

INCOMPRESSIBLE LIMIT FOR NON-ISENTROPIC COMPRESSIBLE EULER-NAVIER-STOKES TWO-PHASE FLOW MODEL IN $\mathbb{R}^3$

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ABSTRACT. This article concerns the incompressible limit for the two-phase flow model consisting of the compressible isothermal Euler equations coupled with compressible non-isentropic Navier-Stokes equations through a drag forcing term. For the 3D Cauchy problem, we rigorously justify the incompressible limits, which mean that the solutions converge to that of a compressible Euler/incompressible Navier-Stokes system and a compressible Euler/non-homogeneous incompressible Navier-Stokes system as Mach number goes to zero, under the small and large temperature variations, respectively.

1. MAIN RESULTS

The two-phase fluid system consisting of the compressible isothermal Euler equations coupled with the non-isentropic compressible Navier-Stokes equations through a drag forcing term in the whole space $\mathbb{R}^3$ is as follows

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \\ \rho_t + \operatorname{div}(\rho u) &= 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P &= \operatorname{div} \mathbb{S}(u) + \sigma n(w - u), \\ [\rho(e + \frac{|u|^2}{2})]_t + \operatorname{div}[\rho u(e + \frac{|u|^2}{2}) + Pu] &= \kappa \Delta \theta + \operatorname{div}(\mathbb{S}(u)u) + \sigma n(w - u)u, \end{aligned} \tag{1.1}$$

for $t > 0$ and $x \in \mathbb{R}^3$. Here, the unknown functions are the densities $n(x, t) > 0$ and $\rho(x, t) > 0$. The velocities of the particle and fluid are $w(x, t)$ and $u(x, t)$ respectively. While the temperature is $\theta(x, t) > 0$. Also, the pressure $P = P(\rho, \theta)$ and internal energy $e = e(\rho, \theta)$ are the smooth functions of ρ and θ : $\mathbb{D}(w)$, and $\mathbb{S}(u)$ are the deformation and the viscous stress tensors given by

$$\mathbb{D}(w) = \frac{1}{2} \left(\nabla w + \nabla^T w \right) \quad \text{and} \quad \mathbb{S}(u) = 2\mu \mathbb{D}(u) + \lambda \operatorname{div} u \mathbb{I}, \tag{1.2}$$

respectively, where T represents the transpose of a matrix, $\mathbb{I}$ is the identity matrix, and μ and λ are the constant shear and bulk viscosity coefficients, respectively, satisfying $\mu > 0$ and $3\lambda + 2\mu \geq 0$ which follow from the 2nd law of thermodynamics, $\kappa \geq 0$ denotes the heat-conductivity, and $\sigma > 0$ is the drag coefficient.

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System (1.1) is the non-isothermal version of the coupled Euler/Navier-Stokes two-phase flow model

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \\ \rho_t + \operatorname{div}(\rho u) &= 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) &= \operatorname{div} \mathbb{S}(u) + \sigma n(w - u), \end{aligned} \quad (1.3)$$

which can be formally derived from the barotropic compressible Navier-Stokes-Vlasov-Fokker-Planck equation with a local alignment force as follows (see [7]):

$$\begin{aligned} f_t + \xi \cdot \nabla_x f &= \operatorname{div}_\xi((\xi - u)f) + \frac{1}{\eta} \operatorname{div}_\xi((\xi - w)f + \nabla_\xi f), \\ \rho_t + \operatorname{div}_x(\rho u) &= 0, \\ (\rho u)_t + \operatorname{div}_x(\rho u \otimes u) + \nabla_x P(\rho) &= \operatorname{div} \mathbb{S}(u) + \sigma n(w - u), \quad \xi, x \in \mathbb{R}^3, t > 0, \end{aligned} \quad (1.4)$$

where $f = f(\xi, x, t)$ is the distribution function associated with the particles, $\eta > 0$, and n and nw are the macroscopical density and momentum defined by

$$n(x, t) := \int_{\mathbb{R}^3} f(\xi, x, t) d\xi, \quad nw(x, t) := \int_{\mathbb{R}^3} \xi f(\xi, x, t) d\xi. \quad (1.5)$$

The physical motivation of Euler or Navier-Stokes two-phase flow models shows that the individual particle velocity ξ is tracked statistically at the kinetic level (Vlasov-Fokker-Planck equation (1.4)₁) and integrating (1.5) over velocity ξ yields macroscopic equations for the particle density $n(x, t)$ and velocity

$$w(x, t) = \frac{\int_{\mathbb{R}^3} \xi f(\xi, x, t) d\xi}{\int_{\mathbb{R}^3} f(\xi, x, t) d\xi}.$$

Moreover, the isothermal compressible Euler (or Navier-Stokes) equations are coupled with the isentropic (or non-isentropic) compressible Navier-Stokes equations through a drag forcing term $\sigma n(w - u)$.

Recently, Li-Wang-Wang [20] applied the Chapman-Enskog expansion (see Chapman-Colwing [5]) for the fluid-particle model (1.4) around the local Maxwellian

$$\frac{n(x, t)}{\sqrt{(2\pi)^3}} e^{-\frac{|\xi - w(x, t)|^2}{2}}$$

to obtain the two-phase flow equations

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \quad x \in \mathbb{R}^3, t > 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= \eta \operatorname{div}(n \mathbb{D}(w)) - \sigma n(w - u) - \eta \int_{\mathbb{R}^3} \xi \otimes \xi \cdot \nabla_x G d\xi, \\ \rho_t + \operatorname{div}(\rho u) &= 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) &= \operatorname{div} \mathbb{S}(u) + \sigma n(w - u), \end{aligned} \quad (1.6)$$

where the term G is governed by the microscopic part. In the limiting case $\eta \rightarrow 0$, the system (1.6) converges to the coupled Euler/Navier-Stokes two-phase flow model (1.3).

The kinetic-fluid system (1.4) is used to describe the interactions between particles and fluid. Recently, this type of the kinetic-fluid system has attracted growing attention due to a very large range of applications, for example, biotechnology and medicine, and in the study of sedimentation phenomena, compressibility of droplets of the spray, and diesel engines, etc [10, 1, 13]. For the Euler/Navier-Stokes system (1.1), Choi [7] proved the global existence of a unique strong solution in both the whole space $\mathbb{R}^3$ and the periodic domain $\mathbb{T}^3$, and obtained the exponential decay rate in $\mathbb{T}^3$. Later, with some additional conditions on the initial data, Tang-Zhang [25], Wu-Zhang-Zou [28], and Zhang et al. [30] obtained the optimal algebraic decay rates in time of the global solutions in the whole space with Sobolev regularity. Wu-Zhou [29] extended L^2 -estimate in [28] to L^p -estimate with $p > 1$. Recently, Huang-Wang [18] proved the global existence of strong solution to 1-D Euler-Navier-Stokes system by using the standard continuity argument for small

H^1 data while the second order derivative can be large. See also similar results [8, 14] for the Euler-Navier-Stokes system (1.1) in a bounded domain. Also, for the pressureless Euler/Navier-Stokes system (i.e., system (1.1) without ∇n in (1.1)₂), Choi-Kwon [9] proved the global existence of a unique strong solution and obtained the exponential decay rate in $\mathbb{T}^3$. Recently, Li-Wang-Zhang [21] showed that the finite-energy classical solutions to the Cauchy problem of this pressureless Euler/Navier-Stokes system do not exist for any small time. The interested reader can refer to the review papers [12, 27] for the viscous two-fluid equations, while for the fluid-particles models, one can refer to [22, 4, 2, 19, 20] and the references therein. The main novelty of this paper is to justify the existence and low Mach number limit of the smooth solutions for the compressible Euler/full compressible Navier-Stokes system (1.1) by the pure elementary energy method.

For the incompressible limit analysis that we have in mind, it is convenient to introduce the nondimensional form of the system (1.1). For that purpose, we consider the following nondimensionalization:

$$\begin{aligned} \hat{t} &= \frac{t}{T_\infty}, \quad \hat{x} = \frac{x}{L_\infty}, \quad \hat{n} = \frac{n}{n_\infty}, \quad \hat{w} = \frac{w}{U_\infty}, \quad \hat{\rho} = \frac{\rho}{\rho_\infty}, \quad \hat{u} = \frac{u}{U_\infty}, \quad \hat{P} = \frac{P}{P_\infty}, \\ \hat{\theta} &= \frac{\theta}{\theta_\infty}, \quad \hat{e} = \frac{e}{e_\infty}, \quad \hat{\mu} = \frac{\mu}{\mu_\infty}, \quad \hat{\lambda} = \frac{\lambda}{\mu_\infty}, \quad \hat{\kappa} = \frac{\kappa}{\kappa_\infty}, \quad \hat{\sigma} = \frac{\sigma}{\sigma_\infty}, \end{aligned}$$

where $T_\infty, L_\infty, n_\infty, U_\infty, \rho_\infty, P_\infty, \theta_\infty, e_\infty, \mu_\infty, \sigma_\infty$, and κ_∞ denote the reference hydrodynamical quantities (time, length, particle density, mean velocity, fluid density, pressure, temperature, energy, viscosity, drag coefficient and heat-conductivity). Moreover, we denote by

$$S_r := \frac{L_\infty}{T_\infty U_\infty}, \quad M_a := \frac{U_\infty}{\sqrt{P_\infty/\rho_\infty}}, \quad Re := \frac{U_\infty \rho_\infty L_\infty}{\mu_\infty},$$

the Strouhal, Mach, and Reynolds numbers corresponding to hydrodynamics, and by

$$\mathcal{L}_n := \frac{1}{U_\infty^2}, \quad \mathcal{L}_{1\sigma} := \frac{\sigma_\infty L_\infty}{U_\infty}, \quad \mathcal{L}_{2\sigma} := \frac{\sigma_\infty L_\infty n_\infty}{U_\infty \rho_\infty},$$

the parameters corresponding to drag. Using the above scalings, we can rewrite (1.1) for the renormalized unknowns (still denoted by as (n, w, ρ, u, P)) follows :

$$\begin{aligned} S_r n_t + \operatorname{div}(nw) &= 0, \\ S_r (nw)_t + \operatorname{div}(nw \otimes w) + \mathcal{L}_n \nabla n &= -\mathcal{L}_{1\sigma} \sigma n(w - u), \\ S_r \rho_t + \operatorname{div}(\rho u) &= 0, \\ S_r (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \frac{1}{M_a^2} \nabla P &= \frac{1}{Re} \left(\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u \right) + \mathcal{L}_{2\sigma} \sigma n(w - u) \\ S_r (\rho e)_t + \operatorname{div}(\rho e u) + P \operatorname{div} u &= \kappa \Delta \theta + \frac{M_a^2}{Re} \left(2\mu \mathbb{D}(u) : \mathbb{D}(u) + \lambda (\operatorname{div} u)^2 \right). \end{aligned} \tag{1.7}$$

The main purpose of this article is to investigate the low Mach number limit of the system (1.7). Therefore, we set $M_a = \epsilon$ for small positive constant $\epsilon > 0$ and put $S_r = Re = \mathcal{L}_n = \mathcal{L}_{1\sigma} = \mathcal{L}_{2\sigma} = 1$ for the sake of simplicity. Then, we obtain from (1.7) that

$$\begin{aligned} n_t^\epsilon + \operatorname{div}(n^\epsilon w^\epsilon) &= 0, \\ (n^\epsilon w^\epsilon)_t + \operatorname{div}(n^\epsilon w^\epsilon \otimes w^\epsilon) + \nabla n^\epsilon &= -\sigma n^\epsilon (w^\epsilon - u^\epsilon), \\ \rho_t^\epsilon + \operatorname{div}(\rho^\epsilon u^\epsilon) &= 0, \\ (\rho^\epsilon u^\epsilon)_t + \operatorname{div}(\rho^\epsilon u^\epsilon \otimes u^\epsilon) + \frac{1}{\epsilon^2} \nabla P^\epsilon &= \mu \Delta u^\epsilon + (\mu + \lambda) \nabla \operatorname{div} u^\epsilon + \sigma n^\epsilon (w^\epsilon - u^\epsilon), \\ (\rho^\epsilon e^\epsilon)_t + \operatorname{div}(\rho^\epsilon u^\epsilon e^\epsilon) + P^\epsilon \operatorname{div} u^\epsilon &= \kappa \Delta \theta^\epsilon + \epsilon^2 \left(2\mu \mathbb{D}(u^\epsilon) : \mathbb{D}(u^\epsilon) + \lambda (\operatorname{div} u^\epsilon)^2 \right). \end{aligned} \tag{1.8}$$

Accordingly, the parameter ϵ essentially represents the maximum Mach number of the fluid. The limit of solutions of (1.8) as ϵ goes to 0 is usually called as low Mach number limit. The purpose of the low Mach number approximation is to justify that the compression, due to fluid pressure variations, can be neglected. This is a common assumption when discussing the fluid

dynamics of highly subsonic flows. So, we assume that the fluid pressure is a small perturbation of a given constant state $\bar{P}$, *i.e.*

$$P^\epsilon = \bar{P} + O(\epsilon). \quad (1.9)$$

Notation. In this paper, $L^p(\mathbb{R}^3)$ and $W_p^k(\mathbb{R}^3)$ denote the usual Lebesgue and Sobolev spaces on $\mathbb{R}^3$, with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. When $p = 2$, we denote $W_p^k(\mathbb{R}^3)$ by $H^k(\mathbb{R}^3)$ with the norm $\|\cdot\|_{H^k}$ and $\|\cdot\|_{H^0} = \|\cdot\|$ will be used to denote the usual L^2 -norm. The notation $\|(A_1, A_2, \dots, A_l)\|_{H^k}$ means the summation of $\|A_i\|_{H^k}$ from $i = 1$ to $i = l$. For an integer m , the symbol ∇^m denotes the summation of all terms D^α with the multi-index α satisfying $|\alpha| = m$. Then, ∇^{k+1} means $\nabla \nabla^k = \nabla \sum_{|\alpha|=k} D^\alpha$. We omit the spatial domain $\mathbb{R}^3$ in integrals for convenience.

Our first result is to show the incompressible limit for the system (1.8) in 3D with $\kappa > 0$ and the initial data

$$(n^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon)(x, 0) = (n_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)(x) \rightarrow (\bar{n}, 0, \bar{\rho}, 0, \bar{\theta}) \quad \text{as } |x| \rightarrow \infty, \quad (1.10)$$

when the temperature has a small variation, *i.e.*

$$\theta^\epsilon = \bar{\theta} + O(\epsilon), \quad (1.11)$$

where $\bar{n}, \bar{\rho}, \bar{\theta}$ are the given positive constants and $\bar{P} = P(\bar{\rho}, \bar{\theta})$. If we assume (1.9) and (1.11), then it holds that

$$\rho^\epsilon = \bar{\rho} + O(\epsilon) \quad \text{and} \quad e^\epsilon = \bar{e} + O(\epsilon),$$

where $\bar{e} = e(\bar{\rho}, \bar{\theta})$, and formally, as the Mach number tends to zero, the system (1.8) become

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \\ \operatorname{div} u &= 0, \\ u_t + (u \cdot \nabla)u + \frac{1}{\rho} \nabla \Pi &= \frac{\mu}{\rho} \Delta u + \frac{\sigma}{\rho} n(w - u) \end{aligned} \quad (1.12)$$

for a pressure function Π , which is considered by [6], where he investigated the global existence and large-time behavior of classical solutions to the system (1.12) in the periodic domain $\mathbb{T}^3$. The system (1.12) is a two-phase model with the compressible Euler equations and the incompressible Navier-Stokes equations coupled through a drag forcing term, and it is rigorously derived in [3] as the hydrodynamic limit of the Vlasov/incompressible Navier-Stokes system with strong noise and local alignment.

To state our first main result, by setting $\varphi^\epsilon = \ln\left(\frac{n^\epsilon}{\bar{n}}\right)$, we reformulate the Cauchy problem (1.8) with $\kappa > 0$, (1.10) as follows:

$$\begin{aligned} \varphi_t^\epsilon + \operatorname{div} w^\epsilon + w^\epsilon \cdot \nabla \varphi^\epsilon &= 0, \quad x \in \mathbb{R}^3, \quad t > 0, \\ w_t^\epsilon + (w^\epsilon \cdot \nabla)w^\epsilon + \nabla \varphi^\epsilon &= -\sigma(w^\epsilon - u^\epsilon), \\ \rho_t^\epsilon + \operatorname{div}(\rho^\epsilon u^\epsilon) &= 0, \\ \rho^\epsilon u_t^\epsilon + \rho^\epsilon(u^\epsilon \cdot \nabla)u^\epsilon + \frac{1}{\epsilon^2} \nabla P^\epsilon &= \mu \Delta u^\epsilon + (\mu + \lambda) \nabla \operatorname{div} u^\epsilon + \sigma \bar{n} e^{\varphi^\epsilon} (w^\epsilon - u^\epsilon), \\ \rho^\epsilon e_t^\epsilon + \rho^\epsilon u^\epsilon \cdot \nabla e^\epsilon + P^\epsilon \operatorname{div} u^\epsilon &= \kappa \Delta \theta^\epsilon + \epsilon^2 \left(2\mu \mathbb{D}(u^\epsilon) : \mathbb{D}(u^\epsilon) + \lambda (\operatorname{div} u^\epsilon)^2 \right), \end{aligned} \quad (1.13)$$

and

$$(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon)(x, 0) = (\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)(x) \rightarrow (0, 0, \bar{\rho}, 0, \bar{\theta}) \quad \text{as } |x| \rightarrow \infty, \quad (1.14)$$

where $\varphi_0^\epsilon = \ln\left(\frac{n_0^\epsilon}{\bar{n}}\right)$. Then we have the incompressible limit under small temperature variation:

Theorem 1.1. *Assume that*

$$P_\rho(\rho, \theta) > 0, \quad e_\theta(\rho, \theta) > 0 \quad \text{for all } \rho > 0, \theta > 0, \quad (1.15)$$

and

$$(\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon - \bar{\rho}, u_0^\epsilon, \theta_0^\epsilon - \bar{\theta}) \in H^N(\mathbb{R}^3),$$

$$\inf_{x \in \mathbb{R}^3} \rho_0^\epsilon(x) > 0, \quad \inf_{x \in \mathbb{R}^3} \theta_0^\epsilon(x) > 0, \quad \operatorname{div} u_0 = 0$$

hold for an integer $N \geq 3$. Moreover, for small positive constant $\delta_0 > 0$ independent of ϵ , the functions $(\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon, \theta_0^\epsilon)$ are assumed to satisfy

$$\|(\varphi_0^\epsilon, w_0^\epsilon, u_0^\epsilon)\|_{H^3} + \epsilon^{-1} \|(\rho_0^\epsilon - \bar{\rho}, \theta_0^\epsilon - \bar{\theta})\|_{H^3} \leq \delta_0. \quad (1.16)$$

Then the following statements hold.

- (1) (Global solutions and uniform estimate) There exists a constant C , independent of ϵ and δ_0 , such that the Cauchy problem (1.13), (1.14) admits a unique solution $(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon)$ on $[0, \infty)$ satisfying

$$\begin{aligned} (\varphi^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon, \theta^\epsilon - \bar{\theta}) &\in C([0, \infty), H^N(\mathbb{R}^3)), \\ (\nabla \varphi^\epsilon, \nabla \rho^\epsilon) &\in L^2(0, \infty; H^{N-1}(\mathbb{R}^3)), \quad (\nabla u^\epsilon, \nabla \theta^\epsilon, w^\epsilon - u^\epsilon) \in L^2(0, \infty; H^N(\mathbb{R}^3)) \end{aligned}$$

and

$$\begin{aligned} &\sup_{t \in \mathbb{R}_+} \|(\varphi^\epsilon, w^\epsilon, u^\epsilon)(t)\|_{H^N}^2 + \epsilon^{-2} \sup_{t \in \mathbb{R}_+} \|(\rho^\epsilon - \bar{\rho}, \theta^\epsilon - \bar{\theta})(t)\|_{H^N}^2 \\ &+ \int_0^\infty \|\nabla \varphi^\epsilon\|_{H^{N-1}}^2 d\tau + \epsilon^{-2} \int_0^\infty \|\nabla \rho^\epsilon\|_{H^{N-1}}^2 d\tau + \int_0^\infty \|\nabla u^\epsilon\|_{H^N}^2 d\tau \\ &+ \epsilon^{-2} \int_0^\infty \|\nabla \theta^\epsilon\|_{H^N}^2 d\tau + \int_0^\infty \|(w^\epsilon - u^\epsilon)\|_{H^N}^2 d\tau \\ &\leq C \left(\|(\varphi_0^\epsilon, w_0^\epsilon, u_0^\epsilon)\|_{H^N}^2 + \epsilon^{-2} \|(\rho_0^\epsilon - \bar{\rho}, \theta_0^\epsilon - \bar{\theta})\|_{H^N}^2 \right). \end{aligned} \quad (1.17)$$

- (2) (Incompressible limit) There exist the functions $(\varphi, w, u) \in L^\infty(0, \infty; H^N(\mathbb{R}^3)), \nabla u \in L^2(0, \infty; H^N(\mathbb{R}^3))$ such that

$$\begin{aligned} (\varphi^\epsilon, w^\epsilon, u^\epsilon) &\rightarrow (\varphi, w, u) \quad (*\text{-weak}) \text{ in } L^\infty(0, \infty; H^N(\mathbb{R}^3)), \\ \nabla u^\epsilon &\rightarrow \nabla u \quad (\text{weak}) \text{ in } L^2(0, \infty; H^N(\mathbb{R}^3)) \end{aligned}$$

as $\epsilon \rightarrow 0$. Furthermore the function pair (n, w, u, Π) for some $\Pi \in L^\infty(0, \infty; H^{N-1}(\mathbb{R}^3)) \cap L^2(0, \infty; H^N(\mathbb{R}^3))$, where $n = \bar{n}e^\varphi$, is the unique smooth global solution on $[0, \infty)$ to the compressible Euler/incompressible Navier-Stokes system (1.12) with the initial data

$$(n^\epsilon, w^\epsilon, u^\epsilon)(x, 0) = (n_0, w_0, u_0)(x), \quad (1.18)$$

where $(n_0, w_0, u_0) \in H^N(\mathbb{R}^3)$ are the functions such that by setting $\varphi_0 = \ln\left(\frac{n_0}{\bar{n}}\right)$, it holds that

$$(\varphi_0^\epsilon, w_0^\epsilon, u_0^\epsilon) \rightarrow (\varphi_0, w_0, u_0), \quad (\text{weak}) \text{ in } H^N(\mathbb{R}^3) \quad \text{as } \epsilon \rightarrow 0.$$

Remark 1.2. (1) For the global existence of solutions, we only assume that the H^3 -norm of initial perturbations is small, while the higher-order Sobolev norms can be arbitrarily large (see (1.16)).

(2) The same result as in Theorem 1.1 for the compressible Euler/isentropic Navier-Stokes two-phase model (1.3) was obtained in [16]. Therefore, Theorem 1.1 can be regarded as a generalization of [16, Theorem 1.2].

Our second result is to show the incompressible limit for the system (1.8) in 3D with $\kappa = 0$ and the initial data (1.10), in the case of a large temperature variation and ideal polytropic gas law

$$P = R\rho\theta, \quad e = c_v\theta, \quad (1.19)$$

where the parameters $R > 0$ and $c_v > 0$ are the gas constant and the heat capacity at constant volume, respectively. Without the loss of generality, we can assume $R = 1$ and $c_v = 1$. Then, using (1.19), we rewrite (1.8)₅ with $\kappa = 0$ as

$$(P^\epsilon)_t + \operatorname{div}(P^\epsilon u^\epsilon) + P^\epsilon \operatorname{div} u^\epsilon = \epsilon^2 \left(2\mu \mathbb{D}(u^\epsilon) : \mathbb{D}(u^\epsilon) + \lambda (\operatorname{div} u^\epsilon)^2 \right). \quad (1.20)$$

Formally, as the Mach number tends to zero, the system $(1.8)_{1-4}$ and (1.20) becomes

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \\ \rho_t + \operatorname{div}(\rho u) &= 0, \\ \operatorname{div} u &= 0, \\ \rho u_t + \rho(u \cdot \nabla)u + \nabla \Pi &= \Delta u + \sigma n(w - u) \end{aligned} \quad (1.21)$$

for a pressure function Π , which is a two-phase model consisting of the compressible Euler equations coupled with the non-homogeneous incompressible Navier-Stokes equations through a drag forcing term. To state our second result, by setting $\varphi^\epsilon = \ln\left(\frac{n^\epsilon}{n}\right)$, we reformulate the Cauchy problem $(1.8)_{1-4}$, (1.20), (1.10) as follows:

$$\begin{aligned} \varphi_t^\epsilon + \operatorname{div} w^\epsilon + w^\epsilon \cdot \nabla \varphi^\epsilon &= 0, \quad x \in \mathbb{R}^3, \quad t > 0, \\ w_t^\epsilon + (w^\epsilon \cdot \nabla)w^\epsilon + \nabla \varphi^\epsilon &= -\sigma(w^\epsilon - u^\epsilon), \\ \rho_t^\epsilon + \operatorname{div}(\rho^\epsilon u^\epsilon) &= 0, \\ \rho^\epsilon u_t^\epsilon + \rho^\epsilon(u^\epsilon \cdot \nabla)u^\epsilon + \frac{1}{\epsilon^2} \nabla P^\epsilon &= \mu \Delta u^\epsilon + (\mu + \lambda) \nabla \operatorname{div} u^\epsilon + \sigma \bar{n} e^{\varphi^\epsilon} (w^\epsilon - u^\epsilon), \\ (P^\epsilon)_t + \operatorname{div}(P^\epsilon u^\epsilon) + P^\epsilon \operatorname{div} u^\epsilon &= \epsilon^2 \left(2\mu \mathbb{D}(u^\epsilon) : \mathbb{D}(u^\epsilon) + \lambda (\operatorname{div} u^\epsilon)^2 \right), \end{aligned} \quad (1.22)$$

and

$$(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, P^\epsilon)(x, 0) = (\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon, P_0^\epsilon)(x) \rightarrow (0, 0, \bar{\rho}, 0, \bar{P}) \quad \text{as } |x| \rightarrow \infty, \quad (1.23)$$

where $\varphi_0^\epsilon = \ln\left(\frac{n_0^\epsilon}{n}\right)$ and $P_0^\epsilon = P(\rho_0^\epsilon, \theta_0^\epsilon)$. Then we have a result on incompressible limit under large temperature variation.

Theorem 1.3. Assume that (1.19) with $R = c_v = 1$ and

$$\begin{aligned} (\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon - \bar{\rho}, u_0^\epsilon, P_0^\epsilon - \bar{P}) &\in H^N(\mathbb{R}^3), \\ \inf_{x \in \mathbb{R}^3} \rho_0^\epsilon(x) > 0, \quad \inf_{x \in \mathbb{R}^3} P_0^\epsilon(x) > 0, \quad \operatorname{div} u_0 &= 0 \end{aligned} \quad (1.24)$$

hold for an integer $N \geq 3$. Moreover, for a positive constant M_0 independent of ϵ , the functions $(\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon, P_0^\epsilon)$ are assumed to satisfy

$$\|(\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon - \bar{\rho}, u_0^\epsilon)\|_{H^N} + \epsilon^{-1} \|P_0^\epsilon - \bar{P}\|_{H^N} \leq M_0. \quad (1.25)$$

Then the following statements hold.

- (1) (Local solutions and uniform estimates) For all small $\epsilon > 0$, there exist constants T_0 and C , depending on M_0 but independent of ϵ , such that the Cauchy problem (1.22), (1.23) admits a unique solution $(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, P^\epsilon)$ on the time interval $[0, T_0]$ satisfying

$$(\varphi^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon, P^\epsilon) \in C([0, T_0]; H^N(\mathbb{R}^3)), \quad \nabla u^\epsilon \in L^2(0, T_0; H^N(\mathbb{R}^3)) \quad (1.26)$$

and

$$\begin{aligned} &\|(\varphi^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon)(t)\|_{H^N}^2 + \epsilon^{-2} \|(P^\epsilon - \bar{P})(t)\|_{H^N}^2 + \int_0^t \|\nabla \varphi^\epsilon\|_{H^{N-1}}^2 d\tau \\ &+ \epsilon^{-2} \int_0^t \|\nabla P^\epsilon\|_{H^{N-1}}^2 d\tau + \int_0^t \|\nabla u^\epsilon\|_{H^N}^2 d\tau + \int_0^t \|(w^\epsilon - u^\epsilon)\|_{H^N}^2 d\tau \leq C \end{aligned} \quad (1.27)$$

for all $t \in [0, T_0]$.

- (2) (Incompressible limit) There exist the functions $(\varphi, w, u) \in L^\infty(0, T_0; H^N(\mathbb{R}^3)), \nabla u \in L^2(0, T_0; H^N(\mathbb{R}^3))$ such that

$$\begin{aligned} (\varphi^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon) &\rightarrow (\varphi, w, \rho - \bar{\rho}, u) \quad (*\text{-weak}) \text{ in } L^\infty(0, T_0; H^N(\mathbb{R}^3)), \\ \nabla u^\epsilon &\rightarrow \nabla u \quad (\text{weak}) \text{ in } L^2(0, T_0; H^N(\mathbb{R}^3)) \end{aligned} \quad (1.28)$$

as $\epsilon \rightarrow 0$. Furthermore the function pair (n, w, ρ, u, Π) , for some $\Pi \in L^\infty(0, T_0; H^{N-1}(\mathbb{R}^3)) \cap L^2(0, T_0; H^N(\mathbb{R}^3))$, where $n = \bar{n}e^\varphi$, is the unique smooth local solution on the time interval $[0, T_0]$ of the compressible Euler/non-homogeneous incompressible Navier-Stokes system (1.21) with the initial data

$$(n^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon)(x, 0) = (n_0, w_0, \rho_0 - \bar{\rho}, u_0)(x), \quad (1.29)$$

where $(n_0, w_0, \rho_0 - \bar{\rho}, u_0) \in H^N(\mathbb{R}^3)$ are the functions such that by setting $\varphi_0 = \ln\left(\frac{n_0}{\bar{n}}\right)$, it holds that

$$(\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon - \bar{\rho}, u_0^\epsilon) \rightarrow (\varphi_0, w_0, \rho_0 - \bar{\rho}, u_0), \quad (\text{weak}) \text{ in } H^N(\mathbb{R}^3) \quad \text{as } \epsilon \rightarrow 0.$$

Remark 1.4. (1) The result in Theorem 1.3 holds for the general gas law (1.15) including the ideal polytropic gas law (1.19).

(2) We can conjecture that for the system (1.8) with $\kappa > 0$, the low Mach number limit under the large temperature variation becomes

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \\ \rho_t + \operatorname{div}(\rho u) &= 0, \\ 2\bar{P} \operatorname{div} u &= \kappa \Delta \theta, \\ \rho u_t + \rho(u \cdot \nabla)u + \nabla \Pi &= \Delta u + \sigma n(w - u), \end{aligned}$$

and the temperature satisfies $\theta = \frac{\bar{P}}{R\rho}$ and

$$\rho \theta_t + \rho u \cdot \nabla \theta + \bar{P} \operatorname{div} u = \kappa \Delta \theta,$$

which is established by Huang-Wang-Wang [17] in the case of the full compressible Navier-Stokes equations.

2. PRELIMINARIES

We first recall the following useful Lemmas which we will use extensively.

Lemma 2.1 (Gagliardo-Nirenberg inequality [23, Theorem 1]). *Let l, s and k be any real numbers satisfying $0 \leq l, s \leq k$, and let $p, r, q \in [1, \infty]$ and $0 \leq \theta \leq 1$ such that*

$$\frac{l}{3} - \frac{1}{p} = \left(\frac{s}{3} - \frac{1}{r}\right)(1 - \theta) + \left(\frac{k}{3} - \frac{1}{q}\right)\theta.$$

Then, for any $u \in W_q^k(\mathbb{R}^3)$, we have

$$\|\nabla^l u\|_{L^p} \leq C \|\nabla^s u\|_{L^r}^{1-\theta} \|\nabla^k u\|_{L^q}^\theta. \quad (2.1)$$

Here when $p = \infty$ we require that $0 < \theta < 1$.

Lemma 2.2 ([11, Theorem 1.4.4.2]). *Let $\Omega = \mathbb{R}^d$, $s_1 \geq s$ and $s_2 \geq s$ be such that either*

$$s_1 + s_2 - s \geq d\left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q}\right) \geq 0, \quad s_j - s > d\left(\frac{1}{q_j} - \frac{1}{q}\right), \quad j = 1, 2$$

or

$$s_1 + s_2 - s > d\left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q}\right) \geq 0, \quad s_j - s \geq d\left(\frac{1}{q_j} - \frac{1}{q}\right), \quad j = 1, 2,$$

then $(u, v) \mapsto u \cdot v$ is a continuous bilinear map from $W_{q_1}^{s_1}(\Omega) \times W_{q_1}^{s_2}(\Omega)$ into $W_q^s(\Omega)$.

Lemma 2.3 ([24, Lemma 2.5]). *Let $f(\varphi)$ and $f(\varphi, w)$ be smooth functions of φ and (φ, w) , respectively, with bounded derivatives of any order, and $\|\varphi\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)} \leq C$. Then for any integer $m \geq 1$, we have*

$$\begin{aligned} \|\nabla^m f(\varphi)\|_{L^p} &\leq C \|\nabla^m \varphi\|_{L^p}, \\ \|\nabla^m f(\varphi, w)\|_{L^p} &\leq C \|\nabla^m(\varphi, w)\|_{L^p}, \end{aligned} \quad (2.2)$$

for any $1 \leq p \leq \infty$, where C may depend on f and m .

Lemma 2.4 ([24, Lemma 2.6]). *Let α be any multi-index with $|\alpha| = k$ and $1 < p < \infty$. Then there exists some constant $C > 0$ such that*

$$\begin{aligned} \|D^\alpha(fg)\|_{L^p} &\leq C(\|f\|_{L^{p_1}} \|\nabla^k g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \\ \|[D^\alpha, f]g\|_{L^p} &\leq C(\|\nabla f\|_{L^{p_1}} \|\nabla^{k-1} g\|_{L^{p_2}} + \|\nabla^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}), \end{aligned} \quad (2.3)$$

where $f, g \in W_\infty^k(\mathbb{R}^3)$, $1 \leq p_2, p_3 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, and $[D^\alpha, f]g = D^\alpha(fg) - fD^\alpha g$.

3. INCOMPRESSIBLE LIMIT: SMALL TEMPERATURE VARIATION

In this section, we prove Theorem 1.1. For brevity, we denote $(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon)$, $P^\epsilon = P(\rho^\epsilon, \theta^\epsilon)$ and $e^\epsilon = e(\rho^\epsilon, \theta^\epsilon)$ in (1.13) by $(\varphi, w, \rho, u, \theta)$, $P = P(\rho, \theta)$ and $e = e(\rho, \theta)$, respectively.

3.1. A priori estimates. We first recall the following thermodynamical relations (see [15]). Setting $v = \rho^{-1}$, it is well-known that by using any given two of the five thermodynamical variables $(v, P, e, \theta$ and $S)$, the remaining three variables are smooth functions of them, where S is the entropy of the gas. The second law of thermodynamics $\theta dS = de + Pdv$ asserts that, if we choose (v, θ) or (v, S) as independent variables and write $(P, e, S) = (P(v, \theta), e(v, \theta), S(v, \theta))$ or $(P, e, \theta) = (\hat{P}(v, S), \hat{e}(v, S), \hat{\theta}(v, S))$, respectively, then we can deduce that

$$\begin{aligned} S_v(v, \theta) &= P_\theta(v, \theta), \\ S_\theta(v, \theta) &= \frac{e_\theta(v, \theta)}{\theta}, \\ e_v(v, \theta) &= \theta P_\theta(v, \theta) - P(v, \theta), \end{aligned} \quad (3.1)$$

or

$$\begin{aligned} \hat{e}_v(v, S) &= -\hat{P}(v, S), & \hat{e}_S(v, S) &= \theta, \\ \hat{P}_v(v, S) &= P_v(v, \theta) - \frac{\theta(P_\theta(v, \theta))^2}{e_\theta(v, \theta)}, & \hat{P}_S(v, S) &= \frac{\theta P_\theta(v, \theta)}{e_\theta(v, \theta)}, \\ \hat{\theta}_v(v, S) &= -\frac{\theta P_\theta(v, \theta)}{e_\theta(v, \theta)}, & \hat{\theta}_S(v, S) &= \frac{\theta}{e_\theta(v, \theta)}. \end{aligned} \quad (3.2)$$

From (3.1) and (3.2) we deduce that

$$-\rho^2 e_\rho(\rho, \theta) = \theta P_\theta(\rho, \theta) - P(\rho, \theta), \quad \hat{e}_S(\rho, S) = \theta \quad \hat{e}_\rho(\rho, S) = \frac{1}{\rho^2} P(\rho, \theta).$$

Therefore, we can rewrite (1.13)₅ as

$$\begin{aligned} \theta_t + u \cdot \nabla \theta + \frac{\theta P_\theta(\rho, \theta)}{\rho e_\theta(\rho, \theta)} \operatorname{div} u - \frac{\kappa}{\rho e_\theta(\rho, \theta)} \Delta \theta &= \frac{2\mu\epsilon^2 \mathbb{D}(u) : \mathbb{D}(u)}{\rho e_\theta(\rho, \theta)} + \frac{\lambda\epsilon^2 (\operatorname{div} u)^2}{\rho e_\theta(\rho, \theta)}, \\ S_t + u \cdot \nabla S - \frac{\kappa}{\rho \theta} \Delta \theta &= \frac{2\mu\epsilon^2 \mathbb{D}(u) : \mathbb{D}(u)}{\rho \theta} + \frac{\lambda\epsilon^2 (\operatorname{div} u)^2}{\rho \theta}. \end{aligned} \quad (3.3)$$

Setting $\psi = \rho - \bar{\rho}$ and $\zeta = \theta - \bar{\theta}$, we reformulate the system (1.13)₁₋₄ and (3.3) as

$$\begin{aligned} \varphi_t + \operatorname{div} w &= g_1, \\ w_t + \nabla \varphi + \sigma(w - u) &= g_2, \\ \psi_t + \bar{\rho} \operatorname{div} u &= g_3, \\ u_t - \frac{\mu}{\bar{\rho}} \Delta u - \frac{\mu + \lambda}{\bar{\rho}} \nabla \operatorname{div} u + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho} \epsilon^2} \nabla \varphi + \frac{P_\theta(\bar{\rho}, \bar{\theta})}{\bar{\rho} \epsilon^2} \nabla \zeta - \sigma \bar{n}(w - u) &= g_4, \\ \zeta_t + \frac{\bar{\theta} P_\theta(\bar{\rho}, \bar{\theta})}{e_\theta(\bar{\rho}, \bar{\theta})} \operatorname{div} u &= \frac{\kappa}{\bar{\rho} e_\theta(\bar{\rho}, \bar{\theta})} \Delta \theta + g_5, \end{aligned} \quad (3.4)$$

where $g_i (i = 1, \dots, 5)$ are defined by

$$g_1 = -w \cdot \nabla \varphi, \quad (3.5)$$

$$g_2 = -(w \cdot \nabla) w, \quad (3.6)$$

$$g_3 = -\psi \operatorname{div} u - u \cdot \nabla \psi, \quad (3.7)$$

$$g_4 = -(u \cdot \nabla)u + \frac{h_1(\psi, \zeta)}{\epsilon^2} \nabla \psi + \frac{h_2(\psi, \zeta)}{\epsilon^2} \nabla \zeta + \sigma \bar{n} h_3(\varphi)(w - u) - h_4(\psi)(\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u), \quad (3.8)$$

$$h_1(\psi, \zeta) = \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}} - \frac{P_\rho(\rho, \theta)}{\rho}, \quad h_2(\psi, \zeta) = \frac{P_\theta(\bar{\rho}, \bar{\theta})}{\bar{\rho}} - \frac{P_\theta(\rho, \theta)}{\rho}, \quad (3.9)$$

$$h_3(\varphi) = e^\varphi - 1, \quad h_4(\psi) = \frac{1}{\bar{\rho}} - \frac{1}{\rho}, \quad (3.10)$$

$$g_5 = -u \cdot \nabla \zeta - \kappa h_5(\psi, \zeta) \Delta \zeta + \frac{2\mu \epsilon^2 \mathbb{D}(u) : \mathbb{D}(u)}{\rho e_\theta(\rho, \theta)} + \frac{\lambda \epsilon^2 (\operatorname{div} u)^2}{\rho e_\theta(\rho, \theta)} + h_6(\psi, \zeta) \operatorname{div} u, \quad (3.11)$$

$$h_5(\psi, \zeta) = \frac{1}{\bar{\rho} e_\theta(\bar{\rho}, \bar{\theta})} - \frac{1}{\rho e_\theta(\rho, \theta)}, \quad h_6(\psi, \zeta) = \frac{\bar{\theta} P_\theta(\bar{\rho}, \bar{\theta})}{\bar{\rho} e_\theta(\bar{\rho}, \bar{\theta})} - \frac{\theta P_\theta(\rho, \theta)}{\rho e_\theta(\rho, \theta)}. \quad (3.12)$$

We suppose that $(\varphi, w, \psi, u, \zeta) \in X_N(0, T)$ is the solution to the system (3.4) for any constant $T > 0$ and an integer $N \geq 3$, where $X_N(I)$ is defined by

$$X_N(I) = \{(\varphi, w, \psi, u, \zeta) \in C(I; H^N(\mathbb{R}^3)) \mid (\nabla \varphi, \nabla \psi) \in L^2(I; H^{N-1}(\mathbb{R}^3)), \\ (\nabla u, \nabla \zeta, w - u) \in L^2(I; H^N(\mathbb{R}^3))\}$$

for any interval $I \subset [0, \infty)$. Also, we assume that

$$\sup_{0 \leq t \leq T} \|(\varphi, w, \epsilon^{-1} \psi, u, \epsilon^{-1} \zeta)(t)\|_{H^3} \leq \delta_1 \quad (3.13)$$

for sufficiently small $\delta_1 > 0$ independent of ϵ . By using (3.13), we can choose $\delta_1 > 0$ such that

$$c_{\bar{n}} \leq n(x, t) = \bar{n} e^{\varphi(x, t)} \leq C_{\bar{n}}, \quad \frac{\bar{\rho}}{2} \leq \rho(x, t) \leq \frac{3\bar{\rho}}{2} \quad \text{and} \quad \frac{\bar{\theta}}{2} \leq \theta(x, t) \leq \frac{3\bar{\theta}}{2}, \quad (3.14)$$

where $c_{\bar{n}}$ and $C_{\bar{n}}$ are positive constants depending only on $\bar{n}$. We first give the following estimate.

Lemma 3.1. *Under assumption (3.13), it holds that*

$$\begin{aligned} & \frac{d}{dt} \left(\|w\|_{H^l}^2 + \frac{1}{\bar{n}} \|u\|_{H^l}^2 + \|\varphi\|_{H^l}^2 + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2 \bar{n} \epsilon^2} \|\psi\|_{H^l}^2 + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta} \epsilon^2} \|\zeta\|_{H^l}^2 \right) \\ & + \sigma \|(w - u)\|_{H^l}^2 + \frac{\mu}{\bar{\rho} \bar{n}} \|\nabla u\|_{H^l}^2 + \frac{\kappa}{\bar{\rho} \bar{\theta} \bar{n} \epsilon^2} \|\nabla \zeta\|_{H^l}^2 \\ & \leq \delta_1 C \left(\|\nabla \varphi\|_{H^{l-1}}^2 + \epsilon^{-2} \|\nabla \psi\|_{H^{l-1}}^2 \right) \end{aligned} \quad (3.15)$$

for any $t \geq 0$ and $l = 3, \dots, N$, where C is the positive constant independent of ϵ, δ_1 and t .

Proof. Applying ∇^k to (3.4) yields

$$\begin{aligned} & \nabla^k \varphi_t + \operatorname{div} \nabla^k w = \nabla^k g_1, \\ & \nabla^k w_t + \nabla^{k+1} \varphi + \sigma \nabla^k (w - u) = \nabla^k g_2, \\ & \nabla^k \psi_t + \bar{\rho} \operatorname{div} \nabla^k u = \nabla^k g_3, \\ & \nabla^k u_t - \frac{\mu}{\bar{\rho}} \Delta \nabla^k u - \frac{\mu + \lambda}{\bar{\rho}} \nabla \operatorname{div} \nabla^k u + \frac{P'(\bar{\rho}, \bar{\theta})}{\bar{\rho} \epsilon^2} \nabla^{k+1} \psi + \frac{P'(\bar{\rho}, \bar{\theta})}{\bar{\rho} \epsilon^2} \nabla^{k+1} \zeta - \sigma \bar{n} \nabla^k (w - u) = \nabla^k g_4, \\ & \nabla^k \zeta_t + \frac{\bar{\theta} P_\theta(\bar{\rho}, \bar{\theta})}{e_\theta(\bar{\rho}, \bar{\theta})} \operatorname{div} \nabla^k u = \frac{\kappa}{\bar{\rho} e_\theta(\bar{\rho}, \bar{\theta})} \Delta \nabla^k \theta + \nabla^k g_5, \end{aligned} \quad (3.16)$$

where $k = 0, 1, \dots, l$.

Multiplying (3.16)₂ by $\nabla^k w$, and using (3.16)₁, (3.5) and (3.6), we have

$$\frac{1}{2} \frac{d}{dt} (\|\nabla^k w\|^2 + \|\nabla^k \varphi\|^2) + \sigma \int \nabla^k (w - u) \nabla^k w \, dx = I_k^1 + I_k^2, \quad (3.17)$$

where

$$I_k^1 = - \int \nabla^k (w \cdot \nabla \varphi) \nabla^k \varphi \, dx, \quad (3.18)$$

$$I_k^2 = - \int \nabla^k \left((w \cdot \nabla) w \right) \cdot \nabla^k w \, dx. \quad (3.19)$$

Here and after, when the integrals over $\mathbb{R}^3$ have not the meaning, we understand it by the duality pairing between $H^{-1}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$.

Also, multiplying (3.16)₄ and (3.16)₅ by $\nabla^k u$ and $\frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta}} \nabla^k \zeta$, respectively, and using (3.16)₃ and (3.7)-(3.11), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla^k u\|^2 + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2 \epsilon^2} \|\nabla^k \varphi\|^2 + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta} \epsilon^2} \|\nabla^k \zeta\|^2 \right) - \sigma \bar{n} \int \nabla^k (w - u) \nabla^k u \, dx \\ & + \int \left(\frac{\mu}{\bar{\rho}} |\nabla^{k+1} \mathbf{u}|^2 + \frac{(\mu + \lambda)}{\bar{\rho}} |\operatorname{div} \nabla^k \mathbf{u}|^2 \right) dx + \frac{\kappa}{\bar{\rho} \bar{\theta} \epsilon^2} \int |\nabla^{k+1} \zeta|^2 dx \\ & = \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2} I_k^3 + I_k^4 + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta}} I_k^5, \end{aligned} \quad (3.20)$$

where

$$I_k^3 = -\epsilon^{-2} \int \nabla^k (\psi \operatorname{div} u + u \cdot \nabla \psi) \nabla^k \psi \, dx, \quad (3.21)$$

$$\begin{aligned} I_k^4 = & \underbrace{- \int \nabla^k \left((u \cdot \nabla) u \right) \cdot \nabla^k u \, dx}_{I_k^{41}} + \underbrace{\epsilon^{-2} \int \nabla^k \left(h_1(\psi, \zeta) \nabla \psi \right) \cdot \nabla^k u \, dx}_{I_k^{42}} \\ & + \underbrace{\epsilon^{-2} \int \nabla^k \left(h_2(\psi, \zeta) \nabla \zeta \right) \cdot \nabla^k u \, dx}_{I_k^{43}} - \underbrace{\sigma \bar{n} \int \nabla^k \left(h_3(\varphi) (w - u) \right) \cdot \nabla^k u \, dx}_{I_k^{44}} \\ & - \underbrace{\int \nabla^k \left(h_4(\psi) \left(\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u \right) \right) \cdot \nabla^k u \, dx}_{I_k^{45}} \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} I_k^5 = & \underbrace{-\kappa \epsilon^{-2} \int \nabla^k [h_5(\psi, \zeta) \Delta \zeta] \nabla^k \zeta \, dx}_{I_k^{51}} \\ & - \underbrace{\epsilon^{-2} \int \nabla^k [u \cdot \nabla \zeta - h_6(\psi, \zeta) \operatorname{div} u] \nabla^k \zeta \, dx}_{I_k^{52}} \\ & + \underbrace{\int \nabla^k \left[\frac{2\mu \mathbb{D}(u) : \mathbb{D}(u) + \lambda (\operatorname{div} u)^2}{\rho e_\theta(\rho, \theta)} \right] \nabla^k \zeta \, dx}_{I_k^{53}}. \end{aligned} \quad (3.23)$$

Summing (3.17) and $\bar{n}^{-1} \times (3.20)$, and then, adding the resulting equality for $k = 0, 1, \dots, l$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|w\|_{H^l}^2 + \frac{1}{\bar{n}} \|u\|_{H^l}^2 + \|\varphi\|_{H^l}^2 + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2 \bar{n} \epsilon^2} \|\psi\|_{H^l}^2 + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta} \epsilon^2} \|\zeta\|_{H^l}^2 \right) \\ & + \sigma \|(w - u)\|_{H^l}^2 + \frac{\mu}{\bar{\rho} \bar{n}} \|\nabla u\|_{H^l}^2 + \frac{\kappa}{\bar{\rho} \bar{\theta} \bar{n} \epsilon^2} \|\nabla \zeta\|_{H^l}^2 \\ & \leq \sum_{k=0}^l |I_k^1| + \sum_{k=0}^l |I_k^2| + \frac{P_\theta(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2 \bar{n}} \sum_{k=0}^l |I_k^3| + \frac{1}{\bar{n}} \sum_{k=0}^l |I_k^4| + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta} \bar{n}} \sum_{k=0}^l |I_k^5|, \end{aligned} \quad (3.24)$$

Next we estimate the right-hand in (3.24). For I_0^1 , using Hölder inequality, (3.14) and (3.13), from (3.18) we obtain

$$I_0^1 \leq C(\|\varphi\|_{L^6} \|\nabla w\| + \|w\|_{L^6} \|\nabla \varphi\|) \|\varphi\|_{L^3} \leq C \delta_1 \|\nabla \varphi\|^2.$$

For I_k^1 ($k = 1, \dots, l$), using Hölder inequality, (2.3)₂ and (3.13), from (3.18) we obtain

$$I_k^1 = - \int [\nabla^k, w] \nabla \varphi \nabla^k \varphi \, dx + \frac{1}{2} \int \operatorname{div} w |\nabla^k \varphi|^2 \, dx$$

$$\begin{aligned}
&\leq C \left(\|\nabla w\|_{L^\infty} \|\nabla^k \varphi\| + \|\nabla \varphi\|_{L^\infty} \|\nabla^k w\| \right) \|\nabla^k \varphi\| + C \|\nabla w\|_{L^\infty} \|\nabla^k \varphi\|^2 \\
&\leq C \delta_1 \left(\|\nabla^k \varphi\|^2 + \|\nabla^k w\|^2 \right).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^l |I_k^1| \leq C \delta_1 \left(\|\nabla \varphi\|_{H^{l-1}}^2 + \|\nabla w\|_{H^{l-1}}^2 \right). \quad (3.25)$$

For I_0^2 , from (3.19) we obtain

$$I_0^2 \leq C (\|w\|_{L^6} \|\nabla w\| + \|w\|_{L^6} \|\nabla w\|) \|w\|_{L^3} \leq C \delta_1 \|\nabla w\|^2.$$

For $I_k^2 (k = 1, \dots, l)$, from (3.19) we obtain

$$\begin{aligned}
I_k^2 &= - \int [\nabla^k, w] \nabla w \cdot \nabla^k w \, dx + \frac{1}{2} \int \operatorname{div} w |\nabla^k w|^2 \, dx \\
&\leq C \|\nabla w\|_{L^\infty} \|\nabla^k w\|^2 \leq C \delta_1 \|\nabla^k w\|^2.
\end{aligned}$$

Therefore,

$$\sum_{k=0}^l |I_k^2| \leq C \delta_1 \|\nabla w\|_{H^{l-1}}^2. \quad (3.26)$$

For I_0^3 , from (3.21) we obtain

$$I_0^3 \leq C \epsilon^{-2} (\|\psi\|_{L^6} \|\nabla u\| + \|u\|_{L^6} \|\nabla \psi\|) \|\psi\|_{L^3} \leq C \delta_1 \epsilon^{-2} \|\nabla \psi\|^2.$$

For $I_k^3 (k = 1, \dots, l)$, (3.21) we obtain

$$\begin{aligned}
I_k^3 &= \epsilon^{-2} \int \nabla^k (\psi \operatorname{div} u) \nabla^k \psi \, dx - \epsilon^{-2} \int [\nabla^k, u] \nabla \psi \nabla^k \psi \, dx + \frac{1}{2\epsilon^2} \int \operatorname{div} u |\nabla^k \psi|^2 \, dx \\
&\leq C \epsilon^{-2} \left(\|\psi\|_{L^\infty} \|\nabla^{k+1} u\| + \|\nabla u\|_{L^\infty} \|\nabla^k \psi\| \right) \|\nabla^k \psi\| \\
&\quad + C \epsilon^{-2} \left(\|\nabla u\|_{L^\infty} \|\nabla^k \psi\| + \|\nabla \psi\|_{L^3} \|\nabla^k u\|_{L^6} \right) \|\nabla^k \psi\| + C \epsilon^{-2} \|\nabla u\|_{L^\infty} \|\nabla^k \psi\|^2 \\
&\leq C \delta_1 \left(\epsilon^{-2} \|\nabla^k \psi\|^2 + \|\nabla^{k+1} u\|^2 \right).
\end{aligned}$$

Therefore,

$$\sum_{k=0}^l |I_k^3| \leq C \delta_1 \left(\epsilon^{-2} \|\nabla \psi\|_{H^{l-1}}^2 + \|\nabla u\|_{H^l}^2 \right). \quad (3.27)$$

For I_0^4 , using Hölder inequality, (2.1), (3.14), and (3.13), from (3.22) we obtain

$$\begin{aligned}
I_0^{41} &\leq C \|u\|_{L^6} \|\nabla u\| \|u\|_{L^3} \leq C \delta_1 \|\nabla u\|^2, \\
I_0^{42} + I_0^{43} &\leq C \epsilon^{-2} (\|h_1(\psi, \zeta)\|_{L^3} \|\nabla \varphi\| + \|h_2(\psi, \zeta)\|_{L^3} \|\nabla \zeta\|) \|u\|_{L^6} \\
&\leq C \delta_1 \left(\|\nabla u\|^2 + \epsilon^{-2} \|\nabla \varphi\|^2 + \epsilon^{-2} \|\nabla \zeta\|^2 \right), \\
I_0^{44} &\leq C \|h_3(\varphi)\|_{L^3} \|(w - u)\| \|u\|_{L^6} \leq C \delta_1 \left(\|(w - u)\|^2 + \|\nabla u\|^2 \right), \\
I_0^{45} &= - \int [\mu \nabla u \cdot \nabla (h_4(\psi)u) + (\lambda + \mu) \operatorname{div} u \operatorname{div} (h_4(\psi)u)] \, dx \\
&\leq C \|\nabla u\| (\|h_4(\psi)\|_{L^\infty} \|\nabla u\| + \|u\|_{L^\infty} \|\nabla h_4(\psi)\|) \\
&\leq C \delta_1 \left(\|\nabla u\|^2 + \epsilon^{-2} \|\nabla \varphi\|^2 \right),
\end{aligned}$$

which implies

$$I_0^4 \leq C \delta_1 \left(\|\nabla u\|^2 + \|(w - u)\|^2 + \epsilon^{-2} \|\nabla \varphi\|^2 + \epsilon^{-2} \|\nabla \zeta\|^2 \right), \quad (3.28)$$

where we assume $0 < \epsilon \leq 1$ without loss of generality. For $I_k^4 (k = 1, \dots, l)$, we obtain from (3.22) that

$$\begin{aligned}
I_k^{41} &\stackrel{(2.3)}{\leq} C \left(\|u\|_{L^3} \|\nabla^{k+1} u\| + \|\nabla u\|_{L^3} \|\nabla^k u\| \right) \|\nabla^k u\|_{L^6} \\
&\stackrel{(3.13)}{\leq} C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \|\nabla^k u\|^2 \right), \\
I_k^{42} &= -\epsilon^{-2} \int \nabla^{k-1} \left(h_1(\psi, \zeta) \nabla \psi \right) \cdot \nabla^{k+1} u \, dx \\
&\stackrel{(2.3)}{\leq} C \epsilon^{-2} \left(\|h_1(\psi, \zeta)\|_{L^\infty} \|\nabla^k \psi\| + \|\nabla \psi\|_{L^3} \|\nabla^{k-1} h_1(\psi, \zeta)\|_{L^6} \right) \|\nabla^{k+1} u\| \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \epsilon^{-2} \|\nabla^k \psi\|^2 + \epsilon^{-2} \|\nabla^k \zeta\|^2 \right) \\
I_k^{43} &= -\epsilon^{-2} \int \nabla^{k-1} \left(h_2(\psi, \zeta) \nabla \zeta \right) \cdot \nabla^{k+1} u \, dx \\
&\stackrel{(2.3)}{\leq} C \epsilon^{-2} \left(\|h_2(\psi, \zeta)\|_{L^\infty} \|\nabla^k \zeta\| + \|\nabla \zeta\|_{L^3} \|\nabla^{k-1} h_2(\psi, \zeta)\|_{L^6} \right) \|\nabla^{k+1} u\| \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \epsilon^{-2} \|\nabla^k \psi\|^2 + \epsilon^{-2} \|\nabla^k \zeta\|^2 \right) \\
I_k^{44} &\stackrel{(2.3)}{\leq} C \left(\|h_3(\varphi)\|_{L^\infty} \|\nabla^k (w - u)\| + \|w - u\|_{L^\infty} \|\nabla^k h_3(\varphi)\| \right) \|\nabla^k u\| \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \left(\|\nabla^k (w - u)\|^2 + \|\nabla^k \varphi\|^2 + \|\nabla^k u\|^2 \right), \\
I_k^{45} &= \mu \int \nabla^{k-1} \left(\operatorname{div} (h_4(\psi) \nabla u) - \nabla h_4(\psi) \nabla u \right) \cdot \nabla^{k+1} u \, dx \\
&\quad + (\lambda + \mu) \int \nabla^{k-1} \left(\nabla (h_4(\psi) \operatorname{div} u) - \nabla h_4(\psi) \operatorname{div} u \right) \cdot \nabla^{k+1} u \, dx \\
&\stackrel{(2.3)}{\leq} C \left(\|h_4(\psi)\|_{L^\infty} \|\nabla^{k+1} u\| + \|\nabla u\|_{L^\infty} \|\nabla^k h_4(\psi)\| + \|\nabla h_4(\psi)\|_{L^3} \|\nabla^k u\|_{L^6} \right) \|\nabla^{k+1} u\| \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \|\nabla^k \psi\|^2 \right),
\end{aligned}$$

which together with (3.28) implies

$$\sum_{k=0}^l |I_k^4| \leq C \delta_1 \left(\|(\nabla u, w - u)\|_{H^l}^2 + \epsilon^{-2} \|(\nabla \psi, \nabla \zeta)\|_{H^{l-1}}^2 + \|\nabla \varphi\|_{H^{l-1}}^2 \right) \quad (3.29)$$

where we assume $0 < \epsilon \leq 1$ without loss of generality.

For I_0^5 , using Hölder inequality, (3.13) and (2.1), from (3.23) we obtain

$$\begin{aligned}
I_0^{51} &= \epsilon^{-2} \kappa \int \nabla \zeta \cdot \nabla (h_5(\psi, \zeta) \zeta) \, dx \\
&\leq C \epsilon^{-2} \|\nabla \zeta\| \left(\|h_5(\psi, \zeta)\|_{L^\infty} \|\nabla \zeta\| + \|\zeta\|_{L^\infty} \|\nabla h_5(\psi, \zeta)\| \right) \\
&\leq C \delta_1 \epsilon^{-2} \left(\|\nabla \zeta\|^2 + \|\nabla \psi\|^2 \right), \\
I_0^{52} &\leq C \epsilon^{-2} (\|u\|_{L^3} \|\nabla \zeta\| + \|h_6(\psi, \zeta)\|_{L^3} \|\nabla u\|) \|\zeta\|_{L^6} \\
&\leq C \delta_1 \left(\epsilon^{-2} \|\nabla \zeta\|^2 + \|\nabla u\|^2 \right), \\
I_0^{53} &\leq C (\|\mathbb{D}(u)\|_{L^3} \|\mathbb{D}(u)\| + \|\operatorname{div} u\|_{L^3} \|\operatorname{div} u\|) \|\zeta\|_{L^6} \\
&\leq C \delta_1 \left(\|\nabla u\|^2 + \|\nabla \zeta\|^2 \right),
\end{aligned}$$

which implies

$$I_0^5 \leq C \delta_1 \left(\|\nabla u\|^2 + \epsilon^{-2} \|\nabla \psi\|^2 + \epsilon^{-2} \|\nabla \zeta\|^2 \right), \quad (3.30)$$

where we assume $0 < \epsilon \leq 1$ without loss of generality. For $I_k^5 (k = 1, \dots, l)$, from (3.23) we obtain

$$\begin{aligned}
I_k^{51} &= \kappa \epsilon^{-2} \int \nabla^{k-1} [\operatorname{div}(h_5(\varphi, \zeta) \nabla \zeta) - \nabla h_5(\psi, \zeta) \nabla \zeta] \cdot \nabla^{k+1} \zeta \, dx \\
&\stackrel{(2.3)}{\leq} C \epsilon^{-2} \left(\|h_5(\psi, \zeta)\|_{L^\infty} \|\nabla^{k+1} \zeta\| + \|\nabla \zeta\|_{L^\infty} \|\nabla^k h_5(\psi, \zeta)\| \right. \\
&\quad \left. + \|\nabla h_5(\psi, \zeta)\|_{L^3} \|\nabla^k \zeta\|_{L^6} \right) \|\nabla^{k+1} \zeta\| \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \epsilon^{-2} \left(\|\nabla^{k+1} \zeta\|^2 + \|\nabla^k \zeta\|^2 + \|\nabla^k \psi\|^2 \right), \\
I_k^{52} &\stackrel{(2.3)}{\leq} C \epsilon^{-2} \left(\|u\|_{L^3} \|\nabla^{k+1} \zeta\| + \|\nabla \zeta\|_{L^3} \|\nabla^k u\| \right) \|\nabla^k \zeta\|_{L^6} \\
&\quad + \left(\|h_6(\psi, \zeta)\|_{L^3} \|\nabla^{k+1} u\| + \|\nabla u\|_{L^3} \|\nabla^k h_6(\psi, \zeta)\| \right) \|\nabla^k \zeta\|_{L^6} \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \|\nabla^k u\|^2 + \epsilon^{-2} \|\nabla^{k+1} \zeta\|^2 + \epsilon^{-2} \|\nabla^k(\psi, \zeta)\|^2 \right), \\
I_k^{53} &\stackrel{(2.3)}{\leq} C \left(\left\| \frac{\mathbb{D}(u)}{\rho e_\theta(\rho, \theta)} \right\|_{L^3} \|\nabla^k \mathbb{D}(u)\| + \|\mathbb{D}(u)\|_{L^3} \left\| \nabla^k \left(\frac{\mathbb{D}(u)}{\rho e_\theta(\rho, \theta)} \right) \right\| \right) \|\nabla^k \zeta\|_{L^6} \\
&\quad + \left(\left\| \frac{\operatorname{div} u}{\rho e_\theta(\rho, \theta)} \right\|_{L^3} \|\nabla^k \operatorname{div} u\| + \|\operatorname{div} u\|_{L^3} \left\| \nabla^k \left(\frac{\operatorname{div} u}{\rho e_\theta(\rho, \theta)} \right) \right\| \right) \|\nabla^k \zeta\|_{L^6} \\
&\stackrel{(3.13), (2.2)}{\leq} C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \epsilon^{-2} \|\nabla^{k+1} \zeta\|^2 + \epsilon^{-2} \|\nabla^k(\psi, \zeta)\|^2 \right),
\end{aligned}$$

which together with (3.30) implies

$$\sum_{k=0}^l |I_k^3| \leq C \delta_1 \left(\|\nabla u\|_{H^l}^2 + \epsilon^{-2} \|\nabla \psi\|_{H^{l-1}}^2 + \epsilon^{-2} \|\nabla \zeta\|_{H^l}^2 \right). \quad (3.31)$$

Substituting (3.25)-(3.27), (3.29), (3.31) into (3.24), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|w\|_{H^l}^2 + \frac{1}{\bar{n}} \|u\|_{H^l}^2 + \|\varphi\|_{H^l}^2 + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2 \bar{n} \epsilon^2} \|\psi\|_{H^l}^2 + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta} \epsilon^2} \|\zeta\|_{H^l}^2 \right) \\
&+ \sigma \|(w - u)\|_{H^l}^2 + \frac{\mu}{\bar{\rho} \bar{n}} \|\nabla u\|_{H^l}^2 + \frac{\kappa}{\bar{\rho} \bar{\theta} \bar{n} \epsilon^2} \|\nabla \zeta\|_{H^l}^2 \\
&\leq C \delta_1 \left(\|\nabla \varphi\|_{H^{l-1}}^2 + \|(\nabla u, w - u)\|_{H^l}^2 + \epsilon^{-2} \|\nabla \psi\|_{H^{l-1}}^2 + \epsilon^{-2} \|\nabla \zeta\|_{H^l}^2 \right),
\end{aligned}$$

which gives us (3.15), using the smallness of $\delta_1 > 0$. The proof of Lemma 3.1 is completed. $\square$

Lemma 3.2. *Under assumption (3.13), it holds that*

$$\begin{aligned}
&\frac{d}{dt} \sum_{k=0}^{l-1} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \frac{d}{dt} \sum_{k=0}^{l-1} \int \nabla^k u \cdot \nabla^{k+1} \psi \, dx + \frac{1}{4} \|\nabla \varphi\|_{H^{l-1}}^2 + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{2 \bar{\rho} \epsilon^2} \|\nabla \psi\|_{H^{l-1}}^2 \\
&\leq C \left(\|\nabla u\|_{H^l}^2 + \|w - u\|_{H^l}^2 + \epsilon^{-2} \|\nabla \zeta\|_{H^{l-1}}^2 \right) + C \delta_1 \|\nabla \varphi\|_{H^{l-1}}^2
\end{aligned} \quad (3.32)$$

for any $t \geq 0$ and $l = 3, \dots, N$, where C is the positive constant independent of ϵ, δ_1 and t .

Proof. Multiplying (3.16)₂ by $\nabla^{k+1} \varphi$ and using

$$\int \nabla^k w_t \cdot \nabla^{k+1} \varphi \, dx = \frac{d}{dt} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx - \int (\operatorname{div} \nabla^k w)^2 \, dx + \int \nabla^k g_1 \operatorname{div} \nabla^k w \, dx$$

because of (3.16)₁, we have

$$\frac{d}{dt} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \|\nabla^{k+1} \varphi\|^2 - \int (\operatorname{div} \nabla^k w)^2 \, dx = I_k^6, \quad (3.33)$$

where

$$I_k^6 = \int \nabla^k (w \cdot \nabla \varphi) \operatorname{div} \nabla^k w \, dx - \int \nabla^k ((w \cdot \nabla) w) \cdot \nabla^{k+1} \varphi \, dx + \sigma \int \nabla^k (w - u) \cdot \nabla^{k+1} \varphi \, dx.$$

Noticing that

$$\begin{aligned} I_k^6 &\stackrel{(2.3)}{\leq} C \left(\|w\|_{L^\infty} \|\nabla^{k+1} \varphi\| + \|\nabla \varphi\|_{L^3} \|\nabla^k w\|_{L^6} \right) \|\nabla^{k+1} w\| \\ &\quad + C \left(\|w\|_{L^\infty} \|\nabla^{k+1} w\| + \|\nabla w\|_{L^3} \|\nabla^k w\|_{L^6} \right) \|\nabla^{k+1} \varphi\| \\ &\quad + C \|\nabla^k (w - u)\| \|\nabla^{k+1} \varphi\| \\ &\stackrel{(3.13)}{\leq} C \delta_1 \left(\|\nabla^{k+1} w\|^2 + \|\nabla^{k+1} \varphi\|^2 \right) + \alpha \|\nabla^{k+1} \varphi\|^2 + \alpha^{-1} C \|\nabla^k (w - u)\|^2 \end{aligned}$$

for any $\alpha > 0$, from (3.33) we obtain

$$\frac{d}{dt} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \frac{1}{2} \|\nabla^{k+1} \varphi\|^2 \leq C \left(\|\nabla^{k+1} w\|^2 + \|\nabla^k (w - u)\|^2 \right),$$

which implies that

$$\frac{d}{dt} \sum_{k=0}^{l-1} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \frac{1}{2} \|\nabla \varphi\|_{H^{l-1}}^2 \leq C \left(\|\nabla w\|_{H^{l-1}}^2 + \|w - u\|_{H^{l-1}}^2 \right). \quad (3.34)$$

Also, multiplying (3.16)₄ by $\nabla^{k+1} \psi$ and using

$$\int \nabla^k u_t \cdot \nabla^{k+1} \psi \, dx = \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \psi \, dx - \bar{\rho} \int (\operatorname{div} \nabla^k u)^2 \, dx + \int \nabla^k g_3 \operatorname{div} \nabla^k u \, dx$$

from (3.16)₃, we have

$$\begin{aligned} &\frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \psi \, dx + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho} \bar{\epsilon}^2} \|\nabla^{k+1} \psi\|^2 \\ &= \sum_{j=7}^9 I_k^j + \bar{\rho} \int (\operatorname{div} \nabla^k u)^2 \, dx + \sigma \bar{n} \int \nabla^k (w - u) \cdot \nabla^{k+1} \psi \, dx, \end{aligned} \quad (3.35)$$

where

$$\begin{aligned} I_k^7 &= \int \nabla^k \operatorname{div}(\psi u) \operatorname{div} \nabla^k \psi \, dx - \int \nabla^k ((u \cdot \nabla) u) \cdot \nabla^{k+1} u \, dx \\ &\quad + \epsilon^{-2} \int \nabla^k \left(h_1(\psi, \zeta) \nabla \psi + h_2(\psi, \zeta) \nabla \zeta \right) \cdot \nabla^{k+1} \psi \, dx, \end{aligned} \quad (3.36)$$

$$\begin{aligned} I_k^8 &= -\sigma \bar{n} \int \nabla^k \left(h_3(\varphi)(w - u) \right) \cdot \nabla^{k+1} \psi \, dx \\ &\quad - \int \nabla^k \left(h_4(\psi) \left(\mu \Delta u + (\mu + \lambda) \nabla \operatorname{div} u \right) \right) \cdot \nabla^{k+1} \psi \, dx \end{aligned} \quad (3.37)$$

$$\begin{aligned} I_k^9 &= -\frac{\mu}{\bar{\rho}} \int \nabla^k \Delta u \nabla^{k+1} \psi \, dx - \frac{\mu + \lambda}{\bar{\rho}} \int \nabla^{k+1} \operatorname{div} u \nabla^{k+1} \psi \, dx \\ &\quad - \frac{P_\theta(\bar{\rho}, \bar{\theta})}{\bar{\rho} \bar{\epsilon}^2} \int \nabla^{k+1} \zeta \nabla^{k+1} \psi \, dx. \end{aligned} \quad (3.38)$$

Next we estimate the right-hand in (3.35). For I_k^7 , from (3.36) we obtain

$$I_k^7 \leq C \delta_1 \left(\|\nabla^{k+1} u\|^2 + \epsilon^{-2} \|\nabla^{k+1} \psi\|^2 + \epsilon^{-2} \|\nabla^{k+1} \zeta\|^2 \right)$$

assuming $0 < \epsilon \leq 1$ without loss of generality. For I_k^8 , from (3.37) we obtain

$$\begin{aligned} I_k^8 &\stackrel{(2.3)}{\lesssim} \left(\|h_3(\varphi)\|_{L^\infty} \|\nabla^k (w - u)\| + \|(w - u)\|_{L^3} \|\nabla^k h_3(\varphi)\|_{L^6} \right) \|\nabla^{k+1} \psi\| \\ &\quad + \left(\|h_4(\psi)\|_{L^\infty} \|\nabla^{k+2} u\| + \|\nabla^2 u\|_{L^3} \|\nabla^k h_4(\psi)\|_{L^6} \right) \|\nabla^{k+1} \psi\| \end{aligned}$$

$$\stackrel{(3.13),(2.2)}{\lesssim} \delta_1 \left(\|\nabla^{k+2}u\|^2 + \|\nabla^{k+1}\psi\|^2 + \|\nabla^{k+1}\varphi\|^2 + \|\nabla^k(w-u)\|^2 \right).$$

For I_k^9 , from (3.38) we obtain

$$\begin{aligned} I_k^8 &\leq C \|\nabla^{k+2}u\| \|\nabla^{k+1}\psi\| + C\epsilon^{-2} \|\nabla^{k+1}\zeta\| \|\nabla^{k+1}\psi\| \\ &\leq \alpha\epsilon^{-2} \|\nabla^{k+1}\psi\|^2 + \alpha^{-1}C \|\nabla^{k+2}u\|^2 + \alpha^{-1}C\epsilon^{-2} \|\nabla^{k+1}\zeta\|^2 \end{aligned}$$

for any $\alpha > 0$. Substituting the estimates on $I_k^i (i = 6, 7, 8)$ into (3.35), and using the smallness of $\delta_1 > 0$ and $\alpha > 0$, we obtain

$$\begin{aligned} &\frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1}\psi \, dx + \frac{P'(\bar{\rho})}{2\bar{\rho}\epsilon^2} \|\nabla^{k+1}\psi\|^2 \\ &\leq C \left(\|\nabla^{k+1}u\|^2 + \|\nabla^{k+2}u\|^2 + \|\nabla^k(w-u)\|^2 \right) + C\delta_1 \|\nabla^{k+1}\varphi\|^2 + C\epsilon^{-2} \|\nabla^{k+1}\zeta\|^2, \end{aligned}$$

assuming $0 < \epsilon \leq 1$ without loss of generality, which implies that

$$\begin{aligned} &\frac{d}{dt} \sum_{k=0}^{l-1} \int \nabla^k u \cdot \nabla^{k+1}\psi \, dx + \frac{P'(\bar{\rho})}{2\bar{\rho}\epsilon^2} \|\nabla\psi\|_{H^{l-1}}^2 \\ &\leq C \left(\|\nabla u\|_{H^l}^2 + \|(w-u)\|_{H^{l-1}}^2 \right) + C\delta_1 \|\nabla\varphi\|_{H^{l-1}}^2 + C\epsilon^{-2} \|\nabla\zeta\|_{H^{l-1}}^2. \end{aligned} \quad (3.39)$$

By (3.34) and (3.39), we obtain (3.32). The proof of Lemma 3.2 is complete. $\square$

3.2. Proof of Theorem 1.1. We first prove the existence of a global smooth solutions to the reformulated system (1.13), (1.10). To this end, we first prove (3.13) by using a priori estimates in the previous subsection. For $N \geq 3$, let $3 \leq l \leq N$. Then, adding (3.15) and $\beta \times (3.32)$, we obtain

$$\begin{aligned} &\frac{d}{dt} \mathcal{F}^l(t) + \left(\frac{\mu}{\bar{\rho}\bar{n}} - C\beta \right) \|\nabla u\|_{H^l}^2 + \left(\frac{\kappa}{\bar{\rho}\bar{\theta}\bar{n}} - C\beta \right) \frac{1}{\epsilon^2} \|\nabla\zeta\|_{H^l}^2 + \left(\sigma - C\beta \right) \|(w-u)\|_{H^l}^2 \\ &+ \left(\frac{\beta}{4} - C\delta_1 \right) \|\nabla\varphi\|_{H^{l-1}}^2 + \left(\beta \frac{P_\rho(\bar{\rho}, \bar{\theta})}{2\bar{\rho}} - C\delta_1 \right) \frac{1}{\epsilon^2} \|\nabla\psi\|_{H^{l-1}}^2 \leq 0, \end{aligned} \quad (3.40)$$

where $\beta \in (0, 1]$ is a small constant and

$$\begin{aligned} \mathcal{F}^l(t) &:= \|w\|_{H^l}^2 + \beta \sum_{k=0}^{l-1} \int \nabla^k w \cdot \nabla^{k+1}\varphi \, dx + \|\varphi\|_{H^l}^2 + \frac{e_\theta(\bar{\rho}, \bar{\theta})}{\bar{\theta}\epsilon^2} \|\zeta\|_{H^l}^2 \\ &+ \frac{1}{\bar{n}} \|u\|_{H^l}^2 + \beta \sum_{k=0}^{l-1} \int \nabla^k u \cdot \nabla^{k+1}\psi \, dx + \frac{P_\rho(\bar{\rho}, \bar{\theta})}{\bar{\rho}^2 \bar{n} \epsilon^2} \|\nabla\psi\|_{H^{l-1}}^2. \end{aligned} \quad (3.41)$$

By (3.41) and (1.15), we can choose the small $\beta \in (0, 1]$ and δ_1 independent $\epsilon \in (0, 1]$ such that

$$\mathcal{F}^l(t) \leq \|(w, \varphi, u, \epsilon^{-1}\psi, \epsilon^{-1}\zeta)(t)\|_{H^l}^2 \quad (3.42)$$

uniformly for all $t \geq 0$, and

$$\begin{aligned} \frac{\mu}{\bar{\rho}\bar{n}} - C\beta &\geq \frac{\mu}{2\bar{\rho}\bar{n}}, \quad \frac{\kappa}{\bar{\rho}\bar{\theta}\bar{n}} - C\beta \geq \frac{\kappa}{2\bar{\rho}\bar{\theta}\bar{n}}, \quad \sigma - C\beta \geq \frac{\sigma}{2}, \\ \frac{\beta}{4} - C\delta_1 &\geq \frac{\beta}{8}, \quad \beta \frac{P_\rho(\bar{\rho}, \bar{\theta})}{2\bar{\rho}} - C\delta_1 \geq \beta \frac{P_\rho(\bar{\rho}, \bar{\theta})}{4\bar{\rho}}. \end{aligned} \quad (3.43)$$

Integrating (3.40) over $t \in [0, \infty)$, and using (3.42), (3.43) and (1.15), we have

$$\begin{aligned} &\sup_{t \in \mathbb{R}_+} \|(\varphi, w, u)(t)\|_{H^l}^2 + \epsilon^{-2} \sup_{t \in \mathbb{R}_+} \|(\psi, \zeta)(t)\|_{H^l}^2 + \int_0^\infty \|\nabla\varphi\|_{H^{l-1}}^2 d\tau \\ &+ \int_0^\infty \|(\nabla u, w-u)\|_{H^l}^2 d\tau + \epsilon^{-2} \int_0^\infty \|\nabla\psi\|_{H^{l-1}}^2 d\tau + \epsilon^{-2} \int_0^\infty \|\nabla\zeta\|_{H^{l-1}}^2 d\tau \\ &\leq C \left(\|(\varphi, w, u)(0)\|_{H^l}^2 + \epsilon^{-2} \|(\psi, \zeta)(0)\|_{H^l}^2 \right) \end{aligned} \quad (3.44)$$

for $l = 3, \dots, N$. Therefore,

$$\|(w, \varphi, u, \epsilon^{-1}\psi, \epsilon^{-1}\zeta)(t)\|_{H^3}^2 \leq C\|(w, \varphi, u, \epsilon^{-1}\psi, \epsilon^{-1}\zeta)(0)\|_{H^3}^2. \quad (3.45)$$

Using (1.16) and (3.45), we obtain (3.13). This in turn allows us to take $l = N$ in (3.44) and we obtain (1.17).

Now, the existence of global solution is proved by the standard continuity argument for a local solution using a priori estimates (3.45). Also, the uniqueness of global solution is proved by the argument to estimate the norms on the difference of two solutions. So, we omit the details for brevity.

We prove the low Mach number limit. Let the solution to the Cauchy problem (1.13), (1.14) satisfying (1.17) by $(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon)$. Then, using (1.17) and (1.16), there exist the functions $(\varphi, w, u) \in L^\infty(0, \infty; H^N(\mathbb{R}^3))$, $\nabla u \in L^2(0, \infty; H^{N+1}(\mathbb{R}^3))$ such that

$$\begin{aligned} (\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon) &\rightarrow (\varphi, w, \bar{\rho}, u, \bar{\theta}) \text{ (*-weak) in } L^\infty(0, \infty; H^N(\mathbb{R}^3)), \\ \nabla u^\epsilon &\rightarrow \nabla u \text{ (weak) in } L^2(0, \infty; H^N(\mathbb{R}^3)), \\ (\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon) &\rightarrow (\varphi, w, \bar{\rho}, u, \bar{\theta}) \text{ (strong) in } L^2(0, \infty; H_{\text{loc}}^{N-1}(\mathbb{R}^3)), \end{aligned} \quad (3.46)$$

as $\epsilon \rightarrow 0$.

Now, we will prove that the function pair (n, w, u, Π) for some $\Pi \in L^\infty(0, \infty; H^{N-1}(\mathbb{R}^3)) \cap L^2(0, \infty; H^N(\mathbb{R}^3))$, where $n = \bar{n}e^\varphi$, is the unique smooth local solution for the compressible Euler/incompressible Navier-Stokes system (1.12), (1.18).

By using $n = \bar{n}e^\varphi$ and (3.46), we obtain from (1.13)_{1,2} that

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \end{aligned}$$

which implies (1.12)_{1,2}. By (1.13)₃ and (3.46), we obtain $\operatorname{div} u = 0$, which implies (1.12)₃.

Multiplying (1.13)₄ by $v \in [C_0^\infty(\mathbb{R}^3)]^3$, $\operatorname{div} v = 0$, and integrating it over $\mathbb{R}^3$, we have

$$\int \rho^\epsilon u_t^\epsilon \cdot v \, dx + \int \rho^\epsilon (u^\epsilon \cdot \nabla) u^\epsilon \cdot v \, dx = \mu \int \Delta u^\epsilon \cdot v \, dx + \sigma \int n^\epsilon (w^\epsilon - u^\epsilon) \cdot v \, dx. \quad (3.47)$$

By using (3.47), (3.46) and the result of De Rham [26, Proposition 1.1], there exists a function $\Pi \in L^\infty(0, \infty; H^{N-1}(\mathbb{R}^3)) \cap L^2(0, \infty; H^N(\mathbb{R}^3))$ satisfying (1.12)₄. It is easy to show that the initial condition (1.18) holds. The uniqueness of solutions to the system (1.12), (1.18) is proved by the argument to estimate the norms on the difference of two solutions. The proof of Theorem 1.1 is completed.

4. INCOMPRESSIBLE LIMIT: LARGE TEMPERATURE VARIATION

In this section, we prove Theorem 1.3. To this end, we first reformulate the system (1.22) as a new system. For brevity, we denote $(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, \theta^\epsilon, P^\epsilon)$ in (1.22) by $(\varphi, w, \rho, u, \theta, P)$.

4.1. Reformulated system. Setting $\psi = \rho - \bar{\rho}$ and $\chi = P - \bar{P}$, we reformulate (1.22) as

$$\begin{aligned} \varphi_t + \operatorname{div} w + w \cdot \nabla \varphi &= 0, \\ w_t + \nabla \varphi + \sigma(w - u) + (w \cdot \nabla) w &= 0, \\ \psi_t + \bar{\rho} \operatorname{div} u + u \cdot \nabla \psi &= G_1(\psi, u), \\ u_t - \mu \operatorname{div} \left(\frac{\nabla u}{\bar{\rho} + \psi} \right) - (\mu + \lambda) \nabla \left(\frac{\operatorname{div} u}{\bar{\rho} + \psi} \right) + \frac{1}{\bar{\rho} \epsilon^2} \nabla \chi - \sigma \bar{n}(w - u) &= G_2(\psi, u, \chi, \varphi, w), \\ \chi_t + 2\bar{P} \operatorname{div} u + u \cdot \nabla \chi - \frac{\epsilon^2}{\bar{\rho} + \psi} \left(2\mu \mathbb{D}(u) : \mathbb{D}(u) + \lambda (\operatorname{div} u)^2 \right) &= G_3(u, \chi), \end{aligned} \quad (4.1)$$

where

$$G_1(\psi, u) = -\psi \operatorname{div} u, \quad (4.2)$$

$$G_2(\psi, u, \chi, \varphi, w) = -(u \cdot \nabla)u + \frac{h_1(\psi)}{\epsilon^2} \nabla \chi + \sigma \bar{\rho} h_2(\varphi)(w - u) \\ + \mu \nabla u \nabla h_1(\psi) + (\mu + \lambda) \operatorname{div} u \operatorname{div} h_1(\psi), \quad (4.3)$$

$$h_1(\psi) = \frac{1}{\bar{\rho}} - \frac{1}{\rho}, \quad h_2(\varphi) = e^\varphi - 1, \quad (4.4)$$

$$G_3(u, \chi) = -\chi \operatorname{div} u. \quad (4.5)$$

Also, the initial condition (1.23) is reformulated as

$$(\varphi, w, \psi, u, \chi)(x, 0) = (\varphi_0, w_0, \psi_0, u_0, \chi_0)(x) \equiv (\varphi_0^\epsilon, w_0^\epsilon, \rho_0^\epsilon - \bar{\rho}, u_0^\epsilon, P_0^\epsilon - \bar{P})(x). \quad (4.6)$$

The proof of the local existence theorem for the reformulated system (4.1), (4.6) relies upon the study of a linearized system and the iteration argument.

4.2. Estimates for the linearized system. In this subsection, we study the linearized system

$$\begin{aligned} \varphi_t + \operatorname{div} w + A_1 \cdot \nabla \varphi &= 0, \quad x \in \mathbb{R}^3, \quad t > 0, \\ w_t + \nabla \varphi + \sigma(w - u) + (A_1 \cdot \nabla)w &= 0, \\ \psi_t + \bar{\rho} \operatorname{div} u + A_2 \cdot \nabla \psi &= G_1, \\ u_t - \mu \operatorname{div} (A_3 \nabla u) - (\mu + \lambda) \nabla (A_3 \operatorname{div} u) + \frac{1}{\bar{\rho} \epsilon^2} \nabla \chi - \sigma \bar{n}(w - u) &= G_2, \\ \chi_t + 2\bar{P} \operatorname{div} u + A_2 \cdot \nabla \chi - \epsilon^2 \mathbb{B} : \mathbb{D}(u) - \epsilon^2 A_4 \operatorname{div} u &= G_3, \\ (\varphi, w, \psi, u, \chi)(x, 0) &= (\varphi_0, w_0, \psi_0, u_0, \chi_0)(x), \quad x \in \Omega, \end{aligned} \quad (4.7)$$

where $A_i (i = 1, \dots, 4)$ are the scalar functions and $\mathbb{B} = (b_{ij})$ is the matrix function. We first consider an estimate for the linearized system (4.7).

Theorem 4.1. *Let $0 < T < \infty$. Assume that $(\varphi_0, w_0, \psi_0, u_0, \chi_0) \in H^N(\mathbb{R}^3)$ for an integer $N \geq 3$. Also, suppose that*

$$\begin{aligned} A_1, A_2 &\in L^1(0, T; H^N(\mathbb{R}^3)), \quad \mathbb{B}, A_3, A_4 \in L^2(0, T; H^N(\mathbb{R}^3)) \\ G_1, G_3 &\in L^1(0, T; H^N(\mathbb{R}^3)), \quad G_2 \in L^2(0, T; H^{N-1}(\mathbb{R}^3)), \end{aligned} \quad (4.8)$$

and

$$0 < c_0 \leq A_3(x, t) \leq M_0, \quad (4.9)$$

where c_0 and M_0 are positive constants independent of x and t . Let

$$\begin{aligned} (\varphi, w, \psi, u, \chi) &\in C([0, T]; H^N(\mathbb{R}^3)), \\ (\nabla \varphi, \nabla \chi) &\in L^2(0, T; H^{N-1}(\mathbb{R}^3)), \quad (\nabla u, w - u) \in L^2(0, T; H^N(\mathbb{R}^3)) \end{aligned} \quad (4.10)$$

and $(\varphi, w, \psi, u, \chi)$ be a solution of system (4.7). Then there exist positive constants C_0 and c_0 , independent of ϵ and t , such that

$$\begin{aligned} &\|(\varphi, w, \psi, u, \epsilon^{-1} \chi)(t)\|_{H^N}^2 + \int_0^t \|\nabla \varphi\|_{H^{N-1}}^2 d\tau + \epsilon^{-2} \int_0^t \|\nabla \chi\|_{H^{N-1}}^2 d\tau \\ &+ \int_0^t \|\nabla u\|_{H^N}^2 d\tau + \int_0^t \|(w - u)\|_{H^N}^2 d\tau \\ &\leq C_0 \left(\|(\varphi, w, \psi, u, \epsilon^{-1} \chi)(0)\|_{H^N}^2 + \int_0^T \|G_1\|_{H^N}^2 d\tau + \int_0^T \|G_2\|_{H^{N-1}}^2 d\tau \right. \\ &\quad \left. + \epsilon^{-1} \int_0^T \|G_3\|_{H^N}^2 d\tau \right) \left(1 + e^{c_0 F_\epsilon(T)} F_\epsilon(T) \right) \end{aligned} \quad (4.11)$$

for all $t \in [0, T]$, where

$$\begin{aligned} F_\epsilon(t) &= \int_0^t \left(1 + \|(A_1, A_2)\|_{H^N} + \|A_3\|_{H^N}^2 \right) d\tau \\ &+ \int_0^t \left(\|G_1\|_{H^N} + \|G_2\|_{H^{N-1}}^2 + \epsilon^{-1} \|G_3\|_{H^N} + \epsilon^2 \|(\mathbb{B}, A_4)\|_{H^N}^2 \right) d\tau. \end{aligned} \quad (4.12)$$

Proof. Applying ∇^k to (4.7) yields

$$\begin{aligned}
 & \nabla^k \varphi_t + \operatorname{div} \nabla^k w + \nabla^k (A_1 \cdot \nabla \varphi) = 0, \\
 & \nabla^k w_t + \nabla \nabla^k \varphi + \sigma \nabla^k (w - u) + \nabla^k ((A_1 \cdot \nabla) w) = 0, \\
 & \nabla^k \psi_t + \bar{\rho} \operatorname{div} \nabla^k u + \nabla^k (A_2 \cdot \nabla \psi) = \nabla^k G_1, \\
 & \nabla^k u_t - \mu \operatorname{div} \nabla^k (A_3 \nabla u) - (\mu + \lambda) \nabla^{k+1} (A_3 \operatorname{div} u) \\
 & \quad + \frac{1}{\bar{\rho} \epsilon^2} \nabla^{k+1} \chi - \sigma \bar{n} \nabla^k (w - u) = \nabla^k G_2, \\
 & \nabla^k \chi_t + 2\bar{P} \operatorname{div} \nabla^k u + \nabla^k (A_2 \cdot \nabla \chi) - \epsilon^2 \nabla^k (\mathbb{B} : \mathbb{D}(u)) - \epsilon^2 \nabla^k (A_4 \operatorname{div} u) = \nabla^k G_3,
 \end{aligned} \tag{4.13}$$

where $k = 0, 1, \dots, N$.

Multiplying (4.13)₂ by $\nabla^k w$ and using (4.13)₁, we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} (\|\nabla^k w\|^2 + \|\nabla^k \varphi\|^2) + \sigma \int \nabla^k (w - u) \nabla^k w \, dx = J_k^1, \\
 & J_k^1 = -\frac{1}{2} \int \operatorname{div} A_1 |\nabla^k \varphi|^2 \, dx - \int [\nabla^k, A_1] \nabla \varphi \nabla^k \varphi \, dx \\
 & \quad + \frac{1}{2} \int \operatorname{div} A_1 |\nabla^k w|^2 \, dx + \int [\nabla^k, A_1] \nabla w \cdot \nabla^k w \, dx,
 \end{aligned} \tag{4.14}$$

where we used

$$\begin{aligned}
 & \nabla^k (A_1 \cdot \nabla \varphi) = A_1 \cdot \nabla^{k+1} \varphi - [\nabla^k, A_1] \nabla \varphi, \\
 & \nabla^k ((A_1 \cdot \nabla) w) = (A_1 \cdot \nabla) \nabla^k w - [\nabla^k, A_2] \nabla w.
 \end{aligned}$$

Multiplying (4.13)₃ by $\nabla^k \psi$ yields

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\nabla^k \psi\|^2 = \int \nabla^k G_1 \nabla^k \psi \, dx - \bar{\rho} \int \operatorname{div} \nabla^k u \nabla^k \psi \, dx + J_k^2, \\
 & J_k^2 = \frac{1}{2} \int \operatorname{div} A_2 |\nabla^k \psi|^2 \, dx + \int [\nabla^k, A_2] \nabla \psi \nabla^k \psi \, dx,
 \end{aligned} \tag{4.15}$$

where we used

$$\nabla^k (A_2 \cdot \nabla \psi) = A_2 \cdot \nabla^{k+1} \psi - [\nabla^k, A_2] \nabla \psi.$$

Also, multiplying (4.13)₄ by $\nabla^k u$ and using (4.13)₅, we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} (\|\nabla^k u\|^2 + \frac{1}{2\bar{\rho}\bar{P}\epsilon^2} \|\nabla^k \chi\|^2) + \int A_3 (\mu |\nabla^{k+1} u|^2 + (\lambda + \mu) |\operatorname{div} \nabla^k u|^2) \, dx \\
 & \quad - \sigma \bar{n} \int \nabla^k (w - u) \nabla^k u \, dx \\
 & = \int \nabla^k G_2 \cdot \nabla^k u \, dx + \frac{1}{2\bar{\rho}\bar{P}\epsilon^2} \int \nabla^k G_3 \nabla^k \chi \, dx \\
 & \quad + \frac{1}{4\bar{\rho}\bar{P}\epsilon^2} \int \operatorname{div} A_2 |\nabla^k \chi|^2 \, dx + \frac{1}{2\bar{\rho}\bar{P}\epsilon^2} \int [\nabla^k, A_2] \nabla \chi \nabla^k \chi \, dx \\
 & \quad + \mu \int [\nabla^k, A_3] \nabla u : \nabla^k u \, dx + (\mu + \lambda) \int [\nabla^k, A_3] \operatorname{div} u \operatorname{div} \nabla^k u \, dx \\
 & \quad + \frac{1}{2\bar{\rho}\bar{P}} \int (\nabla^k (\mathbb{B} : \mathbb{D}(u)) + \nabla^k (A_4 \operatorname{div} u)) \nabla^k \chi \, dx,
 \end{aligned} \tag{4.16}$$

where we used

$$\begin{aligned}
 & \nabla^k (A_2 \cdot \nabla \chi) = A_2 \cdot \nabla^{k+1} \chi - [\nabla^k, A_2] \nabla \chi, \\
 & \nabla^k (A_3 \nabla u) = A_3 \nabla^{k+1} u - [\nabla^k, A_3] \nabla u,
 \end{aligned}$$

$$\nabla^k (A_3 \operatorname{div} u) = A_3 \nabla^k \operatorname{div} u - [\nabla^k, A_3] \operatorname{div} u.$$

Taking (4.14) + (4.15) + $\frac{1}{\bar{n}} \times (4.16)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\nabla^k w\|^2 + \|\nabla^k \varphi\|^2 + \|\nabla^k \psi\|^2 + \frac{1}{\bar{n}} \|\nabla^k u\|^2 + \frac{1}{2\bar{\rho}\bar{P}\bar{n}\epsilon^2} \|\nabla^k \chi\|^2 \right) \\ & + \frac{1}{\bar{n}} \int A_3 \left(\mu |\nabla^{k+1} u|^2 + (\lambda + \mu) |\operatorname{div} \nabla^k u|^2 \right) dx + \sigma \|\nabla^k (w - u)\|^2 \\ & = J_k^1 + J_k^2 + J_k^3 + J_k^4 + J_k^5, \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} J_k^3 &= \frac{1}{4\bar{\rho}\bar{n}\bar{P}\epsilon^2} \int \operatorname{div} A_2 |\nabla^k \chi|^2 dx + \frac{1}{2\bar{\rho}\bar{n}\bar{P}\epsilon^2} \int [\nabla^k, A_2] \nabla \chi \nabla^k \chi dx \\ & + \frac{\mu}{\bar{n}} \int [\nabla^k, A_3] \nabla u : \nabla^k u dx + \frac{\mu + \lambda}{\bar{n}} \int [\nabla^k, A_3] \operatorname{div} u \operatorname{div} \nabla^k u dx, \\ J_k^4 &= \frac{1}{2\bar{\rho}\bar{n}\bar{P}} \int \left(\nabla^k (\mathbb{B} : \mathbb{D}(u)) + \nabla^k (A_4 \operatorname{div} u) \right) \nabla^k \chi dx - \bar{\rho} \int \operatorname{div} \nabla^k u \nabla^k \psi dx, \\ J_k^5 &= \int \nabla^k G_1 \nabla^k \psi dx + \frac{1}{\bar{n}} \int \nabla^k G_2 \cdot \nabla^k u dx + \frac{1}{2\bar{\rho}\bar{n}\bar{P}\epsilon^2} \int \nabla^k G_3 \nabla^k \chi dx. \end{aligned} \quad (4.18)$$

Summing (4.17) for $k = 0, 1, \dots, N$ and using (4.9), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|w\|_{H^N}^2 + \|\varphi\|_{H^N}^2 + \|\psi\|_{H^N}^2 + \frac{1}{\bar{n}} \|u\|_{H^N}^2 + \frac{1}{2\bar{\rho}\bar{P}\bar{n}\epsilon^2} \|\chi\|_{H^N}^2 \right) \\ & + \frac{\mu c_0}{\bar{n}} \|\nabla u\|_{H^N}^2 + \sigma \|(w - u)\|_{H^N}^2 \\ & \leq \sum_{k=0}^N J_k^1 + \sum_{k=0}^N J_k^2 + \sum_{k=0}^N J_k^3 + \sum_{k=0}^N J_k^4 + \sum_{k=0}^N J_k^5. \end{aligned} \quad (4.19)$$

Using Hölder inequality and (2.3), we obtain from (4.14) and (4.15) that

$$\begin{aligned} \sum_{k=0}^N J_k^1 &\leq C \sum_{k=0}^N \left(\|\nabla A_1\|_{L^\infty} \|\nabla^k \varphi\| + \|\nabla \varphi\|_{L^\infty} \|\nabla^k A_1\| \right) \|\nabla^k \varphi\| \\ &+ C \sum_{k=0}^N \left(\|\nabla A_1\|_{L^\infty} \|\nabla^k w\| + \|\nabla w\|_{L^\infty} \|\nabla^k A_1\| \right) \|\nabla^k w\| \\ &\leq C \|A_1\|_{H^N} \left(\|\varphi\|_{H^N}^2 + \|w\|_{H^N}^2 \right) \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} \sum_{k=0}^N J_k^2 &\leq C \sum_{k=0}^N \left(\|\nabla A_2\|_{L^\infty} \|\nabla^k \psi\| + \|\nabla \psi\|_{L^\infty} \|\nabla^k A_2\| \right) \|\nabla^k \psi\| \\ &\leq C \|A_2\|_{H^N} \|\psi\|_{H^N}^2, \end{aligned} \quad (4.21)$$

respectively. Also, we obtain from (4.18) that

$$\begin{aligned} \sum_{k=0}^N J_k^3 &\leq \frac{C}{\epsilon^2} \sum_{k=0}^N \left(\|\nabla A_2\|_{L^\infty} \|\nabla^k \chi\| + \|\nabla \chi\|_{L^\infty} \|\nabla^k A_1\| \right) \|\nabla^k \chi\| \\ &+ C \sum_{k=0}^N \left(\|\nabla A_3\|_{L^\infty} \|\nabla^k u\| + \|\nabla u\|_{L^\infty} \|\nabla^k A_3\| \right) \|\nabla^k u\| \\ &\leq C \left(\|A_2\|_{H^N} \|\epsilon^{-1} \chi\|_{H^N}^2 + \|A_3\|_{H^N} \|u\|_{H^N}^2 \right), \end{aligned} \quad (4.22)$$

$$\begin{aligned}
& \sum_{k=0}^N J_k^4 \\
& \leq C \sum_{k=0}^N \left(\|\mathbb{B}\|_{L^\infty} \|\nabla^{k+1} u\| + \|\nabla u\|_{L^\infty} \|\nabla^k \mathbb{B}\| \right) \|\nabla^k \chi\| \\
& \quad + C \sum_{k=0}^N \left(\|\nabla A_4\|_{L^\infty} \|\nabla^{k+1} u\| + \|\nabla u\|_{L^\infty} \|\nabla^k A_4\| \right) \|\nabla^k \chi\| + C \sum_{k=0}^N \|\nabla^{k+1} u\| \|\nabla^k \psi\| \\
& \leq C \epsilon \left(\|\mathbb{B}\|_{H^N} + \|A_4\|_{H^N} \right) \|\nabla u\|_{H^N} \|\epsilon^{-1} \chi\|_{H^N} + C \|\nabla u\|_{H^N} \|\psi\|_{H^N},
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
\sum_{k=0}^N J_k^5 & \leq C \sum_{k=0}^N \left(\|\nabla^k G_1\| \|\nabla^k \psi\| + \epsilon^{-2} \|\nabla^k G_3\| \|\nabla^k \chi\| \right) + \left| \int G_2 \cdot u \, dx \right| \\
& \quad + C \sum_{k=1}^N \|\nabla^{k-1} G_2\| \|\nabla^{k+1} u\| \\
& \leq C \|G_1\|_{H^N} \|\psi\|_{H^N} + \epsilon^{-2} \|G_3\|_{H^N} \|\chi\|_{H^N} + C \|G_2\|_{H^{N-1}} \left(\|u\|_{H^N} + \|\nabla u\|_{H^N} \right) \\
& \leq C \left(\|G_1\|_{H^N} + \|G_2\|_{H^{N-1}} + \epsilon^{-1} \|G_3\|_{H^N} \right) \left(1 + \|(\psi \cdot u, \epsilon^{-1} \chi)\|_{H^N}^2 \right) \\
& \quad + \alpha \|\nabla u\|_{H^N}^2 + C \alpha^{-1} \|G_2\|_{H^{N-1}}^2
\end{aligned} \tag{4.24}$$

for all $\alpha > 0$.

Substituting (4.20)-(4.24) into (4.19), we obtain

$$\begin{aligned}
& \frac{d}{dt} \left(\|w\|_{H^N}^2 + \|\varphi\|_{H^N}^2 + \|\psi\|_{H^N}^2 + \frac{1}{n} \|u\|_{H^N}^2 + \frac{1}{2\bar{\rho}\bar{P}\bar{n}\epsilon^2} \|\chi\|_{H^N}^2 \right) \\
& \quad + \frac{\mu c_0}{\bar{n}} \|\nabla u\|_{H^N}^2 + 2\sigma \|(w - u)\|_{H^N}^2 \\
& \leq C \|(A_1, A_2, A_3)\|_{H^N} \|(\varphi, w, \psi, u, \epsilon^{-1} \chi)\|_{H^N}^2 \\
& \quad + C \epsilon^2 \|(\mathbb{B}, A_4)\|_{H^N}^2 \|\epsilon^{-1} \chi\|_{H^N}^2 + C \|(\psi, u, \epsilon^{-1} \chi)\|_{H^N}^2 \\
& \quad + C \left(\|G_1\|_{H^N} + \|G_2\|_{H^{N-1}}^2 + \epsilon^{-1} \|G_3\|_{H^N} + 1 \right) \left(1 + \|(\psi \cdot u, \epsilon^{-1} \chi)\|_{H^N}^2 \right).
\end{aligned} \tag{4.25}$$

On the other hand, noticing that

$$\begin{aligned}
& \int \nabla^k w_t \cdot \nabla^{k+1} \varphi \, dx = \frac{d}{dt} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx - \|\nabla^{k+1} w\|^2 + \int \nabla^k w \cdot \nabla^{k+1} (A_1 \cdot \nabla \varphi) \, dx, \\
& \quad \int \nabla^k u_t \cdot \nabla^{k+1} \chi \, dx \\
& = \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \chi \, dx - 2\bar{P} \|\nabla^{k+1} u\|^2 + \int \nabla^k u \cdot \nabla^{k+1} (A_2 \cdot \nabla \chi) \, dx \\
& \quad - \epsilon^2 \int \nabla^k u \cdot \nabla^{k+1} (\mathbb{B} : \mathbb{D}(u) + A_4 \operatorname{div} u) \, dx - \int \nabla^k u \cdot \nabla^{k+1} G_3 \, dx.
\end{aligned}$$

Using (4.13)₁ and (4.13)₅, and multiplying (4.13)₂ and (4.13)₄ by $\nabla^{k+1} \varphi$ and $\nabla^{k+1} \chi$, respectively, we have

$$\begin{aligned}
& \frac{d}{dt} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \frac{d}{dt} \int \nabla^k u \cdot \nabla^{k+1} \chi \, dx + \|\nabla^{k+1} \varphi\|^2 + \frac{1}{\bar{\rho}\epsilon^2} \|\nabla^{k+1} \chi\|^2 \\
& = J_k^6 + J_k^7 + \|\nabla^{k+1} w\|^2 + 2\bar{P} \|\nabla^{k+1} u\|^2,
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
J_k^6 &= - \int \left[\sigma \nabla^k (w - u) + \nabla^k \left((A_1 \cdot \nabla) w \right) \right] \cdot \nabla^{k+1} \varphi \, dx + \int \operatorname{div} \nabla^k w \nabla^k (A_1 \cdot \nabla \varphi) \, dx, \\
J_k^7 &= \int \nabla^k G_2 \cdot \nabla^{k+1} \chi \, dx + \sigma \bar{\rho} \int \nabla^k (w - u) \cdot \nabla^{k+1} \chi \, dx \\
&\quad + \mu \int \operatorname{div} \nabla^k (A_3 \nabla u) \cdot \nabla^{k+1} \chi \, dx + (\mu + \lambda) \int \nabla^{k+1} (A_3 \operatorname{div} u) \nabla^{k+1} \chi \, dx \\
&\quad + \int \operatorname{div} \nabla^k u \cdot \nabla^k (A_2 \cdot \nabla \chi) \, dx + \int \nabla^k u \cdot \nabla^{k+1} G_3 \, dx \\
&\quad - \epsilon^2 \int \operatorname{div} \nabla^k u \cdot \nabla^k (\mathbb{B} : \mathbb{D}(u) + A_4 \operatorname{div} u) \, dx.
\end{aligned} \tag{4.27}$$

Using Hölder inequality and (2.3), we obtain from (4.27) that

$$\begin{aligned}
\sum_{k=0}^{N-1} J_k^5 &\leq C (\|w - u\|_{H^{N-1}} + \|A_1\|_{H^{N-1}} \|w\|_{H^N}) \|\varphi\|_{H^N} \\
&\leq C \|w - u\|_{H^N}^2 + C(1 + \|A_1\|_{H^N}) \|(\varphi, w)\|_{H^N}^2
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
\sum_{k=0}^{N-1} J_k^6 &\leq C \left(\|G_2\|_{H^{N-1}} + \|w - u\|_{H^{N-1}} + \|A_3\|_{H^N} \|\nabla u\|_{H^N} \right) \|\chi\|_{H^N} \\
&\quad + C \|u\|_{H^N} \|A_2\|_{H^{N-1}} \|\chi\|_{H^N} + C \|u\|_{H^{N-1}} \|G_3\|_{H^N} + C \epsilon^2 \|(\mathbb{B}, A_4)\|_{H^{N-1}} \|u\|_{H^N}^2 \\
&\leq C \|(\nabla u, w - u)\|_{H^N}^2 + C(1 + \|A_3\|_{H^N}^2 + \|A_2\|_{H^{N-1}}) \|(\chi, u)\|_{H^N}^2 \\
&\quad + C \left(\|G_2\|_{H^{N-1}} + \|G_3\|_{H^N} \right) \left(1 + \|(\chi, u)\|_{H^N}^2 \right) + C \epsilon^2 \|(\mathbb{B}, A_4)\|_{H^{N-1}} \|u\|_{H^N}^2.
\end{aligned} \tag{4.29}$$

Summing (4.26) for $k = 0, 1, \dots, N-1$, and using (4.28) and (4.29), we obtain

$$\begin{aligned}
&\frac{d}{dt} \sum_{k=0}^{N-1} \left(\int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \int \nabla^k u \cdot \nabla^{k+1} \chi \, dx \right) + \|\nabla \varphi\|_{H^{N-1}}^2 + \frac{1}{\bar{\rho} \epsilon^2} \|\nabla \chi\|_{H^{N-1}}^2 \\
&\leq C \|(\nabla u, w - u)\|_{H^N}^2 + C \left(1 + \|(A_1, A_2)\|_{H^N} + \|A_3\|_{H^N}^2 \right) \|(\varphi, w, \chi, u)\|_{H^N}^2 \\
&\quad + C \left(\|G_2\|_{H^{N-1}} + \|G_3\|_{H^N} \right) \left(1 + \|(\chi, u)\|_{H^N}^2 \right) + C \epsilon^2 \|(\mathbb{B}, A_4)\|_{H^{N-1}} \|u\|_{H^N}^2.
\end{aligned} \tag{4.30}$$

We can assume $0 < \epsilon \leq 1$ without loss of generality. And we choose $\beta \in (0, 1]$ to be suitably small. Then, adding (4.25) and $\beta \times (4.30)$, we obtain

$$\begin{aligned}
&\frac{d}{dt} E(t) + \beta \|\nabla \varphi\|_{H^{N-1}}^2 + \frac{\beta}{\bar{\rho} \epsilon^2} \|\nabla \psi\|_{H^{N-1}}^2 \\
&\quad + \left(\frac{\mu c_0}{\bar{n}} - C\beta \right) \|\nabla u\|_{H^N}^2 + (2\sigma - C\beta) \|w - u\|_{H^N}^2 \\
&\leq C \left(1 + \|(A_1, A_2, G_1, \epsilon^{-1} G_3)\|_{H^N} + \|A_3\|_{H^N}^2 + \|G_2\|_{H^{N-1}}^2 \right) \|(\varphi, w, \epsilon^{-1} \chi, u, \psi)\|_{H^N}^2 \\
&\quad + C \left(\|G_1\|_{H^N} + \|G_2\|_{H^{N-1}}^2 + \epsilon^{-1} \|G_3\|_{H^N} \right) + C \epsilon^2 \|(\mathbb{B}, A_4)\|_{H^N}^2 \|u, \epsilon^{-1} \chi\|_{H^N}^2,
\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}
E(t) &:= \|w\|_{H^N}^2 + \beta \sum_{k=0}^{N-1} \int \nabla^k w \cdot \nabla^{k+1} \varphi \, dx + \|\varphi\|_{H^N}^2 \\
&\quad + \frac{1}{\bar{n}} \|u\|_{H^N}^2 + \beta \sum_{k=0}^{N-1} \int \nabla^k u \cdot \nabla^{k+1} \chi \, dx + \frac{1}{2\bar{\rho} \bar{P} \bar{n} \epsilon^2} \|\chi\|_{H^N}^2 + \|\psi\|_{H^N}^2.
\end{aligned} \tag{4.32}$$

By (4.32) and (1.2), we can choose a small $\beta \in (0, 1]$ independent of $\epsilon \in (0, 1]$ satisfying

$$E(t) \preceq \|(w, \varphi, u, \psi, \epsilon^{-1} \chi)(t)\|_{H^N}^2 \tag{4.33}$$

uniformly for all $t \in [0, T]$.

Integrating (4.31) over $t \in [0, T]$, and using (4.33), (1.2) and the smallness of β , we have

$$\begin{aligned} & \| (w, \varphi, u, \psi, \epsilon^{-1}\chi)(t) \|_{H^N}^2 + \int_0^t \|\nabla \varphi\|_{H^{N-1}}^2 d\tau \\ & + \int_0^t \|\nabla u\|_{H^N}^2 d\tau + \int_0^t \|w - u\|_{H^N}^2 d\tau + \epsilon^{-2} \int_0^t \|\nabla \chi\|_{H^{N-1}}^2 d\tau \\ & \leq C \| (w, \varphi, u, \psi, \epsilon^{-1}\chi)(0) \|_{H^N}^2 + C \int_0^t \left(\|G_1\|_{H^N} + \|G_2\|_{H^{N-1}}^2 + \epsilon^{-1} \|G_3\|_{H^N} \right) d\tau \\ & + C \int_0^t F_\epsilon(\tau) \| (w, \varphi, u, \psi, \epsilon^{-1}\chi) \|_{H^N}^2 d\tau, \end{aligned} \quad (4.34)$$

where $F_\epsilon(t)$ is the function defined in (4.12). Applying Gronwall inequality to (4.34) yields

$$\| (w, \varphi, u, \psi, \epsilon^{-1}\chi)(t) \|_{H^N}^2 \leq C \Lambda_1(T) e^{C\Lambda_2(T)}, \quad (4.35)$$

where

$$\Lambda_1(T) = \| (w, \varphi, u, \psi, \epsilon^{-1}\chi)(0) \|_{H^N}^2 + \int_0^T \left(\|G_1\|_{H^N} + \|G_2\|_{H^{N-1}}^2 + \epsilon^{-1} \|G_3\|_{H^N} \right) d\tau.$$

By (4.34) and (4.35), we obtain (4.11). The proof of Theorem 4.1 is complete. $\square$

Next, applying Theorem 4.1, it is easy to obtain the existence and uniqueness of the smooth solution for the linearized system (4.7) by the standard methods. We will omit the proof for brevity.

Theorem 4.2. *Let $0 < T < \infty$. Assume that (4.8) and $(\varphi_0, w_0, \psi_0, u_0, \chi_0) \in H^N(\mathbb{R}^3)$ for an integer $N \geq 3$. Then, there exists a unique solution $(\varphi, w, \psi, u, \chi)$ of the linearized system (4.7) satisfying (4.10) and (4.11).*

4.3. Proof of Theorem 1.3. We first prove the existence of the local smooth solutions to the reformulated system (4.1), (4.6) using Theorem 4.1. To this end, we define the set

$$\begin{aligned} \mathbf{Z}_M(0, t_0) = \{ & (\varphi, w, \psi, u, \chi) \in X_N(0, t_0) \mid \|(\varphi, w, \psi, u, \chi)\|_{X_N(0, t_0)}^2 \leq M, \\ & 0 < m_0^{-1} \leq \bar{\rho} + \psi(x, t), \quad \bar{P} + \chi(x, t) \leq m_0 \} \end{aligned}$$

for a positive constant $m_0 > 1$ and an integer $N \geq 3$, where

$$\begin{aligned} M &= 4C_0 \|\varphi_0, w_0, \psi_0, u_0, \epsilon^{-1}\chi_0\|_{H^N}^2, \quad C_0 \text{ is the constant determined in (4.11),} \\ X_N(0, t_0) &= \{(\varphi, w, \psi, u, \chi) \in C([0, t_0]; H^N(\mathbb{R}^3)) : \nabla u \in L^2(0, t_0; H^N(\mathbb{R}^3))\}, \end{aligned} \quad (4.36)$$

and

$$\begin{aligned} \|(\varphi, w, \psi, u, \chi)\|_{X_N(0, t_0)}^2 &= \|(\varphi, w, \psi, u, \epsilon^{-1}\chi)(t)\|_{H^N}^2 + \int_0^{t_0} \|\nabla \varphi\|_{H^{N-1}}^2 d\tau \\ &+ \epsilon^{-2} \int_0^{t_0} \|\nabla \chi\|_{H^{N-1}}^2 d\tau + \int_0^{t_0} \|\nabla u\|_{H^N}^2 d\tau + \int_0^t \|w - u\|_{H^N}^2 d\tau. \end{aligned} \quad (4.37)$$

For fixed $(\varphi^{j-1}, w^{j-1}, \psi^{j-1}, u^{j-1}, \chi^{j-1}) \in \mathbf{Z}_N(0, t_0)$, $j = 1, 2, \dots$, we consider the linearized system

$$\begin{aligned}
 \varphi_t + \operatorname{div} w + w^{j-1} \cdot \nabla \varphi &= 0, \quad x \in \mathbb{R}^3, \quad t > 0, \\
 w_t + \nabla \varphi + \sigma(w - u) + (w^{j-1} \cdot \nabla)w &= 0, \\
 \psi_t + \bar{\rho} \operatorname{div} u + u^{j-1} \cdot \nabla \psi &= G_1(\psi^{j-1}, u^{j-1}), \\
 u_t - \mu \operatorname{div} \left(\frac{\nabla u}{\bar{\rho} + \psi^{j-1}} \right) - (\mu + \lambda) \nabla \left(\frac{\operatorname{div} u}{\bar{\rho} + \psi^{j-1}} \right) + \frac{1}{\bar{\rho} \epsilon^2} \nabla \psi - \sigma \bar{n}(w - u) \\
 &= G_2(\psi^{j-1}, u^{j-1}, \chi^{j-1}, \varphi^{j-1}, w^{j-1}), \\
 \chi_t + 2\bar{P} \operatorname{div} u + u^{j-1} \cdot \nabla \chi - \frac{2\mu \epsilon^2 \mathbb{D}(u^{j-1})}{\bar{\rho} + \psi^{j-1}} : \mathbb{D}(u) - \frac{\lambda \epsilon^2 \operatorname{div} u^{j-1}}{\bar{\rho} + \psi^{j-1}} \operatorname{div} u \\
 &= G_3(u^{j-1}, \chi^{j-1}), \\
 (\varphi, w, \psi, u, \chi)(x, 0) &= (\varphi_0, w_0, \psi_0, u_0, \chi_0)(x), \quad x \in \Omega.
 \end{aligned} \tag{4.38}$$

where $(\varphi^0, w^0, \psi^0, u^0, \chi^0) = (\varphi_0, w_0, \psi_0, u_0, \chi_0)$, $G_1(\psi^{j-1}, u^{j-1})$, $G_2(\psi^{j-1}, u^{j-1}, \chi^{j-1}, \varphi^{j-1}, w^{j-1})$, and $G_3(u^{j-1}, \chi^{j-1})$ are defined by (4.2) and (4.3), respectively.

By using Lemma 2.2, we obtain from (4.2) and (4.5) that

$$\begin{aligned}
 G_1(\psi^{j-1}, u^{j-1}) &= -\psi^{j-1} \operatorname{div} u^{j-1} \in L^1(0, t_0; H^N(\mathbb{R}^3)), \\
 \int_0^{t_0} \|G_1(\psi^{j-1}, u^{j-1})\|_{H^N} d\tau &\leq C \sup_{0 \leq t \leq t_0} \|\psi^{j-1}\|_{H^N} \int_0^{t_0} \|\nabla u^{j-1}\|_{H^N} d\tau \leq CM\sqrt{t_0}
 \end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
 G_3(u^{j-1}, \chi^{j-1}) &= -\chi^{j-1} \operatorname{div} u^{j-1} \in L^1(0, t_0; H^N(\mathbb{R}^3)), \\
 \int_0^{t_0} \epsilon^{-1} \|G_3(u^{j-1}, \chi^{j-1})\|_{H^N} d\tau &\leq C \sup_{0 \leq t \leq t_0} \|\epsilon^{-1} \chi^{j-1}\|_{H^N} \int_0^{t_0} \|\nabla u^{j-1}\|_{H^N} d\tau \leq CM\sqrt{t_0},
 \end{aligned} \tag{4.40}$$

respectively. By using Lemma 2.2 and (2.2), we have

$$\begin{aligned}
 &\int_0^{t_0} \|(u^{j-1} \cdot \nabla)u^{j-1}\|_{H^{N-1}}^2 d\tau \\
 &\leq C \sup_{0 \leq t \leq t_0} \|u^{j-1}\|_{H^{N-1}}^2 \int_0^{t_0} \|\nabla u^{j-1}\|_{H^{N-1}}^2 d\tau \leq CMt_0, \\
 &\epsilon^{-2} \int_0^{t_0} \|h_1(\psi^{j-1}) \nabla \chi^{j-1}\|_{H^{N-1}}^2 d\tau \\
 &\leq C \epsilon^{-2} \sup_{0 \leq t \leq t_0} \|\chi^{j-1}\|_{H^{N-1}}^2 \int_0^{t_0} \|\nabla \psi^{j-1}\|_{H^{N-1}}^2 d\tau \leq CMt_0, \\
 &\int_0^{t_0} \|h_2(\varphi^{j-1})(w^{j-1} - u^{j-1})\|_{H^{N-1}}^2 d\tau \\
 &\leq C \sup_{0 \leq t \leq t_0} \|\varphi^{j-1}\|_{H^{N-1}}^2 \int_0^{t_0} \|w^{j-1} - u^{j-1}\|_{H^{N-1}}^2 d\tau \leq CMt_0 \\
 &\int_0^{t_0} \|\mu \nabla u^{j-1} \nabla h_1(\psi^{j-1}) + (\mu + \lambda) \operatorname{div} u^{j-1} \operatorname{div} h_1(\psi^{j-1})\|_{H^{N-1}}^2 d\tau \\
 &\leq C \sup_{0 \leq t \leq t_0} \|\psi^{j-1}\|_{H^N}^2 \int_0^{t_0} \|\nabla u^{j-1}\|_{H^{N-1}}^2 d\tau \leq CMt_0
 \end{aligned}$$

for $(\varphi^{j-1}, w^{j-1}, \psi^{j-1}, u^{j-1}, \chi^{j-1}) \in \mathbf{Z}_N(0, t_0)$, $j = 1, 2, \dots$. Therefore, we obtain from (4.3) that

$$\begin{aligned}
 G_2(\psi^{j-1}, u^{j-1}, \chi^{j-1}, \varphi^{j-1}, w^{j-1}) &\in L^2(0, t_0; H^{N-1}(\mathbb{R}^3)), \\
 \int_0^{t_0} \|G_2(\psi^{j-1}, u^{j-1}, \chi^{j-1}, \varphi^{j-1}, w^{j-1})\|_{H^{N-1}}^2 d\tau &\leq CMt_0.
 \end{aligned} \tag{4.41}$$

Also, we have

$$\begin{aligned} A_1 &= w^{j-1} \in L^1(0, t_0; H^N(\mathbb{R}^3)), \quad \int_0^{t_0} \|A_1\|_{H^N} d\tau \leq CM\sqrt{t_0}, \\ A_2 &= u^{j-1} \in L^1(0, t_0; H^N(\mathbb{R}^3)), \quad \int_0^{t_0} \|A_2\|_{H^N} d\tau \leq CM\sqrt{t_0}, \end{aligned} \quad (4.42)$$

and

$$\begin{aligned} A_3 &= (\bar{\rho} + \psi^{j-1})^{-1} \in L^2(0, t_0; H^N(\mathbb{R}^3)), \\ 0 \leq c_0 \leq A_3(x, t) \leq M_0, \quad \int_0^{t_0} \|A_3\|_{H^N}^2 d\tau &\leq CM\sqrt{t_0} \end{aligned} \quad (4.43)$$

for small $t_0 > 0$. Moreover,

$$\begin{aligned} \mathbb{B} &= \frac{2\mu\mathbb{D}(u^{j-1})}{\bar{\rho} + \psi^{j-1}} \in L^2(0, t_0; H^N(\mathbb{R}^3)), \quad \int_0^{t_0} \|\mathbb{B}\|_{H^N}^2 d\tau \leq CM, \\ A_4 &= \frac{\lambda \operatorname{div} u^{j-1}}{\bar{\rho} + \psi^{j-1}} \in L^2(0, t_0; H^N(\mathbb{R}^3)), \quad \int_0^{t_0} \|A_4\|_{H^N}^2 d\tau \leq CM. \end{aligned} \quad (4.44)$$

By using (4.39)-(4.44) and Theorem 4.2, there exists a unique solution $(\varphi^j, w^j, \psi^j, u^j, \chi^j) \in X_N(0, t_0)$ to the linearized system (4.38) satisfying

$$\begin{aligned} &\|(\varphi^j, w^j, \psi^j, u^j, \epsilon^{-1}\chi^j)(t)\|_{H^N}^2 + \int_0^t \|\nabla \varphi^j\|_{H^{N-1}}^2 d\tau + \epsilon^{-2} \int_0^t \|\nabla \chi^j\|_{H^{N-1}}^2 d\tau \\ &+ \int_0^t \|\nabla u^j\|_{H^N}^2 d\tau + \int_0^t \|w^j - u^j\|_{H^N}^2 d\tau \\ &\leq C_0 \left(\|(\varphi_0, w_0, \psi_0, u_0, \epsilon^{-1}\chi_0)\|_{H^N}^2 + \int_0^{t_0} \|G_1^{j-1}\|_{H^N}^2 d\tau + \int_0^{t_0} \|G_2^{j-1}\|_{H^{N-1}}^2 d\tau \right. \\ &\quad \left. + \int_0^{t_0} \|\epsilon^{-1}G_3^{j-1}\|_{H^N}^2 d\tau \right) \left(1 + e^{c_0 F_\epsilon^{j-1}(t_0)} F_\epsilon^{j-1}(t_0) \right) \end{aligned} \quad (4.45)$$

for all $t \in [0, t_0]$, where $G_1^{j-1} = G_1(\psi^{j-1}, u^{j-1})$, $G_2^{j-1} = G_2(\psi^{j-1}, u^{j-1}, \chi^{j-1}, \varphi^{j-1}, w^{j-1})$, $G_3^{j-1} = G_3(u^{j-1}, \chi^{j-1})$ and

$$\begin{aligned} F_\epsilon^{j-1}(t_0) &= \int_0^{t_0} \left(1 + \|(A_1, A_2)\|_{H^N} + \|A_3\|_{H^N}^2 \right) d\tau \\ &\quad + \int_0^{t_0} \left(\|G_1^{j-1}\|_{H^N} + \|G_2^{j-1}\|_{H^{N-1}}^2 + \epsilon^{-1}\|G_3^{j-1}\|_{H^N} + \epsilon^2\|(\mathbb{B}, A_4)\|_{H^N}^2 \right) d\tau. \end{aligned}$$

Choosing t_0 and ϵ to be small such that $e^{c_0 F_\epsilon^{j-1}(t_0)} F_\epsilon^{j-1}(t_0) \leq 1$ and

$$\int_0^{t_0} \|G_1^{j-1}\|_{H^N}^2 d\tau + \int_0^{t_0} \|G_2^{j-1}\|_{H^{N-1}}^2 d\tau + \int_0^{t_0} \|\epsilon^{-1}G_3^{j-1}\|_{H^N}^2 d\tau \leq \|(\varphi_0, w_0, \psi_0, u_0, \epsilon^{-1}\chi_0)\|_{H^N}^2,$$

which is possible because of (4.39)-(4.44), we obtain from (4.45) that

$$\begin{aligned} &\|(\varphi^j, w^j, \psi^j, u^j, \epsilon^{-1}\chi^j)(t)\|_{H^N}^2 + \int_0^t \|\nabla \varphi^j\|_{H^{N-1}}^2 d\tau + \epsilon^{-2} \int_0^t \|\nabla \chi^j\|_{H^{N-1}}^2 d\tau \\ &+ \int_0^t \|\nabla u^j\|_{H^N}^2 d\tau + \int_0^t \|w^j - u^j\|_{H^N}^2 d\tau \\ &\leq 4C_0 \|(\varphi_0, w_0, \psi_0, u_0, \epsilon^{-1}\chi_0)\|_{H^N}^2 \end{aligned} \quad (4.46)$$

for all $t \in [0, t_0]$. Also, by (1.24), $\bar{\rho} + \psi^j(x, 0) = \rho_0(x)$ and $\bar{P} + \chi^j(x, 0) = P_0(x)$, we know that there exists a positive constant $m_0 > 1$ such that

$$0 < m_0^{-1} \leq \bar{\rho} + \psi^j(x, t), \quad \bar{P} + \chi^j(x, t) \leq m_0$$

for small $t_0 > 0$. Therefore, $(\varphi^j, w^j, \psi^j, u^j, \chi^j) \in \mathbf{Z}_N(0, t_0)$ for $j = 1, 2, \dots$ due to (4.36) and (4.37). Moreover, by using (4.46) and (1.25), we have

$$\begin{aligned} \{(\varphi^j, w^j, \psi^j, u^j, \chi^j)\}_{j=1}^\infty &\text{ is bounded in } L^\infty(0, t_0; H^N(\mathbb{R}^3)), \\ \{\nabla u^j\}_{j=1}^\infty &\text{ is bounded in } L^2(0, t_0; H^N(\mathbb{R}^3)) \end{aligned} \quad (4.47)$$

for the fixed $\epsilon > 0$. Also, by using (4.47) and (4.41)-(4.42), we obtain from (4.38) that

$$\{\partial_t(\varphi^j, w^j, \psi^j, u^j, \chi^j)\}_{j=1}^\infty \text{ is bounded in } L^2(0, t_0; H^{N-1}(\mathbb{R}^3)) \quad (4.48)$$

for the fixed $\epsilon > 0$.

By using (4.47), (4.48) and the compactness result [26, Theorem 2.1 in Ch.III], there exists the subsequences of $\{(\varphi^j, w^j, \psi^j, u^j, \chi^j)\}$ (denoting them as $(\varphi^j, w^j, \psi^j, u^j, \chi^j)$ still) such that when $j \rightarrow \infty$, it holds that

$$\begin{aligned} (\varphi^j, w^j, \psi^j, u^j, \chi^j) &\rightarrow (\varphi, w, \psi, u, \chi) \text{ (*-weak) in } L^\infty(0, t_0; H^N(\mathbb{R}^3)), \\ \nabla u^j &\rightarrow \nabla u \text{ (weak) in } L^2(0, t_0; H^N(\mathbb{R}^3)), \\ (\varphi^j, w^j, \psi^j, u^j, \chi^j) &\rightarrow (\varphi, w, \psi, u, \chi) \text{ (strong) in } L^2(0, t_0; H_{\text{loc}}^{N-1}(\mathbb{R}^3)), \end{aligned} \quad (4.49)$$

where $H_{\text{loc}}^{N-1}(\mathbb{R}^3)$ denote the space $\{v \mid v \in H^{N-1}(\Omega)\}$ for any bounded domain Ω of $\mathbb{R}^3$, and

$$(\varphi, w, \psi, u, \chi) \in L^\infty(0, t_0; H^N(\mathbb{R}^3)), \quad \nabla u \in L^2(0, t_0; H^N(\mathbb{R}^3)). \quad (4.50)$$

By using (4.49) and (4.2)-(4.5), we have

$$\begin{aligned} \int_0^{t_0} \int G_1(\psi^{j-1}, u^{j-1}) \zeta \eta(t) dx dt &\rightarrow \int_0^{t_0} \int G_1(\psi, u) \zeta \eta(t) dx dt, \\ \int_0^{t_0} \int G_2(\psi^{j-1}, u^{j-1}, \chi^{j-1}, \varphi^{j-1}, w^{j-1}) \cdot v \eta(t) dx dt &\rightarrow \int_0^{t_0} \int G_2(\psi, u, \chi, \varphi, w) \cdot v \eta(t) dx dt, \\ \int_0^{t_0} \int G_3(u^{j-1}, \chi^{j-1}) \zeta \eta(t) dx dt &\rightarrow \int_0^{t_0} \int G_3(u, \chi) \zeta \eta(t) dx dt \end{aligned} \quad (4.51)$$

for all $\zeta \in C_0^\infty(\mathbb{R}^3)$, $v \in [C_0^\infty(\mathbb{R}^3)]^3$ and $\eta \in C_0^\infty(0, t_0)$. Moreover, by using (4.49) and (4.51), it is easy to check that $(\varphi, w, \psi, u, \chi)$ is a solution to system (4.1)-(4.6). Then, setting

$$\rho = \bar{\rho} + \psi \quad \text{and} \quad P = \bar{P} + \chi,$$

and by using (4.1)-(4.6) and (4.50), we know that $(\varphi, w, \rho, u, \chi)$ is a solution to the Cauchy problem (1.22), (1.23) satisfying (1.26). Also, the estimate (1.27) follows from (4.46) and (4.49). The proof for the uniqueness of the local solution is proved by the argument to estimate the norms on the difference of two solutions, so we will omit it for brevity.

Now we prove the low Mach number limit. Let the solution to the Cauchy problem (1.22), (1.23) satisfying (1.26) and (1.27) by $(\varphi^\epsilon, w^\epsilon, \rho^\epsilon, u^\epsilon, P^\epsilon)$. Then, using (1.27), Lemma 2.2 and (1.13), there exist the functions $(\varphi, w, \rho - \bar{\rho}, u) \in L^\infty(0, t_0; H^N(\mathbb{R}^3))$, $\nabla u \in L^2(0, t_0; H^{N+1}(\mathbb{R}^3))$ such that

$$\begin{aligned} (\varphi^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon, P^\epsilon) &\rightarrow (\varphi, w, \rho - \bar{\rho}, u, \bar{P}) \text{ (*-weak) in } L^\infty(0, t_0; H^N(\mathbb{R}^3)), \\ \nabla u^\epsilon &\rightarrow \nabla u \text{ (weak) in } L^2(0, t_0; H^N(\mathbb{R}^3)), \\ (\varphi^\epsilon, w^\epsilon, \rho^\epsilon - \bar{\rho}, u^\epsilon, P^\epsilon) &\rightarrow (\varphi, w, \rho - \bar{\rho}, u, \bar{P}) \text{ (strong) in } L^2(0, t_0; H_{\text{loc}}^{N-1}(\mathbb{R}^3)), \end{aligned} \quad (4.52)$$

as $\epsilon \rightarrow 0$, which implies (1.28).

Now, we will prove that the function pair (n, w, ρ, u, Π) for some $\Pi \in L^\infty(0, T_0; H^{N-1}(\mathbb{R}^3)) \cap L^2(0, T_0; H^N(\mathbb{R}^3))$, where $n = \bar{n}e^\varphi$, is the unique smooth local solution for the compressible Euler/non-homogeneous incompressible Navier-Stokes system (1.21), (1.29).

By (1.22)₁₋₃ and (4.52), we obtain

$$\begin{aligned} n_t + \operatorname{div}(nw) &= 0, \\ (nw)_t + \operatorname{div}(nw \otimes w) + \nabla n &= -\sigma n(w - u), \\ \rho_t + \operatorname{div}(\rho u) &= 0, \end{aligned}$$

which implies $(1.21)_{1-3}$.

By $(1.22)_5$ and (4.52) , we obtain $\operatorname{div} u = 0$, which implies $(1.21)_4$. Multiplying $(1.22)_4$ by $v \in [C_0^\infty(\mathbb{R}^3)]^3$, $\operatorname{div} v = 0$, and integrating it over $\mathbb{R}^3$, we have

$$\int \rho^\epsilon u_t^\epsilon \cdot v \, dx + \int \rho^\epsilon (u^\epsilon \cdot \nabla) u^\epsilon \cdot v \, dx = \mu \int \Delta u^\epsilon \cdot v \, dx + \sigma \int n^\epsilon (w^\epsilon - u^\epsilon) \cdot v \, dx. \quad (4.53)$$

By using (4.53) , (4.52) and the result of De Rham [26, Proposition 1.1], there exists a function $\Pi \in L^\infty(0, t_0; H^{N-1}(\mathbb{R}^3)) \cap L^2(0, t_0; H^N(\mathbb{R}^3))$ satisfying $(1.21)_5$. It is easy to show that the initial condition (1.29) holds. The uniqueness of solutions to the system (1.21) , (1.29) is proved by a standard argument. The proof of Theorem 1.3 is complete.

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