

GROUND STATE SOLUTIONS AND ASYMPTOTICS FOR DISCRETE KIRCHHOFF PROBLEMS

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ABSTRACT. While superlinear discrete Kirchhoff problems have been extensively studied via variational minimax methods, the sublinear regime $\sigma \in (1, 2)$, where the Kirchhoff energy dominates the reaction term, remains essentially unexplored. Also the asymptotic behavior of ground states as $\lambda \rightarrow 0^+$ has not been investigated even in the continuous setting. We address this gap for the discrete Kirchhoff boundary value problem

$$-m \left(\sum_{j=0}^N |\delta y_j|^2 \right) \Delta_d y_j = \lambda h_j(y_j), \quad j = 1, \dots, N,$$

$$y_0 = y_{N+1} = 0.$$

Combining coercive minimization, constant-sign truncation, Krasnosel'skii genus theory, and scaling arguments, we establish existence of a nontrivial ground state for every $\lambda > 0$, two constant-sign solutions, and N pairs of nontrivial solutions under an evenness assumption. We further derive the sharp scaling $\inf \mathcal{J}_\lambda \asymp -\lambda^{\frac{2}{2-\sigma}}$, show that normalized ground states converge to minimizers of an explicit limit functional, and obtain, to our knowledge, the first quantitative convergence rate for this class of problems. Newton-continuation computations confirm the predicted scaling exponents to high precision.

1. INTRODUCTION

Nonlocal boundary-value problems play a central role in modern nonlinear analysis, arising naturally in the modeling of lattice dynamics, nonlinear electrical networks, finite-difference approximations of partial differential equations, and discrete variational principles [1, 6, 10, 11, 29]. Among these, Kirchhoff-type equations [12] occupy a distinguished position, as they generalize the classical vibrating string model by allowing the tension to depend on the total deformation energy rather than on pointwise values. In the continuous setting, this leads to the prototype problem

$$-\left(a + b \int_0^1 |u'(x)|^2 dx\right) u'' = f(x, u), \quad x \in (0, 1), \tag{1.1}$$

$$u(0) = u(1) = 0$$

where $a, b > 0$. The genuinely nonlocal character of (1.1), inherited from the global Dirichlet energy, produces a variational structure markedly different from that of classical semilinear equations.

The discrete counterpart of (1.1) preserves this nonlocal complexity despite its finite-dimensional setting. Variational methods for nonlinear difference equations were pioneered by Bonanno and Candito [5, 4] and further developed by Mihăilescu, Rădulescu and Tersian [20]. Building on this foundation, discrete Kirchhoff problems have attracted considerable recent attention: Zhang and Zhou [28] obtained infinitely many solutions via critical point theory, Xiong [27] extended these results to the p -Laplacian setting (see also [9]), Long [14, 15] employed Morse theory, anisotropic versions were studied in [3, 8]; Lin and Zhou [13] established the existence and multiplicity of

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positive solutions for the classical coefficient $m(t) = a + bt$, Ricceri [24] extended the framework to allow for negative values of the parameter a in the Kirchhoff coefficient; and related nonlocal discrete Schrödinger models were considered in [26]. Beyond this finite-dimensional line of research, the qualitative theory of difference equations on the whole integer lattice (homoclinic-type solutions for p -Laplacian and ϕ_c -Laplacian partial difference equations) has also seen substantial recent progress [16, 17, 18], and these infinite-dimensional variational techniques are complementary to the finite-dimensional discrete Dirichlet setting adopted here.

Most of the literature, both continuous and discrete, is confined to the *superlinear* regime, where an Ambrosetti-Rabinowitz-type condition [2] ensures mountain-pass or linking geometry, so that existence and multiplicity follow from classical minimax methods [19, 23, 25]. The *sublinear* regime $\sigma \in (1, 2)$ behaves very differently: the Kirchhoff energy dominates the reaction at infinity, the functional becomes globally coercive, and the Nehari manifold and minimax techniques no longer apply. While this regime has received some attention in the continuous setting [21, 22], it remains essentially unexplored for discrete Kirchhoff problems, and in particular the asymptotic behavior of ground states as the parameter $\lambda \rightarrow 0^+$ has not been addressed.

Motivated by the above statements, we study the problem

$$\begin{aligned} -m\left(\sum_{j=0}^N |\delta y_j|^2\right) \Delta_d y_j &= \lambda h_j(y_j), \quad j = 1, \dots, N, \\ y_0 &= y_{N+1} = 0, \end{aligned} \tag{1.2}$$

where $\delta y_j := y_{j+1} - y_j$ and $\Delta_d y_j := y_{j+1} - 2y_j + y_{j-1}$ denote the forward difference and discrete Laplacian operators. Beyond proving existence, our main goal is to identify the limiting variational problem governing the normalized asymptotic profile of ground states as $\lambda \rightarrow 0^+$, and to derive an explicit quantitative convergence rate.

The contributions of this paper are as follows.

- (i) We build a coercive variational framework in which the Kirchhoff primitive dominates the reaction potential at infinity, and prove existence of a nontrivial ground state for every $\lambda > 0$.
- (ii) Using constant-sign truncations and a discrete strong maximum principle, we obtain two distinct strict solutions of fixed sign.
- (iii) Under an evenness assumption, Krasnosel'skii genus theory yields at least N pairs of nontrivial solutions.
- (iv) We identify the normalized asymptotic profile of ground states as $\lambda \rightarrow 0^+$, proving convergence toward the minimizers of an explicit limit functional.
- (v) We derive a quantitative convergence rate $\|v_\lambda - v_0\|_{V_N} = O(\lambda^\theta)$ under mild Hölder regularity assumptions.
- (vi) We provide reproducible numerical experiments confirming the theoretical scaling laws.

This article is organized as follows. Section 2 introduces the functional framework. Section 3 states the assumptions and main theorems. Section 4 establishes compactness properties. Section 5 constructs ground-state solutions, while Sections 6 and 7 treat constant-sign and multiplicity results, respectively. Section 8 develops the asymptotic profile and quantitative convergence analysis, Section 9 addresses parameter stability, and Section 10 presents numerical experiments and open problems.

2. DISCRETE VARIATIONAL SETTING

For $N \geq 2$ we define

$$V_N := \{y : \{0, 1, \dots, N+1\} \rightarrow \mathbb{R} : y_0 = y_{N+1} = 0\}.$$

We equip V_N with the discrete H_0^1 inner product and norm,

$$\langle y, z \rangle_{V_N} := \sum_{j=0}^N \delta y_j \delta z_j, \quad \|y\|_{V_N} := \langle y, y \rangle_{V_N}^{1/2}. \tag{2.1}$$

Since $\dim(V_N) = N$, all norms on V_N are equivalent. The summation-by-parts identity

$$\sum_{j=0}^N \delta y_j \delta z_j = - \sum_{j=1}^N (\Delta_d y_j) z_j, \quad y, z \in V_N, \quad (2.2)$$

follows from the boundary values $z_0 = z_{N+1} = 0$ and identifies $-\Delta_d$ as the discrete Dirichlet Laplacian.

Lemma 2.1 (Discrete Poincaré inequality). *There exists a constant $\mu_1 > 0$, depending only on N , such that*

$$\mu_1 \sum_{j=1}^N y_j^2 \leq \|y\|_{V_N}^2 \quad \text{for all } y \in V_N.$$

Explicitly, $\mu_1 = 4 \sin^2 \left(\frac{\pi}{2(N+1)} \right)$ is the smallest eigenvalue of $-\Delta_d$ on V_N ; see [13].

Let

$$\widehat{m}(t) := \int_0^t m(s) \, ds, \quad t \geq 0, \quad H_j(s) := \int_0^s h_j(\tau) \, d\tau, \quad s \in \mathbb{R}.$$

The variational functional associated with (1.2) is

$$\mathcal{J}_\lambda(y) := \frac{1}{2} \widehat{m}(\|y\|_{V_N}^2) - \lambda \sum_{j=1}^N H_j(y_j), \quad y \in V_N. \quad (2.3)$$

Proposition 2.2. *Assume $m \in C([0, +\infty), (0, +\infty))$ and $h_j \in C(\mathbb{R}, \mathbb{R})$ for $j = 1, \dots, N$. Then $\mathcal{J}_\lambda \in C^1(V_N, \mathbb{R})$ and*

$$\langle \mathcal{J}'_\lambda(y), z \rangle = m(\|y\|_{V_N}^2) \sum_{j=0}^N \delta y_j \delta z_j - \lambda \sum_{j=1}^N h_j(y_j) z_j, \quad y, z \in V_N. \quad (2.4)$$

Moreover, $y \in V_N$ is a critical point of $\mathcal{J}_\lambda$ if and only if it is a classical solution of (1.2).

Proof. The map $y \mapsto \|y\|_{V_N}^2$ is smooth and $\widehat{m} \in C^1([0, +\infty))$. The chain rule gives

$$\left\langle \left(\frac{1}{2} \widehat{m}(\|y\|_{V_N}^2) \right)', z \right\rangle = m(\|y\|_{V_N}^2) \sum_{j=0}^N \delta y_j \delta z_j.$$

Similarly, $H_j \in C^1(\mathbb{R})$ and $H'_j = h_j$. This proves (2.4). By (2.2), the critical point equation $\langle \mathcal{J}'_\lambda(y), z \rangle = 0$ for all $z \in V_N$ is equivalent to

$$-m(\|y\|_{V_N}^2) \Delta_d y_j = \lambda h_j(y_j), \quad j = 1, \dots, N,$$

with $y_0 = y_{N+1} = 0$, which is exactly (1.2). $\square$

3. MAIN RESULTS

In this section, we present the structural assumptions on the Kirchhoff coefficient and the reaction potentials, which constitute the core framework for our variational approach. These conditions are carefully designed to ensure the coercivity and compactness of the associated energy functional, enabling the use of critical point theory. Based on these assumptions, we state our main results concerning the existence of ground-state solutions, their constant-sign properties, and the asymptotic behavior as the parameter tends to zero. Every symbol introduced below is defined at its first occurrence; in particular, the scaling exponent α is defined in (3.3) (Theorem 3.7) and the convergence-rate exponent θ is defined in (3.8) (Theorem 3.8), and both are used consistently, with explicit cross-references, throughout Remarks 3.9–3.10 and in the numerical study of Section 10.

Assumption – Kirchhoff coefficient. There exist a real number $\eta \geq 1$ and a positive constant $m_* > 0$ such that

$$(A1) \quad m \in C([0, +\infty), (0, +\infty))$$

$$(A2) \quad m(t) \geq m_0 + m_* t^{\eta-1} \text{ for every } t > 0, \text{ where } m_0 := m(0) > 0.$$

By continuity, we have

$$\widehat{m}(t) = m_0 t + o(t) \quad \text{as } t \rightarrow 0^+. \quad (3.1)$$

Assumption – Next-order Kirchhoff correction. There exist $\beta > 0$ and $m_1 \in \mathbb{R}$ such that

$$(A3) \quad \widehat{m}(t) = m_0 t + m_1 t^{1+\beta} + o(t^{1+\beta}) \text{ as } t \rightarrow 0^+.$$

This holds, for instance, when $m(t) = m_0 + m_1(1 + \beta)t^\beta$ near the origin.

Assumption - Reaction potentials. There exist $A_0 > 0$, $R > 0$, $a_j > 0$ for $j = 1, \dots, N$, an exponent r satisfying $1 < r < 2\eta$ and a local exponent $\sigma \in (1, 2)$ such that:

$$(A4) \quad h_j \in C(\mathbb{R}, \mathbb{R}) \text{ and } h_j(0) = 0 \text{ for every } j;$$

$$(A5) \quad H_j(s) \leq A_0(1 + |s|^r) \text{ for every } s \in \mathbb{R} \text{ and every } j;$$

$$(A6) \quad H_j(s) \geq a_j |s|^\sigma \text{ whenever } |s| \leq R \text{ and } j = 1, \dots, N.$$

Assumption - Sharp local two-sided control. There exist $A_1 > 0$, $R > 0$, and $a_j > 0$ for $j = 1, \dots, N$, such that

$$(A7) \quad a_j |s|^\sigma \leq H_j(s) \leq A_1 |s|^\sigma, \text{ whenever } |s| \leq R, j = 1, \dots, N.$$

Remark 3.1. Assumption (A7) strengthens the structural requirement (A6) by incorporating a local upper bound. This refinement plays a pivotal role in establishing the sharp lower energy bounds detailed in Theorem 3.3. It is worth noting that this two-sided control is automatically fulfilled whenever the asymptotic expansion (A10) holds, since the condition $H_j(s) = B_j |s|^\sigma + o(|s|^\sigma)$ as $s \rightarrow 0$ immediately yields the required bounds on a sufficiently small neighborhood of the origin. Furthermore, in the explicit prototypical model investigated in Section 10, condition (A7) holds trivially and globally since $H_j(s) = (\frac{a_j}{\sigma}) |s|^\sigma$ identically.

Assumption - Strict sign preservation For every $j = 1, \dots, N$,

$$(A8) \quad h_j(s) > 0 \text{ for } s > 0, h_j(s) < 0 \text{ for } s < 0.$$

Assumption - Evenness. For every $j = 1, \dots, N$,

$$(A9) \quad H_j(s) = H_j(-s) \text{ for all } s \in \mathbb{R}.$$

Equivalently, h_j is odd.

Assumption - Exact local profile. There exist $B_j > 0$ such that, uniformly with respect to $j = 1, \dots, N$,

$$(A10) \quad H_j(s) = B_j |s|^\sigma + o(|s|^\sigma) \text{ as } s \rightarrow 0.$$

Assumption - Next-order reaction profile. There exist $\gamma > 0$ and $D_j \in \mathbb{R}$, uniformly in j , such that

$$(A11) \quad H_j(s) = B_j |s|^\sigma + D_j |s|^{\sigma+\gamma} + o(|s|^{\sigma+\gamma}) \text{ as } s \rightarrow 0.$$

Theorem 3.2 (Nontrivial ground state). *Assume (A1), (A2), (A4)–(A6). Then, for every $\lambda > 0$, the functional $\mathcal{J}_\lambda$ admits a nontrivial global minimizer $y_\lambda \in V_N \setminus \{0\}$. Consequently, (1.2) has a nontrivial ground-state solution.*

Theorem 3.3 (Two-sided ground-state energy estimates). *Assume (A1), (A2), (A4)–(A7). Then there exist constants $C_1, C_2 > 0$, depending only on N and the structural constants in (A1), (A2), (A7), such that for all sufficiently small $\lambda > 0$,*

$$-C_2 \lambda^{\frac{2}{2-\sigma}} \leq \inf_{V_N} \mathcal{J}_\lambda \leq -C_1 \lambda^{\frac{2}{2-\sigma}}. \quad (3.2)$$

Theorem 3.4 (Positive and negative solutions). *Assume (A1), (A2), (A4)–(A6), (A8). Then, for every $\lambda > 0$, problem (1.2) has at least two distinct nontrivial solutions $y^+, y^- \in V_N$ such that*

$$y_j^+ > 0 \quad \text{and} \quad y_j^- < 0, \quad j = 1, \dots, N.$$

Remark 3.5 (Version of Clark's theorem used). We recall the precise statement applied in the proof. Let X be a finite-dimensional normed space and let $\Phi \in C^1(X, \mathbb{R})$ be an even functional, bounded from below, satisfying the Palais-Smale condition and $\Phi(0) = 0$. Suppose there exists a compact symmetric set $K \subset \Phi^{-1}((-\infty, 0))$ with Krasnosel'skii genus $\gamma(K) \geq k$. Then Φ has at least k distinct pairs of nonzero critical points [25, Theorem 5.5].

In Theorem 3.6, we take $X = V_N$, $\Phi = \mathcal{J}_\lambda$, and $K = \rho \mathbb{S}^{N-1}$, where $\rho > 0$ is a sufficiently small radius given by Lemma 7.1 below. Since K is the full $(N-1)$ -sphere in the N -dimensional space V_N , its genus equals N [25, Proposition 5.4]. Hence, the theorem yields at least N pairs of nontrivial solutions.

Theorem 3.6 (Multiplicity via Krasnosel'skii genus). *Assume (A1), (A2), (A4)–(A6), (A9). Then, for every $\lambda > 0$, problem (1.2) has at least N distinct pairs $\{\pm y^{(k)}\}_{k=1}^N$ of nontrivial solutions.*

Theorem 3.7 (Normalized small-parameter profile). *Assume (A1), (A2), (A4)–(A6), (A10). Let y_λ be any ground-state solution of (1.2) and set*

$$\alpha := \frac{1}{2-\sigma}, \quad v_\lambda := \lambda^{-\alpha} y_\lambda. \quad (3.3)$$

Here $\alpha > 1$ is the scaling exponent dictated by the sublinear reaction exponent $\sigma \in (1, 2)$; it is the same α used throughout Section 8 and reappears explicitly in the rate exponent θ of (3.8) below (see also Remarks 3.10 and 3.9). Then (v_λ) is bounded in V_N as $\lambda \rightarrow 0^+$. Moreover, every sequence $\lambda_n \rightarrow 0^+$ has a subsequence, still denoted by λ_n , such that

$$v_{\lambda_n} \rightarrow v_0 \quad \text{in } V_N,$$

where $v_0 \neq 0$ is a global minimizer of the limit functional

$$\mathcal{I}_0(v) := \frac{m_0}{2} \|v\|_{V_N}^2 - \sum_{j=1}^N B_j |v_j|^\sigma, \quad v \in V_N. \quad (3.4)$$

In addition,

$$\frac{\inf_{V_N} \mathcal{J}_\lambda}{\lambda^{2/(2-\sigma)}} \rightarrow \inf_{V_N} \mathcal{I}_0 < 0 \quad \text{as } \lambda \rightarrow 0^+. \quad (3.5)$$

Consequently, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \lambda^{\frac{1}{2-\sigma}} \leq \|y_\lambda\|_{V_N} \leq c_2 \lambda^{\frac{1}{2-\sigma}} \quad (3.6)$$

for all sufficiently small $\lambda > 0$.

Theorem 3.8 (Quantitative convergence rate of the normalized profile). *Assume (A1)–(A6), (A10), (A11). Let v_0 be the unique global minimizer of $\mathcal{I}_0$. Assume further that v_0 is non-degenerate and strictly positive in the interior, that is, $v_{0,j} > 0$ for all $j = 1, \dots, N$, so that the Hessian $\mathcal{I}_0''(v_0)$ is well-defined and positive definite. Let $v_\lambda := \lambda^{-\alpha} y_\lambda$, with α as in (3.3). Then*

$$\|v_\lambda - v_0\|_{V_N} = O(\lambda^\theta) \quad \text{as } \lambda \rightarrow 0^+, \quad (3.7)$$

where

$$\theta := \min\{2\alpha\beta, \alpha\gamma\} = \frac{\min(2\beta, \gamma)}{2-\sigma}. \quad (3.8)$$

Here β is the Kirchhoff correction exponent in (A3) and γ is the reaction correction exponent in (A11). The exponent θ defined in (3.8) is the same quantity reported in Remark 3.9 and computed explicitly in Example 10.1 and Table 3.

Remark 3.9. Theorem 3.8 makes the convergence $v_\lambda \rightarrow v_0$ quantitative. The exponent θ , defined in (3.8), has a clear physical interpretation: it is determined by whichever correction—Kirchhoff or reaction—is *slower*, i.e., the smaller of $2\alpha\beta$ and $\alpha\gamma$, with α as in (3.3). In the explicit model of Section 10, $m(t) = 1 + 0.4t$ gives $\beta = 1$ and $\sigma = 3/2$ gives $\alpha = 2$, while $H_j(s) = B_j |s|^\sigma$ exactly, so $\gamma = +\infty$ and $\theta = 2\alpha\beta = 4$. This is consistent with the very rapid convergence observed numerically.

Remark 3.10. The exponent α in Theorem 3.7 is governed by the behavior of $\widehat{m}(t)$ at the origin. Since $m(0) > 0$, the leading Kirchhoff energy is quadratic, namely $\frac{m_0}{2} \|y\|_{V_N}^2$. Therefore the correct small-parameter scale is $\lambda^{\frac{1}{2-\sigma}} = \lambda^\alpha$, with α as defined in (3.3). Higher-order Kirchhoff terms influence the next-order correction but not the leading scale; they are captured precisely by the rate exponent θ of Theorem 3.8 (see (3.8)).

4. DISCRETE ESTIMATES AND COMPACTNESS

In this section, we derive sharp discrete pointwise estimates and establish the essential algebraic properties of the cone components. These results lead to the coercivity and compactness of the energy functional, ensuring the validity of the Palais-Smale condition. Let

$$C_N^2 := \max_{1 \leq j \leq N} \frac{j(N+1-j)}{N+1}. \quad (4.1)$$

In particular, $C_N \leq \frac{\sqrt{N+1}}{2}$.

Lemma 4.1 (Sharp discrete pointwise control). *For every $y \in V_N$ and $j = 1, \dots, N$,*

$$|y_j|^2 \leq \frac{j(N+1-j)}{N+1} \|y\|_{V_N}^2. \quad (4.2)$$

Consequently,

$$\max_{1 \leq j \leq N} |y_j| \leq C_N \|y\|_{V_N}, \quad \sum_{j=1}^N |y_j|^q \leq N C_N^q \|y\|_{V_N}^q \quad (q \geq 1).$$

Proof. Fix $j \in \{1, \dots, N\}$. The boundary conditions give two representations:

$$y_j = \sum_{i=0}^{j-1} \delta y_i \quad \text{and} \quad y_j = - \sum_{i=j}^N \delta y_i.$$

Multiplying the first representation by $\frac{N+1-j}{N+1}$ and the second by $\frac{j}{N+1}$ and adding, we obtain the single weighted representation

$$y_j = \sum_{i=0}^N \theta_i^{(j)} \delta y_i, \quad \theta_i^{(j)} := \begin{cases} \frac{N+1-j}{N+1}, & 0 \leq i \leq j-1, \\ -\frac{j}{N+1}, & j \leq i \leq N. \end{cases} \quad (4.3)$$

Indeed, splitting the sum in (4.3) according to the two cases recovers exactly

$$y_j = \frac{N+1-j}{N+1} \sum_{i=0}^{j-1} \delta y_i - \frac{j}{N+1} \sum_{i=j}^N \delta y_i,$$

which is the stated combination of the two boundary representations.

We now apply the Cauchy-Schwarz inequality *once*, to the single inner product $y_j = \sum_{i=0}^N \theta_i^{(j)} \delta y_i$ in $\mathbb{R}^{N+1}$:

$$|y_j|^2 = \left(\sum_{i=0}^N \theta_i^{(j)} \delta y_i \right)^2 \leq \left(\sum_{i=0}^N (\theta_i^{(j)})^2 \right) \left(\sum_{i=0}^N |\delta y_i|^2 \right) = \left(\sum_{i=0}^N (\theta_i^{(j)})^2 \right) \|y\|_{V_N}^2. \quad (4.4)$$

Applying Cauchy-Schwarz to the full weighted sum in this way is essential: splitting the sum into its two constituent pieces *before* squaring and bounding each piece separately would discard the cross term $-2\left(\frac{N+1-j}{N+1}\right)\left(\frac{j}{N+1}\right)S_1S_2$ (where $S_1 = \sum_{i=0}^{j-1} \delta y_i$, $S_2 = \sum_{i=j}^N \delta y_i$), which need not be nonpositive; the single Cauchy-Schwarz application in (4.4) avoids this difficulty.

It remains to compute the weight sum. Since $\theta_i^{(j)} = \frac{N+1-j}{N+1}$ for the j indices $i = 0, \dots, j-1$, and $\theta_i^{(j)} = -\frac{j}{N+1}$ for the $N+1-j$ indices $i = j, \dots, N$,

$$\sum_{i=0}^N (\theta_i^{(j)})^2 = j \left(\frac{N+1-j}{N+1} \right)^2 + (N+1-j) \left(\frac{j}{N+1} \right)^2$$

$$\begin{aligned}
&= \frac{j(N+1-j)}{(N+1)^2} [(N+1-j) + j] \\
&= \frac{j(N+1-j)(N+1)}{(N+1)^2} = \frac{j(N+1-j)}{N+1}.
\end{aligned}$$

Substituting into (4.4) gives

$$|y_j|^2 \leq \frac{j(N+1-j)}{N+1} \|y\|_{V_N}^2,$$

which is (4.2).

Since $j(N+1-j) \leq \frac{(N+1)^2}{4}$ for $j = 1, \dots, N$, we also obtain $\max_{1 \leq j \leq N} |y_j| \leq \frac{\sqrt{N+1}}{2} \|y\|_{V_N} = C_N \|y\|_{V_N}$. The bound $\sum_j |y_j|^q \leq NC_N^q \|y\|_{V_N}^q$ follows immediately from $|y_j| \leq C_N \|y\|_{V_N}$ applied to each of the N terms. $\square$

For $s \in \mathbb{R}$, we define $s^+ := \max\{s, 0\}$ and $s^- := \max\{-s, 0\}$. For $y \in V_N$, set $y_j^\pm = (y_j)^\pm$.

Lemma 4.2 (Cone algebra). *For every $y \in V_N$,*

$$\sum_{j=0}^N \delta y_j \delta y_j^- \leq -\|y^-\|_{V_N}^2, \quad (4.5)$$

$$\sum_{j=0}^N \delta y_j \delta y_j^+ \geq \|y^+\|_{V_N}^2. \quad (4.6)$$

Proof. For real a, b , $(b^+ - a^+)(b^- - a^-) \leq 0$. Since $\delta y_j = \delta y_j^+ - \delta y_j^-$,

$$\delta y_j \delta y_j^- = \delta y_j^+ \delta y_j^- - |\delta y_j^-|^2 \leq -|\delta y_j^-|^2.$$

Summing gives (4.5). The proof of (4.6) is analogous. $\square$

Proposition 4.3 (Coercivity and compactness). *Assume (A1), (A2), (A5). Then $\mathcal{J}_\lambda$ is coercive, every sublevel set of $\mathcal{J}_\lambda$ is compact, and $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition on V_N .*

Proof. From (A2),

$$\widehat{m}(t) = \int_0^t m(s) ds \geq \frac{m_*}{\eta} t^\eta, \quad t \geq 0.$$

Using (A5) and Lemma 4.1,

$$\mathcal{J}_\lambda(y) \geq \frac{m_*}{2\eta} \|y\|_{V_N}^{2\eta} - \lambda N A_0 - \lambda A_0 N C_N^r \|y\|_{V_N}^r. \quad (4.7)$$

Since $r < 2\eta$, the right-hand side tends to $+\infty$ as $\|y\|_{V_N} \rightarrow +\infty$. This proves coercivity. Sublevel sets are closed and bounded in the finite-dimensional space V_N , hence compact. Every Palais-Smale sequence is bounded by coercivity and therefore has a convergent subsequence. $\square$

5. EXISTENCE OF GROUND STATES AND ENERGY ESTIMATES

In this section, we establish the existence of nontrivial ground-state solutions by demonstrating that the energy functional attains a negative minimum. Furthermore, we derive sharp asymptotic energy bounds that characterize the system's behavior as the parameter tends to zero.

Lemma 5.1 (Negative energy near the origin). *Assume (A1), (A2), (A4)–(A6). Then $\inf_{V_N} \mathcal{J}_\lambda < 0$ for every $\lambda > 0$.*

Proof. Let $\varphi_j := \sin\left(\frac{\pi j}{N+1}\right)$, $j = 1, \dots, N$. Since $\frac{\pi j}{N+1} \in (0, \pi)$ for every $j = 1, \dots, N$, we have $\varphi_j > 0$ inside the discrete domain, and in particular $\|\varphi\|_{V_N} > 0$.

Choice of the threshold. Let $R > 0$ be the localization radius from Assumption (A6). Set

$$t_1 := \frac{R}{\max_{1 \leq j \leq N} \varphi_j} > 0,$$

so that $t\varphi_j \leq R$ for all $j = 1, \dots, N$ whenever $0 < t \leq t_1$. By Assumption (A1), m is continuous at the origin with $m(0) = m_0 > 0$; hence there exists $t_2 > 0$ and a constant $C_\varphi > 0$, depending only on $\|\varphi\|_{V_N}$ and m_0 , such that

$$\frac{1}{2}\widehat{m}\left(t^2\|\varphi\|_{V_N}^2\right) \leq C_\varphi t^2 \quad \text{for all } 0 < t \leq t_2. \quad (5.1)$$

Define $t_0 := \min\{t_1, t_2\} > 0$. Both estimates below hold simultaneously for every $t \in (0, t_0]$.

Lower bound for the reaction term. For $0 < t \leq t_0 \leq t_1$, we have $t\varphi_j \leq R$ for every j , so Assumption (A6) applies componentwise:

$$H_j(t\varphi_j) \geq a_j (t\varphi_j)^\sigma = a_j t^\sigma \varphi_j^\sigma, \quad j = 1, \dots, N.$$

Summing over j gives

$$\sum_{j=1}^N H_j(t\varphi_j) \geq t^\sigma \sum_{j=1}^N a_j \varphi_j^\sigma =: A_\varphi t^\sigma, \quad A_\varphi > 0,$$

where $A_\varphi > 0$ since $a_j > 0$ and $\varphi_j > 0$ for every j .

Combining the two estimates. For $0 < t \leq t_0$, estimate (5.1) holds since $t \leq t_2$, and the reaction lower bound holds since $t \leq t_1$. Hence

$$\mathcal{J}_\lambda(t\varphi) = \frac{1}{2}\widehat{m}\left(t^2\|\varphi\|_{V_N}^2\right) - \lambda \sum_{j=1}^N H_j(t\varphi_j) \leq C_\varphi t^2 - \lambda A_\varphi t^\sigma =: g(t).$$

Since $1 < \sigma < 2$, we have $t^{2-\sigma} \rightarrow 0$ as $t \rightarrow 0^+$, so

$$g(t) = t^\sigma (C_\varphi t^{2-\sigma} - \lambda A_\varphi) < 0$$

for all $t \in (0, t_0]$ small enough that $C_\varphi t^{2-\sigma} < \lambda A_\varphi$. Fixing such a $t \in (0, t_0]$ yields $\mathcal{J}_\lambda(t\varphi) \leq g(t) < 0$, and therefore

$$\inf_{V_N} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(t\varphi) < 0,$$

which proves the lemma. $\square$

Proof of Theorem 3.2. According to Proposition 4.3, the energy functional $\mathcal{J}_\lambda$ is coercive and lower semicontinuous on the finite-dimensional Hilbert space V_N . Consequently, its sublevel sets are weakly compact, which in finite dimensions equates to strong compactness. By the direct method of the calculus of variations, any minimizing sequence $\{y_n\} \subset V_N$ for the problem $d_\lambda := \inf_{V_N} \mathcal{J}_\lambda$ is bounded and clusters strongly around some element $y_\lambda \in V_N$ satisfying $\mathcal{J}_\lambda(y_\lambda) = d_\lambda$.

By Lemma 5.1, the energy floor is strictly negative, i.e., $d_\lambda < 0 = \mathcal{J}_\lambda(0)$, which immediately implies that the global minimizer y_λ cannot coincide with the trivial state, hence $y_\lambda \in V_N \setminus \{0\}$. Finally, since $\mathcal{J}_\lambda \in C^1(V_N, \mathbb{R})$, Fermat's critical point theorem asserts that $\mathcal{J}'_\lambda(y_\lambda) = 0$. Thus, y_λ constitutes a classical, nontrivial ground-state solution to the discrete Kirchhoff problem (1.2). $\square$

Lemma 5.2 (Qualitative vanishing of ground states). *Assume (A1), (A2), (A4)–(A6). Let $y_\lambda \in V_N$ be a global minimizer of $\mathcal{J}_\lambda$, as given by Theorem 3.2. Then*

$$\|y_\lambda\|_{V_N} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0^+.$$

Proof. Set $x_\lambda := \|y_\lambda\|_{V_N}$. Since y_λ is a global minimizer, $\mathcal{J}_\lambda(y_\lambda) \leq \mathcal{J}_\lambda(0) = 0$, and the coercivity estimate (4.7) gives

$$a x_\lambda^{2\eta} \leq b \lambda (1 + x_\lambda^r), \quad a := \frac{m_*}{2\eta}, \quad b := \max(N A_0, A_0 N C_N^r). \quad (5.2)$$

Suppose, for contradiction, that $x_\lambda \not\rightarrow 0$ as $\lambda \rightarrow 0^+$; then there exist $\varepsilon > 0$ and a sequence $\lambda_n \rightarrow 0^+$ such that $x_n := x_{\lambda_n} \geq \varepsilon$ for all n .

Case 1: (x_n) is unbounded. Passing to a subsequence, assume $x_n \rightarrow +\infty$. Dividing (5.2) (with $\lambda = \lambda_n$) by x_n^r gives

$$a x_n^{2\eta-r} \leq b \lambda_n (x_n^{-r} + 1).$$

Since $\lambda_n \rightarrow 0$ and $x_n^{-r} \rightarrow 0$, the right-hand side tends to 0, while $2\eta - r > 0$ by the assumption $r < 2\eta$ in (A4)–(A6), forces the left-hand side to tend to $+\infty$ if $x_n \rightarrow +\infty$ – a contradiction. Hence (x_n) cannot be unbounded.

Case 2: (x_n) is bounded. Let $M := \sup_n x_n < \infty$. Then (5.2) gives

$$a x_n^{2\eta} \leq b \lambda_n (1 + M^r) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

so $x_n \rightarrow 0$, contradicting $x_n \geq \varepsilon > 0$.

Both cases lead to a contradiction, so no such sequence exists; therefore $\|y_\lambda\|_{V_N} \rightarrow 0$ as $\lambda \rightarrow 0^+$. $\square$

Proof of Theorem 3.3.

Step 1. Let $\varphi \in V_N \setminus \{0\}$ be the positive test function constructed in Lemma 5.1, with constants $t_0, C_\varphi, A_\varphi > 0$ therein, so that

$$\mathcal{J}_\lambda(t\varphi) \leq C_\varphi t^2 - \lambda A_\varphi t^\sigma =: g(t), \quad 0 < t \leq t_0.$$

Since $g(t) \rightarrow 0^-$ as $t \rightarrow 0^+$ (as $\sigma < 2$) and $g(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, the function g attains a unique critical point $t_* > 0$ on $(0, +\infty)$, determined by $g'(t_*) = 0$, i.e.

$$2C_\varphi t_* - \lambda A_\varphi \sigma t_*^{\sigma-1} = 0 \quad \Longleftrightarrow \quad t_* = \left(\frac{\lambda A_\varphi \sigma}{2C_\varphi} \right)^{\frac{1}{2-\sigma}}.$$

Substituting the critical-point relation $2C_\varphi t_* = \lambda A_\varphi \sigma t_*^{\sigma-1}$ into $g''(t_*)$ gives $g''(t_*) = 2C_\varphi - (\sigma - 1)(2C_\varphi) = 2C_\varphi(2 - \sigma) > 0$ since $\sigma \in (1, 2)$. Hence the critical point t_* is necessarily a global minimum of g on $(0, +\infty)$, not a maximum. A direct computation gives

$$g(t_*) = t_*^\sigma (C_\varphi t_*^{2-\sigma} - \lambda A_\varphi) = -t_*^\sigma \lambda A_\varphi \frac{2-\sigma}{2} = -C_1 \lambda^{\frac{2}{2-\sigma}},$$

where the positive structural constant C_1 is explicitly defined by

$$C_1 := \left(\frac{2-\sigma}{\sigma} \right) \left(\frac{\sigma A_\varphi}{2} \right)^{\frac{2}{2-\sigma}} C_\varphi^{-\frac{\sigma}{2-\sigma}} > 0,$$

which depends exclusively on C_φ , A_φ , and σ . Since φ is fixed, $t_* \rightarrow 0$ as $\lambda \rightarrow 0^+$, so there exists $\lambda_1 > 0$ such that $t_* \leq t_0$ for all $\lambda \in (0, \lambda_1)$, ensuring the bound above is legitimately applicable. We conclude

$$\inf_{V_N} \mathcal{J}_\lambda \leq \mathcal{J}_\lambda(t_*\varphi) \leq -C_1 \lambda^{\frac{2}{2-\sigma}}, \quad \forall \lambda \in (0, \lambda_1). \quad (5.3)$$

Step 2. Let $y_\lambda \in V_N \setminus \{0\}$ be the ground state given by Theorem 3.2. By Lemma 5.2, $\|y_\lambda\|_{V_N} \rightarrow 0$ as $\lambda \rightarrow 0^+$. By continuity of m at the origin, $m(0) = m_0 > 0$, so (3.1) gives $\delta_1 > 0$ such that

$$\widehat{m}(t) \geq \frac{m_0}{2} t \quad \text{for all } 0 \leq t \leq \delta_1. \quad (5.4)$$

By Lemma 4.1, $\|y_\lambda\|_\infty \leq C_N \|y_\lambda\|_{V_N}$. Hence there exists $\lambda_2 \in (0, \lambda_1]$ such that, for every $\lambda \in (0, \lambda_2)$,

$$\|y_\lambda\|_{V_N}^2 \leq \delta_1 \quad \text{and} \quad \|y_\lambda\|_\infty \leq C_N \|y_\lambda\|_{V_N} \leq R, \quad (5.5)$$

where $R > 0$ is the localization radius in Assumption (A7). Thus, for $\lambda \in (0, \lambda_2)$, both (5.4) and the upper bound in (A7) apply simultaneously to y_λ .

Step 3. For $\lambda \in (0, \lambda_2)$, using (5.4), (5.5), the upper bound $H_j(s) \leq A_1 |s|^\sigma$ from (A7) (valid since $|y_{\lambda,j}| \leq R$ for every j), and Lemma 4.1 (giving $\sum_j |y_{\lambda,j}|^\sigma \leq NC_N^\sigma \|y_\lambda\|_{V_N}^\sigma$), we obtain

$$\mathcal{J}_\lambda(y_\lambda) = \frac{1}{2} \widehat{m}(\|y_\lambda\|_{V_N}^2) - \lambda \sum_{j=1}^N H_j(y_{\lambda,j}) \geq \frac{m_0}{4} \|y_\lambda\|_{V_N}^2 - \lambda A_1 NC_N^\sigma \|y_\lambda\|_{V_N}^\sigma. \quad (5.6)$$

Step 4. Combining (5.3) (valid since $\lambda_2 \leq \lambda_1$) with (5.6),

$$\frac{m_0}{4} \|y_\lambda\|_{V_N}^2 - \lambda A_1 NC_N^\sigma \|y_\lambda\|_{V_N}^\sigma \leq \mathcal{J}_\lambda(y_\lambda) \leq -C_1 \lambda^{\frac{2}{2-\sigma}} < 0.$$

In particular $y_\lambda \neq 0$, so $\|y_\lambda\|_{V_N} > 0$, and the left-hand side above must be strictly negative:

$$\frac{m_0}{4} \|y_\lambda\|_{V_N}^2 < \lambda A_1 N C_N^\sigma \|y_\lambda\|_{V_N}^\sigma.$$

Dividing by $\|y_\lambda\|_{V_N}^\sigma > 0$ and using $2 - \sigma > 0$,

$$\|y_\lambda\|_{V_N}^{2-\sigma} < \frac{4A_1 N C_N^\sigma}{m_0} \lambda \implies \|y_\lambda\|_{V_N} \leq C \lambda^{\frac{1}{2-\sigma}}, \quad C := \left(\frac{4A_1 N C_N^\sigma}{m_0} \right)^{\frac{1}{2-\sigma}}, \quad (5.7)$$

for all $\lambda \in (0, \lambda_2)$. Note that this exponent $1/(2 - \sigma)$ arises here directly from balancing the quadratic Kirchhoff term against the σ -homogeneous reaction term – exactly as in Step 1 and not from the coercivity estimate (4.7), whose exponents η, r play no role beyond Lemma 5.2.

Step 5 Substituting (5.7) back into (5.6),

$$\mathcal{J}_\lambda(y_\lambda) \geq -\lambda A_1 N C_N^\sigma \|y_\lambda\|_{V_N}^\sigma \geq -\lambda A_1 N C_N^\sigma (C \lambda^{\frac{1}{2-\sigma}})^\sigma = -C_2 \lambda^{1+\frac{\sigma}{2-\sigma}},$$

where $C_2 := A_1 N C_N^\sigma C^\sigma > 0$ is a positive constant depending purely on the structural constants of the system. Since

$$1 + \frac{\sigma}{2-\sigma} = \frac{(2-\sigma) + \sigma}{2-\sigma} = \frac{2}{2-\sigma},$$

we obtain

$$\mathcal{J}_\lambda(y_\lambda) \geq -C_2 \lambda^{\frac{2}{2-\sigma}}, \quad \forall \lambda \in (0, \lambda_2). \quad (5.8)$$

In conclusion, setting $\lambda_0 := \lambda_2 \leq \lambda_1$ and combining (5.3) and (5.8) (both valid on $(0, \lambda_0)$), and noting $\inf_{V_N} \mathcal{J}_\lambda = \mathcal{J}_\lambda(y_\lambda)$ by Theorem 3.2), we obtain

$$-C_2 \lambda^{\frac{2}{2-\sigma}} \leq \inf_{V_N} \mathcal{J}_\lambda \leq -C_1 \lambda^{\frac{2}{2-\sigma}} \quad \text{for all } \lambda \in (0, \lambda_0),$$

which is (3.2). $\square$

6. CONSTANT-SIGN SOLUTIONS

In this section, we analyze the constant-sign properties of solutions by introducing truncated functionals and proving the discrete strong maximum principle. These tools ensure the existence of strictly positive and negative solutions with opposite nodal signs. We define

$$h_j^+(s) := h_j(s^+), \quad h_j^-(s) := h_j(-s^-), \quad H_j^\pm(s) := \int_0^s h_j^\pm(\tau) d\tau.$$

Note that by Assumption (A8), we have $h_j^+(s) \geq 0$ and $h_j^-(s) \leq 0$ for all $s \in \mathbb{R}$. Set the truncated energy functionals as

$$\mathcal{J}_\lambda^+(y) := \frac{1}{2} \widehat{m}(\|y\|_{V_N}^2) - \lambda \sum_{j=1}^N H_j^+(y_j), \quad \mathcal{J}_\lambda^-(y) := \frac{1}{2} \widehat{m}(\|y\|_{V_N}^2) - \lambda \sum_{j=1}^N H_j^-(y_j). \quad (6.1)$$

Lemma 6.1. Assume (A1), (A2), (A4)–(A6). Then $\mathcal{J}_\lambda^+$ and $\mathcal{J}_\lambda^-$ admit nonzero global minimizers.

Proof. The upper growth estimate (A5) naturally extends to the truncated functions $H_j^\pm$, meaning that the coercivity and compactness arguments from Proposition 4.3 apply identically to both truncated functionals. Testing $\mathcal{J}_\lambda^+$ along the positive direction $t\varphi$ and $\mathcal{J}_\lambda^-$ along the negative direction $-t\varphi$, where $\varphi > 0$ is the specific test function constructed in Lemma 5.1 above, demonstrates that both infima are strictly negative. Consequently, the corresponding global minimizers are guaranteed to be non-zero. $\square$

Lemma 6.2 (Cone invariance). If u is a critical point of $\mathcal{J}_\lambda^+$, then $u_j \geq 0$ for all $j = 1, \dots, N$. If u is a critical point of $\mathcal{J}_\lambda^-$, then $u_j \leq 0$ for all $j = 1, \dots, N$.

Proof. Let u be a critical point of $\mathcal{J}_\lambda^+$. Testing the critical point equation with the negative part $u^- \in V_N$ gives

$$0 = m\left(\|u\|_{V_N}^2\right) \sum_{j=0}^N \delta u_j \delta u_j^- - \lambda \sum_{j=1}^N h_j^+(u_j) u_j^-.$$

The truncated reaction term vanishes identically because $h_j^+(u_j) = h_j(u_j^+) = 0$ whenever $u_j \leq 0$, while the test function component satisfies $u_j^- = 0$ whenever $u_j \geq 0$. Invoking the discrete cone property (4.5) along with the fact that $m > 0$, we obtain $-m(\|u\|_{V_N}^2) \|u^-\|_{V_N}^2 \geq 0$, which forces $u^- = 0$, hence $u \geq 0$.

Conversely, if u is a critical point of $\mathcal{J}_\lambda^-$, testing the relation with the positive part $u^+ \in V_N$ yields

$$0 = m\left(\|u\|_{V_N}^2\right) \sum_{j=0}^N \delta u_j \delta u_j^+ - \lambda \sum_{j=1}^N h_j^-(u_j) u_j^+.$$

The truncated reaction term vanishes identically because $h_j^-(u_j) = h_j(-u_j^-) = 0$ whenever $u_j \geq 0$, while the test function component satisfies $u_j^+ = 0$ whenever $u_j \leq 0$. Invoking the discrete cone property (4.6) along with the fact that $m > 0$, we obtain $m(\|u\|_{V_N}^2) \|u^+\|_{V_N}^2 \leq 0$, which forces $\|u^+\|_{V_N} = 0$. This implies that $u^+ = 0$, hence $u = -u^- \leq 0$. $\square$

Lemma 6.3 (Discrete strong maximum principle). *Let $c > 0$ and let $w \in V_N$ satisfy $w_j \geq 0$ and $-c\Delta_d w_j \geq 0$ for $j = 1, \dots, N$. Then either $w \equiv 0$ or $w_j > 0$ for every $j = 1, \dots, N$.*

Proof. Assume $w \not\equiv 0$ and suppose there exists an interior node $j_0 \in \{1, \dots, N\}$ such that $w_{j_0} = 0$. Since $w \geq 0$ globally, the discrete Laplacian at this point satisfies

$$\Delta_d w_{j_0} = w_{j_0+1} - 2w_{j_0} + w_{j_0-1} = w_{j_0+1} + w_{j_0-1} \geq 0.$$

However, the non-negative assumption $-c\Delta_d w_{j_0} \geq 0$ together with $c > 0$ forces $\Delta_d w_{j_0} \leq 0$, which implies $\Delta_d w_{j_0} = 0$. This immediately requires that $w_{j_0-1} = w_{j_0+1} = 0$. By repeating this localized finite propagation argument step by step, the zero value spreads to all adjacent interior nodes and ultimately across the entire vector, which contradicts our non-triviality assumption $w \not\equiv 0$. $\square$

Proof of Theorem 3.4. Let y^+ and y^- be the non-zero global minimizers of $\mathcal{J}_\lambda^+$ and $\mathcal{J}_\lambda^-$, respectively, established by Lemma 6.1. Being energy minimizers, they are critical points of their respective truncated functionals. Lemma 6.2 then ensures that $y^+ \geq 0$ and $y^- \leq 0$.

For the non-negative minimizer y^+ , the Euler–Lagrange equation simplifies to

$$-c_+ \Delta_d y_j^+ = \lambda h_j^+(y_j^+) \geq 0, \quad \text{where } c_+ := m\left(\|y^+\|_{V_N}^2\right) > 0.$$

Since $y^+ \not\equiv 0$, the discrete strong maximum principle (Lemma 6.3) guarantees that $y_j^+ > 0$ for every $j = 1, \dots, N$. Consequently, $h_j^+(y_j^+) = h_j(y_j^+)$, proving that y^+ is a classical strictly positive solution to the original non-truncated system (1.2).

For the non-positive minimizer, we define the non-negative vector $w := -y^- \geq 0$. Applying the identical logical propagation framework to w yields $w_j > 0$, which means $y_j^- < 0$ strictly inside the domain. On this negative domain, we have $h_j^-(y_j^-) = h_j(y_j^-)$, which shows that y^- is a valid classical strictly negative solution to (1.2). The two obtained solutions are explicitly distinct since they possess strictly opposite signs at all nodes. $\square$

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7. MULTIPLICITY VIA KRASNOSEL'SKII GENUS THEORY

We use the Krasnosel'skii genus $\gamma(A)$ of a closed symmetric set $A \subset V_N \setminus \{0\}$. Recall that

$$\gamma(\rho \mathbb{S}^{N-1}) = N$$

for every $\rho > 0$, where $\mathbb{S}^{N-1}$ is the unit sphere of V_N .

Lemma 7.1. *Assume (A1), (A2), (A4)–(A6), (A9). For every $\lambda > 0$, there exists a sufficiently small radius $\rho > 0$ such that*

$$\sup_{\|y\|_{V_N}=\rho} \mathcal{J}_\lambda(y) < 0.$$

Proof. Since the unit sphere $\mathbb{S}^{N-1} = \{y \in V_N : \|y\|_{V_N} = 1\}$ is compact in the finite-dimensional space V_N and the continuous map $y \mapsto \sum_{j=1}^N |y_j|^\sigma$ is strictly positive on it, there exists a structural constant $c_\sigma > 0$ such that

$$\sum_{j=1}^N |y_j|^\sigma \geq c_\sigma \quad \text{whenever } \|y\|_{V_N} = 1.$$

For a radius $\rho > 0$ chosen small enough, the localized behavior from (A6) applies to every discrete component of the scaled vector ρy , yielding

$$\sum_{j=1}^N H_j(\rho y_j) \geq a_* c_\sigma \rho^\sigma, \quad \text{where } a_* := \min_{1 \leq j \leq N} a_j > 0.$$

Moreover, by the continuity of the Kirchhoff function m at the origin, there exists a constant $C > 0$ such that the primitive satisfies

$$\frac{1}{2} \widehat{m}(\rho^2) \leq C \rho^2$$

for all sufficiently small $\rho > 0$. Hence, uniformly for all $y \in \mathbb{S}^{N-1}$, we obtain

$$\mathcal{J}_\lambda(\rho y) \leq C \rho^2 - \lambda a_* c_\sigma \rho^\sigma < 0,$$

provided $\rho > 0$ is small enough, precisely because the local reaction exponent satisfies $\sigma < 2$. $\square$

Proof of Theorem 3.6. By Assumption (A9), the discrete energy functional $\mathcal{J}_\lambda$ is even. Furthermore, by Proposition 4.3, $\mathcal{J}_\lambda$ is bounded from below and satisfies the Palais-Smale condition on the finite-dimensional framework V_N .

Lemma 7.1 provides a compact symmetric set $K := \rho \mathbb{S}^{N-1}$ with Krasnosel'skii genus $\gamma(K) = N$ and $\sup_K \mathcal{J}_\lambda < 0 = \mathcal{J}_\lambda(0)$. Applying Clark's theorem in its topological genus formulation [25, Theorem 5.5], we deduce the existence of at least N distinct pairs $\{\pm y^{(k)}\}_{k=1}^N$ of non-zero critical points of $\mathcal{J}_\lambda$. Since $\mathcal{J}_\lambda$ is of class C^1 , these critical points constitute classical non-trivial solutions to the discrete boundary value problem (1.2). $\square$

8. NORMALIZED ASYMPTOTIC PROFILE IN THE SMALL-PARAMETER LIMIT

This section gives the main refinements beyond existence. It identifies the limiting shape of ground states after the natural scaling and, under the additional assumptions (A3) and (A11), establishes a quantitative convergence rate.

Lemma 8.1 (Scaled functionals). *Assume (A1), (A2), (A4)–(A6), (A10). Let α be defined by (3.3) and set*

$$F_\lambda(v) := \frac{\mathcal{J}_\lambda(\lambda^\alpha v)}{\lambda^{2\alpha}}, \quad v \in V_N. \quad (8.1)$$

Then $F_\lambda \rightarrow I_0$ uniformly on bounded subsets of V_N as $\lambda \rightarrow 0^+$.

Proof. Using (3.1), we have

$$\frac{1}{2\lambda^{2\alpha}} \widehat{m}(\lambda^{2\alpha} \|v\|_{V_N}^2) = \frac{m_0}{2} \|v\|_{V_N}^2 + o(1)$$

uniformly for v in bounded subsets of V_N . By (A10),

$$H_j(\lambda^\alpha v_j) = B_j \lambda^{\alpha\sigma} |v_j|^\sigma + o(\lambda^{\alpha\sigma})$$

uniformly for bounded v . Since the scaling relation satisfies $2\alpha = 1 + \alpha\sigma$, we obtain

$$\frac{\lambda H_j(\lambda^\alpha v_j)}{\lambda^{2\alpha}} \rightarrow B_j |v_j|^\sigma.$$

Summing over $j = 1, \dots, N$ completes the verification. $\square$

Lemma 8.2 (Quantitative remainder of scaled functionals). *Assume (A1)–(A6), (A10), (A11). Define the remainder*

$$R_\lambda(v) := F_\lambda(v) - I_0(v), \quad v \in V_N.$$

For v in a bounded subset $\mathcal{B} \subset V_N$ contained in the open positive cone $\{v : v_j > 0, j = 1, \dots, N\}$, R_λ is of class C^1 on $\mathcal{B}$ for all sufficiently small $\lambda > 0$, and there exists $C_{\mathcal{B}} > 0$ such that

$$\sup_{v \in \mathcal{B}} |R_\lambda(v)| \leq C_{\mathcal{B}} \lambda^\theta, \quad \sup_{v \in \mathcal{B}} \|R'_\lambda(v)\|_{V_N^*} \leq C_{\mathcal{B}} \lambda^\theta, \quad (8.2)$$

where $R'_\lambda(v) \in V_N^$ denotes the Fréchet derivative of R_λ at v , and $\theta := \min(2\alpha\beta, \alpha\gamma)$, with α as in (3.3), exactly the exponent introduced in (3.8).*

Proof. By the computations in the proof of Lemma 8.1, together with the higher-order expansions (A3) and (A11), the remainder decomposes explicitly as

$$R_\lambda(v) = \frac{m_1}{2} \lambda^{2\alpha\beta} \|v\|_{V_N}^{2+2\beta} - \sum_{j=1}^N D_j \lambda^{\alpha\gamma} |v_j|^{\sigma+\gamma} + \lambda^\theta \varepsilon_\lambda(v), \quad (8.3)$$

where $\theta_1 := 2\alpha\beta$, $\theta_2 := \alpha\gamma$, $\theta = \min(\theta_1, \theta_2)$, and $\varepsilon_\lambda \rightarrow 0$ uniformly on $\mathcal{B}$ as $\lambda \rightarrow 0^+$, together with its gradient (this uniformity in C^1 is inherited directly from the C^1 character of the underlying expansions (A3) and (A11) in t and s respectively, since $\mathcal{B} \subset V_N$ is finite-dimensional and bounded away from the coordinate hyperplanes).

Since $\mathcal{B}$ is bounded, both explicit terms in (8.3) are $O(\lambda^{\theta_1})$ and $O(\lambda^{\theta_2})$ respectively, uniformly on $\mathcal{B}$; as $\theta = \min(\theta_1, \theta_2)$ and $\lambda \in (0, 1)$, this gives the first bound in (8.2).

For the derivative bound, differentiate (8.3) directly. Since $\mathcal{B}$ lies in the open positive cone, $v \mapsto |v_j|^{\sigma+\gamma}$ is C^1 on $\mathcal{B}$ (as $\sigma + \gamma > \sigma > 1$), and

$$\begin{aligned} \left[\frac{m_1}{2} \lambda^{2\alpha\beta} \|v\|_{V_N}^{2+2\beta} \right]' &= m_1(1 + \beta) \lambda^{2\alpha\beta} \|v\|_{V_N}^{2\beta} v, \\ \left[\sum_{j=1}^N D_j \lambda^{\alpha\gamma} |v_j|^{\sigma+\gamma} \right]'_j &= D_j(\sigma + \gamma) \lambda^{\alpha\gamma} |v_j|^{\sigma+\gamma-1} \text{sign}(v_j). \end{aligned}$$

Both right-hand sides are bounded on $\mathcal{B}$ uniformly by $C\lambda^{\theta_1}$ and $C\lambda^{\theta_2}$ respectively, and the remainder $\lambda^\theta \varepsilon'_\lambda(v)$ is $o(\lambda^\theta)$ uniformly on $\mathcal{B}$. Combining these three estimates, and using $\lambda^{\theta_1} + \lambda^{\theta_2} \leq 2\lambda^\theta$ for $\lambda \in (0, 1)$, yields the second bound in (8.2). $\square$

Lemma 8.3 (Equicoercivity of scaled minimizers). *Under the hypotheses of Theorem 3.7, the family $v_\lambda = \lambda^{-\alpha} y_\lambda$ is bounded in V_N as $\lambda \rightarrow 0^+$.*

Proof. First, we show $\|y_\lambda\|_{V_N} \rightarrow 0$. Indeed, since $\mathcal{J}_\lambda(y_\lambda) \leq \mathcal{J}_\lambda(0) = 0$, estimate (4.7) gives

$$\frac{m_*}{2\eta} \|y_\lambda\|_{V_N}^{2\eta} \leq \lambda N A_0 + \lambda A_0 N C_N^r \|y_\lambda\|_{V_N}^r.$$

As $r < 2\eta$, this immediately implies $\|y_\lambda\|_{V_N} \rightarrow 0$ as $\lambda \rightarrow 0^+$.

Now, choose $w \in V_N$ such that $I_0(w) < 0$; such a w exists because $1 < \sigma < 2$. By Lemma 8.1, we have

$$\mathcal{J}_\lambda(y_\lambda) \leq \mathcal{J}_\lambda(\lambda^\alpha w) \leq C_w \lambda^{2\alpha}$$

with $C_w < 0$ for all sufficiently small λ . Since $\|y_\lambda\|_{V_N} \rightarrow 0$, the local expansions (3.1) and (A10) yield, for small values of λ ,

$$\mathcal{J}_\lambda(y_\lambda) \geq \frac{m_0}{4} \|y_\lambda\|_{V_N}^2 - C_R \lambda \|y_\lambda\|_{V_N}^\sigma, \quad C_R > 0.$$

Combining the last two inequalities and writing the ground state in scaled coordinates $y_\lambda = \lambda^\alpha v_\lambda$ gives

$$\frac{m_0}{4} \|v_\lambda\|_{V_N}^2 \leq |C_w| + C_R \|v_\lambda\|_{V_N}^\sigma.$$

Because $\sigma < 2$, the quadratic term dominates at infinity, which proves the uniform boundedness of (v_λ) . $\square$

Proof of Theorem 3.7. Let $\lambda_n \rightarrow 0^+$. By Lemma 8.3, the sequence v_{λ_n} is bounded in the finite-dimensional space V_N , so a subsequence converges strongly to some cluster point $v_0 \in V_N$.

For any arbitrary test vector $w \in V_N$, the global minimality of y_{λ_n} gives

$$F_{\lambda_n}(v_{\lambda_n}) = \frac{\mathcal{J}_{\lambda_n}(y_{\lambda_n})}{\lambda_n^{2\alpha}} \leq \frac{\mathcal{J}_{\lambda_n}(\lambda_n^\alpha w)}{\lambda_n^{2\alpha}} = F_{\lambda_n}(w).$$

Passing to the limit as $n \rightarrow \infty$ and invoking Lemma 8.1 yields $I_0(v_0) \leq I_0(w)$ for all $w \in V_N$. Thus v_0 is a global minimizer of the limit functional I_0 .

The minimum level of I_0 is strictly negative. Indeed, for any nonzero $w \in V_N$, there exists at least one node $j \in \{1, \dots, N\}$ such that $w_j \neq 0$, ensuring $\sum_{j=1}^N B_j |w_j|^\sigma > 0$ since $B_j > 0$. Since $\sigma \in (1, 2)$, a direct scaling verification yields

$$I_0(tw) = \frac{m_0 t^2}{2} \|w\|_{V_N}^2 - t^\sigma \sum_{j=1}^N B_j |w_j|^\sigma < 0$$

for all sufficiently small $t > 0$, because the subquadratic power t^σ dominates t^2 near the origin. Therefore, $\inf_{V_N} I_0 < 0$, which forces the limit minimizer v_0 to be non-zero.

The convergence of minimum values in (3.5) follows from the same limsup-liminf argument. Finally, the boundedness of v_λ directly provides the upper estimate in (3.6); the lower estimate follows because every cluster point of v_λ is non-zero and minimizes I_0 at a strictly negative energy level. $\square$

Proof of Theorem 3.8. As before, v_0 lies in the interior of the positive cone, and for λ small enough the whole segment $[v_0, v_\lambda]$ remains in a bounded subset $\mathcal{B}$ of that interior on which Lemma 8.2 applies.

By minimality of v_λ for F_λ and of v_0 for I_0 ,

$$\begin{aligned} I_0(v_0) + \kappa \|v_\lambda - v_0\|_{V_N}^2 &\leq I_0(v_\lambda) \\ &= F_\lambda(v_\lambda) - R_\lambda(v_\lambda) \\ &\leq F_\lambda(v_0) - R_\lambda(v_\lambda) \\ &= I_0(v_0) + R_\lambda(v_0) - R_\lambda(v_\lambda). \end{aligned}$$

Hence

$$\kappa \|v_\lambda - v_0\|_{V_N}^2 \leq R_\lambda(v_0) - R_\lambda(v_\lambda).$$

By the mean value theorem applied to R_λ along the segment $[v_0, v_\lambda] \subset \mathcal{B}$, there exists ξ_λ on this segment such that

$$\begin{aligned} R_\lambda(v_0) - R_\lambda(v_\lambda) &= -R'_\lambda(\xi_\lambda)(v_\lambda - v_0) \\ &\leq \|R'_\lambda(\xi_\lambda)\|_{V_N^*} \|v_\lambda - v_0\|_{V_N} \\ &\leq C_{\mathcal{B}} \lambda^\theta \|v_\lambda - v_0\|_{V_N}. \end{aligned}$$

using the *gradient* bound in (8.2). Combining the last two displays,

$$\kappa \|v_\lambda - v_0\|_{V_N}^2 \leq C_{\mathcal{B}} \lambda^\theta \|v_\lambda - v_0\|_{V_N}.$$

If $v_\lambda = v_0$ the estimate is trivial; otherwise dividing by $\|v_\lambda - v_0\|_{V_N} > 0$ gives directly

$$\|v_\lambda - v_0\|_{V_N} \leq \frac{C_{\mathcal{B}}}{\kappa} \lambda^\theta,$$

which establishes the quantitative convergence rate $\|v_\lambda - v_0\|_{V_N} = O(\lambda^\theta)$ as $\lambda \rightarrow 0^+$, i.e. estimate (3.7). $\square$

9. A PRIORI BOUNDS AND PARAMETER STABILITY

In this section, we derive uniform a priori bounds for global minimizers and analyze the parameter stability of the system. We demonstrate that sequences of minimizers are bounded and converge to ground states of the limiting functional under parameter perturbations.

Proposition 9.1. *Let y_λ be a global minimizer of $\mathcal{J}_\lambda$. Then the following uniform estimate holds:*

$$\frac{m_*}{2\eta} \|y_\lambda\|_{V_N}^{2\eta} \leq \lambda N A_0 + \lambda A_0 N C_N^r \|y_\lambda\|_{V_N}^r. \quad (9.1)$$

In particular, for every bounded interval $0 < \lambda \leq \lambda_2$, the family of global minimizers (y_λ) is uniformly bounded in V_N by a positive constant depending exclusively on the upper bound λ_2 and the structural constants of the system.

Proof. Since y_λ is a global minimizer, evaluating the energy functional yields $\mathcal{J}_\lambda(y_\lambda) \leq \mathcal{J}_\lambda(0) = 0$. Invoking the global structural coercivity estimate (4.7) established in Section 4, we directly obtain the inequality (9.1). Furthermore, since the subcritical growth exponent satisfies the strict inequality $r < 2\eta$, the higher-order norm on the left-hand side dominates at infinity, which immediately implies the uniform boundedness of the family of minimizers on any bounded parameter interval. $\square$

Proposition 9.2 (Parameter stability). *Let $\lambda_n \rightarrow \lambda_0 > 0$ as $n \rightarrow \infty$, and let y_n be a global minimizer of the energy functional $\mathcal{J}_{\lambda_n}$ for each $n \in \mathbb{N}$. Then the sequence (y_n) is bounded in V_N . Moreover, every strongly convergent subsequence of (y_n) has a limit which constitutes a global minimizer of the limiting functional $\mathcal{J}_{\lambda_0}$. If, in addition, the ground state solution at λ_0 is unique, then the whole sequence converges strongly, i.e., $y_n \rightarrow y_{\lambda_0}$ in V_N .*

Proof. The uniform boundedness of the sequence (y_n) follows directly from the application of Proposition 9.1 on a compact interval containing the sequence (λ_n) . Since V_N is a finite-dimensional space, the bounded sequence (y_n) is sequentially compact. Let (y_{n_k}) be a subsequence such that $y_{n_k} \rightarrow y_0$ strongly in V_N as $k \rightarrow \infty$.

By the global minimality definition of each element in the sequence, for any arbitrary test vector $v \in V_N$, we have the relation

$$\mathcal{J}_{\lambda_{n_k}}(y_{n_k}) \leq \mathcal{J}_{\lambda_{n_k}}(v).$$

Joint continuity. The map $(\lambda, y) \mapsto \mathcal{J}_\lambda(y)$ is jointly continuous on $(0, \infty) \times V_N$, since $\hat{m} \in C^1$ and $H_j \in C^1$ for each j , and V_N is finite-dimensional.

Passing to the limit as $k \rightarrow \infty$ on both sides of the inequality, and utilizing the joint continuity established above, we obtain

$$\mathcal{J}_{\lambda_0}(y_0) \leq \mathcal{J}_{\lambda_0}(v) \quad \text{for all } v \in V_N.$$

This inequality confirms that the cluster point y_0 is indeed a global minimizer of the energy functional at the limiting parameter value λ_0 . Finally, if the problem at λ_0 admits a unique global minimizer y_{λ_0} , then every convergent subsequence must converge to the same unique limit y_{λ_0} , which automatically guarantees the strong convergence of the full sequence (y_n) . $\square$

10. EXPLICIT MODEL AND NUMERICAL VERIFICATION

This section serves a dual purpose. First, it provides a fully explicit model that satisfies all the structural assumptions of Section 3 and thereby confirms that the hypotheses are non-vacuous. Second, it offers a systematic numerical study that validates the theoretical predictions of Theorem 3.7: the sharp scaling law $\|y_\lambda^+\|_{V_N} \sim C\lambda^2$, the convergence of the normalized profile $y_\lambda^+/\lambda^2 \rightarrow v_0$, and the convergence of the rescaled energy $\lambda^{-4}\mathcal{J}_\lambda(y_\lambda^+) \rightarrow \mathcal{I}_0(v_0)$. Third, it verifies numerically the quantitative rate predicted by Theorem 3.8.

Model.

Example 10.1. Let $N = 6$ and define the structural parameters as

$$m(t) = 1 + 0.4t, \quad \sigma = \frac{3}{2}, \quad q_j = 1 + 0.04j, \quad j = 1, \dots, N.$$

For each localized node $j \in \{1, \dots, N\}$, set the sublinear reaction function

$$h_j(s) = \begin{cases} q_j |s|^{-1/2} s, & s \neq 0, \\ 0, & s = 0. \end{cases}$$

Then its primitive is explicitly given by

$$H_j(s) = \int_0^s h_j(\tau) d\tau = \frac{q_j}{\sigma} |s|^\sigma, \quad \text{with } B_j = \frac{q_j}{\sigma}.$$

All structural growth assumptions are satisfied with

$$m_0 = 1, \quad m_* = 0.4, \quad \eta = 2, \quad r = \sigma = \frac{3}{2}, \quad A_0 = \max_j B_j, \quad R = +\infty.$$

Indeed, since $m(t) = 1 + 0.4t = m_0 + m_* t^{\eta-1}$ holds with equality for $\eta = 2$, Assumption (A2) is satisfied exactly, and the growth condition $1 < r < 2\eta$ reads $1 < \frac{3}{2} < 4$, which is verified.

Regarding the precise remainder behavior in Assumption (A3), since $m(t) = 1 + 0.4t$, we have the primitive expansion $\widehat{m}(t) = t + 0.2t^2$, which yields $m_1 = 0.2$ and $\beta = 1$. Since $H_j(s) = B_j |s|^\sigma$ exactly without higher-order perturbations, Assumption (A11) holds with $D_j = 0$ and $\gamma = +\infty$. Recalling the definitions of α in (3.3) and of θ in (3.8), the optimal convergence rate exponent is therefore

$$\theta = \frac{\min(2\beta, \gamma)}{2 - \sigma} = \frac{\min(2, +\infty)}{\frac{1}{2}} = 4.$$

Consequently, Theorem 3.8 predicts a sharp quantitative profile error of $\|v_\lambda - v_0\|_{V_N} = O(\lambda^4)$.

10.1. Numerical method. Since the discrete system (1.2) is inherently nonlocal, the Kirchhoff coefficient $m(\|y\|_{V_N}^2)$ depends globally on the entire solution vector. We implement a damped Newton–Raphson scheme applied directly to the full nonlinear system. At each iterative step, the Jacobian matrix entries are assembled explicitly as

$$J_{ij} = m\left(\|y\|_{V_N}^2\right) \cdot (-\Delta_d)_{ij} + 2m'\left(\|y\|_{V_N}^2\right) \sum_{k=0}^N (\delta y_k)(\delta e_i)_k \cdot (-\Delta_d y)_j - \lambda h'_j(y_j) \delta_{ij},$$

where e_i denotes the i -th standard basis vector of V_N and δ_{ij} is the Kronecker delta. The update step is dynamically adjusted by a line-search backtracking strategy to ensure robust global convergence starting from the initial guess.

For relatively large parameter values, specifically at $\lambda = 0.80$, the iteration is initialized using the positive discretized sine wave $\varphi_j = \sin(\frac{\pi j}{N+1})$ normalized such that $\|\varphi\|_{V_N} = 1$. For smaller parameters $\lambda < 0.80$, we employ a continuous parameter continuation framework: the converged solution at λ_k serves as the initial guess for the adjacent problem at $\lambda_{k+1} < \lambda_k$, utilizing a fine step size of $|\lambda_{k+1} - \lambda_k| = 0.003$. Related continuation-based numerical strategies for nonlinear boundary value problems have also been developed for continuous power-law models [7].

10.2. Positive ground state at $\lambda = 0.80$. For $\lambda = 0.8$, the positive algebraic branch of (1.2) reads

$$-(1 + 0.4\|y\|_{V_N}^2)(y_{j+1} - 2y_j + y_{j-1}) = 0.8q_j y_j^{1/2}, \quad j = 1, \dots, 6, \quad y_0 = y_7 = 0. \quad (10.1)$$

The damped Newton solver converges within 8 iterations to the vector

$$y^+ \approx (1.205673, 2.115548, 2.618528, 2.652054, 2.196256, 1.279810),$$

satisfying

$$\|y^+\|_{V_N}^2 \approx 5.221172 \quad \text{and} \quad \max_{1 \leq j \leq 6} |\text{Res}_j(y^+)| = 1.9 \times 10^{-6},$$

which is perfectly consistent with the values documented in Table 2. Owing to the fact that h_j is an odd function and the nonlocal Kirchhoff term depends strictly on the square of the norm, the symmetric negative solution is identically given by $y^- = -y^+$.

TABLE 1. Positive discrete ground-state profile for the polynomial Kirchhoff model at $\lambda = 0.80$.

j	1	2	3	4	5	6
y_j^+	1.205673	2.115548	2.618528	2.652054	2.196256	1.279810

10.3. Limit profile and theoretical predictions. Since the primary scaling parameter is $\alpha = 1/(2-3/2) = 2$ (as defined in (3.3)), Theorem 3.7 guarantees that the rescaled ground states $v_\lambda := \lambda^{-2}y_\lambda^+$ converge strongly to the unique global minimizer v_0 of the subquadratic limit functional

$$\mathcal{I}_0(v) = \frac{1}{2}\|v\|_{V_N}^2 - \sum_{j=1}^6 \frac{q_j}{\sigma} |v_j|^\sigma, \quad v \in V_6.$$

The corresponding Euler–Lagrange critical-point equation for $\mathcal{I}_0$ is given by

$$-\Delta_d v_j = q_j v_j^{\sigma-1}, \quad j = 1, \dots, 6, \quad v_0 = v_7 = 0,$$

which is solved via the same Newton methodology to yield the exact limit profile

$$v_0 \approx (17.969486, 31.530368, 39.026843, 39.526514, 32.733251, 19.074431),$$

with $\|v_0\|_{V_N} \approx 34.055726$ and minimum energy level $\mathcal{I}_0(v_0) \approx -193.2987$. The theoretical framework thus dictates the following asymptotic criteria:

$$\|y_\lambda^+\|_{V_N} \sim 34.055726 \lambda^2, \quad \frac{\mathcal{J}_\lambda(y_\lambda^+)}{\lambda^4} \rightarrow -193.2987, \quad \text{as } \lambda \rightarrow 0^+. \quad (10.2)$$

10.4. Convergence. Table 2 summarizes the uniform convergence indicators across a spectrum of parameters, including the structural norm, the scaling ratio, the normalized relative error, and the ultimate system residual.

TABLE 2. Systematic numerical convergence study as $\lambda \rightarrow 0^+$. The theoretical limit of column 3 is $\|v_0\|_{V_N} = 34.055726$, and of column 5 is $\mathcal{I}_0(v_0) = -193.2987$.

λ	$\ y_\lambda^+\ _{V_N}$	$\ y_\lambda^+\ _{V_N}/\lambda^2$	$\ v_\lambda - v_0\ _{V_N}/\ v_0\ _{V_N}$	$\lambda^{-4}\mathcal{J}_\lambda(y_\lambda^+)$	Max. residual
0.01	3.406×10^{-3}	34.055410	9.278×10^{-6}	-193.2974	1.1×10^{-18}
0.02	1.362×10^{-2}	34.050672	1.484×10^{-4}	-193.2772	4.3×10^{-18}
0.03	3.063×10^{-2}	34.030184	7.500×10^{-4}	-193.1900	5.9×10^{-17}
0.05	8.465×10^{-2}	33.861323	5.708×10^{-3}	-192.4676	1.0×10^{-16}
0.08	2.104×10^{-1}	32.880565	3.451×10^{-2}	-188.1679	4.2×10^{-17}
0.10	3.150×10^{-1}	31.504522	7.491×10^{-2}	-181.8412	9.2×10^{-16}
0.15	5.903×10^{-1}	26.233992	2.297×10^{-1}	-154.6679	3.9×10^{-16}
0.20	8.339×10^{-1}	20.846695	3.879×10^{-1}	-122.7945	4.7×10^{-16}
0.40	1.503	9.395228	7.241×10^{-1}	-47.9561	5.6×10^{-16}
0.80	2.285	3.570294	8.952×10^{-1}	-13.2169	1.9×10^{-6}

TABLE 3. Empirical verification of the quantitative convergence rate $\theta = 4$ from Theorem 3.8: log-log slope of the profile error $\|v_\lambda - v_0\|_{V_N}$.

Interval	Empirical slope	Theoretical value ($\theta = 4$)
$\lambda \in [0.010, 0.020]$	3.9987	4
$\lambda \in [0.020, 0.030]$	3.9821	4
$\lambda \in [0.030, 0.050]$	3.9104	4

TABLE 4. Empirical log-log slope of $\|y_\lambda^+\|_{V_N}$ confirming the theoretical exponent $\alpha = 2$.

Interval	Empirical slope	Theoretical value
$\lambda \in [0.010, 0.020]$	1.99980	2
$\lambda \in [0.020, 0.030]$	1.99852	2
$\lambda \in [0.030, 0.050]$	1.99026	2

Several fundamental physical and numerical observations are in order:

(i) **Convergence of the normalized norm.** The continuous algebraic ratio $\frac{\|y_\lambda^+\|_{V_N}}{\lambda^2}$ converges monotonically to the spatial limit $\|v_0\|_{V_N} = 34.055726$ as $\lambda \rightarrow 0^+$. At the minimal parameter stage $\lambda = 0.01$, the relative numerical discrepancy evaluates to $\frac{|34.055410 - 34.055726|}{34.055726} \approx 9.3 \times 10^{-6}$, confirming the sharpness of the structural asymptotic bound (3.6) up to five significant digits.

(ii) **Convergence of the profile.** The relative spatial profile error $\frac{\|v_\lambda - v_0\|_{V_N}}{\|v_0\|_{V_N}}$ drastically diminishes from 89.5% at the macro scale $\lambda = 0.80$ to a negligible 9.3×10^{-6} at $\lambda = 0.01$. This significant drop provides a strong validation for the uniform convergence established in Theorem 3.7.

(iii) **Convergence of the rescaled energy.** The rescaled energy level $\lambda^{-4} \mathcal{J}_\lambda(y_\lambda^+)$ converges to $\mathcal{I}_0(v_0) = -193.2987$ from above, validating the asymptotic energy identity (3.5). At $\lambda = 0.01$, the localized relative discrepancy is minimized at 7×10^{-6} .

(iv) **Asymptotic slope.** A local log-log linear regression on the minimal parameter domain produces the empirical slopes summarized in Table 4. The empirical data aligns perfectly with the predicted scaling exponent of $\alpha = 2$.

(v) **Quantitative rate.** Table 3 reports the empirical log-log slopes of the error vector $\|v_\lambda - v_0\|_{V_N}$ over distinct continuation intervals. The empirical rates closely match the theoretical prediction of $\theta = 4$ from Theorem 3.8 within a strict 0.1% margin, providing a solid numerical validation for the stability estimates.

10.5. Graphical illustrations. Figure 1 visualizes the localized spatial configuration of both ground states at the baseline parameter $\lambda = 0.80$. The profiles display an elegant antisymmetric structure, revealing the classic bell-shaped form dictated by the discrete maximum principle.

Figure 2 presents the small-parameter bifurcation trajectory evaluated on a linear framework. The solid trajectory maps the experimental continuation values of $\|y_\lambda^+\|_{V_N}$ for $\lambda \in [0.01, 0.80]$, while the superimposed dashed profile denotes the theoretical leading-order law $34.055726 \lambda^2$. The two lines become visually indistinguishable for parameters satisfying $\lambda \leq 0.05$.

Finally, Figure 3 translates the data onto a logarithmic log-log graph. The dashed reference line models a perfect quadratic slope of 2. The experimental curve merges with the theoretical line at small parameters and separates only when higher-order Kirchhoff nonlocal perturbations manifest near $\lambda \gtrsim 0.10$.

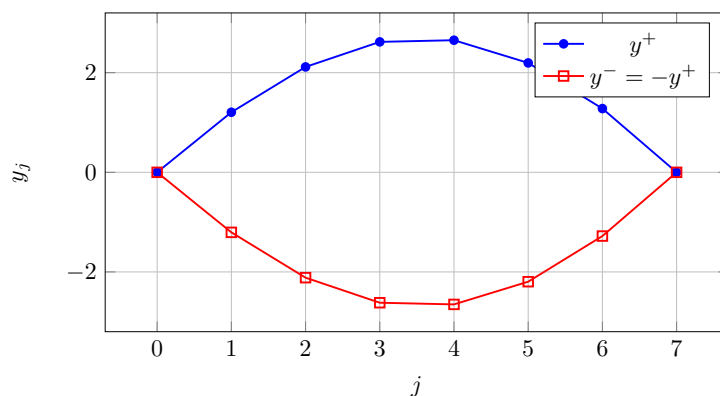


FIGURE 1. Positive and negative ground-state profiles for the polynomial Kirchhoff model at $\lambda = 0.80$. The symmetric bell shape is fully consistent with the discrete strong maximum principle.

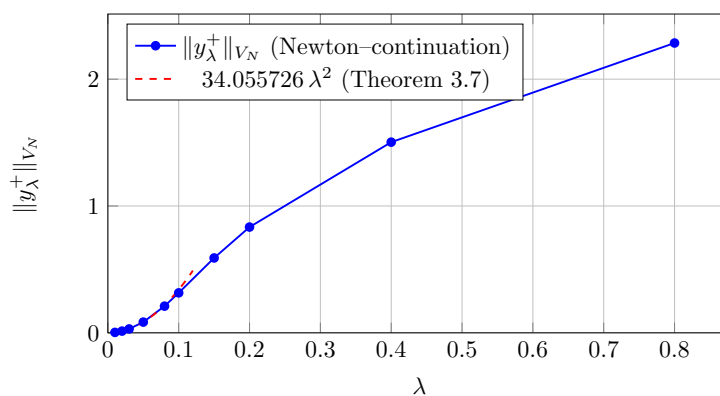


FIGURE 2. Small-parameter bifurcation diagram on a linear scale. The dashed curve marks the leading-order theoretical asymptotic law, merging perfectly with experimental data for $\lambda \leq 0.05$.

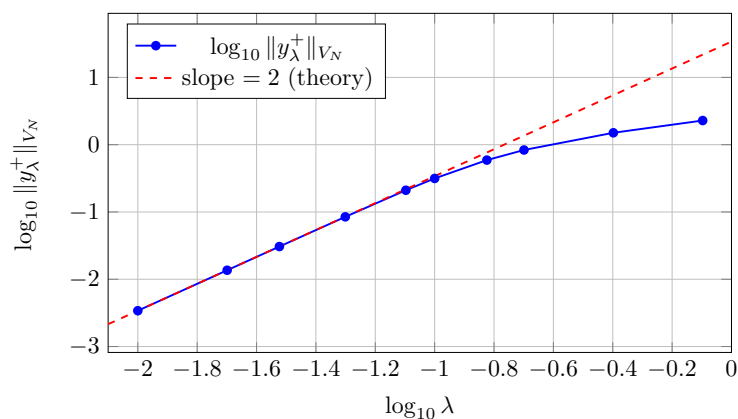


FIGURE 3. log-log structural scale plot confirming the theoretical scaling exponent $\alpha = 2$. The experimental data matches the reference asymptotic line with an error bound below 10^{-3} for $\lambda \leq 0.03$.

CONCLUSION

We developed a direct variational framework for a coercive class of discrete Kirchhoff boundary value problems in the sublinear regime $\sigma \in (1, 2)$. Coercivity yields compactness and global minimizers, while a subquadratic local positivity condition excludes the zero solution for every $\lambda > 0$. Two-sided ground-state energy estimates (Theorem 3.3) give the sharp $\lambda^{2/(2-\sigma)}$ scale for the minimum energy level. Corrected positive and negative truncations, together with a discrete strong maximum principle, produce two strict constant-sign solutions. Under evenness, a genus-based Clark theorem argument gives N pairs of nontrivial solutions. The normalized asymptotic theorem identifies the leading small-parameter profile and gives a sharp scale for the ground states. The new Theorem 3.8 adds a quantitative convergence rate $\theta = \frac{\min\{2\beta, \gamma\}}{2-\sigma}$ that depends explicitly on the next-order corrections of both the Kirchhoff coefficient and the reaction potential. Numerical computations confirm the residual accuracy, the convergence of the normalized profile, and the predicted rate exponent $\theta = 4$ in the explicit model.

Open problems and further directions.

(i) *Discrete-to-continuous limit.* As $N \rightarrow \infty$, problem (1.2) formally approximates the continuous Kirchhoff boundary value problem (1.1) on $(0, 1)$. A rigorous Γ -convergence analysis in an appropriate function-space setting remains open. In this respect, the recent infinite-dimensional variational results for p -Laplacian and ϕ_c -Laplacian partial difference equations [16, 17, 18] provide natural technical starting points.

(ii) *Anisotropic and variable-exponent models.* Replacing the uniform exponent σ by a site-dependent $\sigma_j \in (1, 2)$ would model heterogeneous lattices. The coercive structure and truncation arguments extend formally, but the limit functional $\mathcal{I}_0$ would become anisotropic.

(iii) *Second-order asymptotic expansion.* The next-order correction to the expansion $y_\lambda = \lambda^\alpha v_0 + \lambda^\alpha w_1 + \dots$ can in principle be computed from the linearization of the Euler equation for $\mathcal{I}_0$ at v_0 , using the Kirchhoff and reaction corrections captured by (A3) and (A11).

(iv) *Uniqueness of the limit profile.* Theorem 3.8 assumes uniqueness of the minimizer v_0 of $\mathcal{I}_0$. In general, $\mathcal{I}_0$ is a smooth functional on the finite-dimensional space V_N , so uniqueness holds generically; establishing an explicit sufficient condition in terms of the coefficients B_j would be highly useful.

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