

MULTI-POINT CONCENTRATION EXPANSIONS FOR THE TWO-DIMENSIONAL KELLER-SEGEL FREE ENERGY

JIAPING YANG

ABSTRACT. We study concentration expansions for the two-dimensional Keller-Segel free energy. The main result concerns semi-discrete quadratic optimal transport interpolations from a bounded compactly supported density to a prescribed finite atomic measure. Along these paths the energy admits a logarithmic expansion whose leading coefficient depends only on the masses assigned to the atoms. This coefficient captures the entropy-attraction balance associated with a prescribed finite splitting of the mass. Partial-concentration families with a diffuse remainder are analyzed by the same method. We also quantify how prescribed packet merging changes the leading coefficient, analyze critical profile relaxation and mesoscopic center coalescence, and construct explicit one-step JKO competitors.

1. INTRODUCTION

The Keller-Segel model was introduced to describe chemotactic aggregation and the onset of concentration in biological populations [21]. In the two-dimensional parabolic-elliptic setting, the whole-plane model admits a distinguished mass scale: the value 8π marks the boundary between the subcritical regime and regimes in which concentration may occur. This threshold appears in the logarithmic Hardy-Littlewood-Sobolev inequality, in the scaling of the entropy-interaction free energy, and in the classical virial identity.

Critical-mass theory is linked to the logarithmic HLS inequality and to the entropy-interaction structure of the free energy [6, 9, 12, 16]. Classical PDE works developed existence, blow-up, and qualitative properties of Keller-Segel solutions [4, 17, 19, 25]; more recent work addresses stability refinements, finite-time blow-up constructions, and critical or near-critical concentration dynamics [8, 11, 14, 18]. In particular, recent finite-time constructions include solutions concentrating at prescribed finitely many points [8]. The present paper takes a complementary static variational viewpoint: for prescribed concentration families, it identifies the singular and finite parts of the free energy near atomic limits.

The Wasserstein gradient-flow formulation offers a variational framework for the model. The general metric theory of gradient flows and the JKO minimizing-movement scheme were developed in [1, 20, 23, 26, 27, 28, 29]; Keller-Segel-specific estimates and minimizing-movement arguments appear in [5, 10, 13, 15]. Here the Wasserstein viewpoint serves to organize explicit concentration paths. The geometric input is the classical Laguerre-cell description of semi-discrete optimal transport [2, 3, 22, 24].

We work on $\mathbb{R}^2$ and use the convention

$$c(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\rho(y) \, dy, \quad -\Delta c = \rho.$$

The parabolic-elliptic Keller-Segel equation is

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\rho \nabla c), \tag{1.1}$$

2020 *Mathematics Subject Classification.* 35K55, 35Q92, 49Q22.

Key words and phrases. Keller-Segel equation; critical mass; Wasserstein distance; JKO scheme; semi-discrete optimal transport.

©2026. This work is licensed under a CC BY 4.0 license.

Submitted May 20, 2026. Published October 6, 2026.

and its free energy is

$$\mathcal{F}(\rho) = \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x) \log |x - y| \rho(y) \, dx \, dy.$$

Formally,

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho} \right).$$

We address the associated static variational problem: given a finite atomic splitting of the mass, we examine how the balance between entropy and logarithmic attraction changes along the corresponding semi-discrete Wasserstein interpolation.

Let $\mu = \rho \, dx$ be a bounded compactly supported density of mass M , and let

$$\nu = \sum_{i=1}^k m_i \delta_{z_i}, \quad z_i \neq z_j, \quad \sum_{i=1}^k m_i = M.$$

If T is the quadratic-cost optimal map from μ to ν , we consider

$$\mu_s = ((1-s)\text{Id} + sT)_{\#} \mu, \quad 0 \leq s < 1.$$

On the Laguerre cell $A_i = \{T = z_i\}$ this map reads

$$x \mapsto z_i + (1-s)(x - z_i),$$

so all cells contract with microscopic scale $1-s$. The entropy contributes $2M \log(1/(1-s))$, the self-interaction of the i -th cell contributes $-m_i^2 (4\pi)^{-1} \log(1/(1-s))$, and the interactions between distinct cells remain finite. The main expansion is

$$\mathcal{F}(\mu_s) = \left(2M - \frac{1}{4\pi} \sum_{i=1}^k m_i^2 \right) \log \frac{1}{1-s} + C_{\rho, T, z} + o(1). \quad (1.2)$$

We denote the leading coefficient by

$$\Lambda_M(m) := 2M - \frac{1}{4\pi} \sum_{i=1}^k m_i^2.$$

The principal contribution of this paper is the semi-discrete expansion (1.2) for a prescribed finite atomic splitting. Its logarithmic coefficient depends only on the masses assigned to the atoms, while the finite part depends on the Laguerre cells and target locations. The sign of this coefficient leads to the observations recorded below, including the effect of merging packets, critical profile relaxation, mesoscopic coalescence, the scale-neutral role of 8π , and explicit JKO competitors. The expansion complements the log-HLS critical-mass theory and the dynamical blow-up literature by describing the free energy along prescribed concentration paths. Connecting this description with Keller-Segel trajectories requires stability analysis and information on blow-up-point selection.

We also treat partial concentration in the presence of a diffuse background. If packets of masses m_i concentrate while a bounded compactly supported background σ remains diffuse, the logarithmic coefficient is

$$\Lambda_c(m) = 2M_c - \frac{1}{4\pi} \sum_i m_i^2, \quad M_c = \sum_i m_i.$$

In the common-scale critical case $\Lambda_c(m) = 0$, the point-dependent finite term is

$$\mathcal{E}_{\text{eff}}(z; m, \sigma) = \frac{1}{2\pi} \sum_{1 \leq i < j \leq k} m_i m_j \log |z_i - z_j| + \frac{1}{2\pi} \sum_i m_i \int_{\mathbb{R}^2} \log |z_i - y| \sigma(y) \, dy.$$

The comparable-scale variant yields the single-packet coefficient

$$\lambda(m) = 2m - \frac{m^2}{4\pi} = \frac{m(8\pi - m)}{4\pi},$$

so 8π is the unique positive scale-neutral packet mass. At this mass the sharp logarithmic HLS inequality gives a lower bound for the local profile contribution. Relaxing over profiles leaves an explicit constant per packet and the point-interaction energy. We also allow concentration centers

to approach each other at mesoscopic rates. The resulting coefficients interpolate between the coefficient of separated packets and that of merged clusters. If pairwise separation exponents are prescribed, the logarithmic coefficient acquires an explicit pairwise attraction correction. As the prescribed mesoscopic coalescence exponents increase, the logarithmic coefficient decreases. For 8π packets, every positive coalescence exponent gives a negative correction to the separated-packet coefficient.

Throughout, by an atomic endpoint we mean a singular endpoint of the finite-entropy absolutely continuous class inside Wasserstein space. Atomic measures are not metric boundary points of $(\mathcal{P}_2(\mathbb{R}^2), W_2)$; the singularity considered here is the loss of absolute continuity and finite entropy as a regular density approaches an atomic measure in the Wasserstein distance.

For the prescribed families studied below, boundedness and compact support make the logarithmic self-interactions finite, while separation keeps the finite cross-interactions distinct from the singular local terms. More general concentration patterns would bring compactness and uniform integrability into the analysis.

The paper is organized as follows. Section 2 fixes notation and records the semi-discrete density formula and logarithmic integrability estimates. Section 3 proves the multi-point concentration expansion. Section 4 treats partial concentration, critical finite terms, scale variations, profile relaxation, mesoscopic coalescence, and fixed-profile uniform convergence. Section 5 applies the expansions to the one-step JKO functional and recalls the standard virial identity as a PDE comparison. Section 6 contains concluding remarks and further directions.

2. PRELIMINARIES

Let $\mathcal{M}_M(\mathbb{R}^2)$ denote the set of nonnegative Borel measures on $\mathbb{R}^2$ with total mass M . If $\mu \in \mathcal{M}_M(\mathbb{R}^2)$ is absolutely continuous, we write $\mu = \rho dx$. For measures of the same total mass M , we define

$$W_{2,M}^2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 d\pi(x, y).$$

If μ/M and ν/M are probability measures, then

$$W_{2,M}^2(\mu, \nu) = MW_2^2(\mu/M, \nu/M).$$

We write

$$\text{Ent}(\rho) = \int_{\mathbb{R}^2} \rho \log \rho dx, \quad \mathcal{W}(\rho) = \frac{1}{4\pi} \iint \rho(x) \log |x - y| \rho(y) dx dy,$$

so that $\mathcal{F} = \text{Ent} + \mathcal{W}$. When $\mu = \rho dx$ is absolutely continuous, we also write $\mathcal{F}(\mu) = \mathcal{F}(\rho)$ and $\text{Ent}(\mu) = \text{Ent}(\rho)$. We adopt the convention $0 \log 0 = 0$. Sums indexed by $i < j$ run over unordered pairs. We use the convention $\text{dist}(z, \emptyset) = +\infty$.

The following notation is used throughout:

$$\Lambda_M(m) = 2M - \frac{1}{4\pi} \sum_i m_i^2, \quad M_c = \sum_i m_i, \quad \Lambda_c(m) = 2M_c - \frac{1}{4\pi} \sum_i m_i^2.$$

Here Λ_M is used when all mass M collapses to the atomic endpoint, while Λ_c is used for partial concentration of mass M_c in the presence of a diffuse remainder.

The term “atomic endpoint” is used in this relative sense throughout; it refers to the energy domain rather than the metric boundary of $(\mathcal{P}_2(\mathbb{R}^2), W_2)$.

Assumption 2.1. Unless otherwise stated, the source density ρ is nonnegative, bounded, compactly supported, and has total mass $M > 0$.

Lemma 2.2. If $f, g \in L^\infty(\mathbb{R}^2)$ are nonnegative and compactly supported, then

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) |\log |x - y|| g(y) dx dy < \infty.$$

Proof. Choose $R > 0$ so that both supports are contained in B_R . The negative singularity is integrable in two dimensions since

$$\int_{B_1(0)} |\log |z|| \, dz < \infty.$$

For $|x - y| \geq 1$, the logarithm is bounded by a constant depending on R . Boundedness and compact support of f and g complete the proof. $\square$

Let

$$\nu = \sum_{i=1}^k m_i \delta_{z_i}, \quad m_i > 0, \quad z_i \neq z_j \ (i \neq j), \quad \sum_i m_i = M. \quad (2.1)$$

Since $\mu = \rho \, dx$ is absolutely continuous, Brenier's theorem [7] yields a quadratic-cost optimal map T from μ to ν . In the semi-discrete case, this map is described by Laguerre cells. More precisely, there exist weights $\alpha_i \in \mathbb{R}$ such that, up to negligible boundaries,

$$A_i = \{x : |x - z_i|^2 - \alpha_i \leq |x - z_j|^2 - \alpha_j \text{ for all } j\},$$

$$T(x) = z_i \quad \text{for } x \in A_i, \quad \int_{A_i} \rho(x) \, dx = m_i.$$

Geometrically, the cells A_i form a weighted Voronoi, or Laguerre, partition of the source support. The weights α_i are chosen so that the ρ -mass of A_i is exactly the prescribed atom mass m_i . Along the interpolation (2.2), each point in A_i moves on the straight segment from x to the target point z_i ; near $s = 1$, the whole cell is therefore compressed around z_i at the common scale $1 - s$. This compression mechanism is the geometric input from semi-discrete optimal transport used in the energy expansion.

The cell boundaries are contained in finitely many affine lines and hence are Lebesgue negligible. This standard semi-discrete representation is reviewed, for example, in [2, 3, 22, 24].

For $0 \leq s < 1$, we set

$$X_s(x) = (1 - s)x + sT(x), \quad \mu_s = (X_s)_\# \mu. \quad (2.2)$$

Writing $a = 1 - s$, on A_i one has

$$X_s(x) = z_i + a(x - z_i). \quad (2.3)$$

Lemma 2.3 (Density formula). *Under Assumption 2.1, let T be the optimal map from $\mu = \rho \, dx$ to ν . For every $0 \leq s < 1$, X_s is one-to-one outside negligible sets, and μ_s is absolutely continuous. If ρ_s is the density of μ_s and $a = 1 - s$, then*

$$\rho_s(X_s(x)) = a^{-2} \rho(x) \quad (2.4)$$

for Lebesgue a.e. $x \in A_i$, $i = 1, \dots, k$.

Proof. Let $T = \nabla \varphi$ at the differentiability points of a convex Brenier potential φ . Define

$$\psi_s(x) = \frac{1-s}{2} |x|^2 + s\varphi(x).$$

Then ψ_s is $(1-s)$ -strongly convex. More explicitly, if $p \in \partial \psi_s(x)$ and $q \in \partial \psi_s(y)$, the strong monotonicity of the subdifferential gives

$$(p - q) \cdot (x - y) \geq (1 - s) |x - y|^2.$$

At points where ψ_s is differentiable, $p = X_s(x)$ and $q = X_s(y)$. Hence, on the common differentiability set, $X_s(x) = X_s(y)$ implies $x = y$. Since a convex function is differentiable Lebesgue almost everywhere, X_s is injective on a set of full Lebesgue measure.

We next compute the Jacobian on each cell. The Laguerre cell boundary satisfies

$$\partial A_i \subset \cup_{j \neq i} \{x : |x - z_i|^2 - \alpha_i = |x - z_j|^2 - \alpha_j\}.$$

Each set on the right is an affine line since the quadratic terms cancel. Hence the cell boundaries have zero Lebesgue measure. In the interior of A_i the map is

$$X_s(x) = z_i + a(x - z_i), \quad a = 1 - s,$$

so $DX_s = aI_2$ and $|\det DX_s| = a^2$. The injectivity established above implies that the images of the full-measure portions of the cells overlap only on negligible sets. Applying the area formula on each cell and summing over i yields, for every bounded Borel function h ,

$$\int h(y)\rho_s(y) \, dy = \sum_i \int_{A_i} h(X_s(x))\rho(x) \, dx = \sum_i \int_{X_s(A_i)} h(y)a^{-2}\rho(X_s^{-1}y) \, dy.$$

This proves (2.4). $\square$

3. MULTI-POINT CONCENTRATION ALONG SEMI-DISCRETE TRANSPORT

We begin with the single-packet scaling identity and then turn to the semi-discrete multi-point expansion.

Remark 3.1. For a single concentrating packet, the standard scaling already reveals the critical mass. If

$$\rho_a(x) = a^{-2}\rho(x/a),$$

then a change of variables yields

$$\begin{aligned} \int \rho_a \log \rho_a &= \int \rho \log \rho + 2M \log \frac{1}{a}, \\ \mathcal{W}(\rho_a) &= \mathcal{W}(\rho) - \frac{M^2}{4\pi} \log \frac{1}{a}. \end{aligned}$$

Thus

$$\mathcal{F}(\rho_a) = \mathcal{F}(\rho) + \left(2M - \frac{M^2}{4\pi}\right) \log \frac{1}{a}.$$

In particular, $M = 8\pi$ is the leading-order critical value for one-point concentration.

Theorem 3.2 (Multi-point semi-discrete expansion). *Assume Assumption 2.1. Let ν be the atomic measure (2.1), T be the optimal map from $\mu = \rho \, dx$ to ν , and μ_s be given by (2.2). Then, as $s \uparrow 1$,*

$$\mathcal{F}(\mu_s) = \Lambda_M(m) \log \frac{1}{1-s} + C_{\rho,T,z} + o(1), \quad (3.1)$$

where

$$\begin{aligned} C_{\rho,T,z} &= \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{1}{4\pi} \sum_{i=1}^k \iint_{A_i \times A_i} \rho(x) \log |x-y| \rho(y) \, dx \, dy \\ &\quad + \frac{1}{2\pi} \sum_{1 \leq i < j \leq k} m_i m_j \log |z_i - z_j|. \end{aligned} \quad (3.2)$$

The constant $C_{\rho,T,z}$ is finite. The coefficient of the logarithmic divergence depends only on the mass vector $m = (m_1, \dots, m_k)$. The finite term depends on the Laguerre cell decomposition and on the target locations.

Proof. Set $a = 1-s$. We decompose the energy into entropy, self-interaction, and cross-interaction terms. By Lemma 2.3 and the change of variables on each cell,

$$\text{Ent}(\mu_s) = \sum_{i=1}^k \int_{X_s(A_i)} \rho_s(y) \log \rho_s(y) \, dy \quad (3.3)$$

$$= \sum_{i=1}^k \int_{A_i} \rho(x) \log(a^{-2}\rho(x)) \, dx \quad (3.4)$$

$$= \text{Ent}(\rho) + 2M \log \frac{1}{a}. \quad (3.5)$$

For the self-interaction terms, define

$$I_{ij}(a) = \iint_{A_i \times A_j} \rho(x)\rho(y) \log |X_s(x) - X_s(y)| \, dx \, dy.$$

Then $\mathcal{W}(\mu_s) = (4\pi)^{-1} \sum_{i,j} I_{ij}(a)$. If $i = j$, then, by (2.3),

$$X_s(x) - X_s(y) = a(x - y).$$

Consequently

$$\begin{aligned} I_{ii}(a) &= \iint_{A_i \times A_i} \rho(x)\rho(y) \{\log a + \log |x - y|\} \, dx \, dy \\ &= \iint_{A_i \times A_i} \rho(x)\rho(y) \log |x - y| \, dx \, dy - m_i^2 \log \frac{1}{a}. \end{aligned} \quad (3.6)$$

The finite part of (3.6) is finite by Lemma 2.2.

It remains to handle the cross-interaction terms. When $k = 1$, the cross-interaction sum is empty. We may therefore assume $k \geq 2$ and, for $i \neq j$, write

$$X_s(x) - X_s(y) = z_i - z_j + aB_{ij}(x, y), \quad B_{ij}(x, y) = (x - z_i) - (y - z_j).$$

Let $K = \text{supp } \rho$. Since K is compact, there exists $C_K < \infty$ such that

$$|B_{ij}(x, y)| \leq C_K, \quad x, y \in K, \quad i \neq j.$$

Let $d = \min_{i \neq j} |z_i - z_j| > 0$. If $C_K > 0$ and $0 < a < d/(2C_K)$, then

$$|z_i - z_j + aB_{ij}(x, y)| \geq |z_i - z_j| - aC_K \geq \frac{1}{2}|z_i - z_j|. \quad (3.7)$$

If $C_K = 0$, the same lower bound is immediate. Also,

$$|z_i - z_j + aB_{ij}(x, y)| \leq |z_i - z_j| + C_K. \quad (3.8)$$

Equations (3.7) and (3.8) imply that

$$|\log |z_i - z_j + aB_{ij}(x, y)|| \leq C_{ij}$$

for all sufficiently small a and all $x, y \in K$. Hence $C_{ij}\rho(x)\rho(y)$ is an integrable dominating function on $A_i \times A_j$. Since the integrand converges pointwise to $\log |z_i - z_j|$, the dominated convergence theorem yields

$$I_{ij}(a) = m_i m_j \log |z_i - z_j| + o(1), \quad i \neq j. \quad (3.9)$$

Since the energy sum contains both (i, j) and (j, i) , the cross terms produce the unordered-pair contribution in (3.2). Combining (3.5), (3.6), and (3.9) yields (3.1). The constant $C_{\rho, T, z}$ is finite by Lemma 2.2 for the self-interaction terms and since $z_i \neq z_j$ for the cross-interaction terms. $\square$

Corollary 3.3. *Under the hypotheses of Theorem 3.2, the following hold.*

- (i) *The sign of $\Lambda_M(m)$ determines the leading behavior: $\mathcal{F}(\mu_s)$ tends to $-\infty$, approaches the finite limit $C_{\rho, T, z}$, or tends to $+\infty$, according as $\sum_i m_i^2$ is larger than, equal to, or smaller than $8\pi M$. For k equal packets, the supercritical condition is $M > 8\pi k$; this is the threshold for the prescribed equal splitting, rather than a global critical mass for (1.1).*
- (ii) *The expansion can be expressed in terms of the distance to the atomic endpoint:*

$$\mathcal{F}(\mu_s) = \Lambda_M(m) \log \frac{W_{2, M}(\mu, \nu)}{W_{2, M}(\mu_s, \nu)} + C_{\rho, T, z} + o(1), \quad s \uparrow 1.$$

- (iii) *If two packets of masses m_i and m_j are merged while the total mass is kept fixed, then*

$$\Lambda_M(\dots, m_i + m_j, \dots) - \Lambda_M(\dots, m_i, m_j, \dots) = -\frac{m_i m_j}{2\pi} < 0. \quad (3.10)$$

The identity therefore shows that merging the two packets lowers the logarithmic coefficient and increases the leading attractive contribution.

- (iv) *For s sufficiently close to 1, the map $s \mapsto \mathcal{F}(\mu_s)$ is twice differentiable, and*

$$\frac{d^2}{ds^2} \mathcal{F}(\mu_s) = \frac{\Lambda_M(m)}{(1-s)^2} + O(1). \quad (3.11)$$

If $\Lambda_M(m) < 0$, this second derivative tends to $-\infty$. Consequently, no uniform finite lower bound on the convexity of $\mathcal{F}$ along displacement interpolations can hold on any class containing such tails of supercritical concentration paths.

Proof. The three leading alternatives follow immediately from Theorem 3.2. Since μ_s is the constant-speed Wasserstein geodesic from μ to ν ,

$$W_{2,M}(\mu_s, \nu) = (1-s)W_{2,M}(\mu, \nu),$$

and the distance form follows upon substituting this identity into (3.1). Formula (3.10) follows from

$$(m_i + m_j)^2 - m_i^2 - m_j^2 = 2m_i m_j.$$

For the second derivative, we do not differentiate the remainder in (3.1); instead we use the explicit decomposition from the proof of Theorem 3.2. The entropy and self-interaction terms contribute the exact singular part

$$\Lambda_M(m) \log \frac{1}{1-s}.$$

It remains to control the cross terms. For $i \neq j$, write

$$w_{ij}(s, x, y) = z_i - z_j + (1-s)B_{ij}(x, y), \quad G_{ij}(s, x, y) = \log |w_{ij}(s, x, y)|.$$

Bounds (3.7)–(3.8) show that $|w_{ij}|$ is bounded above and bounded away from zero for s close to 1 and $x, y \in \text{supp } \rho$. Since B_{ij} is bounded,

$$\partial_s G_{ij} = -\frac{w_{ij} \cdot B_{ij}}{|w_{ij}|^2},$$

and

$$\partial_s^2 G_{ij} = \frac{|B_{ij}|^2}{|w_{ij}|^2} - 2 \frac{(w_{ij} \cdot B_{ij})^2}{|w_{ij}|^4}$$

are uniformly bounded. The dominated convergence theorem justifies differentiation under the integral sign, so the cross part contributes $O(1)$ to the second derivative. This proves (3.11); the final assertion follows for $\Lambda_M(m) < 0$. $\square$

Remark 3.4 (Relation with Keller-Segel dynamics). *The coefficient $\Lambda_M(m)$ describes the leading singular energy variation along the semi-discrete interpolation. The concentration analysis for Keller-Segel trajectories also uses dynamical information, notably the virial identity and stability estimates.*

Remark 3.5. In Theorem 3.2, boundedness and compact support ensure finite entropy and logarithmic self-interaction, while compact support and separation provide the domination needed for the cross terms. The cellwise density formula follows from the semi-discrete transport structure. The same argument applies to the partial-concentration families with fixed separated centers and to the mesoscopic regimes of Section 4, where inter-center distances remain larger than packet radii. Packet-scale collisions, noncompact tails, and loss of mass at infinity require additional compactness and uniform integrability estimates.

4. PARTIAL CONCENTRATION AND CRITICAL FINITE TERMS

We now consider explicit families in which part of the mass concentrates while a diffuse background persists. The formulas in this section separate the contribution of the concentrating packets from the background interaction.

Assumption 4.1. Let $z_1, \dots, z_k \in \mathbb{R}^2$. Let $\eta_i \geq 0$ be bounded compactly supported densities with masses $m_i > 0$. Let $\sigma \geq 0$ be bounded and compactly supported. Assume

$$z_i \neq z_j \quad (i \neq j), \quad \text{dist}(z_i, \text{supp } \sigma) > 0 \quad \text{for each } i. \quad (4.1)$$

The separation hypothesis keeps the finite term explicit and prevents overlap among the concentrating packets and between the packets and the diffuse background. For $a > 0$, we define

$$\rho_a(x) = \sigma(x) + \sum_{i=1}^k a^{-2} \eta_i \left(\frac{x - z_i}{a} \right). \quad (4.2)$$

Set $M_c = \sum_i m_i$ and $M_r = \int \sigma$.

Theorem 4.2 (Partial-concentration expansion). *Under Assumption 4.1, as $a \downarrow 0$,*

$$\mathcal{F}(\rho_a) = \Lambda_c(m) \log \frac{1}{a} + C_{\eta, \sigma, z} + o(1), \quad (4.3)$$

where

$$\begin{aligned} C_{\eta, \sigma, z} = & \sum_i \int \eta_i \log \eta_i + \frac{1}{4\pi} \sum_i \iint \eta_i(u) \log |u - v| \eta_i(v) \, du \, dv \\ & + \frac{1}{2\pi} \sum_{1 \leq i < j \leq k} m_i m_j \log |z_i - z_j| + \frac{1}{2\pi} \sum_i m_i \int_{\mathbb{R}^2} \log |z_i - y| \sigma(y) \, dy. \end{aligned} \quad (4.4)$$

The coefficient Λ_c involves only the concentrating mass M_c , not the total mass $M_c + M_r$.

Proof. We decompose the free energy. We choose $R_i > 0$ so that $\text{supp } \eta_i \subset B_{R_i}(0)$. The i -th packet is supported in $B_{aR_i}(z_i)$. By (4.1), there exists $a_0 > 0$ such that for $0 < a < a_0$ these balls are mutually disjoint and disjoint from $\text{supp } \sigma$. Hence the entropy splits exactly:

$$\begin{aligned} \text{Ent}(\rho_a) &= \text{Ent}(\sigma) + \sum_i \int_{B_{aR_i}(z_i)} a^{-2} \eta_i\left(\frac{x - z_i}{a}\right) \log \left[a^{-2} \eta_i\left(\frac{x - z_i}{a}\right) \right] \, dx \\ &= \text{Ent}(\sigma) + \sum_i \int \eta_i \log \eta_i + 2M_c \log \frac{1}{a}. \end{aligned} \quad (4.5)$$

For the self-interaction of packet i , the change of variables $x = z_i + au$, $y = z_i + av$ gives

$$\begin{aligned} & \frac{1}{4\pi} \iint a^{-2} \eta_i\left(\frac{x - z_i}{a}\right) \log |x - y| a^{-2} \eta_i\left(\frac{y - z_i}{a}\right) \, dx \, dy \\ &= \frac{1}{4\pi} \iint \eta_i(u) \log |u - v| \eta_i(v) \, du \, dv - \frac{m_i^2}{4\pi} \log \frac{1}{a}. \end{aligned} \quad (4.6)$$

For interactions between different packets, if $i \neq j$, then $x = z_i + au$ and $y = z_j + av$ yield

$$\log |x - y| = \log |z_i - z_j + a(u - v)|.$$

The profiles have fixed compact supports and the points z_i are separated, so the lower and upper bounds used in the proof of Theorem 3.2 hold uniformly for the rescaled profiles. Dominated convergence therefore applies. Each unordered pair is counted twice in the double integral. Equivalently, its total contribution to the energy is

$$\frac{1}{2\pi} m_i m_j \log |z_i - z_j| + o(1).$$

Summing over $i < j$ yields the fourth term in (4.4).

Finally, for the interaction between packet i and the background,

$$\begin{aligned} & \frac{1}{4\pi} \left[\iint \eta_{i,a}(x) \log |x - y| \sigma(y) \, dx \, dy + \iint \sigma(x) \log |x - y| \eta_{i,a}(y) \, dx \, dy \right] \\ &= \frac{1}{2\pi} \iint \eta_{i,a}(x) \log |x - y| \sigma(y) \, dx \, dy, \end{aligned}$$

where $\eta_{i,a}(x) = a^{-2} \eta_i((x - z_i)/a)$. Setting $x = z_i + au$, this becomes

$$\frac{1}{2\pi} \iint \eta_i(u) \log |z_i + au - y| \sigma(y) \, du \, dy.$$

Since $\text{dist}(z_i, \text{supp } \sigma) > 0$ and η_i is compactly supported, the logarithm is uniformly bounded for small a and converges pointwise to $\log |z_i - y|$. The dominated convergence theorem yields

$$\frac{1}{2\pi} m_i \int \log |z_i - y| \sigma(y) \, dy + o(1).$$

Adding (4.5), (4.6), the packet-packet terms, and the packet-background terms proves (4.3). $\square$

Corollary 4.3. *For the family (4.2), the logarithmic coefficient is negative, zero, or positive according to*

$$\sum_i m_i^2 \gtrless 8\pi M_c.$$

It is a coefficient for the prescribed concentration family rather than a threshold for solution trajectories of (1.1).

4.1. Critical finite parts. We now assume that

$$\Lambda_c(m) = 0, \quad \text{that is,} \quad \sum_i m_i^2 = 8\pi M_c. \quad (4.7)$$

For $z \notin \text{supp } \sigma$, set

$$\Phi_\sigma(z) = \int_{\mathbb{R}^2} \log |z - y| \sigma(y) \, dy.$$

Define

$$\mathcal{E}_{\text{eff}}(z; m, \sigma) = \frac{1}{2\pi} \sum_{1 \leq i < j \leq k} m_i m_j \log |z_i - z_j| + \frac{1}{2\pi} \sum_i m_i \Phi_\sigma(z_i) \quad (4.8)$$

and

$$\mathcal{C}_{\text{prof}}(\eta, \sigma) = \mathcal{F}(\sigma) + \sum_i \int \eta_i \log \eta_i + \frac{1}{4\pi} \sum_i \iint \eta_i(u) \log |u - v| \eta_i(v) \, du \, dv.$$

Corollary 4.4. *Under Assumption 4.1 and condition (4.7), the family (4.2) satisfies*

$$\mathcal{F}(\rho_a) = \mathcal{C}_{\text{prof}}(\eta, \sigma) + \mathcal{E}_{\text{eff}}(z; m, \sigma) + o(1), \quad a \downarrow 0.$$

After cancellation of the logarithmic divergence, the point-dependent finite term is given by the explicit logarithmic expression (4.8).

Proof. This follows from Theorem 4.2 with $\Lambda_c(m) = 0$. In this case $C_{\eta, \sigma, z} = \mathcal{C}_{\text{prof}}(\eta, \sigma) + \mathcal{E}_{\text{eff}}(z; m, \sigma)$. $\square$

The above formula gives the finite part for a separated concentration family. The condition $\Lambda_c(m) = 0$ is a common-scale cancellation condition. If relative packet scales are allowed to vary, additional finite terms appear; see Proposition 4.7 and Corollary 4.8.

Proposition 4.5. *Fix $\delta > 0$ and consider separated configurations satisfying*

$$|z_i - z_j| \geq \delta \quad (i \neq j), \quad \text{dist}(z_i, \text{supp } \sigma) \geq \delta.$$

Then $\mathcal{E}_{\text{eff}}$ is smooth in z on this set, and

$$\nabla_{z_i} \mathcal{E}_{\text{eff}}(z; m, \sigma) = \frac{m_i}{2\pi} \sum_{j \neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^2} + \frac{m_i}{2\pi} \int_{\mathbb{R}^2} \frac{z_i - y}{|z_i - y|^2} \sigma(y) \, dy. \quad (4.9)$$

If the point derivative vanishes at an interior separated configuration, then

$$\sum_{j \neq i} m_j \frac{z_i - z_j}{|z_i - z_j|^2} + \int_{\mathbb{R}^2} \frac{z_i - y}{|z_i - y|^2} \sigma(y) \, dy = 0, \quad i = 1, \dots, k. \quad (4.10)$$

Proof. On the separated configuration set, $|z_i - z_j| \geq \delta$ and $|z_i - y| \geq \delta$ for $y \in \text{supp } \sigma$. Therefore the maps

$$z_i \mapsto \log |z_i - z_j|, \quad z_i \mapsto \log |z_i - y|$$

are smooth, and their derivatives are bounded by constants depending only on δ on the relevant set. Since $\sigma \in L_c^1(\mathbb{R}^2)$, dominated differentiation yields

$$\nabla_{z_i} \Phi_\sigma(z_i) = \int_{\mathbb{R}^2} \frac{z_i - y}{|z_i - y|^2} \sigma(y) \, dy.$$

For the unordered-pair term, differentiating

$$\frac{1}{2\pi} m_i m_j \log |z_i - z_j|$$

with respect to z_i gives

$$\frac{m_i m_j}{2\pi} \frac{z_i - z_j}{|z_i - z_j|^2}.$$

Adding the background derivative yields (4.9). Since $m_i > 0$, the vanishing of the point derivative is equivalent to (4.10). $\square$

Remark 4.6 (Critical finite terms in the whole plane). The finite term $\mathcal{E}_{\text{eff}}$ is not coercive in general. If two points collide, the pairwise logarithmic term tends to $-\infty$, and in the whole plane compactness may also be lost at infinity. Hence (4.10) should be interpreted as the point-derivative equation within a separated configuration regime. In the special case $m_i = 8\pi$ for $i = 1, \dots, k$ and $\sigma = 0$,

$$\mathcal{E}_{\text{eff}}(z; 8\pi, 0) = 32\pi \sum_{1 \leq i < j \leq k} \log |z_i - z_j|,$$

which, for $k \geq 2$, admits no finite minimizing configuration without confinement or background.

4.2. Comparable packet scales. The next variant records the effect of changing the relative microscopic scales of the packets. For a packet of mass m , set

$$\lambda(m) := 2m - \frac{m^2}{4\pi} = \frac{m(8\pi - m)}{4\pi}. \quad (4.11)$$

Proposition 4.7. *Let Assumption 4.1 hold, and fix numbers $r_i > 0$. Define*

$$\rho_a^{(r)}(x) = \sigma(x) + \sum_{i=1}^k (ar_i)^{-2} \eta_i \left(\frac{x - z_i}{ar_i} \right).$$

Then, as $a \downarrow 0$,

$$\mathcal{F}(\rho_a^{(r)}) = \Lambda_c(m) \log \frac{1}{a} + C_{\eta, \sigma, z} + \sum_{i=1}^k \lambda(m_i) \log \frac{1}{r_i} + o(1), \quad (4.12)$$

where $C_{\eta, \sigma, z}$ is the constant in (4.4).

Proof. Only the entropy and the self-interaction of each packet change. For the entropy of packet i ,

$$\begin{aligned} & \int (ar_i)^{-2} \eta_i \left(\frac{x - z_i}{ar_i} \right) \log \left[(ar_i)^{-2} \eta_i \left(\frac{x - z_i}{ar_i} \right) \right] dx \\ &= \int \eta_i \log \eta_i + 2m_i \log \frac{1}{a} + 2m_i \log \frac{1}{r_i}. \end{aligned}$$

For its self-interaction,

$$\begin{aligned} & \frac{1}{4\pi} \iint \eta_i(u) \log(ar_i |u - v|) \eta_i(v) du dv \\ &= \frac{1}{4\pi} \iint \eta_i(u) \log |u - v| \eta_i(v) du dv - \frac{m_i^2}{4\pi} \log \frac{1}{a} - \frac{m_i^2}{4\pi} \log \frac{1}{r_i}. \end{aligned}$$

The additional finite contribution from the i -th scale is

$$\left(2m_i - \frac{m_i^2}{4\pi} \right) \log \frac{1}{r_i}.$$

The packet-packet and packet-background terms are treated exactly as in Theorem 4.2, because $ar_i \rightarrow 0$ and all r_i are fixed positive constants. Summing over i proves (4.12). $\square$

Corollary 4.8. *The value $m = 8\pi$ is the unique positive zero of λ in (4.11). For a multi-packet family,*

$$\Lambda_c(m) = \sum_{i=1}^k \lambda(m_i).$$

If $0 < m_i \leq 8\pi$ for every i , then

$$\Lambda_c(m) = 0 \iff m_i = 8\pi \text{ for every } i.$$

Proof. The formula for λ is immediate from the entropy and self-interaction contributions of one packet. Its only positive zero is $m = 8\pi$. The identity $\Lambda_c(m) = \sum_i \lambda(m_i)$ follows from the definitions. If $0 < m_i \leq 8\pi$, then each $\lambda(m_i) \geq 0$, so the sum can vanish only when every term vanishes. $\square$

Remark 4.9. Corollary 4.8 provides a static interpretation of the 8π mass scale: a packet of mass 8π is neutral under changes of its microscopic scale. The common-scale condition $\Lambda_c(m) = 0$ is weaker. It may arise from cancellation between packets with $m_i < 8\pi$ and packets with $m_j > 8\pi$, and such a cancellation is destroyed by relative rescaling. In the comparable-scale expansion (4.12), a packet with $m_i > 8\pi$ has $\lambda(m_i) < 0$, so decreasing its relative scale lowers the finite part.

The scale dependence is equally transparent for hierarchical concentrations. If the i -th packet concentrates at scale a^{γ_i} with fixed exponent $\gamma_i > 0$, then the leading logarithmic part becomes

$$\sum_{i=1}^k \gamma_i \left(2m_i - \frac{m_i^2}{4\pi} \right) \log \frac{1}{a}.$$

Only the entropy and the self-interaction terms carry the microscopic scale; packet-packet and packet-background interactions remain finite under the same separation hypotheses.

4.3. Critical profile relaxation. The fixed-profile finite term still contains the local profile energies. At the scale-neutral mass 8π , the sharp logarithmic Hardy-Littlewood-Sobolev inequality [6, 12] gives a lower bound for these local terms. This identifies the profile-relaxed finite term within the separated critical packet class.

For a nonnegative density η of mass 8π , we define

$$\mathcal{S}_{\text{prof}}(\eta) = \int_{\mathbb{R}^2} \eta \log \eta \, dx + \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \eta(x) \log |x - y| \eta(y) \, dx \, dy,$$

whenever the two terms are finite.

Lemma 4.10. *Let $\eta \geq 0$ have mass 8π and assume that $\mathcal{S}_{\text{prof}}(\eta)$ is well defined. Then*

$$\mathcal{S}_{\text{prof}}(\eta) \geq \mathcal{C}_{8\pi}, \quad \mathcal{C}_{8\pi} := 8\pi(\log 8 - 1). \quad (4.13)$$

Equality is attained in the finite-energy class by translations and dilations of

$$U(x) = \frac{8}{(1 + |x|^2)^2}.$$

Proof. This is the sharp logarithmic Hardy-Littlewood-Sobolev inequality in the present normalization [6, 12]. The value of the constant follows by evaluating the optimizer U . Since

$$\begin{aligned} \int_{\mathbb{R}^2} U \, dx &= 8\pi, \quad \int_{\mathbb{R}^2} U \log U \, dx = 8\pi(\log 8 - 2), \\ \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} U(x) \log |x - y| U(y) \, dx \, dy &= 8\pi, \end{aligned}$$

one obtains $\mathcal{S}_{\text{prof}}(U) = 8\pi(\log 8 - 1)$. Translations and dilations preserve the mass and the value of $\mathcal{S}_{\text{prof}}$ at mass 8π . $\square$

Proposition 4.11 (Critical profile relaxation). *Let Assumption 4.1 hold with*

$$m_i = 8\pi, \quad i = 1, \dots, k.$$

Put $m^{8\pi} = (8\pi, \dots, 8\pi)$. For every choice of bounded compactly supported profiles η_i of mass 8π , the family (4.2) satisfies

$$\liminf_{a \downarrow 0} \{ \mathcal{F}(\rho_a) - \mathcal{E}_{\text{eff}}(z; m^{8\pi}, \sigma) \} \geq \mathcal{F}(\sigma) + k\mathcal{C}_{8\pi}. \quad (4.14)$$

Conversely, for every $\varepsilon > 0$ there exist bounded compactly supported profiles η_i of mass 8π such that the corresponding family (4.2) satisfies

$$\limsup_{a \downarrow 0} \{ \mathcal{F}(\rho_a) - \mathcal{E}_{\text{eff}}(z; m^{8\pi}, \sigma) \} \leq \mathcal{F}(\sigma) + k\mathcal{C}_{8\pi} + \varepsilon. \quad (4.15)$$

Thus, within separated critical-packet families, relaxation over the local profiles replaces $\mathcal{C}_{\text{prof}}(\eta, \sigma)$ by $\mathcal{F}(\sigma) + k\mathcal{C}_{8\pi}$.

Proof. When $m_i = 8\pi$ for every i , Corollary 4.8 gives $\Lambda_c(m) = 0$. Corollary 4.4 and the definition of $\mathcal{C}_{\text{prof}}$ yield

$$\mathcal{F}(\rho_a) = \mathcal{F}(\sigma) + \sum_{i=1}^k \mathcal{S}_{\text{prof}}(\eta_i) + \mathcal{E}_{\text{eff}}(z; m^{8\pi}, \sigma) + o(1).$$

Applying Lemma 4.10 to each profile gives (4.14).

For the converse statement, use the critical log-HLS optimizer $U(x) = 8(1 + |x|^2)^{-2}$. Let

$$U_R = \frac{8\pi}{\int_{B_R} U} U \mathbf{1}_{B_R}.$$

Then U_R is bounded, compactly supported, and has mass 8π . Since U has finite entropy and finite logarithmic interaction, truncation and normalization give

$$\mathcal{S}_{\text{prof}}(U_R) \rightarrow \mathcal{S}_{\text{prof}}(U) = \mathcal{C}_{8\pi}.$$

Indeed, the entropy term converges by dominated convergence. The logarithmic interaction term also converges by dominated convergence, since $|\log |x - y|| U(x)U(y)$ is integrable after splitting the regions $|x - y| \leq 1$ and $|x - y| > 1$. Choosing R large enough and taking $\eta_i = U_R$ for every i , Corollary 4.4 gives (4.15). $\square$

Proposition 4.11 carries out the profile relaxation within the separated class. For critical 8π packets, the sharp inequality underlying the classical Keller-Segel critical mass removes the remaining dependence on the local profiles.

4.4. Mesoscopic packet coalescence. The preceding results keep the concentration centers separated at the macroscopic scale. We next allow several centers to approach one another at an intermediate scale between the packet radius and the macroscopic separation. The resulting expansion quantifies the change in the logarithmic coefficient as packets merge on this mesoscopic scale.

Let $\mathcal{P}$ be a partition of $\{1, \dots, k\}$. For each cluster $C \in \mathcal{P}$, choose $q_C \in \mathbb{R}^2$ and $\beta_C \in (0, 1)$. Assume that

$$q_C \neq q_D \quad (C \neq D), \quad \text{dist}(q_C, \text{supp } \sigma) > 0 \quad (C \in \mathcal{P}).$$

For $i \in C$, choose offsets $\xi_i \in \mathbb{R}^2$ satisfying $\xi_i \neq \xi_j$ whenever $i \neq j$ belong to the same cluster. Define

$$z_i(a) = q_C + a^{\beta_C} \xi_i, \quad i \in C,$$

and

$$\rho_a^{\mathcal{P}}(x) = \sigma(x) + \sum_{i=1}^k a^{-2} \eta_i \left(\frac{x - z_i(a)}{a} \right). \quad (4.16)$$

Set $M_C = \sum_{i \in C} m_i$.

Proposition 4.12 (Mesoscopic coalescence expansion). *Assume that the profiles η_i and the background σ satisfy the boundedness and compact support hypotheses in Assumption 4.1. With the moving centers $z_i(a)$ defined above, the family (4.16) satisfies*

$$\mathcal{F}(\rho_a^{\mathcal{P}}) = \Lambda_{\mathcal{P}, \beta}(m) \log \frac{1}{a} + C_{\eta, \sigma, \mathcal{P}, q, \xi} + o(1), \quad a \downarrow 0, \quad (4.17)$$

where

$$\begin{aligned} \Lambda_{\mathcal{P}, \beta}(m) &= \Lambda_c(m) - \frac{1}{2\pi} \sum_{C \in \mathcal{P}} \beta_C \sum_{\substack{i < j \\ i, j \in C}} m_i m_j \\ &= \sum_{C \in \mathcal{P}} \left\{ (1 - \beta_C) \sum_{i \in C} \lambda(m_i) + \beta_C \lambda(M_C) \right\}, \end{aligned} \quad (4.18)$$

and

$$\begin{aligned}
C_{\eta, \sigma, \mathcal{P}, q, \xi} &= \mathcal{F}(\sigma) + \sum_i \int \eta_i \log \eta_i + \frac{1}{4\pi} \sum_i \iint \eta_i(u) \log |u - v| \eta_i(v) \, du \, dv \\
&\quad + \frac{1}{2\pi} \sum_{C \in \mathcal{P}} \sum_{\substack{i < j \\ i, j \in C}} m_i m_j \log |\xi_i - \xi_j| + \frac{1}{2\pi} \sum_{\substack{\{C, D\} \subset \mathcal{P} \\ C \neq D}} M_C M_D \log |q_C - q_D| \\
&\quad + \frac{1}{2\pi} \sum_{C \in \mathcal{P}} M_C \int_{\mathbb{R}^2} \log |q_C - y| \sigma(y) \, dy.
\end{aligned} \tag{4.19}$$

Proof. Choose $R_\eta > 0$ such that all profiles are supported in $B_{R_\eta}(0)$. Since $0 < \beta_C < 1$, the packet radius aR_η is negligible with respect to the intracluster distance $a^{\beta_C} |\xi_i - \xi_j|$. The assumptions on the cluster centers also keep distinct clusters and the background separated. Hence, for all sufficiently small a , the supports of the packet densities are mutually disjoint and disjoint from $\text{supp } \sigma$. The entropy and packet self-interaction terms are therefore the same as in the proof of Theorem 4.2; they contribute

$$\sum_i \lambda(m_i) \log \frac{1}{a} + \sum_i \int \eta_i \log \eta_i + \frac{1}{4\pi} \sum_i \iint \eta_i(u) \log |u - v| \eta_i(v) \, du \, dv$$

together with $\mathcal{F}(\sigma)$.

Consider two packets i, j in the same cluster C . With $x = z_i(a) + au$ and $y = z_j(a) + av$,

$$x - y = a^{\beta_C} (\xi_i - \xi_j + a^{1-\beta_C} (u - v)).$$

Thus

$$\log |x - y| = -\beta_C \log \frac{1}{a} + \log |\xi_i - \xi_j + a^{1-\beta_C} (u - v)|.$$

The second logarithm is uniformly bounded on the fixed profile supports for all small a and converges to $\log |\xi_i - \xi_j|$. Dominated convergence gives the intracluster contribution

$$\frac{1}{2\pi} m_i m_j \left[-\beta_C \log \frac{1}{a} + \log |\xi_i - \xi_j| \right] + o(1)$$

for each unordered pair $i < j$ in C .

If $i \in C$, $j \in D$, and $C \neq D$, then

$$x - y = q_C - q_D + a^{\beta_C} \xi_i - a^{\beta_D} \xi_j + a(u - v),$$

which remains uniformly bounded away from zero and converges uniformly on the profile supports to $q_C - q_D$. Hence the intercluster contribution is

$$\frac{1}{2\pi} m_i m_j \log |q_C - q_D| + o(1).$$

After summing over $i \in C$ and $j \in D$, this gives the second line of (4.19).

Finally, for packet-background interactions,

$$z_i(a) + au \rightarrow q_C$$

uniformly in u on the support of η_i , and the distance from q_C to $\text{supp } \sigma$ is positive. Dominated convergence yields

$$\frac{1}{2\pi} m_i \int_{\mathbb{R}^2} \log |q_C - y| \sigma(y) \, dy + o(1).$$

Summing the preceding contributions gives (4.17). The first expression in (4.18) is obtained by collecting logarithmic terms. The second follows from

$$M_C^2 = \sum_{i \in C} m_i^2 + 2 \sum_{i < j, i, j \in C} m_i m_j.$$

□

Corollary 4.13. *For a fixed cluster C ,*

$$\frac{\partial}{\partial \beta_C} \Lambda_{\mathcal{P}, \beta}(m) = -\frac{1}{2\pi} \sum_{\substack{i < j \\ i, j \in C}} m_i m_j \leq 0,$$

with strict inequality whenever C contains at least two packets. Thus, for a nontrivial cluster, the logarithmic coefficient decreases from the separated-packet value toward the merged-cluster value as β_C increases. In particular, if $m_i = 8\pi$ for every packet in a nontrivial cluster, then any $\beta_C > 0$ makes the corresponding cluster contribution negative.

Proof. The derivative follows directly from (4.18). If all masses in a nontrivial cluster are equal to 8π , then each separated packet has $\lambda(8\pi) = 0$, whereas

$$-\frac{\beta_C}{2\pi} \sum_{i < j, i, j \in C} m_i m_j < 0.$$

□

The preceding proposition treats one mesoscopic scale inside each cluster. The next statement records the corresponding expansion when the separation rate is prescribed pair by pair.

Proposition 4.14 (Pairwise mesoscopic rates). *Assume that the profiles η_i and the background σ satisfy the boundedness and compact support hypotheses in Assumption 4.1. Let $z_i(a)$ be moving centers such that $z_i(a) \rightarrow q_i$ as $a \downarrow 0$, with*

$$\text{dist}(q_i, \text{supp } \sigma) > 0 \quad \text{for each } i.$$

Assume that for every $i < j$ there exist $\beta_{ij} \in [0, 1)$ and $\zeta_{ij} \in \mathbb{R}^2 \setminus \{0\}$ such that

$$a^{-\beta_{ij}} (z_i(a) - z_j(a)) \rightarrow \zeta_{ij}. \quad (4.20)$$

The pairwise exponents and limiting vectors are induced by the chosen moving centers; no independent compatibility condition is imposed. Define

$$\rho_a^{\text{pw}}(x) = \sigma(x) + \sum_{i=1}^k a^{-2} \eta_i \left(\frac{x - z_i(a)}{a} \right). \quad (4.21)$$

Then, as $a \downarrow 0$,

$$\mathcal{F}(\rho_a^{\text{pw}}) = \Lambda_{\text{pw}}(m, \beta) \log \frac{1}{a} + C_{\eta, \sigma, q, \zeta}^{\text{pw}} + o(1), \quad (4.22)$$

where

$$\Lambda_{\text{pw}}(m, \beta) = \sum_{i=1}^k \lambda(m_i) - \frac{1}{2\pi} \sum_{1 \leq i < j \leq k} \beta_{ij} m_i m_j, \quad (4.23)$$

and

$$\begin{aligned} C_{\eta, \sigma, q, \zeta}^{\text{pw}} = & \mathcal{F}(\sigma) + \sum_i \int \eta_i \log \eta_i + \frac{1}{4\pi} \sum_i \iint \eta_i(u) \log |u - v| \eta_i(v) \, du \, dv \\ & + \frac{1}{2\pi} \sum_{1 \leq i < j \leq k} m_i m_j \log |\zeta_{ij}| + \frac{1}{2\pi} \sum_i m_i \int_{\mathbb{R}^2} \log |q_i - y| \sigma(y) \, dy. \end{aligned} \quad (4.24)$$

Proof. Let $R_\eta > 0$ contain the supports of all profiles. By (4.20) and $\beta_{ij} < 1$,

$$|z_i(a) - z_j(a)| \geq c_{ij} a^{\beta_{ij}}, \quad \frac{a}{a^{\beta_{ij}}} \rightarrow 0,$$

for all sufficiently small a . Hence the packet supports are mutually disjoint. Since $\text{dist}(q_i, \text{supp } \sigma) > 0$, they are also disjoint from the background support for small a . The entropy and self-interaction terms are therefore the same as in Theorem 4.2, and give the coefficient $\sum_i \lambda(m_i)$ together with the profile terms in (4.24).

For the interaction of packets $i < j$, set $x = z_i(a) + au$ and $y = z_j(a) + av$. Then

$$x - y = a^{\beta_{ij}} \{ a^{-\beta_{ij}} (z_i(a) - z_j(a)) + a^{1-\beta_{ij}} (u - v) \}.$$

The expression in braces converges uniformly on the fixed profile supports to ζ_{ij} . Therefore

$$\log |x - y| = -\beta_{ij} \log \frac{1}{a} + \log |\zeta_{ij}| + o(1),$$

uniformly on those supports. Its contribution to the free energy is

$$\frac{1}{2\pi} m_i m_j \left(-\beta_{ij} \log \frac{1}{a} + \log |\zeta_{ij}| \right) + o(1).$$

The packet-background interaction converges, by the same separated dominated-convergence argument as before, to

$$\frac{1}{2\pi} m_i \int_{\mathbb{R}^2} \log |q_i - y| \sigma(y) dy.$$

Summing over all packets and unordered pairs gives (4.22). $\square$

Corollary 4.15. *The coefficient (4.23) is nonincreasing in every pair exponent β_{ij} , and*

$$\frac{\partial}{\partial \beta_{ij}} \Lambda_{\text{pw}}(m, \beta) = -\frac{m_i m_j}{2\pi} < 0.$$

In particular, if $m_i = 8\pi$ for all i , then

$$\Lambda_{\text{pw}}(m, \beta) = -32\pi \sum_{1 \leq i < j \leq k} \beta_{ij},$$

so any nonzero pairwise coalescence exponent makes the logarithmic coefficient strictly negative.

Both assertions in the corollary follow immediately from (4.23) and $\lambda(8\pi) = 0$.

4.5. Uniform convergence on separated configuration sets. The expansion is pointwise in the concentration locations. On compact separated configuration sets the error estimates are uniform, yielding the following fixed-profile finite-dimensional consequence.

Fix $R > 0$ and $\delta > 0$, and define the separated compact configuration set

$$\mathcal{K}_{R,\delta} := \{z = (z_1, \dots, z_k) \in (\overline{B_R})^k : |z_i - z_j| \geq \delta \ (i \neq j), \text{ dist}(z_i, \text{supp } \sigma) \geq \delta\}.$$

We consider only choices of R and δ for which $\mathcal{K}_{R,\delta}$ is nonempty. For $z \in \mathcal{K}_{R,\delta}$, set

$$\rho_a^z(x) = \sigma(x) + \sum_{i=1}^k a^{-2} \eta_i \left(\frac{x - z_i}{a} \right), \quad (4.25)$$

for $a > 0$ small enough that the packets are separated from one another and from $\text{supp } \sigma$. Define

$$\mathcal{G}_a(z) := \mathcal{F}(\rho_a^z) - \Lambda_c(m) \log \frac{1}{a}, \quad \mathcal{G}_0(z) := \mathcal{C}_{\text{prof}}(\eta, \sigma) + \mathcal{E}_{\text{eff}}(z; m, \sigma).$$

Proposition 4.16. *On the nonempty set $\mathcal{K}_{R,\delta}$, endowed with the Euclidean topology,*

$$\sup_{z \in \mathcal{K}_{R,\delta}} |\mathcal{G}_a(z) - \mathcal{G}_0(z)| \rightarrow 0, \quad a \downarrow 0. \quad (4.26)$$

Equivalently, $\mathcal{G}_a$ converges uniformly to $\mathcal{G}_0$.

Moreover, let $\tau > 0$, set $M_{\text{tot}} = M_c + M_r$, and let $\mu_0 \in \mathcal{M}_{M_{\text{tot}}}(\mathbb{R}^2)$ have finite second moment. In the following display $\mathcal{W}_{2,M}$ is the transport distance for measures of mass M_{tot} . Put

$$\bar{\nu}_z := \sigma dx + \sum_{i=1}^k m_i \delta_{z_i}.$$

Then the fixed-profile JKO-renormalized functionals

$$\mathcal{H}_a(z) := \mathcal{G}_a(z) + \frac{1}{2\tau} \mathcal{W}_{2,M}^2(\rho_a^z dx, \mu_0)$$

converge uniformly on $\mathcal{K}_{R,\delta}$ to

$$\mathcal{H}_0(z) := \mathcal{G}_0(z) + \frac{1}{2\tau} \mathcal{W}_{2,M}^2(\bar{\nu}_z, \mu_0).$$

Proof. Let $R_\eta > 0$ be such that $\text{supp } \eta_i \subset B_{R_\eta}(0)$ for every i . If $aR_\eta < \delta/4$, then all packets in (4.25) are mutually disjoint and remain separated from $\text{supp } \sigma$, uniformly in $z \in \mathcal{K}_{R,\delta}$. The entropy and self-interaction terms are independent of z except through the fixed masses and profiles, so their expansions are uniform.

For the packet-packet interactions, setting $x = z_i + au$ and $y = z_j + av$, one has

$$|z_i - z_j + a(u - v)| \geq \delta/2$$

for all $z \in \mathcal{K}_{R,\delta}$ and all u, v in the profile supports, provided a is sufficiently small. The same expression is also bounded above uniformly because $z_i, z_j \in \overline{B_R}$ and the profile supports are bounded. Hence the logarithms are uniformly bounded and converge uniformly to $\log |z_i - z_j|$. The packet-background interactions are treated similarly: $|z_i + au - y| \geq \delta/2$ for $y \in \text{supp } \sigma$, while the upper bound is uniform because z_i and $\text{supp } \sigma$ are bounded. All dominated-convergence remainders in Theorem 4.2 are therefore uniform in $z \in \mathcal{K}_{R,\delta}$, which gives (4.26).

It remains to handle the JKO term. We couple σdx with itself and the i -th packet with $m_i \delta_{z_i}$. Uniformly in $z \in \mathcal{K}_{R,\delta}$,

$$\mathbf{W}_{2,M}^2(\rho_a^z dx, \bar{\nu}_z) \leq a^2 \sum_i \int |u|^2 \eta_i(u) du.$$

It follows that $\rho_a^z dx$ converges to $\bar{\nu}_z$ in $\mathbf{W}_{2,M}$ uniformly in z . Since $\mathcal{K}_{R,\delta}$ is bounded and σ is compactly supported, the measures $\bar{\nu}_z$ have uniformly bounded second moments. Because μ_0 has finite second moment, the quantities $\mathbf{W}_{2,M}(\bar{\nu}_z, \mu_0)$ are uniformly bounded. The triangle inequality also bounds $\mathbf{W}_{2,M}(\rho_a^z dx, \mu_0)$ uniformly for all small a . Therefore, using $|A^2 - B^2| \leq (A+B)|A-B|$ and the triangle inequality,

$$\sup_{z \in \mathcal{K}_{R,\delta}} |\mathbf{W}_{2,M}^2(\rho_a^z dx, \mu_0) - \mathbf{W}_{2,M}^2(\bar{\nu}_z, \mu_0)| \rightarrow 0.$$

Combining this with (4.26) yields uniform convergence of $\mathcal{H}_a$ to $\mathcal{H}_0$. On the compact metric space $\mathcal{K}_{R,\delta}$, uniform convergence also yields the corresponding finite-dimensional variational convergence. $\square$

Proposition 4.16 is a fixed-profile finite-dimensional consequence of the expansion. Measure-space variational limits involve concentration-compactness and local log-HLS lower bounds.

5. JKO COMPARISON AND STANDARD VIRIAL IDENTITY

We next apply the expansions to the one-step JKO functional and compare them with the classical virial identity. Along a prescribed supercritical concentration direction, the Wasserstein penalty in the JKO functional remains bounded while the free energy tends to $-\infty$, yielding explicit competitors for which the one-step functional is unbounded below. We then recall the classical virial identity, which gives the second-moment law for the PDE and serves as a standard comparison.

5.1. One-step JKO competitors. Let $M_* > 0$ and let $\mu_0 \in \mathcal{M}_{M_*}(\mathbb{R}^2)$ have finite second moment. For $\tau > 0$, define the one-step JKO functional on absolutely continuous candidates $\mu = \rho dx \in \mathcal{M}_{M_*}(\mathbb{R}^2)$ by

$$\mathcal{J}_\tau(\mu \mid \mu_0) = \frac{1}{2\tau} \mathbf{W}_{2,M}^2(\mu, \mu_0) + \mathcal{F}(\mu),$$

whenever $\mathcal{F}(\mu)$ is finite. In this subsection, the notation $\mathbf{W}_{2,M}$ always refers to the transport distance at the common mass of the two measures under consideration.

Proposition 5.1. *Let μ_s be the semi-discrete interpolation of Theorem 3.2, with atomic endpoint $\nu = \sum_i m_i \delta_{z_i}$. Then, for every $\tau > 0$ and every finite-second-moment reference measure $\mu_0 \in \mathcal{M}_M(\mathbb{R}^2)$,*

$$\mathcal{J}_\tau(\mu_s \mid \mu_0) = \Lambda_M(m) \log \frac{1}{1-s} + C_{\rho,T,z} + \frac{1}{2\tau} \mathbf{W}_{2,M}^2(\nu, \mu_0) + o(1) \quad (5.1)$$

as $s \uparrow 1$. In particular, if $\Lambda_M(m) < 0$, then

$$\mathcal{J}_\tau(\mu_s \mid \mu_0) \rightarrow -\infty.$$

Proof. The free-energy part is given by Theorem 3.2. Since $\mu_s \rightarrow \nu$ in $W_{2,M}$ along the displacement interpolation and μ_0 has finite second moment, the triangle inequality yields

$$|W_{2,M}(\mu_s, \mu_0) - W_{2,M}(\nu, \mu_0)| \leq W_{2,M}(\mu_s, \nu) \rightarrow 0.$$

The squared transport penalty therefore converges to $(2\tau)^{-1}W_{2,M}^2(\nu, \mu_0)$. Adding this to the free-energy expansion gives (5.1). If $\Lambda_M(m) < 0$, the logarithmic term tends to $-\infty$ while the penalty remains finite. Thus the displayed competitors make the functional unbounded below in the supercritical case. Compactness and selection of minimizing sequences are separate questions. $\square$

Remark 5.2. If $M > 8\pi$, the one-atom direction has $\Lambda_M(M) < 0$. Hence the unrestricted one-step JKO functional is unbounded below along an explicit absolutely continuous concentrating sequence. This yields the corresponding unconstrained supercritical competitor family.

Proposition 5.3. *Let ρ_a be the partial-concentration family (4.2), and set*

$$\bar{\nu} = \sigma \, dx + \sum_{i=1}^k m_i \delta_{z_i}.$$

Set $M_{\text{tot}} = M_c + M_r$, and let $\mu_0 \in \mathcal{M}_{M_{\text{tot}}}(\mathbb{R}^2)$ have finite second moment. In this proposition, $W_{2,M}$ is understood at the common mass M_{tot} . Then

$$\mathcal{J}_\tau(\rho_a \, dx \mid \mu_0) = \Lambda_c(m) \log \frac{1}{a} + C_{\eta, \sigma, z} + \frac{1}{2\tau} W_{2,M}^2(\bar{\nu}, \mu_0) + o(1). \quad (5.2)$$

If $\Lambda_c(m) < 0$, then $\mathcal{J}_\tau(\rho_a \, dx \mid \mu_0) \rightarrow -\infty$. If $\Lambda_c(m) = 0$, then

$$\mathcal{J}_\tau(\rho_a \, dx \mid \mu_0) \rightarrow \mathcal{C}_{\text{prof}}(\eta, \sigma) + \mathcal{E}_{\text{eff}}(z; m, \sigma) + \frac{1}{2\tau} W_{2,M}^2(\bar{\nu}, \mu_0). \quad (5.3)$$

Proof. The free-energy expansion is given by Theorem 4.2. It remains to prove convergence of the transport penalty. We couple the diffuse part $\sigma \, dx$ with itself and the i -th packet with the atom $m_i \delta_{z_i}$. This gives

$$W_{2,M}^2(\rho_a \, dx, \bar{\nu}) \leq \sum_i \int |x - z_i|^2 a^{-2} \eta_i\left(\frac{x - z_i}{a}\right) dx = a^2 \sum_i \int |u|^2 \eta_i(u) du \rightarrow 0,$$

where the compact support of the profiles ensures finite second moments. Therefore $\rho_a \, dx \rightarrow \bar{\nu}$ in $W_{2,M}$. By the triangle inequality,

$$W_{2,M}(\rho_a \, dx, \mu_0) \rightarrow W_{2,M}(\bar{\nu}, \mu_0),$$

and hence the distance term converges. Combining this with Theorem 4.2 gives (5.2); the two stated consequences follow from the sign of Λ_c and from Corollary 4.4 in the critical case. $\square$

Remark 5.4. Equation (5.3) is an expansion along separated families. Together with Proposition 4.16, it gives a fixed-profile finite-dimensional variational limit on compact configuration sets. Questions about minimizing sequences involve additional compactness, lower-bound, and profile-relaxation issues.

5.2. Standard virial comparison. We next recall the classical virial identity in a form that uses the same Wasserstein-square notation. This standard computation provides a dynamical point of comparison for the variational expansions above.

Let $\rho(t)$ be a smooth solution of (1.1) with finite second moment and mass M . Its center of mass

$$\bar{x} = \frac{1}{M} \int_{\mathbb{R}^2} x \rho(t, x) \, dx$$

is constant in time. Define

$$I(t) = \int_{\mathbb{R}^2} |x - \bar{x}|^2 \rho(t, x) \, dx.$$

Proposition 5.5. *Let ρ be a smooth solution of (1.1) with sufficient spatial decay so that the integrations by parts below are justified, and assume that ρ has finite second moment and mass M . Then*

$$\frac{d}{dt}I(t) = 4M\left(1 - \frac{M}{8\pi}\right). \quad (5.4)$$

If $M > 8\pi$, such a solution cannot persist beyond

$$T_{\max} \leq \frac{I(0)}{4M(M/(8\pi) - 1)} = \frac{W_2^2(\rho_0/M, \delta_{\bar{x}})}{4(M/(8\pi) - 1)}. \quad (5.5)$$

Equivalently,

$$T_{\max} \leq \frac{W_{2,M}^2(\rho_0 \, dx, M\delta_{\bar{x}})}{4M(M/(8\pi) - 1)}.$$

Proof. We present the computation for smooth, rapidly decaying solutions. Under weaker decay assumptions, one inserts a radial cutoff in $|x|^2$, performs the same calculation, and lets the cutoff radius tend to infinity; the finite second moment and the standard integrability of the interaction term justify the limit.

By translating coordinates, we may assume $\bar{x} = 0$. Differentiating $I(t) = \int |x|^2 \rho(t, x) \, dx$ and using (1.1), we have

$$\frac{d}{dt}I(t) = \int |x|^2 \Delta \rho \, dx - \int |x|^2 \nabla \cdot (\rho \nabla c) \, dx.$$

The diffusion contribution is

$$\int |x|^2 \Delta \rho \, dx = \int \rho \Delta |x|^2 \, dx = 4M.$$

For the drift term, integration by parts gives

$$- \int |x|^2 \nabla \cdot (\rho \nabla c) \, dx = 2 \int x \cdot \nabla c(x) \rho(x) \, dx.$$

Under the convention for c ,

$$\nabla c(x) = -\frac{1}{2\pi} \int \frac{x - y}{|x - y|^2} \rho(y) \, dy,$$

and therefore

$$2 \int x \cdot \nabla c(x) \rho(x) \, dx = -\frac{1}{\pi} \iint \rho(x) \rho(y) \frac{x \cdot (x - y)}{|x - y|^2} \, dx \, dy. \quad (5.6)$$

Let

$$A = \iint \rho(x) \rho(y) \frac{x \cdot (x - y)}{|x - y|^2} \, dx \, dy.$$

Interchanging x and y gives

$$A = \iint \rho(x) \rho(y) \frac{y \cdot (y - x)}{|x - y|^2} \, dx \, dy.$$

Adding these two expressions yields

$$2A = \iint \rho(x) \rho(y) \frac{|x - y|^2}{|x - y|^2} \, dx \, dy = M^2,$$

so that $A = M^2/2$. Substituting into (5.6) gives

$$\frac{d}{dt}I(t) = 4M - \frac{M^2}{2\pi} = 4M\left(1 - \frac{M}{8\pi}\right),$$

which proves (5.4). If $M > 8\pi$, the right-hand side is negative. Since $I(t) \geq 0$, a smooth finite-moment solution cannot persist beyond the time given in (5.5). Finally,

$$W_2^2(\rho_0/M, \delta_{\bar{x}}) = \frac{1}{M} \int |x - \bar{x}|^2 \rho_0(x) \, dx = \frac{I(0)}{M},$$

which yields the Wasserstein-square form of the estimate. $\square$

The identity also extends to the standard finite-second-moment class under the usual regularity hypotheses by inserting radial cutoffs in the preceding computation and passing to the limit.

The estimate (5.5) concerns the Keller-Segel gradient-flow trajectory. The paths used in Theorem 3.2 are displacement interpolations to atomic endpoints; the virial estimate and the geodesic expansions therefore describe complementary aspects of concentration.

6. CONCLUDING REMARKS AND FURTHER DIRECTIONS

For the prescribed whole-plane families considered above, the concentration expansions separate the logarithmic divergence from a finite term. The concentrating masses and the local logarithmic kernel determine the leading coefficient, whereas the finite term records the Laguerre-cell geometry, the local profiles, and the interactions among packets and with the diffuse background. The JKO results use these expansions to construct one-step competitors, while the virial identity provides a standard PDE comparison. This structure also suggests a formal bounded-domain analogue and raises the open measure-space problem of identifying the finite term for arbitrary concentrating sequences and relating it to concentration locations.

Finite parts for bounded-domain Green functions. For a smooth bounded domain $\Omega \subset \mathbb{R}^2$, let G_Ω be the Dirichlet Green function,

$$-\Delta_x G_\Omega(x, y) = \delta_y, \quad G_\Omega(x, y) = 0 \quad x \in \partial\Omega.$$

Writing

$$G_\Omega(x, y) = -\frac{1}{2\pi} \log |x - y| + H_\Omega(x, y), \quad R_\Omega(x) = H_\Omega(x, x),$$

the local singularity gives the same logarithmic coefficient for interior separated concentration points. For the bounded-domain energy

$$\mathcal{F}_\Omega(\rho) = \int_\Omega \rho \log \rho \, dx - \frac{1}{2} \iint_{\Omega \times \Omega} G_\Omega(x, y) \rho(x) \rho(y) \, dx \, dy,$$

the corresponding formal finite part is

$$\mathcal{E}_{\text{eff}}^\Omega(z; m, \sigma) = -\frac{1}{2} \sum_i m_i^2 R_\Omega(z_i) - \sum_{1 \leq i < j \leq k} m_i m_j G_\Omega(z_i, z_j) - \sum_i m_i \int_\Omega G_\Omega(z_i, y) \sigma(y) \, dy.$$

This is the bounded-domain analogue of (4.8): the pairwise logarithm is replaced by the Green function and the Robin function records the regular part of self-interaction. A full treatment would require boundary and admissible-displacement estimates not used in the whole-plane arguments above.

Beyond prescribed separated classes. The profile-relaxation result in Section 4 identifies the local critical log-HLS constant within separated packet families and suggests a corresponding measure-space problem in which packet profiles and concentration centers emerge from the limiting sequence. Formally, the candidate finite term combines the diffuse energy and the local critical log-HLS constant with pairwise atomic interactions and interactions with the diffuse background.

A rigorous derivation would require compactness estimates that identify the masses, centers, and local profiles of the emerging atoms, together with local lower bounds for the combined entropy and logarithmic interaction and control of the interaction with the diffuse remainder. Such a measure-space theory could also clarify whether the finite-dimensional energy is related to concentration locations arising in Keller-Segel trajectories or in minimizing sequences for the JKO functional.

REFERENCES

- [1] L. Ambrosio, N. Gigli, G. Savaré; *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2008.
- [2] F. Aurenhammer, F. Hoffmann, B. Aronov; Minkowski-type theorems and least-squares clustering, *Algorithmica*, **20** (1998), no. 1, 61–76.
- [3] M. Bansil, J. Kitagawa; Quantitative stability in the geometry of semi-discrete optimal transport, *International Mathematics Research Notices*, **2022** (2022), no. 10, 7354–7389.

- [4] P. Biler; Local and global solvability of some parabolic systems modelling chemotaxis, *Advances in Mathematical Sciences and Applications*, **8** (1998), no. 2, 715–743.
- [5] A. Blanchet; A gradient flow approach to the Keller-Segel systems, *RIMS Kôkyûroku*, **1837** (2013), 52–73.
- [6] A. Blanchet, J. Dolbeault, B. Perthame; Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions, *Electronic Journal of Differential Equations*, **2006** (2006), no. 44, 1–33.
- [7] Y. Brenier; Polar factorization and monotone rearrangement of vector-valued functions, *Communications on Pure and Applied Mathematics*, **44** (1991), 375–417.
- [8] F. Buseghin, J. Dávila, M. del Pino, M. Musso; Existence of finite time blow-up in Keller-Segel system, *Annals of PDE*, **12** (2026), article 17.
- [9] E. Caglioti, P.-L. Lions, C. Marchioro, M. Pulvirenti; A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description, *Communications in Mathematical Physics* **143** (1992), no. 3, 501–525.
- [10] V. Calvez, L. Corrias; The parabolic-parabolic Keller-Segel model in $\mathbb{R}^2$, *Communications in Mathematical Sciences*, **6** (2008), no. 2, 417–447.
- [11] E. A. Carlen; Stability for the logarithmic Hardy-Littlewood-Sobolev inequality with application to the Keller-Segel equation, *Journal of Functional Analysis*, **288** (2025), no. 6, article 110818.
- [12] E. A. Carlen, M. Loss; Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on S^n , *Geometric and Functional Analysis*, **2** (1992), no. 1, 90–104.
- [13] J. A. Carrillo, F. Santambrogio; L^∞ estimates for the JKO scheme in parabolic-elliptic Keller-Segel systems, *Quarterly of Applied Mathematics*, **76** (2018), no. 3, 515–530.
- [14] J. Dávila, M. del Pino, J. Dolbeault, M. Musso, J. Wei; Existence and stability of infinite time blow-up in the Keller-Segel system, *Archive for Rational Mechanics and Analysis*, **248** (2024), no. 4, article 61.
- [15] S. Di Marino, F. Santambrogio; JKO estimates in linear and non-linear Fokker-Planck equations, and Keller-Segel: L^p and Sobolev bounds, *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, **39** (2022), no. 6, 1485–1517.
- [16] J. Dolbeault, B. Perthame; Optimal critical mass in the two dimensional Keller-Segel model in $\mathbb{R}^2$, *Comptes Rendus Mathématique*, **339** (2004), no. 9, 611–616.
- [17] D. Horstmann; From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **105** (2003), no. 3, 103–165.
- [18] C.-Y. Hsieh, Y. Yu; Long-time dynamics of classical Keller-Segel equation, *Nonlinear Analysis: Real World Applications* **76** (2024), article 104026.
- [19] W. Jäger, S. Luckhaus; On explosions of solutions to a system of partial differential equations modelling chemotaxis, *Transactions of the American Mathematical Society*, **329** (1992), no. 2, 819–824.
- [20] R. Jordan, D. Kinderlehrer, F. Otto; The variational formulation of the Fokker-Planck equation, *SIAM Journal on Mathematical Analysis*, **29** (1998), no. 1, 1–17.
- [21] E. F. Keller, L. A. Segel; Initiation of slime mold aggregation viewed as an instability, *Journal of Theoretical Biology*, **26** (1970), no. 3, 399–415.
- [22] J. Kitagawa, Q. Mérigot, B. Thibert; Convergence of a Newton algorithm for semi-discrete optimal transport, *Journal of the European Mathematical Society*, **21** (2019), no. 9, 2603–2651.
- [23] R. J. McCann; A convexity principle for interacting gases, *Advances in Mathematics*, **128** (1997), no. 1, 153–179.
- [24] Q. Mérigot; A multiscale approach to optimal transport, *Computer Graphics Forum*, **30** (2011), no. 5, 1583–1592.
- [25] T. Nagai; Blow-up of radially symmetric solutions to a chemotaxis system, *Advances in Mathematical Sciences and Applications*, **5** (1995), no. 2, 581–601.
- [26] F. Otto; The geometry of dissipative evolution equations: the porous medium equation, *Communications in Partial Differential Equations*, **26** (2001), no. 1–2, 101–174.
- [27] F. Santambrogio; *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling*, Progress in Nonlinear Differential Equations and Their Applications, vol. 87, Birkhäuser, Cham, 2015.
- [28] C. Villani; *Topics in Optimal Transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.
- [29] C. Villani; *Optimal Transport: Old and New*, Grundlehren der Mathematischen Wissenschaften, vol. 338, Springer, Berlin, 2009.

JIAPING YANG

SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, 220 HANDAN ROAD, SHANGHAI 200433, CHINA

Email address: jpyang22@m.fudan.edu.cn