

GROUND STATE SOLUTIONS FOR FRACTIONAL p -LAPLACE EQUATIONS WITH MIXED NONLINEARITIES IN THE CRITICAL AND SUBCRITICAL CASES

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ABSTRACT. In this article, we study fractional p -Laplace equations with mixed nonlinearities. By employing Lieb's compactness theorem and a generalized Sobolev inequality involving Morrey norms in $D^{s,p}(\mathbb{R}^N)$, we establish the existence of ground state solutions under appropriate subcritical and critical growth conditions.

1. INTRODUCTION AND MAIN RESULTS

Consider the fractional p -Laplacian equation

$$(-\Delta)_p^s u + |u|^{p-2}u = f(x, u), \quad x \in \mathbb{R}^N, \quad (1.1)$$

where $s \in (0, 1)$, $p > 1$ and $(-\Delta)_p^s$ denotes the fractional p -Laplace operator defined by

$$(-\Delta)_p^s u(x) = \mathcal{C}_{N,s,p} P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

where $\mathcal{C}_{N,s,p}$ is a normalizing constant. For simplicity, we usually assume that $\mathcal{C}_{N,s,p} = 1$. Equation (1.1) arises in various physical contexts [6, 17, 20]. The fractional p -Laplacian, in conjunction with mixed nonlinearities, captures essential features of various complex phenomena, including the diffusion dynamics of biological populations [10] and anisotropic heat transport in reversed shear magnetic fields [21]. As a result, a thorough understanding of fractional p -Laplacian operators with mixed nonlinear effects has become a subject of significant theoretical and practical interest. In the past decades, such nonlinear problems have attracted growing attention from a broad spectrum of researchers [2, 4, 7, 13, 15, 16, 17, 18, 19, 22, 23, 27, 30, 31, 32].

For the fractional p -Laplacian problems with the critical Sobolev and Hardy exponents, no matter in bounded domains or in the entire spaces, quite a few powerful and innovative methods have been proposed for the existence and multiplicity of solutions [3, 5, 11, 14, 25, 26, 28]. It becomes very natural to ask whether we can obtain the existence of ground state solutions to the fractional p -Laplace equation involving doubly Hardy-Sobolev critical exponents. In this study, we aim to study this question under the given subcritical and critical growth conditions.

We start with the fractional elliptic equation with subcritical mixed nonlinearities

$$(-\Delta)_p^s u + |u|^{p-2}u = K(x)|u|^{q_1-2}u + K(x)|u|^{q_2-2}u, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $N \geq 3$, $s \in (0, 1)$, $p \in (1, \infty)$ and $p < q_1 < q_2 < p^*$. We impose the hypotheses on the nonsymmetric perturbation function $K(x)$ as follows

- (H1) $1 \not\equiv K(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $1 = \lim_{|x| \rightarrow +\infty} K(x) = \inf_{x \in \mathbb{R}^N} K(x)$.
- (H2) There exists $\rho > 0$ such that $K(x) = 1$ in $B(0, \rho)$.

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For convenience of our statement, we denote by $D^{s,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo norm

$$\|u\|_{D^{s,p}(\mathbb{R}^N)}^p := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy.$$

For $s \in (0, 1)$ and $p \in (1, \infty)$, we define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < \infty\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} |u|^p dx.$$

Proposition 1.1 ([12, Theorem 6.5]). Let $s \in (0, 1)$ and $p \in (1, \infty)$. Then $W^{s,p}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [p, p_s^*]$ and $D^{s,p}(\mathbb{R}^N)$ is continuously embedded into $L^{p_s^*}(\mathbb{R}^N)$.

Based on Proposition 1.1, we define the two minimization problems

$$\begin{aligned} \mathcal{S}_q &:= \inf_{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{W^{s,p}(\mathbb{R}^N)}^p}{\left(\int_{\mathbb{R}^N} |u|^q dx\right)^{p/q}}, \\ \mathcal{H}_\alpha &:= \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{D^{s,p}(\mathbb{R}^N)}^p}{\left(\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha} dx\right)^{p/p_s^*(\alpha)}}. \end{aligned} \quad (1.3)$$

Definition 1.2. We say $u \in W^{s,p}(\mathbb{R}^N)$ to be a weak solution of equation (1.2), if for every $\varphi \in W^{s,p}(\mathbb{R}^N)$ it holds

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy + \int_{\mathbb{R}^N} |u|^{p-2} u \varphi dx \\ &= \int_{\mathbb{R}^N} K(x) |u|^{q_1-2} u \varphi dx + \int_{\mathbb{R}^N} K(x) |u|^{q_2-2} u \varphi dx. \end{aligned}$$

The corresponding energy functional of equation (1.2) is

$$I_q(u) = \frac{1}{p} \|u\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{1}{q_1} \int_{\mathbb{R}^N} K(x) |u|^{q_1} dx - \frac{1}{q_2} \int_{\mathbb{R}^N} K(x) |u|^{q_2} dx.$$

We can see that $I_q \in C^1(W^{s,p}(\mathbb{R}^N), \mathbb{R})$ and

$$\begin{aligned} \langle I'_q(u), \varphi \rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \\ &\quad - \int_{\mathbb{R}^N} K(x) |u|^{q_1} dx - \int_{\mathbb{R}^N} K(x) |u|^{q_2} dx, \quad \varphi \in W^{s,p}(\mathbb{R}^N). \end{aligned}$$

Our first main result is regarding the existence of ground state solutions to the fractional p -Laplace equation with two nonlinear terms of the subcritical growth.

Theorem 1.3. Assume that $N \geq 3$, $p < q_1 < q_2 < p_s^*$ and conditions (H1) and (H2) hold. Then (1.2) has a ground state solution.

It is notable that equation (1.2) contains the fractional p -Laplacian, so the localization methods become invalid [9] and we shall study the problem in the nonlocal situation. Although the exponents q_1 and q_2 are subcritical, the nonsymmetric perturbation function $K(x)$ and the whole space $\mathbb{R}^N$ make the functional I_q not satisfy the (PS) condition. To overcome these difficulties, we start with the geometrical structure of I_q . By applying the refined Sobolev's inequality, we obtain a generalized version of Lieb's compactness theorem, which plays a crucial role in the compactness analysis. We then derive an explicit estimate of the mountain pass threshold and establish the existence result of equation (1.2).

Next we consider the doubly critical case

$$(-\Delta)_p^s u = K(x) \frac{|u|^{p_s^*(\alpha_1)-2} u}{|x|^{\alpha_1}} + K(x) \frac{|u|^{p_s^*(\alpha_2)-2} u}{|x|^{\alpha_2}}, \quad x \in \mathbb{R}^N, \quad (1.4)$$

where $N \geq 3$, $s \in (0, 1)$, $p \in (1, \infty)$ and $0 < \alpha_1 < \alpha_2 < p$.

Definition 1.4. We say $u \in D^{s,p}(\mathbb{R}^N)$ is a weak solution of (1.4), if for every $\varphi \in D^{s,p}(\mathbb{R}^N)$ it holds

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy \\ &= \int_{\mathbb{R}^N} K(x) \frac{|u|^{p_s^*(\alpha_1)-2} u \varphi}{|x|^{\alpha_1}} dx + \int_{\mathbb{R}^N} K(x) \frac{|u|^{p_s^*(\alpha_2)-2} u \varphi}{|x|^{\alpha_2}} dx. \end{aligned}$$

Consider the functional $I_\alpha : D^{s,p}(\mathbb{R}^N) \rightarrow \mathbb{R}$ associated with equation (1.4) as

$$I_\alpha(u) = \frac{1}{p} \|u\|_{D^{s,p}(\mathbb{R}^N)}^p - \frac{1}{p_s^*(\alpha_1)} \int_{\mathbb{R}^N} K(x) \frac{|u|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx - \frac{1}{p_s^*(\alpha_2)} \int_{\mathbb{R}^N} K(x) \frac{|u|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx.$$

It is not difficult to see that I_α is well-defined, belongs to C^1 , and satisfies

$$\begin{aligned} \langle I'_\alpha(u), \varphi \rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dx dy \\ &\quad - \int_{\mathbb{R}^N} K(x) \frac{|u|^{p_s^*(\alpha_1)-2} u \varphi}{|x|^{\alpha_1}} dx - \int_{\mathbb{R}^N} K(x) \frac{|u|^{p_s^*(\alpha_2)-2} u \varphi}{|x|^{\alpha_2}} dx, \quad \varphi \in D^{s,p}(\mathbb{R}^N). \end{aligned}$$

Theorem 1.5. Assume that $N \geq 3$, $s \in (0, 1)$, $p \in (1, \infty)$, $0 < \alpha_1 < \alpha_2 < p$ and conditions (H1) and (H2) hold. Then equation (1.4) admits a ground state solution.

To prove Theorem 1.5, we note that the combination of two critical exponents induces more difficulties in terms of the compactness analysis. Specifically, we have trouble in ruling out the “vanishing” of the corresponding (PS) sequence due to an asymptotic competition between the two critical nonlinearities. The energy functional I_α is lack of translation and scaling invariance when we analyze the property of the (PS) sequence. The lack of an explicit formula for a minimizer of $\mathcal{H}_\alpha$ makes the estimation of the mountain pass threshold more complicated.

The paper is organized as follows. We prove Theorems 1.3 and 1.5 in Sections 2 and 3, respectively. Throughout this paper, we denote by C and C_i positive constants whose values may vary from line to line. Following the tradition, we use $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ to represent a ball with center x_0 and radius $r > 0$.

2. PROOF OF THEOREM 1.3

The proof is straightforward, after the following technical lemmas.

Lemma 2.1 ([1, Theorem 2.1]). Let X be a Banach space and $I \in C^1(X, \mathbb{R})$. If there exists $\rho > 0$ such that $I|_{\partial B_\rho(0)} > 0$, and there exists a function $e \in X$, such that $\|e\|_X > \rho$ and $I(e) < 0$, Then there exist a Palais-Smale sequence at level c and a sequence $\{u_n\}$ such that

$$I(u_n) \rightarrow c \quad \text{and} \quad I'(u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)) > 0, \quad \Gamma := \{\gamma \in C([0,1], W^{s,p}(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

We denote the Nehari manifold of equation (1.2) by

$$\mathcal{N}_q := \{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} : \langle I'_q(u), u \rangle = 0\}.$$

Lemma 2.2 ([29, Theorem 4.2]). Assume that the conditions in Theorem 1.3 hold. Then

- (1) the functional I_q possesses the mountain pass geometry;
- (2) for any $u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_q$ and $I_q(t_u u) = \max_{t>0} I_q(tu)$; and

$$(3) \quad c_q = \bar{c}_q = \bar{\bar{c}}_q > 0,$$

where

$$\bar{c}_q = \inf_{u \in \mathcal{N}_q} I_q(u), \quad \bar{\bar{c}}_q = \inf_{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} I_q(tu).$$

Lemma 2.3. For $u \in \mathcal{N}_q$, we have $\Phi'_q(u) \neq 0$, where $\Phi_q(u) := \langle I'_q(u), u \rangle$. Moreover, if $I_q(u) = c_q$, then u is a ground state solution of equation (1.2).

Proof. For $u \in \mathcal{N}_q$, it follows that

$$\begin{aligned} \langle \Phi'_q(u), u \rangle &= \langle \Phi'_q(u), u \rangle - q_1 \Phi_q(u) \\ &= p \|u\|_{W^{s,p}(\mathbb{R}^N)}^p - q_1 \int_{\mathbb{R}^N} K(x) |u|^{q_1} dx - q_2 \int_{\mathbb{R}^N} K(x) |u|^{q_2} dx \\ &\quad - q_1 \|u\|_{W^{s,p}(\mathbb{R}^N)}^p + q_1 \int_{\mathbb{R}^N} K(x) |u|^{q_1} dx + q_1 \int_{\mathbb{R}^N} K(x) |u|^{q_2} dx \\ &= (p - q_1) \|u\|_{W^{s,p}(\mathbb{R}^N)}^p + (q_1 - q_2) \int_{\mathbb{R}^N} K(x) |u|^{q_2} dx < 0. \end{aligned}$$

Thus, $\Phi'_q(u) \neq 0$ for $u \in \mathcal{N}_q$.

Suppose that $u \in \mathcal{N}_q$ and $I_q(u) = c_q$. By the Lagrange multiplier theorem, there exists a $\lambda \in \mathbb{R}$ such that $I'_q(u) = \lambda \Phi'_q(u)$. Thus we have

$$\lambda \langle \Phi'_q(u), u \rangle = \langle I'_q(u), u \rangle = \Phi_q(u) = 0.$$

This implies that $\lambda = 0$ and $I'_q(u) = 0$. □

Lemma 2.4 (Refined Sobolev's inequality). Let $r \in (p, p_s^*)$. Then

$$\int_{\mathbb{R}^N} |u|^r dx \leq C \left(\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u|^r dx \right)^{\frac{r-2}{r}} \|u\|_{W^{s,p}(\mathbb{R}^N)}^p, \quad u \in W^{s,p}(\mathbb{R}^N). \quad (2.1)$$

Proof. Let $u \in D^{s,p}(\mathbb{R}^N)$ and $r \in (p, p_s^*)$. From Hölder's and Sobolev's inequalities it follows that

$$\begin{aligned} \int_{B_1(z)} |u|^r dx &\leq \left(\int_{B_1(z)} |u|^p dx \right)^{\frac{p_s^* - r}{p_s^* - p}} \left(\int_{B_1(z)} |u|^{p_s^*} dx \right)^{\frac{r-p}{p_s^* - p}} \\ &\leq C \left(\|u\|_{D^{s,p}(B_1(z))}^p + \int_{B_1(z)} |u|^p dx \right)^{\frac{p_s^* - r}{p_s^* - p}} \|u\|_{D^{s,p}(\mathbb{R}^N)}^{\frac{p_s^* (r-p)}{p(p_s^* - p)}} \\ &\leq C \left(\|u\|_{D^{s,p}(B_1(z))}^p + \int_{B_1(z)} |u|^p dx \right)^{\frac{r}{p}}. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{B_1(z)} |u|^r dx &= \left(\int_{B_1(z)} |u|^r dx \right)^{p/r} \left(\int_{B_1(z)} |u|^r dx \right)^{\frac{r-p}{r}} \\ &\leq C \left(\|u\|_{D^{s,p}(B_1(z))}^p + \int_{B_1(z)} |u|^p dx \right) \left(\int_{B_1(z)} |u|^r dx \right)^{\frac{r-p}{r}}. \end{aligned}$$

Covering $\mathbb{R}^N$ by balls of radius 1, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N + 1$ balls, we obtain the desired result (2.1). □

Lemma 2.5 (Lieb's compactness theorem). Let $r \in (p, p_s^*)$ and $\{u_n\}$ be a bounded sequence in $W^{s,p}(\mathbb{R}^N)$. If $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^r dx > 0$ and $u_n \rightharpoonup u \in W^{s,p}(\mathbb{R}^N)$, then there exists $\{z_n\} \subset \mathbb{R}^N$ such that $\bar{u}_n := u_n(x + z_n) \rightarrow \bar{u} \neq 0$ in $L^r_{\text{loc}}(\mathbb{R}^N)$. Moreover, if $\{z_n\}$ is bounded, then $u_n \rightarrow u \neq 0$ in $L^r_{\text{loc}}(\mathbb{R}^N)$.

Proof. Note that $\{u_n\}$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$. Up to a subsequence, we assume that there exists u satisfying

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } W^{s,p}(\mathbb{R}^N), \\ u_n &\rightarrow u \quad \text{a.e. in } \mathbb{R}^N, \end{aligned}$$

$$u_n \rightarrow u \quad \text{in } L^r_{\text{loc}}(\mathbb{R}^N).$$

By Lemma 2.4 there exists $C > 0$ such that

$$\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^r dx \geq C.$$

Since $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$ and $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $r \in (p, p_s^*)$, we obtain

$$\sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^r dx \leq \int_{\mathbb{R}^N} |u_n|^r dx \leq C.$$

So there exists $C_1 > 0$ such that

$$C_1 \leq \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^r dx \leq C_1^{-1}.$$

This indicates that we can find $z_n \in \mathbb{R}^N$ and $C_2 > 0$ to satisfy

$$\int_{B_1(z_n)} |u_n|^r dx \geq \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |u_n|^r dx - \frac{C}{2n} \geq C_2.$$

We set $\bar{u}_n := u_n(x + z_n)$. Then $\|\bar{u}_n\|_{W^{s,p}(\mathbb{R}^N)} = \|u_n\|_{W^{s,p}(\mathbb{R}^N)}$ and

$$\int_{B_1(0)} |\bar{u}_n|^r dx \geq C_2.$$

Up to a subsequence, there exists $\bar{u}$ such that

$$\bar{u}_n \rightharpoonup \bar{u} \quad \text{in } W^{s,p}(\mathbb{R}^N) \quad \text{and} \quad \bar{u}_n \rightarrow \bar{u} \quad \text{a.e. in } \mathbb{R}^N.$$

Since the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^N)$ is compact, we deduce that $\bar{u} \not\equiv 0$.

If $\{z_n\}$ is bounded, then there exists $C > 0$ such that $B_1(z_n) \subset B_C(0)$ and

$$\int_{B_C(0)} |u_n|^r dx \geq \int_{B_1(z_n)} |u_n|^r dx \geq C_2.$$

This leads to $u \not\equiv 0$. □

Lemma 2.6. *We have $c_q < c_q^*$, where*

$$c_q^* := \min \left\{ \left(\frac{1}{p} - \frac{1}{q_1} \right) \mathcal{S}_{q_1}^{\frac{q_1}{q_1-p}}, \left(\frac{1}{p} - \frac{1}{q_2} \right) \mathcal{S}_{q_2}^{\frac{q_2}{q_2-p}} \right\}.$$

Proof. Let v be the minimizer of $\mathcal{S}_{q_1}$. Then

$$\int_{\mathbb{R}^N} K(x) |v|^{q_1} dx > \int_{\mathbb{R}^N} |v|^{q_1} dx.$$

We set

$$g(t) = \frac{t^p}{p} \|v\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{t^{q_1}}{q_1} \int_{\mathbb{R}^N} K(x) |v|^{q_1} dx.$$

It is not difficult to show that

$$g'(t) \begin{cases} > 0, & \text{if } t \in (0, t_q), \\ = 0, & \text{if } t = t_q, \\ < 0, & \text{if } t \in (t_q, \infty), \end{cases}$$

where

$$t_q = \left[\frac{\|v\|_{W^{s,p}(\mathbb{R}^N)}^p}{\int_{\mathbb{R}^N} K(x) |v|^{q_1} dx} \right]^{\frac{1}{q_1-p}}.$$

From condition (H1) it follows that

$$\begin{aligned} \max_{t>0} I_q(tv) &\leq \frac{t_q^p}{p} \|v\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{t_q^{q_1}}{q_1} \int_{\mathbb{R}^N} K(x) |v|^{q_1} dx \\ &= \left(\frac{1}{p} - \frac{1}{q_1} \right) \left[\frac{\|v\|_{W^{s,p}(\mathbb{R}^N)}^p}{\left(\int_{\mathbb{R}^N} K(x) |v|^{q_1} dx \right)^{p/q_1}} \right]^{\frac{q_1}{q_1-p}} \end{aligned}$$

$$\begin{aligned}
&< \left(\frac{1}{p} - \frac{1}{q_1}\right) \left[\frac{\|v\|_{W^{s,p}(\mathbb{R}^N)}^p}{\left(\int_{\mathbb{R}^N} |v|^{q_1} dx\right)^{p/q_1}} \right]^{\frac{q_1}{q_1-p}} \\
&= \left(\frac{1}{p} - \frac{1}{q_1}\right) \mathcal{S}_{q_1}^{\frac{q_1}{q_1-p}}.
\end{aligned}$$

By letting w be the minimizer of $\mathcal{S}_{q_2}$, we can obtain

$$\max_{t>0} I_q(tw) < \left(\frac{1}{p} - \frac{1}{q_2}\right) \mathcal{S}_{q_2}^{\frac{q_2}{q_2-p}}.$$

□

Lemma 2.7. I_q satisfies the $(PS)_c$ condition under the level $c \in (0, c_q^*)$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence at level $c \in (0, c_q^*)$. Then

$$\begin{aligned}
c &= I_q(u_n) - \frac{1}{q_1} \langle I'_q(u_n), u_n \rangle \\
&= \left(\frac{1}{p} - \frac{1}{q_1}\right) \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p + \left(\frac{1}{q_1} - \frac{1}{q_2}\right) \int_{\mathbb{R}^N} K(x) |u_n|^{q_2} dx \\
&\geq \left(\frac{1}{p} - \frac{1}{q_1}\right) \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

This indicates that $\{u_n\}$ is bounded. We now split our proof into five steps.

Step 1. If $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_2} dx = 0$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_1} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_1} dx.$$

Clearly, we have

$$\int_{\mathbb{R}^N} [K(x) - 1] |u_n|^{q_1} dx = \int_{|x| \leq r} [K(x) - 1] |u_n|^{q_1} dx + \int_{|x| > r} [K(x) - 1] |u_n|^{q_1} dx.$$

Given that $W^{s,p}(\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^{q_2}(\mathbb{R}^N)$ is compact, by Holder's inequality we have

$$\begin{aligned}
\int_{|x| \leq r} [K(x) - 1] |u_n|^{q_1} dx &\leq \left(\int_{|x| \leq r} [K(x) - 1] |u_n|^p dx \right)^{\frac{q_2 - q_1}{q_2 - p}} \left(\int_{|x| \leq r} [K(x) - 1] |u_n|^{q_2} dx \right)^{\frac{q_1 - p}{q_2 - p}} \\
&\leq C \left(\int_{|x| \leq r} [K(x) - 1] |u_n|^{q_2} dx \right)^{\frac{q_1 - p}{q_2 - p}} = o(1).
\end{aligned}$$

Using condition (H1) and letting $r > 0$ be large, we obtain

$$\lim_{n \rightarrow \infty} \int_{|x| > r} [K(x) - 1] |u_n|^{q_1} dx = o(1).$$

That is,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_1} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_1} dx.$$

Similarly, if $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_1} dx = 0$, we can obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_2} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_2} dx.$$

Step 2. To prove $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_1} dx > 0$, by way of contradiction, we assume that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_1} dx = 0.$$

Taking the limit as $n \rightarrow \infty$ yields

$$c = \frac{1}{p} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{1}{q_2} \int_{\mathbb{R}^N} K(x) |u_n|^{q_2} dx,$$

$$o(1) = \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p - \int_{\mathbb{R}^N} K(x)|u_n|^{q_2} dx,$$

which lead to

$$c = \left(\frac{1}{p} - \frac{1}{q_2}\right) \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p, \quad \text{as } n \rightarrow \infty.$$

From Step 1, we obtain

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} K(x)|u_n|^{q_2} dx = \int_{\mathbb{R}^N} |u_n|^{q_2} dx \leq \mathcal{S}_{q_2}^{-\frac{q_2}{p}} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^{q_2},$$

which implies that

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p \geq \mathcal{S}_{q_2}^{\frac{q_2}{q_2-p}}.$$

Then

$$c \geq \left(\frac{1}{p} - \frac{1}{q_2}\right) \mathcal{S}_{q_2}^{\frac{q_2}{q_2-p}}.$$

This contradicts $c < c_q^*$.

Similarly, to show $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x)|u_n|^{q_2} dx > 0$, on the contrary, we assume that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x)|u_n|^{q_2} dx = 0.$$

By passing the limits as $n \rightarrow \infty$, it holds

$$\begin{aligned} c &= \frac{1}{p} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{1}{q_1} \int_{\mathbb{R}^N} K(x)|u_n|^{q_1} dx, \\ o(1) &= \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p - \int_{\mathbb{R}^N} K(x)|u_n|^{q_1} dx, \end{aligned}$$

which give

$$c = \left(\frac{1}{p} - \frac{1}{q_1}\right) \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p, \quad \text{as } n \rightarrow \infty.$$

Then we deduce that

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} K(x)|u_n|^{q_1} dx = \int_{\mathbb{R}^N} |u_n|^{q_1} dx \leq \mathcal{S}_{q_1}^{-\frac{q_1}{p}} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^{q_1}.$$

That is,

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p \geq \mathcal{S}_{q_1}^{\frac{q_1}{q_1-p}}.$$

Thus it holds

$$c \geq \left(\frac{1}{p} - \frac{1}{q_1}\right) \mathcal{S}_{q_1}^{\frac{q_1}{q_1-p}}.$$

This contradicts $c < c_q^*$.

Step 3. From $K(x) \in L^\infty(\mathbb{R}^N)$ and Step 2, it follows that

$$\begin{aligned} 0 &< \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x)|u_n|^{q_1} dx \leq \|K\|_{L^\infty(\mathbb{R}^N)} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_1} dx, \\ 0 &< \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x)|u_n|^{q_2} dx \leq \|K\|_{L^\infty(\mathbb{R}^N)} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_2} dx, \end{aligned}$$

which yield

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_1} dx > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{q_2} dx > 0.$$

From Lemma 2.5, we can know that there exists $\{z_n\} \subset \mathbb{R}^N$ such that

$$\begin{aligned} \bar{u}_n &:= u_n(x + z_n) \rightarrow \bar{u} \neq 0 \quad \text{in } L_{\text{loc}}^{q_1}(\mathbb{R}^N), \\ \int_{B_1(0)} |\bar{u}_n|^{q_1} dx &= \int_{B_1(z_n)} |u_n|^{q_1} dx \geq C. \end{aligned}$$

We claim that

$$\bar{I}_q(\bar{u}_n) = I_q(u_n) \rightarrow c \quad \text{and} \quad \bar{I}'_q(\bar{u}_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where

$$\bar{I}_q(\bar{u}_n) = \frac{1}{p} \|\bar{u}_n\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{1}{q_1} \int_{\mathbb{R}^N} K(x+z_n) |\bar{u}_n|^{q_1} dx - \frac{1}{q_2} \int_{\mathbb{R}^N} K(x+z_n) |\bar{u}_n|^{q_2} dx. \quad (2.2)$$

Then, it follows that

$$|\langle \bar{I}'_q(\bar{u}_n), \varphi \rangle| = |\langle I'_q(u_n), \varphi_n \rangle| \leq \|I'_q(u_n)\|_{(W^{s,p}(\mathbb{R}^N))^{-1}} \|\varphi_n\|_{W^{s,p}(\mathbb{R}^N)} = o_n(1), \quad \varphi \in W^{s,p}(\mathbb{R}^N),$$

where $\varphi_n = \varphi(x - z_n)$ and $\|\varphi_n\|_{W^{s,p}(\mathbb{R}^N)} = \|\varphi\|_{W^{s,p}(\mathbb{R}^N)}$. Thus we have $\bar{I}'_q(\bar{u}_n) \rightarrow 0$, which means that $\{\bar{u}_n\}$ is a $(PS)_c$ sequence of $\bar{I}_q$. It is easy to see that $\{\bar{u}_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$. We now assume that there exists $\bar{u}$ such that $\bar{u}_n \rightharpoonup \bar{u}$ in $W^{s,p}(\mathbb{R}^N)$ and $\bar{u}_n \rightarrow \bar{u}$ a.e. in $\mathbb{R}^N$.

Step 4. In this step, we show $\{z_n\}$ is bounded in $\mathbb{R}^N$. If $\lim_{n \rightarrow \infty} |z_n| = \infty$, then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [K(x+z_n) - 1] |\bar{u}_n|^{q_1-1} |\varphi| dx \\ & \leq \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} [K(x+z_n) - 1] |\bar{u}_n|^{q_1} dx \right)^{\frac{q_1-1}{q_1}} \left(\int_{\mathbb{R}^N} [K(x+z_n) - 1] |\varphi|^{q_1} dx \right)^{1/q_1} \\ & \leq C \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} [K(x+z_n) - 1] |\varphi|^{q_1} dx \right)^{1/q_1} = 0. \end{aligned}$$

Thus we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x+z_n) |\bar{u}_n|^{q_1-2} \bar{u}_n \varphi dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^{q_1-2} \bar{u}_n \varphi dx, \quad (2.3)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x+z_n) |\bar{u}_n|^{q_2-2} \bar{u}_n \varphi dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^{q_2-2} \bar{u}_n \varphi dx. \quad (2.4)$$

It follows from $\bar{u}_n \rightharpoonup \bar{u}$ in $W^{s,p}(\mathbb{R}^N)$ and (2.3)-(2.4) that

$$\langle \bar{I}'_q(\bar{u}_n), \varphi \rangle = \langle I'_{q,\infty}(\bar{u}), \varphi \rangle = 0,$$

which implies that $\bar{u}$ is a weak solution of the equation

$$(-\Delta)_p^s u + |u|^{p-2} u = |u|^{q_1-2} u + |u|^{q_2-2} u, \quad x \in \mathbb{R}^N, \quad (2.5)$$

and the associated energy functional with equation (2.5) is

$$I_{q,\infty}(u) = \frac{1}{p} \|u\|_{W^{s,p}(\mathbb{R}^N)}^p - \frac{1}{q_1} \int_{\mathbb{R}^N} |u|^{q_1} dx - \frac{1}{q_2} \int_{\mathbb{R}^N} |u|^{q_2} dx.$$

Let $c_{q,\infty}$ be the mountain pass level of $I_{q,\infty}$. From condition (H1), it is not difficult to see that $c < c_{q,\infty}$. It follows from (2.2), Brézis-Lieb's lemma [8] and $\langle I'_{q,\infty}(\bar{u}), \varphi \rangle = 0$ that

$$\begin{aligned} c_{q,\infty} &> c = \bar{I}_q(\bar{u}_n) - \frac{1}{p} \langle \bar{I}'_q(\bar{u}_n), \bar{u}_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{q_1} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x+z_n) |\bar{u}_n|^{q_1} dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{q_2} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x+z_n) |\bar{u}_n|^{q_2} dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q_1} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^{q_1} dx + \left(\frac{1}{p} - \frac{1}{q_2} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\bar{u}_n|^{q_2} dx \\ &\geq \left(\frac{1}{p} - \frac{1}{q_1} \right) \int_{\mathbb{R}^N} |\bar{u}|^{q_1} dx + \left(\frac{1}{p} - \frac{1}{q_2} \right) \int_{\mathbb{R}^N} |\bar{u}|^{q_2} dx \\ &= I_{q,\infty}(\bar{u}) - \frac{1}{p} \langle I'_{q,\infty}(\bar{u}), \bar{u} \rangle \\ &= I_{q,\infty}(\bar{u}) \geq c_{q,\infty}, \end{aligned}$$

which yields a contradiction. Hence, $\{z_n\}$ is bounded in $\mathbb{R}^N$. By using Lemma 2.5 again, up to a subsequence, we have $u_n(x) \rightarrow u \not\equiv 0$ in $L_{\text{loc}}^{q_1}(\mathbb{R}^N)$.

Step 5. To prove $u_n \rightarrow u$ in $W^{s,p}(\mathbb{R}^N)$, from $u_n \rightharpoonup u \neq 0$ in $W^{s,p}(\mathbb{R}^N)$, we have

$$\lim_{n \rightarrow \infty} \langle I'_q(u_n), \varphi \rangle = \langle I'_q(u), \varphi \rangle.$$

Then $u \in \mathcal{N}_q$. According to Brézis-Lieb's lemma [8], we deduce that

$$\begin{aligned} c &\leq I_q(u) = I_q(u) - \frac{1}{q_1} \langle I'_q(u), u \rangle \\ &= \left(\frac{1}{p} - \frac{1}{q_1} \right) \|u\|_{W^{s,p}(\mathbb{R}^N)}^p + \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \int_{\mathbb{R}^N} K(x) |u|^{q_2} dx \\ &\leq \left(\frac{1}{p} - \frac{1}{q_1} \right) \lim_{n \rightarrow \infty} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p + \left(\frac{1}{q_1} - \frac{1}{q_2} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) |u_n|^{q_2} dx \\ &= \lim_{n \rightarrow \infty} I_q(u_n) - \frac{1}{q_1} \lim_{n \rightarrow \infty} \langle I'_q(u_n), u_n \rangle = c. \end{aligned}$$

Thus, we obtain $I_q(u) = c$ and $\lim_{n \rightarrow \infty} \|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p = \|u\|_{W^{s,p}(\mathbb{R}^N)}^p$. $\square$

Proof of Theorem 1.3. From Lemmas 2.3 and 2.7, Theorem 1.3 follows immediately. $\square$

3. PROOF OF THEOREM 1.5

To construct the geometrical structure of I_α , we set

$$\mathcal{N}_\alpha := \{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\} : \langle I'_\alpha(u), u \rangle = 0\}.$$

Similar to Lemmas 2.2 and 2.3 respectively, we can obtain the following.

Lemma 3.1. *Assume that all conditions described in Theorem 1.5 hold. Then*

- (1) *the functional I_α possesses the mountain pass geometry;*
- (2) *for any $u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\alpha$ and $I_\alpha(t_u u) = \max_{t>0} I_\alpha(tu)$; and*
- (3) $c_\alpha = \tilde{c}_\alpha = \tilde{\tilde{c}}_\alpha > 0$,

where

$$\tilde{c}_\alpha = \inf_{u \in \mathcal{N}_\alpha} I_\alpha(u), \quad \tilde{\tilde{c}}_\alpha = \inf_{u \in D^{s,p}(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} I_\alpha(tu).$$

Lemma 3.2. *For $u \in \mathcal{N}_\alpha$, we have $\Phi'_\alpha(u) \neq 0$, where $\Phi_\alpha(u) := \langle I'_\alpha(u), u \rangle$. Moreover, if $I_\alpha(u) = c_\alpha$, then u is a ground state solution of equation (1.4).*

We recall the definition of the homogeneous Morrey space $\mathcal{M}^{r,\varpi}(\mathbb{R}^N)$. A measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ belongs to the Morrey space $\mathcal{M}^{r,\varpi}(\mathbb{R}^N)$ with the norm $\|u\|_{\mathcal{M}^{r,\varpi}(\mathbb{R}^N)}$, where $r \in [1, \infty)$ and $\varpi \in (0, N]$ if and only if

$$\|u\|_{\mathcal{M}^{r,\varpi}(\mathbb{R}^N)} := \sup_{R>0, x \in \mathbb{R}^N} R^{\frac{N}{\varpi} - \frac{N}{r}} \|u\|_{L^r(B_R(x))} < \infty.$$

Lemma 3.3 ([25, Proposition 4]). *For $N \geq 3$, there exists $C > 0$ such that for $\theta \in [\frac{p}{p_s^*}, 1)$ it holds*

$$\|u\|_{L^{p_s^*}(\mathbb{R}^N)} \leq C \|u\|_{D^{s,p}(\mathbb{R}^N)}^\theta \|u\|_{\mathcal{M}^{p,p_s^*}(\mathbb{R}^N)}^{1-\theta}, \quad u \in D^{s,p}(\mathbb{R}^N).$$

As we see, when $r = p_s^*$ in Lemma 2.5 it becomes invalid. To fill this gap, we establish the following generalized version of Lieb's compactness theorem.

Lemma 3.4 (Lieb's compactness theorem). *Let $\{u_n\}$ be a bounded sequence in $D^{s,p}(\mathbb{R}^N)$. If*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx > 0 \quad \text{and} \quad u_n \rightharpoonup u \in D^{s,p}(\mathbb{R}^N),$$

then

- (1) *There exist $\sigma_n > 0$ and $\{z_n\} \subset \mathbb{R}^N$ such that*

$$\tilde{u}_n := \sigma_n^{\frac{N-p_s}{p}} u_n(\sigma_n x + z_n) \rightarrow \tilde{u} \neq 0 \quad \text{in } L_{\text{loc}}^p(\mathbb{R}^N).$$

(2) If $\{\frac{z_n}{\sigma_n}\}$ is bounded in $\mathbb{R}^N$, then

$$\hat{u}_n := \sigma_n^{\frac{N-ps}{p}} u_n(\sigma_n x) \rightarrow \hat{u} \neq 0 \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N).$$

(3) Up to a subsequence, if $\{z_n\}$ is bounded in $\mathbb{R}^N$ and there exists $\tilde{\sigma} > 0$ such that $\sigma_n \rightarrow \tilde{\sigma}$, as $n \rightarrow \infty$. Then

$$u_n \rightarrow u \neq 0 \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N).$$

Proof. (1) Note that $\{u_n\}$ is bounded in $D^{s,p}(\mathbb{R}^N)$. Up to a subsequence, we assume that there exists u such that

$$u_n \rightharpoonup u \text{ in } D^{s,p}(\mathbb{R}^N), \quad u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N \quad \text{and} \quad u_n \rightarrow u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N).$$

According to Lemma 3.3, we can find $C > 0$ such that

$$\|u_n\|_{\mathcal{M}^{p,p^*}(\mathbb{R}^N)} \geq C, \quad n \in \mathbb{N}^*.$$

By the definition of Morrey norm, we can find $\sigma_n > 0$, $z_n \in \mathbb{R}^N$ and $C_3 > 0$ such that

$$\sigma_n^{-ps} \int_{B_{\sigma_n}(z_n)} |u_n|^p dy \geq \|u_n\|_{\mathcal{M}^{p,p^*}(\mathbb{R}^N)}^p - \frac{C}{2n} \geq C_3.$$

Let $\tilde{u}_n = \sigma_n^{\frac{N-ps}{p}} u_n(\sigma_n x + z_n)$. Then

$$\int_{B_1(0)} |\tilde{u}_n|^p dy = \sigma_n^{-ps} \int_{B_{\sigma_n}(z_n)} |u_n|^p dy \geq C_3. \quad (3.1)$$

Given that the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N)$ is compact, we obtain $\tilde{u}_n \rightarrow \tilde{u} \neq 0$.

(2) Since $\{\frac{z_n}{\sigma_n}\}$ is bounded in $\mathbb{R}^N$, by (3.1) it is not difficult to see that

$$\int_{B_C(0)} |\hat{u}_n|^p dx \geq \int_{B_1(\frac{z_n}{\sigma_n})} |\hat{u}_n|^p dx = \sigma_n^{-ps} \int_{B_{\sigma_n}(z_n)} |u_n|^p dx \geq C_3.$$

Thus, we have $\hat{u}_n \rightarrow \hat{u} \neq 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$.

(3) Given that $\{z_n\}$ is bounded in $\mathbb{R}^N$, there exists $\tilde{C} > 0$ such that $|z_n| \leq \tilde{C}$. Using $\lim_{n \rightarrow \infty} \sigma_n = \tilde{\sigma} \neq 0$, up to a subsequence, we have

$$B_{\sigma_n}(z_n) \subset B_{|z_n|+\sigma_n}(0) \subset B_{\tilde{C}+2\tilde{\sigma}}(0).$$

It follows from (3.1) that

$$0 < C\tilde{\sigma}^{ps} = C \lim_{n \rightarrow \infty} \sigma_n^{ps} \leq \lim_{n \rightarrow \infty} \int_{B_{\tilde{C}+2\tilde{\sigma}}(0)} |u_n|^p dy.$$

Since the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N)$ is compact, we obtain $u \neq 0$. $\square$

Lemma 3.5. We have $c_\alpha < c_\alpha^*$, where

$$c_\alpha^* := \left\{ \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \mathcal{H}_{\alpha_1}^{\frac{p_s^*(\alpha_1)}{p_s^*(\alpha_1)-p}}, \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \mathcal{H}_{\alpha_2}^{\frac{p_s^*(\alpha_2)}{p_s^*(\alpha_2)-p}} \right\}.$$

Proof. Let $\tilde{v}$ be the minimizer of $\mathcal{H}_{\alpha_1}$ in (1.3). By condition (H1) we have

$$\int_{\mathbb{R}^N} K(x) \frac{|\tilde{v}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx > \int_{\mathbb{R}^N} \frac{|\tilde{v}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx.$$

Let

$$h(t) = \frac{t^p}{p} \|\tilde{v}\|_{D^{s,p}(\mathbb{R}^N)}^p - \frac{t^{p_s^*(\alpha_1)}}{p_s^*(\alpha_1)} \int_{\mathbb{R}^N} K(x) \frac{|\tilde{v}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx.$$

A direct computation gives

$$h'(t) \begin{cases} > 0, & t \in (0, t_\alpha), \\ = 0, & t = t_\alpha, \\ < 0, & t \in (t_\alpha, \infty), \end{cases}$$

where

$$t_\alpha = \left[\frac{\|\tilde{v}\|_{D^{s,p}(\mathbb{R}^N)}^p}{\int_{\mathbb{R}^N} K(x) \frac{|\tilde{v}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx} \right]^{\frac{1}{p_s^*(\alpha_1)-p}}.$$

Then

$$\begin{aligned} \max_{t>0} I_\alpha(t\tilde{v}) &\leq \max_{t>0} h(t) \\ &= \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \left[\frac{\|\tilde{v}\|_{D^{s,p}(\mathbb{R}^N)}^p}{\int_{\mathbb{R}^N} K(x) \frac{|\tilde{v}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx} \right]^{\frac{p_s^*(\alpha_1)}{p_s^*(\alpha_1)-p}} \\ &< \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \left[\frac{\|\tilde{v}\|_{D^{s,p}(\mathbb{R}^N)}^p}{\int_{\mathbb{R}^N} \frac{|\tilde{v}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx} \right]^{\frac{p_s^*(\alpha_1)}{p_s^*(\alpha_1)-p}} \\ &= \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \mathcal{H}_{\alpha_1}^{\frac{p_s^*(\alpha_1)}{p_s^*(\alpha_1)-p}}. \end{aligned}$$

Let $\tilde{w}$ be the minimizer of $\mathcal{H}_{\alpha_2}$. Similarly, we obtain

$$0 < \max_{t>0} I_\alpha(t\tilde{w}) < \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \mathcal{H}_{\alpha_2}^{\frac{p_s^*(\alpha_2)}{p_s^*(\alpha_2)-p}}.$$

The proof is complete. $\square$

Lemma 3.6. I_α satisfies the $(PS)_c$ condition under the level $c \in (0, c_\alpha^*)$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence at level $c \in (0, c_\alpha^*)$. Since $\alpha_1 < \alpha_2$, it follows that $p_s^*(\alpha_1) > p_s^*(\alpha_2)$ and

$$\begin{aligned} c &= I_\alpha(u_n) - \frac{1}{p_s^*(\alpha_2)} \langle I'_\alpha(u_n), u_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p + \left(\frac{1}{p_s^*(\alpha_2)} - \frac{1}{p_s^*(\alpha_1)} \right) \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \\ &\geq \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which implies that $\{u_n\}$ is bounded. We now divide the proof into six steps.

Step 1. If $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx = 0$, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx.$$

Choosing $\alpha_3 \in (\alpha_2, p)$ we have

$$\begin{aligned} \left(\int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_3)}}{|x|^{\alpha_3}} dx \right)^{\frac{p}{p_s^*(\alpha_3)}} &\leq \|K\|_{L^\infty(\mathbb{R}^N)}^{\frac{p}{p_s^*(\alpha_3)}} \left(\int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_3)}}{|x|^{\alpha_3}} dx \right)^{\frac{p}{p_s^*(\alpha_3)}} \\ &\leq \|K\|_{L^\infty(\mathbb{R}^N)}^{\frac{p}{p_s^*(\alpha_3)}} \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p \leq C. \end{aligned}$$

In view of condition (H1) and $0 < \alpha_1 < \alpha_2 < \alpha_3 < p$, then $p_s^*(\alpha_3) < p_s^*(\alpha_2) < p_s^*(\alpha_1)$ and we obtain

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} [K(x) - 1] \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx \\ &\leq \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx \\ &\leq \left(\int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_3)}}{|x|^{\alpha_3}} dx \right)^{\frac{p_s^*(\alpha_1)-p_s^*(\alpha_2)}{p_s^*(\alpha_1)-p_s^*(\alpha_3)}} \left(\int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \right)^{\frac{p_s^*(\alpha_2)-p_s^*(\alpha_3)}{p_s^*(\alpha_1)-p_s^*(\alpha_3)}} \end{aligned}$$

$$\leq C \left(\int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \right)^{\frac{p_s^*(\alpha_2) - p_s^*(\alpha_3)}{p_s^*(\alpha_1) - p_s^*(\alpha_3)}} = o(1), \quad \text{as } n \rightarrow \infty.$$

Similarly, if $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx = 0$, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx.$$

Step 2. We prove $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx > 0$, by way of contradiction, assume that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx = 0.$$

Taking the limits as $n \rightarrow \infty$ gives

$$\begin{aligned} c &= \frac{1}{p} \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p - \frac{1}{p_s^*(\alpha_2)} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx, \\ o(1) &= \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p - \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx, \end{aligned}$$

which lead to

$$c = \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p, \quad \text{as } n \rightarrow \infty.$$

From Step 1, we have

$$\|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx = \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx \leq \mathcal{H}_{\alpha_2}^{-\frac{p_s^*(\alpha_2)}{p}} \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^{p_s^*(\alpha_2)},$$

which implies

$$\|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p \geq \mathcal{H}_{\alpha_2}^{\frac{p_s^*(\alpha_2)}{p_s^*(\alpha_2) - p}}.$$

Then we have

$$c \geq \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \mathcal{H}_{\alpha_2}^{\frac{p_s^*(\alpha_2)}{p_s^*(\alpha_2) - p}},$$

which yields a contradiction with $c < c_\alpha^*$. Thus,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx > 0.$$

Similarly, we deduce that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx > 0$.

Step 3. From condition (H1) and Step 2, we obtain

$$\begin{aligned} 0 &< \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \leq \|K\|_{L^\infty(\mathbb{R}^N)} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx, \\ 0 &< \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K(x) \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx \leq \|K\|_{L^\infty(\mathbb{R}^N)} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx, \end{aligned}$$

which lead to

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx > 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx > 0.$$

It follows from Hölder's and Hardy's inequalities that

$$\int_{\mathbb{R}^N} \frac{|u_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \leq \left(\int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^{ps}} dx \right)^{\frac{\alpha_1}{ps}} \left(\int_{\mathbb{R}^N} |u_n|^{p_s^*} dx \right)^{\frac{ps - \alpha_1}{ps}} \leq C \left(\int_{\mathbb{R}^N} |u_n|^{p_s^*} dx \right)^{\frac{ps - \alpha_1}{ps}}.$$

Then we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx > 0.$$

From Lemma 3.4, there exist $\{z_n\} \subset \mathbb{R}^N$ and $\sigma_n \in \mathbb{R}^+$ such that

$$\begin{aligned}\tilde{u}_n &:= \sigma_n^{\frac{N-ps}{p}} u_n(\sigma_n x + z_n) \rightarrow \tilde{u} \neq 0 \quad \text{in } L_{\text{loc}}^p(\mathbb{R}^N), \\ \|\tilde{u}_n\|_{D^{s,p}(\mathbb{R}^N)} &= \|u_n\|_{D^{s,p}(\mathbb{R}^N)} \\ \int_{\mathbb{R}^N} |\tilde{u}_n|^{p_s^*} dx &= \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx.\end{aligned}$$

We claim that

$$\lim_{n \rightarrow \infty} \tilde{I}_\alpha(\tilde{u}_n) = \lim_{n \rightarrow \infty} I_\alpha(u_n) = c \quad \text{and} \quad \tilde{I}'_\alpha(\tilde{u}_n) \rightarrow 0,$$

where

$$\begin{aligned}\tilde{I}_\alpha(\tilde{u}_n) &= \frac{1}{p} \|\tilde{u}_n\|_{D^{s,p}(\mathbb{R}^N)}^p - \frac{1}{p_s^*(\alpha_1)} \int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\tilde{u}_n|^{p_s^*(\alpha_1)}}{|x + \frac{z_n}{\sigma_n}|^{\alpha_1}} dx \\ &\quad - \frac{1}{p_s^*(\alpha_2)} \int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\tilde{u}_n|^{p_s^*(\alpha_2)}}{|x + \frac{z_n}{\sigma_n}|^{\alpha_2}} dx.\end{aligned}$$

This is so because

$$|\langle \tilde{I}'_\alpha(\tilde{u}_n), \varphi \rangle| = |\langle I'_\alpha(u_n), \varphi_n \rangle| \leq \|I'_\alpha(u_n)\|_{(D^{s,p}(\mathbb{R}^N))^{-1}} \|\varphi_n\|_{D^{s,p}(\mathbb{R}^N)} = o_n(1), \quad \varphi \in D^{s,p}(\mathbb{R}^N),$$

where $\varphi_n = \sigma_n^{-\frac{N-ps}{p}} \varphi(\frac{x-z_n}{\sigma_n})$ and $\|\varphi_n\|_{D^{s,p}(\mathbb{R}^N)} = \|\varphi\|_{D^{s,p}(\mathbb{R}^N)}$. This leads to $\tilde{I}'_\alpha(\tilde{u}_n) \rightarrow 0$. Thus, $\{\tilde{u}_n\}$ is a $(PS)_c$ sequence of $\tilde{I}_\alpha$.

Step 4. To show that $\{\frac{z_n}{\sigma_n}\}$ is bounded in $\mathbb{R}^N$, we suppose, on the contrary, that $|\frac{z_n}{\sigma_n}| \rightarrow \infty$ as $n \rightarrow \infty$. Note that

$$\begin{aligned}& \int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\tilde{u}_n|^{p_s^*(\alpha_1)-1} |\varphi|}{|x + \frac{z_n}{\sigma_n}|^{\alpha_1}} dx \\ & \leq \left[\int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\tilde{u}_n|^{p_s^*(\alpha_1)}}{|x + \frac{z_n}{\sigma_n}|^{\alpha_1}} dx \right]^{\frac{p_s^*(\alpha_1)-1}{p_s^*(\alpha_1)}} \left[\int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\varphi|^{p_s^*(\alpha_1)}}{|x + \frac{z_n}{\sigma_n}|^{\alpha_1}} dx \right]^{\frac{1}{p_s^*(\alpha_1)}} \\ & \leq C \left[\int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\varphi|^{p_s^*(\alpha_1)}}{|x + \frac{z_n}{\sigma_n}|^{\alpha_1}} dx \right]^{\frac{1}{p_s^*(\alpha_1)}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\tilde{u}_n|^{p_s^*(\alpha_1)-2} \tilde{u}_n \varphi}{|x + \frac{z_n}{\sigma_n}|^{\alpha_1}} dx = o(1), \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

Similarly, we can deduce that

$$\int_{\mathbb{R}^N} K\left(x + \frac{z_n}{\sigma_n}\right) \frac{|\tilde{u}_n|^{p_s^*(\alpha_2)-2} \tilde{u}_n \varphi}{|x + \frac{z_n}{\sigma_n}|^{\alpha_2}} dx = o(1), \quad \text{as } n \rightarrow \infty. \quad (3.3)$$

From the fact that $\tilde{u}_n \rightharpoonup \tilde{u}$ in $D^{s,p}(\mathbb{R}^N)$ and (3.2)-(3.3), it follows that $\tilde{u}$ is a weak solution of the equation

$$(-\Delta)_p^s u = 0, \quad x \in \mathbb{R}^N.$$

Since $(-\Delta)_p^s u = 0$ only has trivial solution in $D^{s,p}(\mathbb{R}^N)$. This yields a contradiction with Lemma 3.4. Therefore, we obtain that $\{\frac{z_n}{\sigma_n}\}$ is bounded in $\mathbb{R}^N$.

Step 5. Using Lemma 3.4 and the fact that $\{\frac{z_n}{\sigma_n}\}$ is bounded in $\mathbb{R}^N$, we obtain

$$\begin{aligned}\hat{u}_n &:= \sigma_n^{\frac{N-ps}{p}} u_n(\sigma_n x) \rightarrow \hat{u} \neq 0 \quad \text{in } L_{\text{loc}}^p(\mathbb{R}^N), \\ \|\hat{u}_n\|_{D^{s,p}(\mathbb{R}^N)} &= \|u_n\|_{D^{s,p}(\mathbb{R}^N)}, \\ \int_{\mathbb{R}^N} |\hat{u}_n|^{p_s^*} dx &= \int_{\mathbb{R}^N} |u_n|^{p_s^*} dx.\end{aligned}$$

We claim that

$$\lim_{n \rightarrow \infty} \hat{I}_\alpha(\hat{u}_n) = \lim_{n \rightarrow \infty} I_\alpha(u_n) = c \quad \text{and} \quad \hat{I}'_\alpha(\hat{u}_n) \rightarrow 0,$$

where

$$\begin{aligned}\hat{I}_\alpha(\hat{u}_n) &= \frac{1}{p} \|\hat{u}_n\|_{D^{s,p}(\mathbb{R}^N)}^p - \frac{1}{p_s^*(\alpha_1)} \int_{\mathbb{R}^N} K\left(\frac{x}{\sigma_n}\right) \frac{|\hat{u}_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \\ &\quad - \frac{1}{p_s^*(\alpha_2)} \int_{\mathbb{R}^N} K\left(\frac{x}{\sigma_n}\right) \frac{|\hat{u}_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx.\end{aligned}$$

This is so because

$$|\langle \hat{I}'_\alpha(\hat{u}_n), \varphi \rangle| = |\langle I'_\alpha(u_n), \varphi_n \rangle| \leq \|I'_\alpha(u_n)\|_{(D^{s,p}(\mathbb{R}^N))^{-1}} \|\varphi_n\|_{D^{s,p}(\mathbb{R}^N)} = o_n(1), \quad \varphi \in D^{s,p}(\mathbb{R}^N),$$

where $\varphi_n = \sigma_n^{-\frac{N-ps}{p}} \varphi(\frac{x}{\sigma_n})$ and $\|\varphi_n\|_{D^{s,p}(\mathbb{R}^N)} = \|\varphi\|_{D^{s,p}(\mathbb{R}^N)}$. This leads to $\hat{I}'_\alpha(\hat{u}_n) \rightarrow 0$, as $n \rightarrow \infty$. Consequently, $\{\hat{u}_n\}$ is a $(PS)_c$ sequence of $\hat{I}_\alpha$.

Next, we verify that $\sigma_n \not\rightarrow \infty$. We suppose on the contrary that $\sigma_n \rightarrow \infty$, as $n \rightarrow \infty$. By condition (H2), we have

$$\int_{\mathbb{R}^N} K\left(\frac{x}{\sigma_n}\right) \frac{|\hat{u}_n|^{p_s^*(\alpha_1)-2} \hat{u}_n \varphi}{|x|^{\alpha_1}} dx = \int_{\mathbb{R}^N} \frac{|\hat{u}_n|^{p_s^*(\alpha_1)-2} \hat{u}_n \varphi}{|x|^{\alpha_1}} dx, \quad \text{as } n \rightarrow \infty. \quad (3.4)$$

By Hölder's inequality, we obtain

$$\begin{aligned}& \int_{\mathbb{R}^N} \left[K\left(\frac{x}{\sigma_n}\right) - 1 \right] \frac{|\hat{u}_n|^{p_s^*(\alpha_1)-1} |\varphi|}{|x|^{\alpha_1}} dx \\ & \leq \left\{ \int_{\mathbb{R}^N} \left[K\left(\frac{x}{\sigma_n}\right) - 1 \right] \frac{|\hat{u}_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \right\}^{\frac{p_s^*(\alpha_1)-1}{p_s^*(\alpha_1)}} \left\{ \int_{\mathbb{R}^N} \left[K\left(\frac{x}{\sigma_n}\right) - 1 \right] \frac{|\varphi|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \right\}^{\frac{1}{p_s^*(\alpha_1)}} \\ & \leq C \left\{ \int_{\mathbb{R}^N} \left[K\left(\frac{x}{\sigma_n}\right) - 1 \right] \frac{|\varphi|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \right\}^{\frac{1}{p_s^*(\alpha_1)}} \rightarrow 0, \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Similarly, we can deduce that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K\left(\frac{x}{\sigma_n}\right) \frac{|\hat{u}_n|^{p_s^*(\alpha_2)-2} \hat{u}_n \varphi}{|x|^{\alpha_2}} dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|\hat{u}_n|^{p_s^*(\alpha_2)-2} \hat{u}_n \varphi}{|x|^{\alpha_2}} dx. \quad (3.5)$$

It follows from $\hat{u}_n \rightharpoonup \hat{u}$ in $D^{s,p}(\mathbb{R}^N)$ and (3.4)-(3.5) that

$$\lim_{n \rightarrow \infty} \langle \hat{I}'_\alpha(\hat{u}_n), \varphi \rangle = \langle I'_{\alpha,\infty}(\hat{u}), \varphi \rangle = 0, \quad \varphi \in D^{s,p}(\mathbb{R}^N),$$

which implies that $\hat{u}$ is a weak solution of

$$(-\Delta)_p^s u = \frac{|u|^{p_s^*(\alpha_1)-2} u}{|x|^{\alpha_1}} + \frac{|u|^{p_s^*(\alpha_2)-2} u}{|x|^{\alpha_2}}, \quad x \in \mathbb{R}^N, \quad (3.6)$$

and the corresponding energy functional to equation (3.6) is

$$I_{\alpha,\infty}(u) = \frac{1}{p} \|u\|_{D^{s,p}(\mathbb{R}^N)}^p - \frac{1}{p_s^*(\alpha_1)} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx - \frac{1}{p_s^*(\alpha_2)} \int_{\mathbb{R}^N} \frac{|u|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx.$$

Let $c_{\alpha,\infty}$ be the mountain pass level of $I_{\alpha,\infty}$. From condition (H1), it is easy to see that $c < c_{\alpha,\infty}$. Putting this together with condition (H1), (3.4)-(3.5) and Brézis-Lieb's lemma [8], we obtain

$$\begin{aligned}c_{\alpha,\infty} &> c = \hat{I}_\alpha(\hat{u}_n) - \frac{1}{p} \langle \hat{I}'_\alpha(\hat{u}_n), \hat{u}_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K\left(\frac{x}{\sigma_n}\right) \frac{|\hat{u}_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} K\left(\frac{x}{\sigma_n}\right) \frac{|\hat{u}_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx \\ &\geq \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|\hat{u}_n|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx \\ &\quad + \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)} \right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|\hat{u}_n|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx\end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_1)}\right) \int_{\mathbb{R}^N} \frac{|\hat{u}|^{p_s^*(\alpha_1)}}{|x|^{\alpha_1}} dx + \left(\frac{1}{p} - \frac{1}{p_s^*(\alpha_2)}\right) \int_{\mathbb{R}^N} \frac{|\hat{u}|^{p_s^*(\alpha_2)}}{|x|^{\alpha_2}} dx \\
&= I_{\alpha,\infty}(\hat{u}) - \frac{1}{p} \langle I'_{\alpha,\infty}(\hat{u}), \hat{u} \rangle \\
&= I_{\alpha,\infty}(\hat{u}) \\
&\geq c_{\alpha,\infty},
\end{aligned}$$

which yields a contradiction. Hence, we obtain $\sigma_n \not\rightarrow \infty$, as $n \rightarrow \infty$.

On the other hand, we can show that $\sigma_n \not\rightarrow 0$. Otherwise, we suppose that $\sigma_n \rightarrow 0$, as $n \rightarrow \infty$. In view of condition (H1), we can obtain (3.4)-(3.5). Repeating the arguments from line to line under (3.5), we can deduce that $\sigma_n \not\rightarrow 0$, as $n \rightarrow \infty$.

Since $\{\sigma_n\}$ is bounded in $\mathbb{R}^N$, up to a subsequence, we can find that $\tilde{\sigma} > 0$ such that $\sigma_n \rightarrow \tilde{\sigma}$, as $n \rightarrow \infty$. Since $\{\frac{z_n}{\sigma_n}\}$ is bounded in $\mathbb{R}^N$, we know that $\{z_n\}$ is bounded in $\mathbb{R}^N$. Making use of Lemma 3.4, up to a subsequence, we have

$$u_n(x) \rightarrow u \neq 0 \quad \text{in } L_{\text{loc}}^p(\mathbb{R}^N).$$

Step 6. We now prove $u_n \rightarrow u$ in $D^{s,p}(\mathbb{R}^N)$. Since $u_n \rightharpoonup u \neq 0$ in $D^{s,p}(\mathbb{R}^N)$, we have

$$\lim_{n \rightarrow \infty} \langle I'_\alpha(u_n), \varphi \rangle = \langle I'_\alpha(u), \varphi \rangle, \quad \varphi \in D^{s,p}(\mathbb{R}^N).$$

It follows that

$$c \leq I_\alpha(u) = I_\alpha(u) - \frac{1}{p_{s,\alpha_1}^*} \langle I_\alpha(u), u \rangle \leq \lim_{n \rightarrow \infty} I_\alpha(u_n) - \frac{1}{p_{s,\alpha_1}^*} \lim_{n \rightarrow \infty} \langle I'_\alpha(u_n), u_n \rangle = c.$$

This indicates that $I_\alpha(u) = c$ and $\lim_{n \rightarrow \infty} \|u_n\|_{D^{s,p}(\mathbb{R}^N)}^p = \|u\|_{D^{s,p}(\mathbb{R}^N)}^p$. $\square$

Proof of Theorem 1.5. As we can see that Theorem 1.5 follows from Lemmas 3.2 and 3.5-3.6 immediately. $\square$

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REFERENCES

- [1] A. Ambrosetti, P. Rabinowitz; Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381.
- [2] F. Andreu, J. Mazón, J. Rossi, J. Toledo; A nonlocal p -Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions, *SIAM J. Math. Anal.*, **40** (2008/09), 1815-1851.
- [3] M. Bhakta, K. Perera, F. Sk; A system of equations involving the fractional p -Laplacian and doubly critical nonlinearities, *Adv. Nonlinear Stud.*, **23** (2023), Paper No. 20230103, 24.
- [4] D. Blazevski, D. del Castillo-Negrete; Local and nonlocal anisotropic transport in reversed shear magnetic fields: Shearless cantori and nondiffusive transport, *Phys. Rev. E*, **87** (2013), 063106.
- [5] J. Bonder, N. Saintier, A. Silva; The concentration-compactness principle for fractional order Sobolev spaces in unbounded domains and applications to the generalized fractional Brézis-Nirenberg problem, *Nonlinear Differ. Equ. Appl.*, **25** (2018), Paper No. 52, 25.
- [6] L. Brasco, E. Lindgren; Higher Sobolev regularity for the fractional p -Laplace equation in the superquadratic case, *Adv. Math.*, **304** (2017), 300-354.
- [7] L. Brasco, E. Parini; The second eigenvalue of the fractional p -Laplacian, *Adv. Calc. Var.*, **9** (2016), 323-355.
- [8] H. Brézis, E. Lieb; A relation between pointwise convergence of functions and convergence of functionals, *Proceedings of the American Mathematical Society*, **88** (1983), 486-490.
- [9] L. Caffarelli, L. Silvestre; An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations*, **32** (2007), 1245-1260.
- [10] D. Cassani, Z. Liu, G. Romani; Nonlocal planar Schrödinger-Poisson systems in the fractional Sobolev limiting case, *J. Differential Equations*, **383** (2024), 214-269.
- [11] W. Chen, S. Mosconi, M. Squassina; Nonlocal problems with critical Hardy nonlinearity, *J. Funct. Anal.*, **275** (2018), 3065-3114.
- [12] E. Di Nezza, G. Palatucci, E. Valdinoci; Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521-573.

- [13] S. Dipierro, E. Valdinoci; Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes, *Phys. A*, **575** (2021), Paper No. 126052, 20.
- [14] Z. Feng, Y. Su; Ground state solutions of fractional equations with Coulomb potential and critical exponent, *Stud. Appl. Math.*, **153** (2024), no. 2, Paper No. e12723, 25.
- [15] Z. Feng, Y. Su; Traveling wave phenomena of inhomogeneous half-wave equation, *J. Differential Equations*, **400** (2024), 248-277.
- [16] G. Franzina, G. Palatucci; Fractional p -eigenvalues, *Riv. Math. Univ. Parma (N.S.)*, **5** (2014), 373-386.
- [17] P. Garain, E. Lindgren; Higher Hölder regularity for the fractional p -Laplace equation in the subquadratic case. *Math. Ann.*, **390** (2024), 5753-5792.
- [18] A. Iannizzotto, S. Mosconi, M. Squassina; Global Hölder regularity for the fractional p -Laplacian. *Rev. Mat. Iberoam.*, **32** (2016), 1353-1392.
- [19] H. Ishii, G. Nakamura; A class of integral equations and approximation of p -Laplace equations. *Calc. Var. Partial Differential Equations*, **37** (2010), 485-522.
- [20] X. Lin, S. Zheng, Z. Feng; Multiple solutions of p -fractional Schrödinger-Choquard-Kirchhoff equations with Hardy-Littlewood-Sobolev critical exponents. *Adv. Nonlinear Stud.*, **23** (2023), Paper No. 20220059, 21.
- [21] Z. Liu, V. Rădulescu, J. Zhang; Groundstates of the planar Schrödinger-Poisson system with potential well and lack of symmetry, *Proc. Roy. Soc. Edinburgh Sect. A*, **154** (2024), 993-1023.
- [22] S. Mosconi, K. Perera, M. Squassina, Y. Yang; The Brezis-Nirenberg problem for the fractional p -Laplacian, *Calc. Var. Partial Differential Equations*, **55** (2016), Art. 105, 25.
- [23] Y.C. Mo, M. Zhu, B. H. Feng; Blow-up criteria and instability of standing waves for the fractional Schrödinger poisson equation, *Electron. J. Differential Equations*, **2023** (2023) No. 24, 1-23.
- [24] G. Palatucci, A. Pisante; Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces, *Calc. Var. Partial Differential Equations*, **50** (2014), 799-829.
- [25] M. Sakuma; Ground states for p -fractional Choquard-type equations with doubly or triply critical nonlinearity, *Fract. Calc. Appl. Anal.*, (2025).
- [26] Y. Shen; Existence of solutions to elliptic problems with fractional p -Laplacian and multiple critical nonlinearities in the entire space $\mathbb{R}^N$, *Nonlinear Anal.*, **202** (2021), Paper No. 112102, 17.
- [27] Y. Shen; The Brézis-Nirenberg problem for the fractional p -Laplacian in unbounded domains, *Acta Math. Sin. (Engl. Ser.)*, **39** (2023), 2181-2206.
- [28] Y. Su; Fractional p -Laplacian problem with critical Stein-Weiss type term, *J. Geom. Anal.*, **33** (2023), Paper No. 160, 22.
- [29] M. Willem; *Minimax Theorems*, Birkhäuser Boston, Inc., Boston, MA, 24, 1996.
- [30] J. P. Wu, M. Zhu; Initial-boundary value problem of plasma-charge model in the half space, *Electron. J. Differential Equations*, **2025** *2025) No. 106, 1-18.
- [31] M. Zhu, Y. Wang; Blow-up of solutions for an integrable periodic two-component Camassa-Holm system with cubic nonlinearity, *Math. Methods Appl. Sci.*, **46** (2023), 7215-7229.
- [32] X. Zhu, M. Zhu, Y. Wang, K. Wang; Wave-breaking for two-component Fornberg-Whitham systems with dissipation, *Electron. J. Differential Equations*, **2025** (2025) No. 70, 1-23.

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