

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR MIXED LOCAL AND NONLOCAL KIRCHHOFF TYPE PROBLEMS

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ABSTRACT. We study the mixed local and nonlocal Kirchhoff problem

$$\begin{aligned} & M\left(\int_{\Omega} |\nabla u|^p dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy\right) (-\Delta_p u + (-\Delta)_p^s u) \\ &= f(x, u) + \lambda |u|^{p^*-2} u, \quad \text{in } \Omega, \\ & u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 3$, $0 < s < 1 < p < N$, $p^* = Np/(N-p)$ and $\lambda \geq 0$. M is a continuous function that vanishes at the origin. f is a Carathéodory function that is odd in its second variable. When $M(t)$ is comparable to $t^{\alpha-1}$ with $1 < \alpha < p^*/p$, Krasnosel'skii genus theory gives infinitely many pairs of nontrivial weak solutions for $\lambda = 0$ under subcritical growth assumptions on f . If the lower-order growth exponent exceeds p , we combine a truncation with Moser iteration. For each positive integer k , we obtain at least k distinct pairs of nontrivial weak solutions when λ is sufficiently small, with the upper bound on λ depending on k . In the Kirchhoff critical case $\alpha = p^*/p$, we prove the Palais-Smale condition and obtain infinitely many pairs of nontrivial weak solutions for $0 \leq \lambda < AS^{p^*/p}$, where A is the lower bound coefficient of M and S is the mixed Sobolev constant.

1. INTRODUCTION

In this article, we study the problem

$$\begin{aligned} & M\left(\int_{\Omega} |\nabla u|^p dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy\right) (-\Delta_p u + (-\Delta)_p^s u) \\ &= f(x, u) + \lambda |u|^{p^*-2} u, \quad \text{in } \Omega, \\ & u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with a smooth boundary, $N \geq 3$, $\lambda \geq 0$, $0 < s < 1 < p < N$, and

$$p^* = \frac{pN}{N-p}.$$

The function $M \in C(\mathbb{R}, \mathbb{R})$ is a Kirchhoff function and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function. The operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denotes the classical p -Laplacian, while the fractional p -Laplacian is normalized by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B(x, \varepsilon)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy.$$

The principal value is used when it exists. For functions in the energy space, the operator is understood in the weak sense given in Definition 2.1. This normalization agrees with the first variation of the nonlocal energy.

2020 *Mathematics Subject Classification*. 35A15, 35J60, 35R11.

Key words and phrases. Local and nonlocal operators; Kirchhoff problems; multiple solutions; genus theory.

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Submitted December 13, 2025. Published October 11, 2026.

In recent years, much attention has been devoted to mixed local and nonlocal elliptic problems with critical nonlinearities. The celebrated paper of Brézis and Nirenberg [8] initiated the systematic study of elliptic equations involving critical Sobolev growth. Since then, a large number of works have investigated local, nonlocal and mixed local-nonlocal problems in bounded domains and in the whole space. In particular, Biagi, Dipierro, Valdinoci and Vecchi [5] established a Brézis-Nirenberg type result for mixed local and nonlocal operators in bounded domains. Later, da Silva, Fiscella and Vitoria [13] studied mixed local-nonlocal quasilinear problems with critical nonlinearities for $p \neq 2$ and obtained existence and multiplicity results depending on the growth of the lower-order term. For Choquard-type equations involving mixed local and nonlocal operators, we refer to Anthal, Giacomoni and Sreenadh [2].

The study of equations driven by combined local and nonlocal operators is motivated not only by purely mathematical questions but also by applications. For $p = 2$, such operators arise naturally in models of biological populations whose individuals diffuse by both random walks and jump processes. For related models and applications, see [22]. For qualitative properties of solutions, such as maximum principles, regularity and gradient estimates for mixed local and nonlocal equations, we refer to [4, 6, 14, 17] and the references therein.

Kirchhoff-type problems originate from the classical model introduced by Kirchhoff [19] as a generalization of the D'Alembert wave equation. In the last decades, Kirchhoff equations involving pure local or pure fractional operators have been widely studied; see, for example, [1, 3, 11, 12, 15, 16, 18, 21, 24, 25] and the references therein. Figueiredo and Corrêa [12] studied a p -Kirchhoff equation by variational methods and obtained a positive solution under suitable assumptions on M and f . Furtado, Oliveira and Silva [16] considered a Kirchhoff equation with critical growth and proved the existence of multiple solutions. Recently, Jiao and Pei [18] obtained multiplicity results for a fractional p -Kirchhoff problem with the critical Sobolev exponent.

We do not impose the Ambrosetti-Rabinowitz condition. Under the assumptions below, the Kirchhoff energy has order $p\alpha$, whereas the lower-order nonlinearity has growth $q < p\alpha$. This inequality yields coercivity when $\lambda = 0$ and negative energy on small spheres in each finite-dimensional subspace. Oddness then allows us to apply Krasnosel'skii genus theory.

For $\lambda > 0$, the critical term prevents this coercivity argument. We truncate that term and select a finite family of critical points whose energies are uniformly separated from zero. Their norms have a positive lower bound. This controls the degenerate Kirchhoff coefficient in the Moser iteration and allows us to remove the truncation for sufficiently small λ , depending on the prescribed number of solution pairs. When $p\alpha = p^*$, a lower bound on the Kirchhoff coefficient and the Sobolev inequality give coercivity and compactness under an explicit condition on λ .

Throughout this paper, we write $\mathbb{R}_0^+ = [0, \infty)$. We impose the following assumptions.

(A1) $M \in C(\mathbb{R}, \mathbb{R})$, and there exist constants $\alpha \in (1, \frac{p^*}{p})$, $0 < A < B$, such that

$$At^{\alpha-1} \leq M(t) \leq Bt^{\alpha-1} \quad \text{for all } t \in \mathbb{R}_0^+.$$

(A2) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and there exist constants $q \in (1, p\alpha)$, $0 < Q_1 < Q_2$, such that

$$Q_1 t^{q-1} \leq f(x, t) \leq Q_2 t^{q-1} \quad \text{for all } t \in \mathbb{R}_0^+ \quad \text{and for a.e. } x \in \Omega.$$

(A3) $f(x, t)$ is odd with respect to t , that is,

$$f(x, -t) = -f(x, t) \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Assumption (A1) implies $M(0) = 0$ and $M(t) > 0$ for $t > 0$. Thus the Kirchhoff coefficient is degenerate at the origin. Assumptions (A2) and (A3) give two-sided power bounds and oddness. No Ambrosetti-Rabinowitz condition is required. Our first result concerns the case $\lambda = 0$.

Theorem 1.1. *Assume (A1)–(A3) hold. If $\lambda = 0$, then problem (1.1) has infinitely many pairs of nontrivial weak solutions.*

Theorem 1.1 extends the multiplicity result obtained in [11] to the mixed local-nonlocal Kirchhoff framework. Due to the lack of compactness caused by the critical Sobolev term, genus theory

cannot be applied directly to (1.1) when $\lambda > 0$. To overcome this difficulty, we use a truncation argument combined with Moser iteration. In this part, the iteration requires the additional condition $q > p$. Thus we state the following result.

Theorem 1.2. *Assume (A1)–(A3) hold, with $p < q < \alpha p$. Then, for any positive integer k , there exists a constant $\lambda_k > 0$ such that problem (1.1) admits at least k distinct pairs of nontrivial weak solutions for every $\lambda \in (0, \lambda_k)$.*

Also we consider the Kirchhoff critical case

$$\alpha = \frac{p^*}{p}.$$

We assume that M and f satisfy the following conditions.

(A4) $M \in C(\mathbb{R}, \mathbb{R})$, and there exist constants $0 < A < B$ such that

$$At^{(p^*-p)/p} \leq M(t) \leq Bt^{(p^*-p)/p} \quad \text{for all } t \in \mathbb{R}_0^+.$$

(A5) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, $f(x, t)$ is odd with respect to t , and there exists $q \in (p, p^*)$ such that, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying

$$|f(x, t)| \leq p\varepsilon|t|^{p-1} + qC_\varepsilon|t|^{q-1} \quad \text{for a.e. } x \in \Omega \quad \text{and all } t \in \mathbb{R}.$$

(A6) There exist $q_1 \in (p, p^*)$ and $a_0 > 0$ such that

$$F(x, t) \geq a_0|t|^{q_1} \quad \text{for a.e. } x \in \Omega \quad \text{and all } t \in \mathbb{R},$$

where

$$F(x, t) = \int_0^t f(x, \tau) d\tau.$$

Theorem 1.3. *Assume (A4)–(A6) hold. Then problem (1.1) has infinitely many pairs of nontrivial weak solutions for every*

$$\lambda \in [0, AS^{p^*/p}),$$

where S is the mixed Sobolev constant defined in (2.1).

The remainder of the paper is organized as follows. In Section 2, we introduce the variational framework and recall some preliminary results on the Sobolev embedding and Krasnosel'skii genus. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we prove Theorem 1.2 using a truncation argument and a Moser iteration scheme. Finally, Section 5 deals with the Kirchhoff critical case and proves Theorem 1.3.

2. PRELIMINARIES

In this section, we introduce the variational setting for problem (1.1). For $1 \leq r < \infty$, we denote by $L^r(\Omega)$ the usual Lebesgue space with norm $\|\cdot\|_r$. Since the fractional p -Laplacian is defined through interactions in the whole space $\mathbb{R}^N$, it is natural to work in the space

$$X_0(\Omega) = \{u \in W^{1,p}(\mathbb{R}^N) : u|_\Omega \in W_0^{1,p}(\Omega) \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

We endow $X_0(\Omega)$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p}.$$

Since $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, the above norm is equivalent to the standard norm in $W_0^{1,p}(\Omega)$ when Ω is bounded; see [9].

Definition 2.1. We say that $u \in X_0(\Omega)$ is a weak solution of (1.1) if

$$\begin{aligned} & M(\|u\|^p) \left[\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \iint_{\mathbb{R}^{2N}} A_u(x, y)(v(x) - v(y)) dx dy \right] \\ &= \int_\Omega f(x, u)v dx + \lambda \int_\Omega |u|^{p^*-2} uv dx \end{aligned}$$

for all $v \in X_0(\Omega)$, where

$$A_u(x, y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}.$$

The energy functional associated with problem (1.1) is

$$J_\lambda(u) = \frac{1}{p} M^*(\|u\|^p) - \frac{\lambda}{p^*} \int_{\Omega} |u|^{p^*} dx - \int_{\Omega} F(x, u) dx,$$

where

$$M^*(t) = \int_0^t M(\tau) d\tau, \quad F(x, t) = \int_0^t f(x, \tau) d\tau.$$

Then $J_\lambda \in C^1(X_0(\Omega), \mathbb{R})$, and

$$\begin{aligned} \langle J'_\lambda(u), v \rangle &= M(\|u\|^p) \left[\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \iint_{\mathbb{R}^{2N}} A_u(x, y)(v(x) - v(y)) dx dy \right] \\ &\quad - \lambda \int_{\Omega} |u|^{p^*-2} uv dx - \int_{\Omega} f(x, u)v dx. \end{aligned}$$

Thus critical points of J_λ are weak solutions of problem (1.1).

Under (A3), or the oddness assumption in (A5), we have $F(x, -t) = F(x, t)$. Hence J_λ is even and $J'_\lambda(-u) = -J'_\lambda(u)$. Therefore, every nonzero critical point gives two distinct weak solutions u and $-u$. Throughout this paper, a pair of solutions means $\{u, -u\}$ with $u \neq 0$, and distinct pairs satisfy $u_i \neq \pm u_j$ for $i \neq j$.

Lemma 2.2. *The embedding $X_0(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for every $1 \leq r \leq p^*$, and compact for every $1 \leq r < p^*$. Moreover, the Sobolev constant is positive and satisfies*

$$S = \inf_{u \in X_0(\Omega) \setminus \{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}. \quad (2.1)$$

For the Moser iteration in Section 4, we shall use the standard Sobolev inequality in $W_0^{1,p}(\Omega)$. Namely, there exists a constant $C_S > 0$ such that

$$\|u\|_{p^*}^p \leq C_S \int_{\Omega} |\nabla u|^p dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Lemma 2.3. *Let $\xi, \eta \in \mathbb{R}^N$. Then there exists a constant $c_p > 0$ such that*

$$\langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta \rangle \begin{cases} \geq c_p |\xi - \eta|^p & \text{if } p \geq 2, \\ \geq c_p \frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}} & \text{if } 1 < p < 2. \end{cases}$$

For a proof of Lemma 2.3, we refer the reader to the appendix of [12].

Next, we recall some basic facts on Krasnosel'skii genus; see [20, 23]. Let E be a Banach space and define

$$\Lambda(E) = \{Z \subset E \setminus \{0\} : Z \text{ is closed and symmetric}\}.$$

For $Z \in \Lambda(E)$, the genus of Z is defined as

$$\gamma(Z) = \inf \{m \geq 1 : \text{there exists a continuous odd map } \varphi : Z \rightarrow \mathbb{R}^m \setminus \{0\}\}.$$

We set $\gamma(\emptyset) = 0$. If no such integer m exists, then $\gamma(Z) = \infty$.

Theorem 2.4. *Let $E = \mathbb{R}^m$, and let $D \subset E$ be an open, bounded and symmetric set with $0 \in D$. Then $\gamma(\partial D) = m$.*

Corollary 2.5. *If S^{m-1} is the unit sphere in $\mathbb{R}^m$, then $\gamma(S^{m-1}) = m$.*

If E is infinite dimensional and separable, then the unit sphere of E has infinite genus.

Theorem 2.6 ([10]). *Let E be a Banach space and $I \in C^1(E, \mathbb{R})$. Assume that I satisfies the Palais-Smale condition. Suppose also that I is even, bounded from below, and that there exists a compact set $K \in \Lambda(E)$ such that*

$$\gamma(K) = k, \quad \sup_{u \in K} I(u) < I(0).$$

Then I possesses at least k distinct pairs of critical points, and their corresponding critical values are less than $I(0)$.

3. PROOF OF THEOREM 1.1

In this section we consider the case $\lambda = 0$. The corresponding functional is

$$J_0(u) = \frac{1}{p} M^*(\|u\|^p) - \int_{\Omega} F(x, u) dx.$$

Lemma 3.1. *Assume (A1)–(A3) hold. Then J_0 is coercive and bounded from below on $X_0(\Omega)$.*

Proof. By (A1)–(A3) and the Sobolev embedding, there exists $C > 0$ such that

$$J_0(u) \geq \frac{1}{p} \int_0^{\|u\|^p} A t^{\alpha-1} dt - C \int_{\Omega} |u|^q dx \geq \frac{A}{p\alpha} \|u\|^{p\alpha} - C \|u\|^q$$

for all $u \in X_0(\Omega)$. Since $q < \alpha p$, it follows that J_0 is coercive and bounded from below. $\square$

Lemma 3.2. *Assume (A1)–(A3) hold. Then J_0 satisfies the Palais-Smale condition in $X_0(\Omega)$.*

Proof. Let $\{u_n\} \subset X_0(\Omega)$ be a $(PS)_c$ -sequence for J_0 , that is,

$$J_0(u_n) \rightarrow c, \quad J'_0(u_n) \rightarrow 0 \quad \text{in } X_0(\Omega)^*$$

as $n \rightarrow \infty$. By Lemma 3.1, J_0 is coercive and bounded from below. Hence $\{u_n\}$ is bounded in $X_0(\Omega)$. Passing to a subsequence if necessary, there exists $u \in X_0(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } X_0(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^r(\Omega), \quad 1 \leq r < p^*, \\ u_n(x) &\rightarrow u(x) \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

Moreover, up to a subsequence, we can assume that

$$\|u_n\|^p \rightarrow t_0 \geq 0.$$

If $t_0 = 0$, then $\|u_n\| \rightarrow 0$, and therefore $u_n \rightarrow 0$ strongly in $X_0(\Omega)$. Hence the proof is complete in this case. We now assume that $t_0 > 0$. Since M is continuous and satisfies (A1), there exist constants $m_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$M(\|u_n\|^p) \geq m_0 > 0 \quad \text{for all } n \geq n_0.$$

Since $q < p^*$, the compact embedding $X_0(\Omega) \hookrightarrow L^q(\Omega)$ gives

$$u_n \rightarrow u \quad \text{in } L^q(\Omega).$$

By (A2) and the oddness of f , there exists $C > 0$ such that

$$|f(x, t)| \leq C|t|^{q-1} \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Thus, by Hölder's inequality,

$$\left| \int_{\Omega} f(x, u_n)(u_n - u) dx \right| \leq C \int_{\Omega} |u_n|^{q-1} |u_n - u| dx \leq C \|u_n\|_q^{q-1} \|u_n - u\|_q \rightarrow 0.$$

Since $J'_0(u_n) \rightarrow 0$ in $X_0(\Omega)^*$, we have $\langle J'_0(u_n), u_n - u \rangle = o(1)$. Consequently,

$$\begin{aligned} &M(\|u_n\|^p) \left[\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla(u_n - u) dx \right. \\ &\quad \left. + \iint_{\mathbb{R}^{2N}} A_{u_n}(x, y) ((u_n - u)(x) - (u_n - u)(y)) dx dy \right] = o(1). \end{aligned}$$

Since $M(\|u_n\|^p) \geq m_0 > 0$ for all large n , it follows that

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) dx + \iint_{\mathbb{R}^{2N}} A_{u_n}(x, y) ((u_n - u)(x) - (u_n - u)(y)) dx dy = o(1).$$

On the other hand, since $u_n \rightharpoonup u$ in $X_0(\Omega)$, we have

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (u_n - u) dx \rightarrow 0,$$

and

$$\iint_{\mathbb{R}^{2N}} A_u(x, y) ((u_n - u)(x) - (u_n - u)(y)) dx dy \rightarrow 0.$$

Indeed, the mappings

$$v \mapsto \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx$$

and

$$v \mapsto \iint_{\mathbb{R}^{2N}} A_u(x, y) (v(x) - v(y)) dx dy$$

are continuous linear functionals on $X_0(\Omega)$. Therefore,

$$I_n := \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot (\nabla u_n - \nabla u) dx,$$

and

$$H_n := \iint_{\mathbb{R}^{2N}} (A_{u_n}(x, y) - A_u(x, y)) ((u_n - u)(x) - (u_n - u)(y)) dx dy.$$

Then $I_n + H_n \rightarrow 0$. By the monotonicity of the local and nonlocal p -Laplacian operators, both $I_n \geq 0$ and $H_n \geq 0$. Hence

$$I_n \rightarrow 0, \quad H_n \rightarrow 0.$$

We now prove that $u_n \rightarrow u$ strongly in $X_0(\Omega)$. We distinguish two cases.

If $p \geq 2$, Lemma 2.3 gives

$$\begin{aligned} I_n &\geq c_p \int_{\Omega} |\nabla u_n - \nabla u|^p dx, \\ H_n &\geq c_p \iint_{\mathbb{R}^{2N}} \frac{|(u_n - u)(x) - (u_n - u)(y)|^p}{|x - y|^{N+ps}} dx dy. \end{aligned}$$

Since $I_n + H_n \rightarrow 0$, we obtain $\|u_n - u\| \rightarrow 0$.

If $1 < p < 2$, then Lemma 2.3 and Hölder's inequality imply

$$\begin{aligned} &\int_{\Omega} |\nabla u_n - \nabla u|^p dx \\ &\leq C \left[\int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \cdot (\nabla u_n - \nabla u) dx \right]^{p/2} \left(\int_{\Omega} (|\nabla u_n|^p + |\nabla u|^p) dx \right)^{(2-p)/2} \\ &\leq C I_n^{p/2}. \end{aligned}$$

Here we have used the boundedness of $\{u_n\}$ in $X_0(\Omega)$. Similarly,

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|(u_n - u)(x) - (u_n - u)(y)|^p}{|x - y|^{N+ps}} dx dy \\ &\leq C \left[\iint_{\mathbb{R}^{2N}} (A_{u_n}(x, y) - A_u(x, y)) ((u_n - u)(x) - (u_n - u)(y)) dx dy \right]^{p/2} \\ &\quad \times \left[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p + |u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right]^{(2-p)/2} \\ &\leq C H_n^{p/2}. \end{aligned}$$

Since $I_n \rightarrow 0$ and $H_n \rightarrow 0$, we have that $\|u_n - u\| \rightarrow 0$. In both cases, $u_n \rightarrow u$ strongly in $X_0(\Omega)$. Hence J_0 satisfies the Palais-Smale condition. $\square$

Proof of Theorem 1.1. Let $k \in \mathbb{N}$ be fixed. Choose a k -dimensional subspace

$$E_k = \text{span}\{\varphi_1, \dots, \varphi_k\} \subset C_0^\infty(\Omega).$$

Since all norms are equivalent on E_k , there exists a constant $C_k > 0$ such that

$$\int_{\Omega} |u|^q dx \geq C_k \|u\|^q \quad \text{for all } u \in E_k.$$

By (A1)–(A3), we have

$$J_0(u) \leq \frac{B}{p\alpha} \|u\|^{p\alpha} - C_k \|u\|^q \quad \text{for all } u \in E_k,$$

where C_k is a positive constant that may change from line to line.

Since $q < p\alpha$, there exists $\ell > 0$ sufficiently small such that

$$\frac{B}{p\alpha} \ell^{p\alpha} - C_k \ell^q < 0.$$

We set

$$K_{k,\ell} = \{u \in E_k : \|u\| = \ell\}.$$

Then $K_{k,\ell}$ is compact, symmetric, $0 \notin K_{k,\ell}$, and

$$\sup_{u \in K_{k,\ell}} J_0(u) < 0 = J_0(0).$$

Moreover, since E_k is isomorphic to $\mathbb{R}^k$, the set $K_{k,\ell}$ is homeomorphic to S^{k-1} . Hence $\gamma(K_{k,\ell}) = k$.

By Lemmas 3.1 and 3.2, J_0 is even, bounded from below and satisfies the Palais-Smale condition. Therefore, Theorem 2.6 implies that J_0 has at least k distinct pairs of nontrivial critical points. Since k is arbitrary, problem (1.1) with $\lambda = 0$ has infinitely many pairs of nontrivial weak solutions. $\square$

4. PROOF OF THEOREM 1.2

In this section, we assume in addition that

$$p < q < \alpha p.$$

This restriction is used in the Moser iteration below. Since $\lambda > 0$, the term $\lambda|u|^{p^*-2}u$ has critical Sobolev growth, and therefore the compactness argument used for Theorem 1.1 cannot be applied directly. We overcome this difficulty by a truncation argument. Let $K > 0$. Define $g_K : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$g_K(x, t) = \begin{cases} f(x, t) + \lambda|t|^{p^*-2}t, & |t| \leq K, \\ f(x, t) + \lambda K^{p^*-q}|t|^{q-2}t, & |t| > K. \end{cases}$$

Let

$$G_K(x, t) = \int_0^t g_K(x, \tau) d\tau.$$

Since $f(x, t)$ is odd with respect to t , the function $g_K(x, t)$ is also odd with respect to t , and $G_K(x, t)$ is even with respect to t . Moreover, for all $t > 0$ and a.e. $x \in \Omega$,

$$Q_1 t^{q-1} \leq g_K(x, t) \leq \tilde{Q}_2 t^{q-1},$$

where

$$\tilde{Q}_2 = Q_2 + \lambda K^{p^*-q}.$$

We consider the truncated problem

$$\begin{aligned} M(\|u\|^p) (-\Delta_p u + (-\Delta)_p^s u) &= g_K(x, u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

The corresponding energy functional is

$$I_{K,\lambda}(u) = \frac{1}{p} M^*(\|u\|^p) - \int_{\Omega} G_K(x, u) dx.$$

Then $I_{K,\lambda} \in C^1(X_0(\Omega), \mathbb{R})$, $I_{K,\lambda}$ is even, and its critical points are weak solutions of the truncated problem. By Lemmas 3.1 and 3.2, with f replaced by g_K , the functional $I_{K,\lambda}$ is coercive and satisfies the Palais-Smale condition. Theorem 1.1 therefore applies to the truncated problem. Hence, for every positive integer k , the truncated problem has at least k distinct pairs of nontrivial weak solutions. For the fixed integer k , we choose k distinct pairs of critical points obtained by the genus minimax construction and denote them by

$$U_{k,K,\lambda} = \{\pm u_\lambda^1, \dots, \pm u_\lambda^k\}.$$

The dependence of each u_λ^j on K is suppressed. The points are selected at the first k genus minimax levels, as described below.

Lemma 4.1. *For each fixed positive integer k , there exist constants $d_1, d_2 > 0$ such that the following estimates hold for every $u_\lambda \in U_{k,K,\lambda}$. The constants may depend on k , the fixed problem data and the chosen subspace E_k , but are independent of K , λ and the selected critical point:*

$$d_1 \tilde{Q}_2^{-1/q} \leq \|u_\lambda\| \leq d_2 \tilde{Q}_2^{1/(p\alpha-q)}.$$

Proof. Fix a k -dimensional subspace $E_k \subset X_0(\Omega)$ independently of K and λ . Since all norms are equivalent on E_k , there exists $C_k > 0$ such that

$$\int_\Omega |u|^q dx \geq C_k \|u\|^q \quad \text{for all } u \in E_k.$$

For $u \in E_k$, by (A1) and the lower estimate on G_K , we have

$$I_{K,\lambda}(u) \leq \frac{B}{p\alpha} \|u\|^{p\alpha} - C_k \|u\|^q,$$

where $C_k > 0$ may change from line to line. Since $q < p\alpha$, we can choose $\ell > 0$ such that

$$a_k := C_k \ell^q - \frac{B}{p\alpha} \ell^{p\alpha} > 0.$$

Set

$$K_{k,\ell} = \{u \in E_k : \|u\| = \ell\}.$$

Then

$$\sup_{u \in K_{k,\ell}} I_{K,\lambda}(u) \leq -a_k < 0.$$

Let

$$\Sigma_j = \{A \subset X_0(\Omega) : A \text{ is compact, symmetric, } 0 \notin A \text{ and } \gamma(A) \geq j\},$$

and define

$$c_j^{K,\lambda} = \inf_{A \in \Sigma_j} \sup_{u \in A} I_{K,\lambda}(u), \quad j = 1, \dots, k.$$

Since $\gamma(K_{k,\ell}) = k$, we have $K_{k,\ell} \in \Sigma_j$ for $j = 1, \dots, k$. Consequently,

$$c_j^{K,\lambda} \leq \sup_{u \in K_{k,\ell}} I_{K,\lambda}(u) \leq -a_k < 0.$$

Each $c_j^{K,\lambda}$ is a negative critical value by the genus minimax principle; see [23]. If $c_j^{K,\lambda} = \dots = c_{j+r}^{K,\lambda} = c$ with $r \geq 1$, the critical set

$$\mathcal{C}_c = \{u \in X_0(\Omega) : I_{K,\lambda}(u) = c, I'_{K,\lambda}(u) = 0\}$$

satisfies $\gamma(\mathcal{C}_c) \geq r + 1$. Every nonempty finite symmetric set avoiding zero has genus one. Hence a repeated level contains infinitely many distinct pairs of critical points. If there is no repeated level, the first k levels already give k distinct pairs. Thus the family $U_{k,K,\lambda}$ can be selected so that

$$I_{K,\lambda}(u_\lambda) \leq -a_k \quad \text{for every } u_\lambda \in U_{k,K,\lambda}.$$

On the other hand, by (A1), the upper estimate on G_K , and the Sobolev embedding, there exists $C > 0$, independent of K and λ , such that

$$I_{K,\lambda}(u) \geq \frac{A}{p\alpha} \|u\|^{p\alpha} - C \tilde{Q}_2 \|u\|^q \quad \text{for all } u \in X_0(\Omega).$$

Taking $u = u_\lambda$, we obtain

$$\frac{A}{p\alpha} \|u_\lambda\|^{p\alpha} - C\tilde{Q}_2 \|u_\lambda\|^q \leq I_{K,\lambda}(u_\lambda) \leq -a_k < 0.$$

In particular,

$$\frac{A}{p\alpha} \|u_\lambda\|^{p\alpha} < C\tilde{Q}_2 \|u_\lambda\|^q,$$

and hence

$$\|u_\lambda\| \leq \left(\frac{p\alpha C}{A} \right)^{1/(p\alpha-q)} \tilde{Q}_2^{1/(p\alpha-q)}.$$

Thus the upper estimate holds with

$$d_2 = \left(\frac{p\alpha C}{A} \right)^{1/(p\alpha-q)}.$$

For the lower estimate, using again

$$I_{K,\lambda}(u_\lambda) \geq -C\tilde{Q}_2 \|u_\lambda\|^q$$

and $I_{K,\lambda}(u_\lambda) \leq -a_k$, we obtain $C\tilde{Q}_2 \|u_\lambda\|^q \geq a_k$. Therefore,

$$\|u_\lambda\| \geq \left(\frac{a_k}{C} \right)^{1/q} \tilde{Q}_2^{-1/q}.$$

The proof is complete by taking $d_1 = \left(\frac{a_k}{C} \right)^{1/q}$. $\square$

Proof of Theorem 1.2. Let $k \in \mathbb{N}$ be fixed. We shall prove that there exist $K > 0$ and $\lambda_k > 0$ such that every solution $u_\lambda \in U_{k,K,\lambda}$ of the truncated problem satisfies

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq K \quad \text{for all } \lambda \in (0, \lambda_k).$$

Then, for such λ , the truncation is inactive, namely

$$g_K(x, u_\lambda) = f(x, u_\lambda) + \lambda |u_\lambda|^{p^*-2} u_\lambda,$$

and therefore u_λ is a weak solution of the original problem. For simplicity, write $u = u_\lambda$. Constants denoted by C below may depend on k and the fixed problem data, but are independent of K , λ , L , β and the selected critical point. Let $L > 0$ and define

$$u_L = \min\{u^+, L\}, \quad z_L = u_L^{p(\beta-1)} u^+,$$

where $\beta > 1$. Since $z_L \in X_0(\Omega)$, it can be used as a test function in the weak formulation of the truncated problem. Hence

$$M(\|u\|^p) \left[\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla z_L \, dx + \iint_{\mathbb{R}^{2N}} A_u(x, y) (z_L(x) - z_L(y)) \, dx \, dy \right] = \int_\Omega g_K(x, u) z_L \, dx.$$

We first observe that the function $t \mapsto \min\{t^+, L\}^{p(\beta-1)} t^+$ is nondecreasing on $\mathbb{R}$. Therefore,

$$(u(x) - u(y))(z_L(x) - z_L(y)) \geq 0 \quad \text{for a.e. } (x, y) \in \mathbb{R}^{2N},$$

which implies

$$A_u(x, y)(z_L(x) - z_L(y)) \geq 0 \quad \text{for a.e. } (x, y) \in \mathbb{R}^{2N}.$$

Moreover,

$$\begin{aligned} \nabla z_L &= u_L^{p(\beta-1)} \nabla u^+ + p(\beta-1) u^+ u_L^{p(\beta-1)-1} \nabla u_L, \\ \int_\Omega p(\beta-1) |\nabla u|^{p-2} \nabla u \cdot \nabla u_L u^+ u_L^{p(\beta-1)-1} \, dx &\geq 0. \end{aligned}$$

Consequently,

$$M(\|u\|^p) \int_\Omega |\nabla u^+|^p u_L^{p(\beta-1)} \, dx \leq \int_\Omega g_K(x, u) u_L^{p(\beta-1)} u^+ \, dx.$$

By Lemma 4.1 and (A1), we have

$$M(\|u\|^p) \geq A \|u\|^{p(\alpha-1)} \geq A d_1^{p(\alpha-1)} \tilde{Q}_2^{-p(\alpha-1)/q}.$$

Using also the estimate $g_K(x, u)u^+ \leq \tilde{Q}_2(u^+)^q$, we obtain

$$\int_{\Omega} |\nabla u^+|^p u_L^{p(\beta-1)} dx \leq C \tilde{Q}_2^{1+p(\alpha-1)/q} \int_{\Omega} (u^+)^q u_L^{p(\beta-1)} dx,$$

where $C > 0$ is independent of L , K , λ , β and u . Setting $w_L = u^+ u_L^{\beta-1}$, we have

$$|\nabla w_L|^p \leq C \beta^p |\nabla u^+|^p u_L^{p(\beta-1)}.$$

Let $C_S > 0$ be a Sobolev constant such that

$$\|v\|_{p^*}^p \leq C_S \int_{\Omega} |\nabla v|^p dx \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Then

$$\begin{aligned} \|w_L\|_{p^*}^p &\leq C \beta^p \int_{\Omega} |\nabla u^+|^p u_L^{p(\beta-1)} dx \\ &\leq C \beta^p \tilde{Q}_2^{1+p(\alpha-1)/q} \int_{\Omega} (u^+)^q u_L^{p(\beta-1)} dx \\ &= C \beta^p \tilde{Q}_2^{1+p(\alpha-1)/q} \int_{\Omega} (u^+)^{q-p} w_L^p dx. \end{aligned}$$

Since $p < q < p^*$, Hölder's inequality gives

$$\int_{\Omega} (u^+)^{q-p} w_L^p dx \leq \|u^+\|_{p^*}^{q-p} \|w_L\|_{\alpha^*}^p,$$

where

$$\alpha^* = \frac{p^* p}{p^* + p - q}.$$

Clearly, $p < \alpha^* < p^*$. By the Sobolev embedding and Lemma 4.1,

$$\|u^+\|_{p^*}^{q-p} \leq C \|u\|^{q-p} \leq C \tilde{Q}_2^{(q-p)/(p\alpha-q)}.$$

Thus

$$\|w_L\|_{p^*}^p \leq C \beta^p \tilde{Q}_2^{\sigma} \|w_L\|_{\alpha^*}^p,$$

where

$$\sigma = 1 + \frac{p(\alpha-1)}{q} + \frac{q-p}{p\alpha-q}.$$

If $u^+ \in L^{\beta\alpha^*}(\Omega)$, then $w_L \leq (u^+)^{\beta}$. Letting $L \rightarrow \infty$ and using Fatou's lemma, we obtain

$$\|u^+\|_{\beta p^*}^{p\beta} \leq C \beta^p \tilde{Q}_2^{\sigma} \|u^+\|_{\beta\alpha^*}^{p\beta}.$$

Hence

$$\|u^+\|_{\beta p^*} \leq \left(C \beta^p \tilde{Q}_2^{\sigma} \right)^{1/(p\beta)} \|u^+\|_{\beta\alpha^*}.$$

Now we set

$$\theta = \frac{p^*}{\alpha^*} > 1.$$

For $\beta = \theta$, we have $\beta\alpha^* = p^*$, so the preceding estimate applies. It then applies successively with $\beta = \theta^i$, $i = 1, 2, \dots$, and yields

$$\|u^+\|_{\theta^i p^*} \leq \left(C \theta^{ip} \tilde{Q}_2^{\sigma} \right)^{1/(p\theta^i)} \|u^+\|_{\theta^{i-1} p^*}.$$

Iterating this inequality from $i = 1$ to $i = m$, we find that

$$\|u^+\|_{\theta^m p^*} \leq C^{\sum_{i=1}^m \frac{1}{p\theta^i}} \theta^{\sum_{i=1}^m \frac{i}{\theta^i}} \tilde{Q}_2^{\frac{\sigma}{p} \sum_{i=1}^m \frac{1}{\theta^i}} \|u^+\|_{p^*}.$$

Since $\theta > 1$,

$$\sum_{i=1}^{\infty} \frac{1}{\theta^i} < \infty, \quad \sum_{i=1}^{\infty} \frac{i}{\theta^i} < \infty.$$

Letting $m \rightarrow \infty$, we obtain

$$\|u^+\|_{L^{\infty}(\Omega)} \leq C \tilde{Q}_2^{\sigma/(p(\theta-1))} \|u^+\|_{p^*}.$$

Using Lemma 4.1 once again,

$$\|u^+\|_{p^*} \leq C\|u\| \leq C\tilde{Q}_2^{1/(p\alpha-q)}.$$

Therefore,

$$\|u^+\|_{L^\infty(\Omega)} \leq C\tilde{Q}_2^\mu,$$

where

$$\mu = \frac{\sigma}{p(\theta-1)} + \frac{1}{p\alpha-q} > 0.$$

Since $g_K(x, t)$ is odd with respect to t , if u is a solution of the truncated problem, then $-u$ is also a solution. Applying the preceding estimate to $-u$, we also obtain

$$\|u^-\|_{L^\infty(\Omega)} \leq C\tilde{Q}_2^\mu.$$

Thus

$$\|u\|_{L^\infty(\Omega)} \leq C\tilde{Q}_2^\mu = C\left(Q_2 + \lambda K^{p^*-q}\right)^\mu.$$

We now choose $K > 0$ sufficiently large so that

$$\left(\frac{K}{C}\right)^{1/\mu} - Q_2 > 0.$$

Then we define

$$\lambda_k = \frac{\left(\frac{K}{C}\right)^{1/\mu} - Q_2}{K^{p^*-q}} > 0.$$

For every $\lambda \in (0, \lambda_k)$, we have

$$C\left(Q_2 + \lambda K^{p^*-q}\right)^\mu \leq K.$$

Therefore, $\|u\|_{L^\infty(\Omega)} \leq K$ for every $u \in U_{k,K,\lambda}$. Consequently,

$$g_K(x, u) = f(x, u) + \lambda|u|^{p^*-2}u,$$

and hence every $u \in U_{k,K,\lambda}$ is a weak solution of the original problem. Since $U_{k,K,\lambda}$ contains k distinct pairs of nontrivial solutions, problem (1.1) admits at least k distinct pairs of nontrivial weak solutions for every $\lambda \in (0, \lambda_k)$. This completes the proof. $\square$

5. PROOF OF THEOREM 1.3

In this section we consider the Kirchhoff critical case $\alpha = p^*/p$. The corresponding energy functional is still denoted by

$$J_\lambda(u) = \frac{1}{p}M^*(\|u\|^p) - \frac{\lambda}{p^*} \int_\Omega |u|^{p^*} dx - \int_\Omega F(x, u) dx.$$

Lemma 5.1. *Assume (A4), (A5). Then J_λ is coercive and bounded from below on $X_0(\Omega)$ for $0 \leq \lambda < AS^{p^*/p}$.*

Proof. By (A5), for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x, t)| \leq \varepsilon|t|^p + C_\varepsilon|t|^q \quad \text{for a.e. } x \in \Omega \quad \text{and all } t \in \mathbb{R}.$$

Using (A4), the Sobolev inequality and the definition of S , we obtain

$$\begin{aligned} J_\lambda(u) &\geq \frac{A}{p^*}\|u\|^{p^*} - \frac{\lambda}{p^*}\|u\|_{p^*}^{p^*} - C\|u\|^p - C\|u\|^q \\ &\geq \frac{1}{p^*}\left(A - \lambda S^{-p^*/p}\right)\|u\|^{p^*} - C\|u\|^p - C\|u\|^q. \end{aligned}$$

Since $A - \lambda S^{-p^*/p} > 0$ and $p < q < p^*$, it follows that J_λ is coercive and bounded from below. $\square$

Lemma 5.2. *Assume (A4), (A5). Then J_λ satisfies the Palais-Smale condition on $X_0(\Omega)$ for $0 \leq \lambda < AS^{p^*/p}$.*

Proof. Let $\{u_n\} \subset X_0(\Omega)$ be a Palais-Smale sequence for J_λ . Lemma 5.1 gives boundedness. Passing to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } X_0(\Omega), \\ u_n &\rightarrow u \quad \text{in } L^r(\Omega) \quad (1 \leq r < p^*), \\ u_n(x) &\rightarrow u(x) \quad \text{for a.e. } x \in \Omega, \\ \|u_n\|^p &\rightarrow t_0 \geq 0. \end{aligned}$$

If $t_0 = 0$, then $u_n \rightarrow 0$ in $X_0(\Omega)$ and there is nothing to prove. Assume $t_0 > 0$. By continuity and (A4),

$$m_n := M(\|u_n\|^p) \rightarrow M(t_0) > 0.$$

In particular, $m_0 \leq m_n \leq m_1$ for all sufficiently large n , with constants $m_0, m_1 > 0$.

We first prove the almost everywhere convergence of the gradients. Put $w_n = u_n - u$ and, for $\delta > 0$, define

$$T_\delta(t) = \max\{-\delta, \min\{t, \delta\}\}.$$

The Lipschitz property of T_δ gives

$$\|T_\delta(w_n)\| \leq \|w_n\|, \quad \|T_\delta(w_n)\|_p \leq \|w_n\|_p \rightarrow 0.$$

Thus $T_\delta(w_n) \rightarrow 0$ in $X_0(\Omega)$ for each fixed $\delta > 0$. By (A5) and the boundedness of u_n in $L^{p^*}(\Omega)$,

$$h_n = f(x, u_n) + \lambda|u_n|^{p^*-2}u_n, \quad \sup_n \|h_n\|_1 < \infty.$$

Testing $J'_\lambda(u_n) = o(1)$ with $T_\delta(w_n)$ and subtracting the pairing with the fixed local and nonlocal fluxes of u , we obtain

$$\begin{aligned} &\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla T_\delta(w_n) \, dx \\ &+ \iint_{\mathbb{R}^{2N}} (A_{u_n}(x, y) - A_u(x, y)) (T_\delta(w_n(x)) - T_\delta(w_n(y))) \, dx \, dy \\ &= \frac{1}{m_n} \int_\Omega h_n T_\delta(w_n) \, dx + o(1). \end{aligned}$$

Here the pairings with the fluxes of u tend to zero by weak convergence. The nonlocal integrand is nonnegative. The first factor has the sign of $w_n(x) - w_n(y)$, and T_δ is nondecreasing. Set

$$D_n(x) = (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot (\nabla u_n - \nabla u) \geq 0.$$

Since $\nabla T_\delta(w_n) = \nabla w_n$ on $\{|w_n| < \delta\}$ and is zero elsewhere almost everywhere, it follows that

$$\limsup_{n \rightarrow \infty} \int_{\{|w_n| < \delta\}} D_n(x) \, dx \leq C\delta, \quad (5.1)$$

where C is independent of δ .

Fix $h > 0$ and $R > h$. By Lemma 2.3, there exists $d_{R,h} > 0$ such that

$$D_n(x) \geq d_{R,h} \quad \text{if } |\nabla w_n(x)| \geq h \text{ and } |\nabla u_n(x)| + |\nabla u(x)| \leq R.$$

The L^p bounds on the gradients yield

$$|\{| \nabla w_n | \geq h \}| \leq |\{| w_n | \geq \delta \}| + \frac{C}{R^p} + \frac{1}{d_{R,h}} \int_{\{|w_n| < \delta\}} D_n(x) \, dx.$$

First let $n \rightarrow \infty$ and use (5.1). Then let $\delta \downarrow 0$ and $R \rightarrow \infty$. We obtain $\nabla u_n \rightarrow \nabla u$ in measure. After a further subsequence,

$$\nabla u_n(x) \rightarrow \nabla u(x) \quad \text{for a.e. } x \in \Omega.$$

For the zero extensions to $\mathbb{R}^N$, set

$$v_n(x, y) = \frac{u_n(x) - u_n(y)}{|x - y|^{(N+ps)/p}}, \quad v(x, y) = \frac{u(x) - u(y)}{|x - y|^{(N+ps)/p}}.$$

Then $v_n \rightarrow v$ almost everywhere in $\mathbb{R}^{2N}$ and $\{v_n\}$ is bounded in $L^p(\mathbb{R}^{2N})$. The Brézis-Lieb lemma [7], applied to the local gradients, the nonlocal difference quotients and the critical term, gives

$$\begin{aligned} \|u_n\|^p &= \|w_n\|^p + \|u\|^p + o(1), \\ \|u_n\|_{p^*}^{p^*} &= \|w_n\|_{p^*}^{p^*} + \|u\|_{p^*}^{p^*} + o(1). \end{aligned} \quad (5.2)$$

Almost everywhere convergence and boundedness also imply

$$\begin{aligned} |\nabla u_n|^{p-2} \nabla u_n &\rightharpoonup |\nabla u|^{p-2} \nabla u \quad \text{in } L^{p/(p-1)}(\Omega; \mathbb{R}^N), \\ |v_n|^{p-2} v_n &\rightharpoonup |v|^{p-2} v \quad \text{in } L^{p/(p-1)}(\mathbb{R}^{2N}), \\ |u_n|^{p^*-2} u_n &\rightharpoonup |u|^{p^*-2} u \quad \text{in } L^{p^*/(p^*-1)}(\Omega). \end{aligned}$$

Consequently, by (5.2),

$$\begin{aligned} &\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w_n \, dx + \iint_{\mathbb{R}^{2N}} A_{u_n}(x, y) (w_n(x) - w_n(y)) \, dx \, dy \\ &= \|u_n\|^p - \|u\|^p + o(1) = \|w_n\|^p + o(1), \end{aligned}$$

and

$$\int_{\Omega} |u_n|^{p^*-2} u_n w_n \, dx = \|u_n\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*} + o(1) = \|w_n\|_{p^*}^{p^*} + o(1).$$

Moreover, (A5) and strong convergence in $L^p(\Omega)$ and $L^q(\Omega)$ imply

$$\left| \int_{\Omega} f(x, u_n) w_n \, dx \right| \leq C(\|u_n\|_p^{p-1} \|w_n\|_p + \|u_n\|_q^{q-1} \|w_n\|_q) \rightarrow 0.$$

Testing $J'_\lambda(u_n)$ with w_n now yields

$$M(\|u_n\|^p) \|w_n\|^p - \lambda \|w_n\|_{p^*}^{p^*} = o(1). \quad (5.3)$$

By (A4), the definition of S , and (5.3), we have

$$A \|u_n\|^{p^*-p} \|w_n\|^p \leq \lambda S^{-p^*/p} \|w_n\|^{p^*} + o(1).$$

By (5.2),

$$\|w_n\|^p \rightarrow t_0 - \|u\|^p =: \eta^p \geq 0.$$

Passing to the limit, we obtain

$$A(\eta^p + \|u\|^p)^{(p^*-p)/p} \eta^p \leq \lambda S^{-p^*/p} \eta^{p^*}.$$

Thus $(A - \lambda S^{-p^*/p}) \eta^{p^*} \leq 0$. Since $\lambda < A S^{p^*/p}$, we have $\eta = 0$. Therefore $u_n \rightarrow u$ strongly in $X_0(\Omega)$. $\square$

Proof of Theorem 1.3. By Lemmas 5.1 and 5.2, it follows that J_λ is bounded from below, even and satisfies the Palais-Smale condition on $X_0(\Omega)$.

It remains to verify the genus geometry. Let $k \in \mathbb{N}$ be fixed and choose a k -dimensional subspace

$$E_k = \text{span}\{\varphi_1, \dots, \varphi_k\} \subset C_0^\infty(\Omega).$$

Since all norms are equivalent on E_k , there exists $C_k > 0$ such that

$$\int_{\Omega} |u|^{q_1} \, dx \geq C_k \|u\|^{q_1} \quad \text{for all } u \in E_k.$$

By (A6),

$$\int_{\Omega} F(x, u) \, dx \geq a_0 C_k \|u\|^{q_1} \quad \text{for all } u \in E_k.$$

Using (A4), we obtain

$$J_\lambda(u) \leq \frac{B}{p^*} \|u\|^{p^*} - a_0 C_k \|u\|^{q_1} \quad \text{for all } u \in E_k.$$

Since $p < q_1 < p^*$, there exists $\rho > 0$ sufficiently small such that

$$\frac{B}{p^*} \rho^{p^*} - a_0 C_k \rho^{q_1} < 0.$$

Setting $K_{k,\rho} = \{u \in E_k : \|u\| = \rho\}$, we have

$$\sup_{u \in K_{k,\rho}} J_\lambda(u) < 0 = J_\lambda(0).$$

Moreover, $\gamma(K_{k,\rho}) = k$. By Theorem 2.6, J_λ has at least k distinct pairs of nontrivial critical points. Since k is arbitrary, problem (1.1) has infinitely many pairs of nontrivial weak solutions. $\square$

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