle for Fractional Systems). *Consider the fractional-order system*

$${}^C D_t^\gamma \mathbf{u}(t) = \mathbf{f}(\mathbf{u}(t)), \quad \text{where } \mathbf{u} \in \mathbb{R}^n \text{ and } \gamma \in (0, 1).$$

Here $\mathbf{f}$ is continuous and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $V(\mathbf{u})$ is positive definite.
- (ii) The fractional Caputo derivative of V along trajectories satisfies

$${}^C D_t^\gamma V(\mathbf{u}(t)) \leq 0 \quad \text{for all } \mathbf{u} \in \Omega,$$

for some domain $\Omega \subseteq \mathbb{R}^n$ containing the origin.

Define the set

$$\mathcal{E} = \{\mathbf{u} \in \Omega : {}^C D_t^\gamma V(\mathbf{u}) = 0\}.$$

Then every solution $\mathbf{u}(t)$ starting in Ω approaches the largest invariant set $\mathcal{M} \subseteq \mathcal{E}$ as $t \rightarrow \infty$. In particular, if $\mathcal{M} = \{0\}$, then the equilibrium at the origin is asymptotically stable in Ω .

Remark 3.10. Proposition 3.9 is still valid for a general equilibrium point $\mathbf{u}^*$ instead of the origin. To see this, consider the coordinate transformation $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}^*$, where $\mathbf{u}^*$ is the positive equilibrium satisfying $\mathbf{f}(\mathbf{u}^*) = 0$. Under this transformation, the system becomes

$${}^C D_t^\gamma \tilde{\mathbf{u}}(t) = \mathbf{f}(\tilde{\mathbf{u}}(t) + \mathbf{u}^*),$$

and the equilibrium at $\tilde{\mathbf{u}} = \mathbf{0}$ corresponds to the original equilibrium $\mathbf{u} = \mathbf{u}^*$ and we can define a neighborhood $\tilde{\Omega} = \Omega - \mathbf{u}^*$ of the origin. The Generalized LaSalle Invariance Principle guarantees that if $\mathcal{M} = \{\mathbf{u}^*\}$, then the equilibrium $\mathbf{u}^*$ is asymptotically stable in Ω .

Theorem 3.11 (Global stability). *Let $E_2 = (x^*, y^*)$ be an interior equilibrium of the model (1.2). Define*

$$\Omega = \{(x, y) \in \mathbb{R}_{\geq 0}^2 : x \geq x^*, y \geq y^*, y \geq \frac{a-b}{k\beta b}, y(x^* + (\mu-1)x) \leq \mu xy^*\}.$$

Then E_2 is globally asymptotically stable in Ω .

Proof. Consider the Lyapunov function

$$V(x, y) = \left(x - x^* - x^* \ln \frac{x}{x^*}\right) + \left(y - y^* - y^* \ln \frac{y}{y^*}\right),$$

which is positive definite on $\mathbb{R}_{>0}^2$ and vanishes only at the equilibrium (x^*, y^*) . According to the well-known inequality for the Caputo derivative of logarithmic functions, the derivative of V along any positive solution $(x(t), y(t))$ of system (1.2) satisfies

$${}^C D_t^\gamma V(x(t), y(t)) \leq \left(1 - \frac{x^*}{x}\right) {}^C D_t^\gamma x + \left(1 - \frac{y^*}{y}\right) {}^C D_t^\gamma y.$$

Substituting the model (1.2) yields

$$\begin{aligned} {}^C D_t^\gamma V(x, y) &\leq \left(1 - \frac{x^*}{x}\right) \left(\frac{ax}{1+k\beta y} - bx - cx^2 - \frac{pxy^q}{x^2 + \alpha\beta x + d}\right) \\ &\quad + \left(1 - \frac{y^*}{y}\right) \left(-ey + \frac{\mu pxy^q}{x^2 + \alpha\beta x + d}\right) \\ &= (x - x^*) \left(\frac{a}{1+k\beta y} - b\right) - cx(x - x^*) - e(y - y^*) \\ &\quad + \frac{py^{q-1}}{x^2 + \alpha\beta x + d} (y(x^* + (\mu-1)x) - \mu xy^*). \end{aligned}$$

For each $(x, y) \in \Omega$ we take

$$x - x^* \geq 0, \quad y - y^* \geq 0, \quad \frac{a}{1+k\beta y} - b \leq 0, \quad y(x^* + (\mu-1)x) - \mu xy^* \leq 0,$$

hence ${}^C D_t^\gamma V(x, y) \leq 0$ on Ω .

Let

$$\mathcal{E} = \{(x, y) \in \Omega : {}^C D_t^\gamma V(x, y) = 0\}.$$

From the above inequality, ${}^C D_t^\gamma V(x, y) = 0$ implies simultaneously $x = x^*$ and $y = y^*$, so $\mathcal{E} = \{(x^*, y^*)\}$. Therefore the largest invariant set $\mathcal{M} \subseteq \mathcal{E}$ is exactly $\mathcal{M} = \{(x^*, y^*)\}$. By Proposition 3.9, every solution starting in Ω converges to (x^*, y^*) as $t \rightarrow \infty$. $\square$

3.3. Hopf bifurcation analysis. A Hopf bifurcation describes a transition in which an equilibrium loses stability and oscillatory dynamics emerge as a bifurcation parameter varies. For our fractional-order systems, the local stability of an equilibrium depends not only on the Jacobian eigenvalues but also on the fractional order γ through the argument condition $|\arg(\lambda_i)| > \frac{\gamma\pi}{2}$ (see Theorem 3.8). Consequently, varying the fractional order itself may switch stability and induce a Hopf bifurcation. In what follows, we take $\gamma \in (0, 1)$ as the bifurcation parameter and employ the Hopf bifurcation criterion introduced by [20, Theorem 3].

Lemma 3.12. *Let $\gamma \in (0, 1)$ be the bifurcation parameter. When γ passes through a critical value $\gamma^* \in (0, 1)$, the model (1.2) undergoes a Hopf bifurcation at an equilibrium point if the following conditions hold.*

- (i) *The Jacobian matrix at the equilibrium has a pair of complex conjugate eigenvalues $\lambda_{1,2} = u \pm iv$ with $u > 0$;*
- (ii) *$m(\gamma^*) = 0$, where $m(\gamma) = \frac{\gamma\pi}{2} - \min_{1 \leq i \leq 2} |\arg(\lambda_i)|$;*
- (iii) *$\frac{dm(\gamma)}{d\gamma} \Big|_{\gamma=\gamma^*} \neq 0$.*

In the complex-eigenvalue case of Theorem 3.8 (i.e., $\Delta < 0$), the local stability of the equilibrium depends explicitly on the fractional order γ through the argument condition $|\arg(\lambda_i)| > \frac{\gamma\pi}{2}$. Hence, it is natural to regard γ as a bifurcation parameter. Combining Theorem 3.8 with the Hopf bifurcation criterion in Lemma 3.12, we obtain the following result. This result gives a fractional-order Hopf bifurcation criterion for the interior equilibrium of model (1.2).

Theorem 3.13 (Hopf bifurcation). *Let $E_2 = (x^*, y^*)$ be the interior equilibrium of (1.2), and let $J = J(E_2)$ be the Jacobian matrix. Set $\Delta := \text{tr}^2(J) - 4\det(J)$. Assume that $\Delta < 0$ and $\text{tr}(J) > 0$. Define the critical fractional order*

$$\gamma^* = \frac{2}{\pi} \arctan\left(\frac{\sqrt{-\Delta}}{\text{tr}(J)}\right) \in (0, 1).$$

Then the followings hold.

- (i) *The equilibrium E_2 is locally asymptotically stable for $0 < \gamma < \gamma^*$, and unstable for $\gamma^* < \gamma < 1$.*
- (ii) *Moreover, (1.2) undergoes a Hopf bifurcation at E_2 when γ passes through γ^* .*

Proof. Since $\Delta < 0$, the eigenvalues of J form a complex conjugate pair

$$\lambda_{1,2} = \frac{\text{tr}(J)}{2} \pm i \frac{\sqrt{-\Delta}}{2}.$$

Then $\text{Re}(\lambda_{1,2}) = \frac{\text{tr}(J)}{2} > 0$, hence condition (i) in Lemma 3.12 holds.

Moreover, since $\Delta < 0$ and $\text{tr}(J) > 0$,

$$|\arg(\lambda_1)| = |\arg(\lambda_2)| = \arctan\left(\frac{\sqrt{-\Delta}}{\text{tr}(J)}\right) =: \phi, \quad \phi \in \left(0, \frac{\pi}{2}\right).$$

By Theorem 3.8 (iii-b), in the case $\Delta < 0$ and $\text{tr}(J) > 0$ the equilibrium E_2 is locally asymptotically stable if and only if

$$\phi > \frac{\gamma\pi}{2}.$$

Therefore, defining

$$\gamma^* := \frac{2\phi}{\pi} = \frac{2}{\pi} \arctan\left(\frac{\sqrt{-\Delta}}{\text{tr}(J)}\right),$$

we obtain the stability switching: E_2 is locally asymptotically stable for $0 < \gamma < \gamma^*$ and unstable for $\gamma^* < \gamma < 1$. This proves part (i).

To verify the Hopf criterion (ii), we define

$$m(\gamma) = \frac{\gamma\pi}{2} - \min_{1 \leq i \leq 2} |\arg(\lambda_i)| = \frac{\gamma\pi}{2} - \phi.$$

Then $m(\gamma^*) = 0$, so condition (ii) holds, and

$$\left. \frac{dm(\gamma)}{d\gamma} \right|_{\gamma=\gamma^*} = \frac{\pi}{2} \neq 0,$$

so condition (iii) holds. Hence, by Lemma 3.12, the system undergoes a Hopf bifurcation at E_2 when γ passes through γ^* , proving part (ii). $\square$

4. NUMERICAL SIMULATIONS

In this section, we present numerical simulations of the fractional-order model (1.2) implemented in Python using the BrainPy package [28]. The numerical experiments are designed according to the stability classification in Theorem 3.8 and the Hopf bifurcation criterion with respect to the fractional order γ in Theorem 3.13. For each parameter set, we compute the interior equilibrium $E_2 = (x^*, y^*)$, evaluate the Jacobian matrix $J = J(E_2)$ (see (3.2)), and then compute $\text{tr}(J)$ and the discriminant Δ given in (3.3) and (3.4), respectively. Recall that, by Lemma 3.7, the sign of Δ determines whether the eigenvalues are real ($\Delta \geq 0$) or form a complex conjugate pair ($\Delta < 0$).

The parameter sets used in the simulations are listed in Table 4.

TABLE 4. Model parameters used in the five numerical simulations.

Parameter	Sim. 1	Sim. 2	Sim. 3	Sim. 4	Sim. 5
a	1	1	1	1	1
b	0.7	0.7	0.08	0.08	0.08
c	0.7	0.7	0.8	0.8	0.8
d	1	0.12	0.12	0.12	0.12
e	1	0.9	0.64	0.64	0.64
k	0.6	0.6	0.66	0.66	0.66
p	1	1	1	1	1
q	0.97	0.97	0.97	1	1.05
α	0.3	0.3	0.3	0.3	0.3
β	0.1	0.1	0.1	0.1	0.1
μ	1	1	1	1	1

In addition to the five representative simulations listed in Table 4, we also perform a separate numerical experiment in which all parameters are fixed as in Simulation 3 except for the predator-interference exponent q . This additional experiment is used to clarify how changes in q affect the population time series and phase-plane dynamics.

Simulation 1 (real eigenvalues, stable for all $0 < \gamma < 1$). For the parameter values in Simulation 1, the system admits an interior equilibrium E_2 . At E_2 , we obtain $\text{tr}(J) = -0.33 < 0$ and $\Delta = 0.073 > 0$, hence the eigenvalues are real by Lemma 3.7. Since $0 < \Delta < \text{tr}(J)^2$ and $\text{tr}(J) < 0$, Theorem 3.8 (i-b) implies that E_2 is locally asymptotically stable for all $0 < \gamma < 1$. Figure 1 shows convergence of $(x(t), y(t))$ to E_2 for several values of γ .

Simulation 2 (complex eigenvalues with $\text{tr}(J) < 0$). In Simulation 2, the parameters d and e are changed from Simulation 1 (see Table 4). For this parameter set, the Jacobian at E_2 has a complex conjugate pair of eigenvalues

$$\lambda_{1,2} = -0.016 \pm 0.84i, \quad \Delta = -2.85 < 0.$$

In particular, $\text{tr}(J) = 2\text{Re}(\lambda_{1,2}) = -0.032 < 0$. Hence, by Theorem 3.8 (iii-a), the equilibrium E_2 is locally asymptotically stable for all $0 < \gamma < 1$, while the approach to E_2 is oscillatory. This behavior is illustrated in Figure 2.

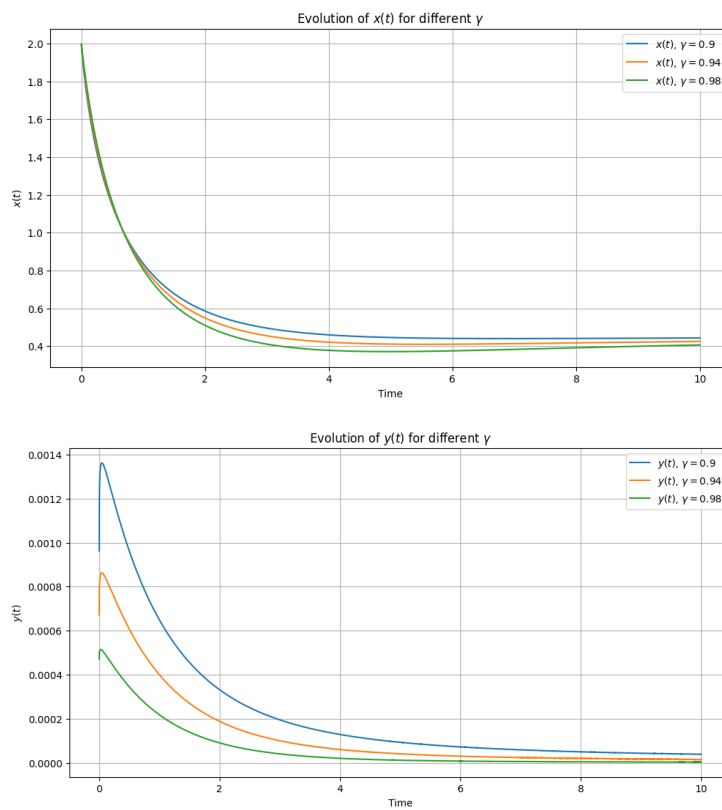


FIGURE 1. Simulation 1: Time series of $x(t)$ and $y(t)$ for several values of the fractional order γ .

Simulation 3 (Hopf bifurcation with respect to γ). In Simulation 3, the parameters b , c , e , and k are updated relative to Simulation 2 (Table 4). The system still admits an interior equilibrium E_2 , and we obtain

$$\lambda_{1,2} = 0.007 \pm 0.69i, \quad \text{tr}(J) = 0.14 > 0, \quad \Delta = -1.88 < 0.$$

Thus the setting of Theorem 3.13 holds. The critical fractional order is computed by

$$\gamma^* = \frac{2}{\pi} \arctan\left(\frac{\sqrt{-\Delta}}{\text{tr}(J)}\right) = 0.971.$$

Accordingly, Theorem 3.13 predicts that E_2 is locally asymptotically stable for $0 < \gamma < \gamma^*$ and becomes unstable for $\gamma^* < \gamma < 1$, with a Hopf bifurcation occurring as γ passes through γ^* . Figure 3 illustrates the transition in phase portraits for $\gamma < \gamma^*$ and $\gamma > \gamma^*$. The time series and bifurcation diagrams in Figs. 4 and 5 further confirm the occurrence of a Hopf bifurcation, characterized by the onset of sustained oscillations and a sharp increase in amplitude as the fractional order γ crosses the critical value γ^* .

Simulations 4 and 5 (effect of $q \geq 1$ on γ^*). Simulations 4 and 5 consider $q = 1$ and $q = 1.05$, respectively, while keeping the remaining parameters as in Simulation 3 (Table 4). In both cases, we remain in the regime $\Delta < 0$ and $\text{tr}(J) > 0$, so the stability of E_2 depends on γ and Theorem 3.13 applies. Numerically, the critical values are found to be approximately $\gamma^* \simeq 0.96$ for $q = 1$ and $\gamma^* \simeq 0.94$ for $q = 1.05$. Figures 6 and 7 show that $x(t)$ transitions from convergence to periodic oscillations as γ crosses γ^* , indicating the emergence of a limit cycle. Comparing Simulations 3–5, we observe that the critical bifurcation value γ^* decreases as q increases.

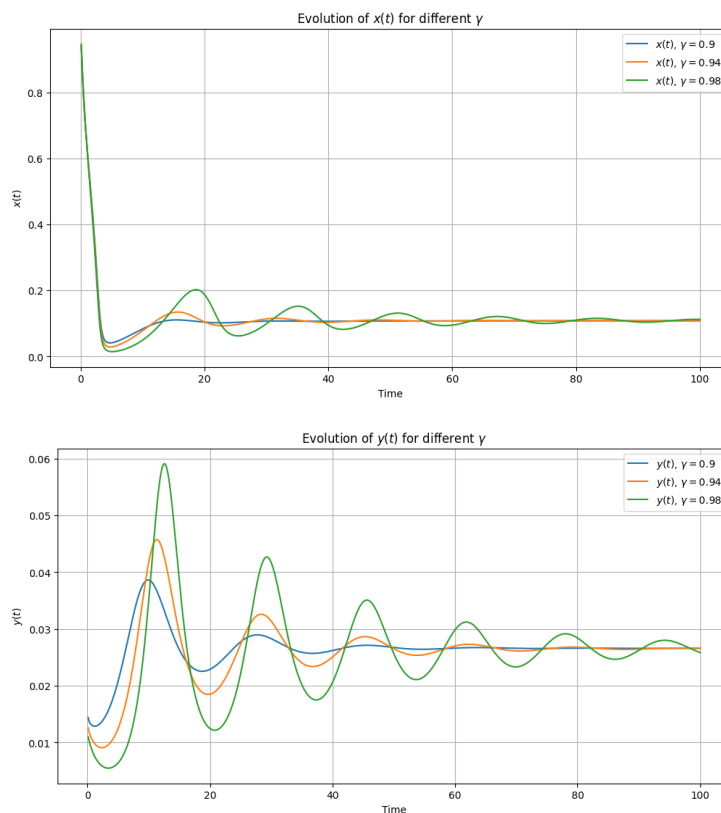


FIGURE 2. Simulation 2: Oscillatory convergence of $x(t)$ and $y(t)$ to the interior equilibrium E_2 for several values of γ .

Population dynamics under variation of q . To further illustrate the dynamical role of the predator-interference exponent q , we fix all parameters as in Simulation 3 and vary

$$q = 0.8, 0.97, 1.0, 1.05,$$

while keeping the fractional order fixed at $\gamma = 0.95$. These values represent the three biologically distinct regimes: predator interference ($0 < q < 1$), independent predation ($q = 1$), and cooperative or aggregated predation ($q > 1$). Figure 8 shows the time series of the prey population $x(t)$ and the corresponding phase portraits in the x - y plane. The separate time series of the predator population $y(t)$ are omitted to avoid redundancy, since they show the same qualitative transition as $x(t)$; the predator dynamics are included in the x - y phase portraits. The simulations demonstrate that increasing q can change the qualitative behavior of the coexistence dynamics. For $q = 0.8$, the trajectory approaches the coexistence equilibrium and no sustained oscillation is observed. For $q = 0.97$ and $q = 1.0$, mild oscillations decay toward the equilibrium, reflecting that γ lies close to γ^* . For $q = 1.05$, since $\gamma = 0.95 > \gamma^* \simeq 0.94$, the coexistence equilibrium loses stability. Consequently, large-amplitude sustained oscillations emerge, as also seen from the phase portrait. Overall, these simulations show that stronger predator interference ($0 < q < 1$) tends to stabilize the coexistence equilibrium, whereas increasing q toward cooperative or aggregated predation lowers the critical value γ^* and makes order-induced oscillatory dynamics more likely.

Dependence of γ^* on q . In addition to the time-series simulations above, Figure 9 depicts the relationship between the predator-interference parameter q and the critical fractional order γ^* , computed from the formula in Theorem 3.13. The numerical results suggest that $\gamma^* < 1$ once q exceeds approximately 0.8, highlighting the influence of predator interference on the stability threshold. In particular, as q decreases toward a value near 0.8, the critical value γ^* approaches 1, indicating that stronger predator interference tends to delay or suppress the onset of oscillatory

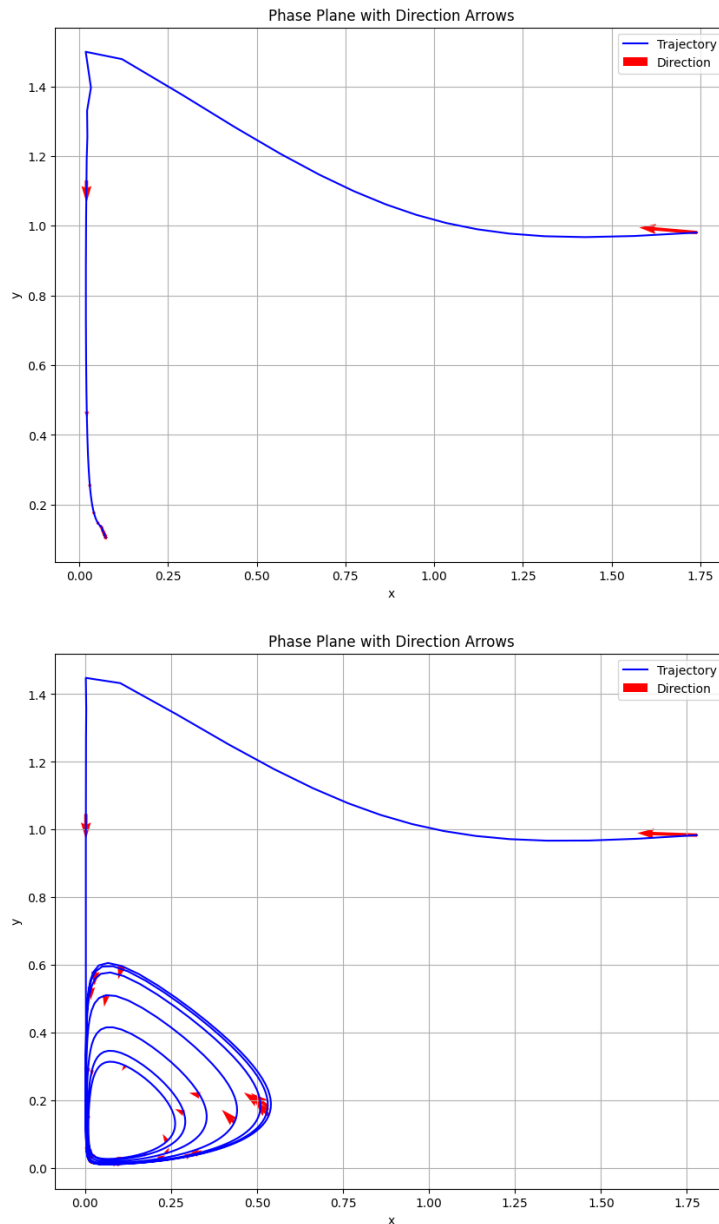


FIGURE 3. Simulation 3: Phase portraits for $\gamma = 0.90$ (top, $\gamma < \gamma^*$) and $\gamma = 0.98$ (bottom, $\gamma > \gamma^*$).

dynamics. Conversely, as q increases, cooperative or aggregated predation lowers the critical value γ^* and makes order-induced oscillations more likely.

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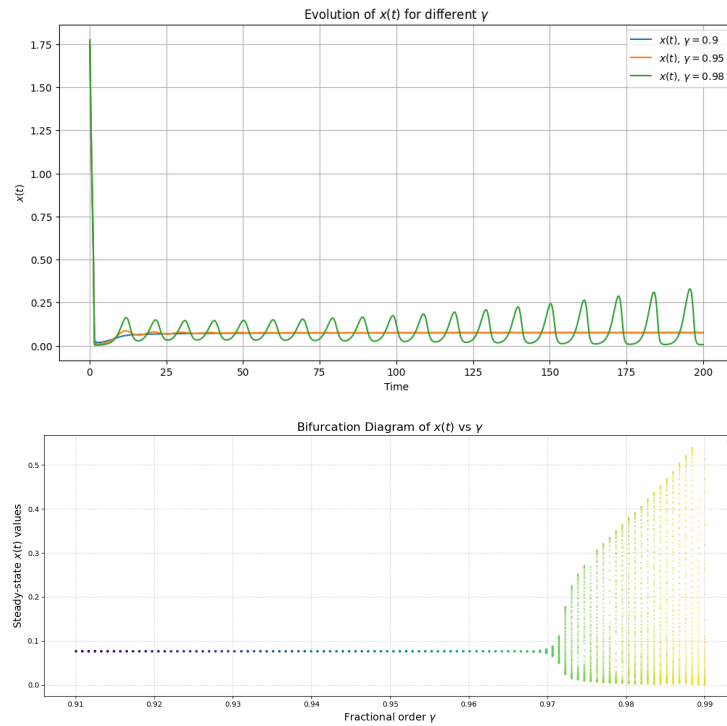


FIGURE 4. Simulation 3 ($0 < q < 1$): Time series of $x(t)$ (top) and bifurcation diagram (bottom) versus γ in the regime $\text{tr}(J) > 0$ and $\Delta < 0$.

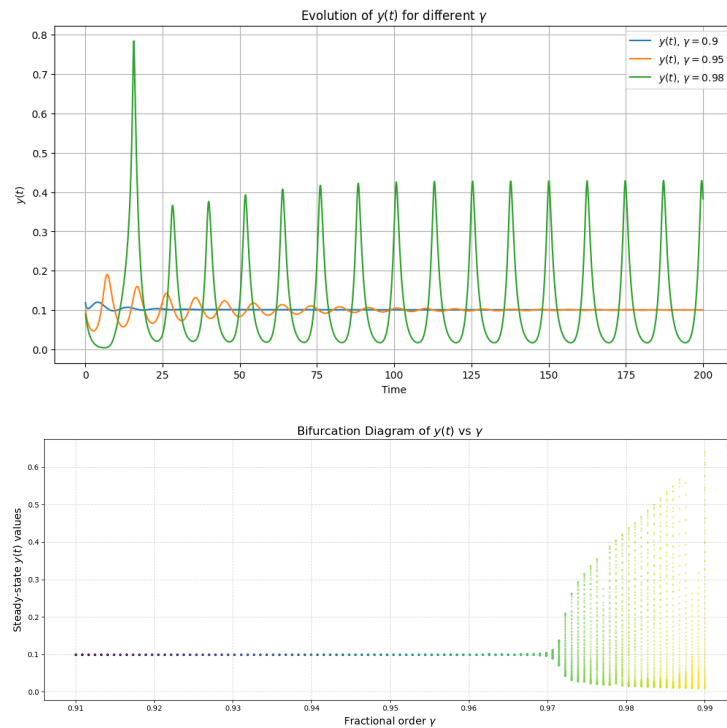


FIGURE 5. Simulation 3 ($0 < q < 1$): Time series of $y(t)$ (top) and bifurcation diagram (bottom) versus γ in the regime $\text{tr}(J) > 0$ and $\Delta < 0$.

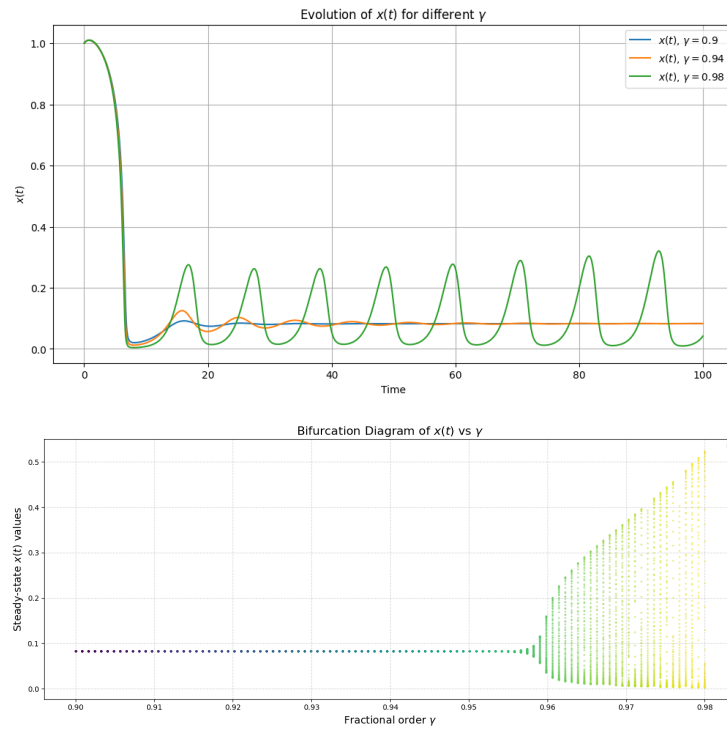


FIGURE 6. Simulation 4 ($q = 1$): Time series of $x(t)$ (top) and bifurcation diagram (bottom) versus γ in the regime $\text{tr}(J) > 0$ and $\Delta < 0$.

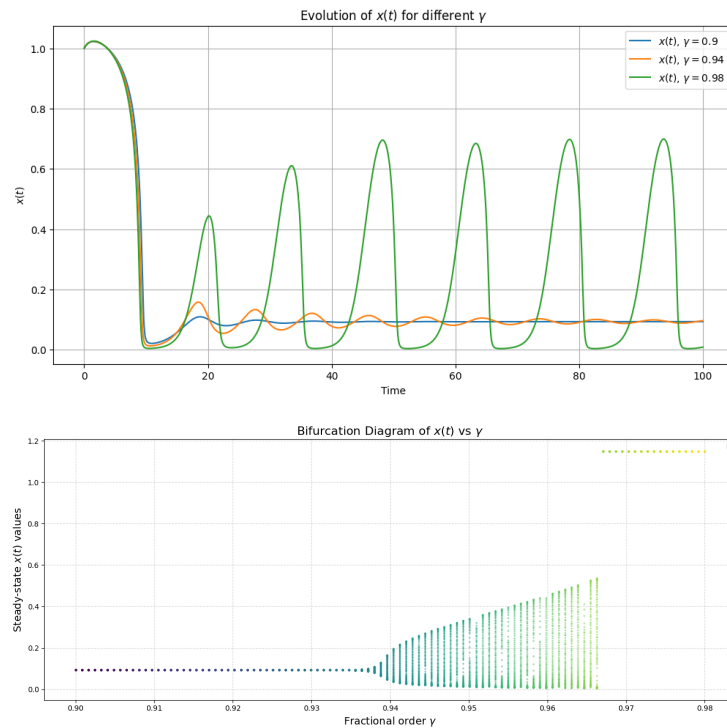


FIGURE 7. Simulation 5 ($q = 1.05$): Time series of $x(t)$ (top) and bifurcation diagram (bottom) versus γ in the regime $\text{tr}(J) > 0$ and $\Delta < 0$.

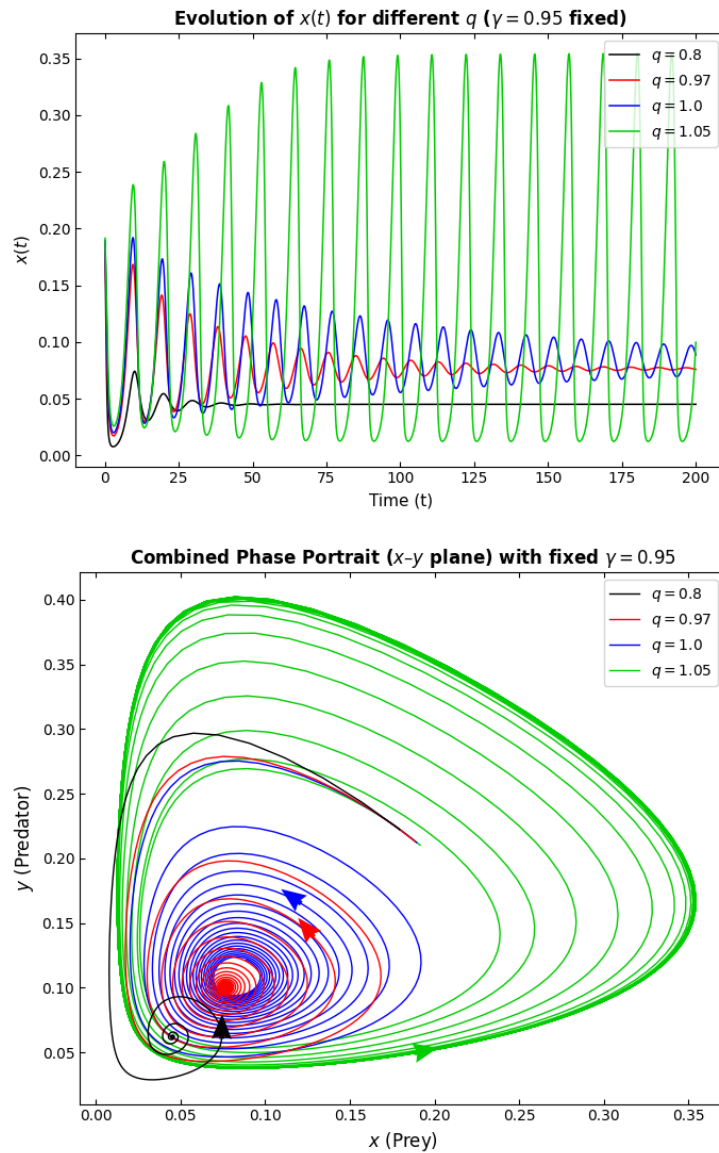


FIGURE 8. Time series of $x(t)$ (top) and phase portraits in the x - y plane (bottom) for $q = 0.8, 0.97, 1.0, 1.05$ with $\gamma = 0.95$ fixed.

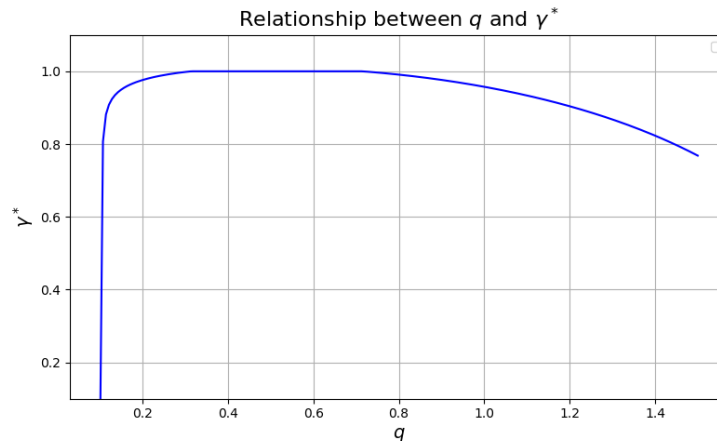


FIGURE 9. Relationship between the predator-interference parameter q and the critical fractional order γ^* .

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