

EXISTENCE OF POSITIVE SOLUTIONS TO SINGULAR ELLIPTIC PROBLEMS WITH GRADIENT TERMS

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ABSTRACT. In this article, we study the existence of classical, positive solutions that vanish asymptotically at infinity to the class of singular elliptic problems with equation

$$\Delta u(x) + f(x, u(x)) - b(x)(u(x))^{-1} \|\nabla u(x)\|^2 + g(x)x \cdot \nabla u(x) = 0$$

in the exterior domain denoted by $\Omega_R = \{x \in \mathbb{R}^n, \|x\| > R > 1\}$, $n > 2$. Moreover, we study the problem with additional boundary condition $u(x) = S$ when $\|x\| = R$. In the first part of the paper, we present auxiliary results concerning existence and multiplicity of solutions for the case when $b \equiv 0$. These results play a crucial role in the proof of the existence of solution for our singular problem in the second part of this paper. We base our approach on the sub-super solution method for bounded sub-domains, as well as the approximation method for the unbounded domain combined with some classical convergence procedure.

1. INTRODUCTION

Elliptic PDEs with singularities arise in various problems concerning physical phenomena and applied mathematics. In paper [5], the authors describe isothermal reaction models in chemical catalyst kinetics using the Langmuir-Hinshelwood model. The analysis starts from a problem depending on a small parameter $\varepsilon > 0$, while the singular elliptic equation is obtained by letting ε go to zero. The model is

$$\begin{aligned} \Delta u &= \lambda u^m \left(\frac{\varepsilon + 1}{\varepsilon + u} \right)^{m+k} \quad \text{in } \Omega, \\ u &\geq 0 \quad \text{in } \Omega, \\ u &= 1 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ - bounded and $u \geq 0$, $\lambda > 0$, $k > 0$, $m \geq 1$, small $\varepsilon > 0$ correspond to the absorption parameters. In such models Ω represents a catalyst pellet or an enzyme membrane [2, 3, 4], u is the concentration of a reactant or a substrate diffusing across the boundary. When $\varepsilon \rightarrow 0$, the equation associated with (1.1) has a form with singularity

$$\Delta u = \lambda u^{-k}, \quad k > 0,$$

which belongs to the class of generalized Lane-Emden-Fowler equations ([2]). The discussion concerning classical solutions of such equations with convection and singular potential is described among others in [12].

In [13], a problem with the van der Waals interactions leads to the elliptic singular model

$$\Delta u + (\lambda u^{-\alpha} - u^{-\beta}) = 0, \quad u > 0 \text{ in } B_1^\beta \text{ and } u = 0 \text{ on } \partial B_1^\beta, \tag{1.2}$$

where $0 < \alpha, \beta < 1$ and $B_1^\beta := \{x \in \mathbb{R}^n, \|x\| < 1\}$, $n \in \mathbb{N} = 1, 2, 3, \dots$. Equations like (1.2) arise in the study of chemical heterogeneous catalysts or steady states of the thin film. The author investigates the existence and multiplicity of radial solutions for the problem applying the shooting

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method for a certain ODE problem corresponding to (1.2). Some other interesting methods for ODEs are described for example in [14].

The development of mathematical methods for elliptic problems with singular part can be traced back to several foundational contributions. A key classical paper is the work of Crandall, Rabinowitz, and Tartar (1977) [9], in which singular elliptic equations posed on bounded domains in $\mathbb{R}^n$ were analyzed. Subsequent studies expanded and refined these early results. In particular, the work by Lazer and McKenna [14] provided further understanding of singular nonlinearities. Both works were focused on the existence and properties of solutions to equations in which no gradient-dependent components are present.

A broader class of problems has also been investigated in bounded domains, where the studied equations include quadratic growth in respect to the gradient. In [18], a following class of quasilinear elliptic problems with boundary condition was studied

$$\begin{aligned} -\operatorname{div} a(Du) &= \lambda[\xi(z)u^{-\eta} + f(z, u)] \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &> 0, \quad \lambda > 0, \quad 0 < \eta < 1. \end{aligned}$$

In that study, $a(\cdot)$ provided many differential operators such as the p -Laplacian ($1 < p < \infty$) and the (p, q) -Laplacian ($1 < q < p$, so called double phase problems).

The case of unbounded domain has been addressed in the literature, notably by Covei [7], where the author examines classical solutions, asymptotically vanishing at infinity to the equation

$$-\Delta u + c(x)u^{-1}\|\nabla u\|^2 = a(x) \quad \text{for } x \in \mathbb{R}^n, \quad u > 0.$$

Applying a suitable transformation to such a solution leads to the existence of blow up solutions to the class of equations of the form $-\Delta u = a(x)h(u)$ in Ω , $u > 0$ (see e.g. [7, Remarks 1,2]). There, the classical approach of approximation is applied. First, the positive solution is constructed in some arbitrary bounded domain with boundary of $C^{2+\gamma}$ class and further, the author establishes existence results for a positive solution u of the problem in the whole space $\mathbb{R}^n$ (see e.g. [9] and [19]). Moreover, it is proven that $u(x) = O(\|x\|^{2-n})$ as $\|x\| \rightarrow \infty$.

In this work, we follow the classical procedure with focus put on existence of such solutions in the exterior domain. Here, the following elliptic problem with boundary condition is studied

$$\begin{aligned} \Delta u(x) + f(x, u(x)) - b(x)(u(x))^{-1}\|\nabla u(x)\|^2 + k(x, x \cdot \nabla u(x)) &= 0, \quad x \in \Omega_R, \\ \lim_{\|x\| \rightarrow \infty} u(x) &= 0, \end{aligned} \tag{1.3}$$

where $\Omega_R = \{x \in \mathbb{R}^n, \|x\| > R > 1\}$, $n > 2$, b, f and k are sufficiently smooth functions, $b \geq 0$, $f \geq 0$ and k can change its sign. As a solution of (1.3) we understand a function u such that $u \in C^{2+\gamma}(\overline{M})$ for every bounded subdomain $M \subset \Omega_R$ and u satisfies the equation in (1.3) at every point $x \in \Omega_R$. We also investigate the case of (1.3) with additional boundary condition

$$\begin{aligned} \Delta u(x) + f(x, u(x)) - b(x)(u(x))^{-1}\|\nabla u(x)\|^2 + k(x, x \cdot \nabla u(x)) &= 0, \quad x \in \Omega_R, \\ u(x) &= S \text{ when } \|x\| = R, \\ \lim_{\|x\| \rightarrow \infty} u(x) &= 0. \end{aligned} \tag{1.4}$$

In the first part of this article, we establish that there exists at least one solution of (1.4) for every S in certain interval in case when $b(x) \equiv 0$. In consequence, this result allows us to describe multiplicity of positive solutions for non-singular problem without condition $u(x) = S$ on the boundary of the ball. This leads to the problem formulated by Constantin in 1996 ([6]) and we show that solutions of such problem create a finite strictly increasing sequence and we can establish uncountable set of such finite sequences. Some results of the first part are used in the singular case. There, we study the existence of positive solutions for (1.3) with $b(x) > 0$ and (1.4) with $b \in (0, 1/2]$. First, we prove the existence of solutions in certain annuli. To obtain our main result, we apply the domain approximation method and classical estimates of solutions for elliptic equations to construct the solution of each of our problems. We also investigate the rate of decay of solutions. Similar problems have been discussed in [20, 21, 15], but in more specific cases,

namely when the term associated with convection is linear with respect to the second variable. Here, we relax assumptions concerning both the sign of this term and allow its nonlinearity.

Both in nonsingular and singular case, the sub-super solutions methods are applied as, for example, in [7, 10]. We recall that as a subsolution (supersolution) of the equation in (1.4), we understand a function $w \in C^{2+\gamma}(\overline{M})$ ($v \in C^{2+\gamma}(\overline{M})$) for every bounded subdomain $M \subset \Omega_R$ such that w satisfies

$$\Delta w(x) + f(x, w(x)) - b(x)(w(x))^{-1} \|\nabla w(x)\|^2 + k(x, x \cdot \nabla w(x)) \geq 0$$

(and for being a supersolution, w satisfies

$$(\Delta v(x) + f(x, v(x)) - b(x)(v(x))^{-1} \|\nabla v(x)\|^2 + k(x, x \cdot \nabla v(x)) \leq 0)$$

at every point $x \in \Omega_R$. Another contribution of this paper is that for the problem (1.4), we are able to provide explicit formulas of estimates from both above and below for the solution.

2. NONSINGULAR PROBLEM WITH THE BOUNDARY CONDITION

In this section we consider (1.3) with $b(x) \equiv 0$ and additional boundary condition which gives the following problem

$$\begin{aligned} \Delta u(x) + f(x, u(x)) + k(x, x \cdot \nabla u(x)) &= 0 \quad \text{for } x \in \Omega_R, \\ u(x) &= S \quad \text{when } \|x\| = R, \\ \lim_{\|x\| \rightarrow \infty} u(x) &= 0. \end{aligned} \quad (2.1)$$

We deal with the case when the functions f and k satisfy the following conditions:.

- (A1) $f : \Omega_1 \times \mathbb{R} \rightarrow [0, +\infty)$ is a locally Hölder-continuous function with an exponent $\gamma \in (0, 1)$, where $\Omega_1 = \{x \in \mathbb{R}^n, \|x\| > 1\}$, $n > 2$ and there exist constants $d > 0$ and $0 < c < 2d(n-2)$ such that for all $r \in (1, +\infty)$,

$$\sup_{u \in [0, 4d]} \sup_{\|x\|=r} f(x, u) \leq cr^{-n-1}. \quad (2.2)$$

- (A2) $k : \Omega_1 \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally Hölder-continuous function with an exponent $\gamma \in (0, 1)$, $k(x, 0) = 0$ in Ω_1 and

- (1) for any bounded subset $M \subset \Omega_1$ there exists a constant $C > 0$ such that for all $x \in \overline{M}$ and $\tau \in \mathbb{R}$, $|k(x, \tau)| \leq C(1 + \tau^2)$,
- (2) there exist $g : \Omega_1 \rightarrow \mathbb{R}$ of C^1 class, constants $D \in (0, 2d(n-2))$, $a \in (0, \frac{1}{2})$ and $p > 2n-4$ such that for $\tau \in [-(2d(n-2) + D)\|x\|^{2-n}, 0]$ and $x \in \Omega_1$, we have

$$\begin{aligned} -a^2\|x\|^{-3} \leq g(x) &\leq \frac{p+2-n}{\|x\|^2}, \\ k(x, \tau) &\geq g(x)\tau, \\ k(x, \tau) &\leq D\|x\|^{-1-n}. \end{aligned} \quad (2.3)$$

Remark 2.1. It is obvious that if $k(x, \tau) \leq 0$ for any $x \in \Omega_1$ and any $\tau \in [-(2d(n-2) + D)\|x\|^{2-n}, 0]$, then (2.3) is always held.

We present natural examples of k satisfying (A2).

Remark 2.2. In particular, we can consider k given as follows

- (1) $k(x, \tau) = g(x)\tau$ (as in [15]),
- (2) $k(x, \tau) = g(x)\tau(1 + \frac{[T(x, \tau)]^2}{1+[T(x, \tau)]^2})$, where $T(x, \tau)$ is sufficiently smooth.

Proof. In case when $k(x, \tau) = g(x)\tau$ it suffices to consider $D = 2ad(n-2)$. Then the inequality satisfied by g allows us to derive that for all $\tau \in [-(2d(n-2) + D)\|x\|^{2-n}, 0]$, we have the following chain of estimates

$$\begin{aligned} k(x, \tau) &\leq -a^2\|x\|^{-3}\tau \leq a^2\|x\|^{-3}(2d(n-2) + D)\|x\|^{2-n} \\ &< (da(n-2) + \frac{D}{4})\|x\|^{-1-n} \end{aligned}$$

$$\leq D\|x\|^{-1-n}.$$

To show that $k(x, \tau) = g(x)\tau(1 + \frac{[T(x, \tau)]^2}{1+[T(x, \tau)]^2})$ satisfies (2.3) with $D = 4ad(n-2)$, we consider two cases. In the first one, we assume $g \geq 0$, thus $k(x, \tau) \leq 0$ for all $\tau \in [-(2d(n-2) + D)\|x\|^{2-n}, 0]$ and we note that by Remark 2.1 the proof is immediate. On the other hand, when $-a^2\|x\|^{-3} \leq g(x) < 0$, let us define $\tilde{T}(x, \tau) := -\tau(1 + \frac{[T(x, \tau)]^2}{1+[T(x, \tau)]^2})$, thus $0 \leq \tilde{T}(x, \tau) < -2\tau$ for all $-(2d(n-2) + D)\|x\|^{2-n} \leq \tau \leq 0$. Finally, we estimate

$$\begin{aligned} k(x, \tau) &= -g(x)\tilde{T}(x, \tau) < -2g(x)[2d(n-2) + D]\|x\|^{2-n} \\ &\leq 2a^2\|x\|^{-3}[2d(n-2) + D]\|x\|^{2-n} \\ &= 2a^2(\frac{D}{2a} + D)\|x\|^{-1-n} \\ &= D(a + 2a^2)\|x\|^{-1-n} \\ &< D\|x\|^{-1-n}, \end{aligned}$$

where the last inequality follows from $a < \frac{1}{2}$. We emphasize the fact that the second provided formula for $k(x, \tau)$ is still at least linear in respect to τ . $\square$

To prove the existence of a solution to (2.1) we use the following sub-super solution method

Theorem 2.3 (Noussair-Swanson [19]). *Let us consider an equation*

$$\Delta u(x) = F(x, u(x), \nabla u(x)) \text{ for } x \in \Omega_R, \quad (2.4)$$

where $F: \Omega_1 \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$. Assume that there exists an exponent $\gamma \in (0, 1)$ such that for any bounded sets $M \subset \Omega_1$ and $N \subset \mathbb{R}^n$ and any bounded interval $J \subset \mathbb{R}$, F is a Hölder-continuous function with the exponent γ on the set $(\overline{M} \times \overline{J} \times \overline{N})$. Moreover, assume that for any bounded subset $M \subset \Omega_1$ there exists a nonnegative, continuous function $g_M > 0$ such that $|F(x, u, y)| \leq g_M(|u|)(1 + y^2)$ for all $x \in \overline{M}$ and $y \in \mathbb{R}^n$ (Nagumo condition). Let us also assume that there exist a number $R > 1$, a nonnegative subsolution w and a positive supersolution v of (2.4), such that $w(x) \leq v(x)$ for $x \in \overline{\Omega}_R$. Then (2.4) has a solution u in Ω_R such that $w(x) \leq u(x) \leq v(x)$ for $x \in \overline{\Omega}_R$. In addition, u can be chosen in such a way that $u(x) = v(x)$ when $\|x\| = R$.

Remark 2.4. We mention that the last assertion of Theorem 2.1 follows directly from [15, Lemma 3.1] and from the construction of the solution u given in [16, Theorem 3.3]. In [16, Lemma 3.1] a solution u_j to (2.4) in a certain annulus in Ω_R is presented. This solution is equal to the supersolution on the boundary of the annulus. Further in [16, Theorem 3.3], authors construct the diagonal sequence of solutions $\{u_j^j(x)\}$ which converges to the solution $u(x)$ in Ω_R . This construction guarantees that the solution is equal to the supersolution when $\|x\| = R$ which was mentioned explicitly in the corollary from this theorem [15, Corollary 3.5].

We emphasize that the solution has the same value on the boundary as a supersolution, what is guaranteed by Theorem 2.3. This fact plays the crucial role for the study of the existence and multiplicity of the solutions for the problem without the condition on boundary of the ball. It allows us to describe an iterative procedure to construct a strictly increasing sequence of solutions.

2.1. Sub and supersolution to the problem without singularity. For the purpose of formulating the main theorem for nonsingular boundary value problem, we define

$$A := (c + D) \frac{R^{1-n}[R(n-1) - (n-2)]}{(n-1)(n-2)}, \quad (2.5)$$

$$B := (D + \frac{2(n-2)d + c}{2}) \frac{R^{1-n}[R(n-1) - (n-2)]}{(n-1)(n-2)}. \quad (2.6)$$

Note that the inequality $A < B$ follows directly from $c < 2(n-2)d$. In this part, we show the existence of the solution to (2.1) for each $S \in (A, B]$. Before coming to the theorem, let

us describe the process of constructing the supersolution to (2.1). To this effect, let us define $c_S := \frac{S(n-1)(n-2)}{R^{1-n}[R(n-1)-(n-2)]} - c - D$. It is obvious that for any $n > 2$ and any $R > 1$,

$$c + D < \frac{S(n-1)(n-2)}{R^{1-n}[R(n-1)-(n-2)]} \leq D + \frac{2(n-2)d+c}{2} = c + D + \frac{2(n-2)d-c}{2}.$$

It follows immediately from the monotonicity of c_S with respect to S , that $A < S \leq B$ implies $0 < c_S \leq \frac{2(n-2)d-c}{2}$. Now, let us define

$$\tilde{f}_S(r) := (c + D + c_S)r^{-n-1} = \frac{S(n-1)(n-2)}{r^{n+1}[R^{1-n}[R(n-1)-(n-2)]]}, \quad (2.7)$$

with c satisfying (A1). It is obvious that $cr^{-n-1} < \tilde{f}_S(r)$ and taking the estimate $c < 2(n-2)d$ from (A1), formula for B , the fact that $S \leq B$ and its above implication, we obtain

$$\int_1^\infty r^{n-1} \tilde{f}_S(r) dr < 2d(n-2) + D. \quad (2.8)$$

Remark 2.5. Note that $D < 2d(n-2)$ ((A2), point (2)) implies

$$\int_1^\infty r^{n-1} \tilde{f}_S(r) dr < 4d(n-2).$$

This guarantees that constructed supersolution of (2.1) will be bounded from above by $4d$, which we are going to prove in the Proposition 2.6 below. Such estimate will imply that the solution of (2.1) can only be bounded from above by $4d$.

Now, we construct the supersolution of (2.1) from the following proposition

Proposition 2.6. *Let $\tilde{f}_S$ be defined as in (2.7). Then, for every $S \in (A, B]$, $v_S(x)$ given as*

$$v_S(x) = \frac{1}{(n-2)^2} \int_{1-\|x\|^{2-n}}^1 \int_0^q (1-\xi)^{\frac{2n-2}{2-n}} \tilde{f}_S((1-\xi)^{\frac{1}{2-n}}) d\xi dq, \quad (2.9)$$

is the radial solution of the problem

$$\begin{aligned} -\Delta v_S(x) &= \tilde{f}_S(\|x\|) \quad \text{for } x \in \Omega_1, \\ \lim_{\|x\| \rightarrow \infty} v_S(x) &= 0, \end{aligned} \quad (2.10)$$

such that

$$v_S(x) = S \quad \text{for } \|x\| = R, \quad (2.11)$$

$$0 < v_S(x) < 2d + \frac{D}{(n-2)} \quad \text{for } \|x\| > 1, \quad (2.12)$$

$$-(2d + \frac{D}{(n-2)})(n-2)\|x\|^{2-n} \leq x \cdot \nabla v_S(x) \leq 0, \quad \text{for } \|x\| > 1. \quad (2.13)$$

Moreover, for every S , $v_S(x)$ is the supersolution of (2.1).

Proof. Fix $S \in (A, B]$. First, v_S can be rewritten in shorter way. To this effect, let us define

$$\begin{aligned} h_S(t) &:= \frac{1}{(n-2)^2} (1-t)^{\frac{2n-2}{2-n}} \tilde{f}_S((1-t)^{\frac{1}{2-n}}), \\ z_S(t) &:= \int_t^1 \left[\int_0^q h_S(\xi) d\xi \right] dq. \end{aligned}$$

By substitution $t = 1 - \|x\|^{2-n}$, we obtain

$$v_S(x) = \int_{1-\|x\|^{2-n}}^1 \left[\int_0^q h_S(\xi) d\xi \right] dq = z_S(1 - \|x\|^{2-n}) \quad \text{in } \Omega_1. \quad (2.14)$$

It is easy to show that with $z_S(t)$ and $h_S(t)$ defined as above, we obtain $z_S''(t) = -h_S(t)$ and for every $x \in \Omega_1$, we can calculate the following via substitution $t = 1 - \|x\|^{2-n}$,

$$\begin{aligned} -\Delta v_S(x) &= -\Delta z_S(1 - \|x\|^{2-n}) = -z_S''(1 - \|x\|^{2-n})(n-2)^2\|x\|^{2-2n} \\ &= h_S(1 - \|x\|^{2-n})(n-2)^2\|x\|^{2-2n} = \tilde{f}_S(\|x\|). \end{aligned} \quad (2.15)$$

Thus $v_S(x)$ is indeed the radial solution of (2.10). Next, we calculate for $\|x\| = R$,

$$\begin{aligned} v_S(x) &= z_S(1 - R^{2-n}) = \int_{1-R^{2-n}}^1 \int_0^q h_S(\xi) d\xi dq \\ &= \int_{1-R^{2-n}}^1 \frac{c + D + c_S}{n-2} [1 - (1-q)^{\frac{1}{n-2}}] dq \\ &= \frac{c + D + c_S}{n-2} \left(R^{2-n} - \frac{n-2}{n-1} R^{1-n} \right) \\ &= (c + D + c_S) \frac{R^{1-n} [R(n-1) - (n-2)]}{(n-1)(n-2)} = S. \end{aligned}$$

Therefore $v_S(x)$ satisfies (2.11). To show that (2.12) and (2.13) hold, we first note that by (2.8) we have the following estimate for $\xi \in (0, 1)$,

$$\int_0^1 h_S(\xi) d\xi = \frac{1}{n-2} \int_1^\infty r^{n-1} \tilde{f}_S(r) dr < 2d + \frac{D}{(n-2)}. \quad (2.16)$$

Next, we show that function $z_S(t) = \int_t^1 [\int_0^q h_S(\xi) d\xi] dq$ satisfies the following estimates

$$0 < z_S(t) \text{ in } [0, 1] \text{ and } z_S(t) < 2d + \frac{D}{(n-2)} \text{ in } [0, 1], \quad (2.17)$$

$$-\left(2d + \frac{D}{(n-2)}\right) < z_S'(t) \text{ in } (0, 1) \text{ and } z_S'(t) < 0 \text{ in } (0, 1]. \quad (2.18)$$

We notice immediately that inequalities $0 < z_S(t)$ and $z_S'(t) < 0$ result from non-negativity of $\tilde{f}_S$ and the formula for h_S . Further, from (2.16), we obtain the following chains of inequalities

$$\begin{aligned} z_S(t) &= \int_t^1 \int_0^q h_S(\xi) d\xi dq \leq \int_t^1 \int_0^1 h_S(\xi) d\xi dq < 2d + \frac{D}{(n-2)}, \\ -z_S'(t) &= \int_0^t h_S(\xi) d\xi \leq \int_0^1 h_S(\xi) d\xi < 2d + \frac{D}{(n-2)}. \end{aligned}$$

Therefore, (2.12) follows directly from (2.17). Next, by combining the non-positivity of $z_S'(t)$ for $t \in (0, 1]$ given in (2.18) with the obvious fact that for $x \in \mathbb{R}^n$ such that $\|x\| > 1$ and $1 - \|x\|^{2-n} \in (0, 1)$, we obtain

$$x \cdot \nabla v_S(x) = z_S'(1 - \|x\|^{2-n})(n-2)\|x\|^{2-n} \leq 0.$$

On the other hand, by the first inequality in (2.18), the following holds,

$$x \cdot \nabla v_S(x) \geq -(2d(n-2) + D)\|x\|^{2-n},$$

which proves (2.13). Finally, to show that $v_S(x)$ is also the supersolution of (2.1), we claim that

$$\Delta v_S(x) + f(x, v_S(x)) + k(x, x \cdot \nabla v_S(x)) \leq 0 \quad \text{for } x \in \Omega_R.$$

Indeed, from (2.13), for all $x \in \Omega_1$, we obtain

$$x \cdot \nabla v_S(x) \in [-(2d(n-2) + D)\|x\|^{2-n}, 0].$$

Combining this fact with conditions in the assumption (A2), we conclude that $k(x, x \cdot \nabla v_S(x)) \leq D\|x\|^{-1-n}$. Further, combining (2.12) with $D < 2d(n-2)$ gives as the following estimate

$$v_S(x) < 2d + \frac{D}{(n-2)} < 4d \quad \text{for } \|x\| > 1;$$

thus $f(x, v_S(x))$ satisfies (2.2), that is $f(x, v_S(x)) \leq cr^{-n-1}$ and by the formula of $\tilde{f}_S(\|x\|)$ we obtain

$$\begin{aligned} \Delta v_S(x) + f(x, v_S(x)) + k(x, x \cdot \nabla v_S(x)) &\leq \Delta v_S(x) + c\|x\|^{-n-1} + D\|x\|^{-1-n} \\ &\leq \Delta v_S(x) + \tilde{f}_S(\|x\|) = 0, \end{aligned}$$

where the last equality follows from (2.15). Since $v_S(x)|_{\partial\Omega_R} = S$ follows from (2.11), v_S is a positive supersolution of (2.1), which completes the proof. $\square$

With the construction of a supersolution to (2.1) finalized, we formulate the following proposition regarding the subsolutions.

Proposition 2.7. *Assume (A1) and (A2). Then for all $\delta > 0$, function $w = \delta\|x\|^{-p}$ is a positive subsolution of (2.1). In particular, for every $S \in (A, B]$, there exists the subsolution $w_S(x) = \delta_S\|x\|^{-p}$ given with $\delta_S = z_S(1 - R^{2-n})R^p$, such that $w_S(x) \leq v_S(x)$ in Ω_R .*

Proof. Take $-g(x) \geq -\frac{p+2-n}{\|x\|^2}$, $k(x, \tau) \geq g(x)\tau$, $p > 2n - 4$ and $\delta > 0$. It is obvious that $x \cdot \nabla w(x) = -\delta p\|x\|^{-p}$ and $\Delta w(x) = \delta p(p+2-n)\|x\|^{-p-2}$. Taking into account $f \geq 0$ and the above inequalities and formulas, we estimate

$$\begin{aligned} \Delta w(x) + k(x, x \cdot \nabla w(x)) + f(x, w(x)) &\geq \Delta w(x) + g(x)x \cdot \nabla w(x) \\ &= \delta p(p+2-n)\|x\|^{-p-2} + [-g(x)]\delta p\|x\|^{-p} \\ &\geq \delta p(p+2-n)\|x\|^{-p-2} - \frac{p+2-n}{\|x\|^2}\delta p\|x\|^{-p} = 0. \end{aligned}$$

Now we fix $S \in (A, B]$ and consider $w_S(x)$ and $v_S(x)$ in Ω_R . It is obvious that positivity of $w_S(x) = z_S(1 - R^{2-n})R^p\|x\|^{-p}$, where $R > 1$ follows from the positivity of $z_S(t)$ for any $t \in [0, 1]$. Our task is to show $w_S(x) \leq v_S(x)$ in Ω_R . Indeed, by the virtue of (2.10), formula for $\tilde{f}_S$, properties of $v_S(x)$ and $w_S(x)$, we obtain

$$-\Delta[v_S(x) - w_S(x)] = -\Delta v_S(x) + \Delta w_S(x) \geq D\|x\|^{-n-1} + \delta_S p(p+2-n)\|x\|^{-p-2} \geq 0,$$

where the last inequality follows from $p > 2n - 4$ and $n > 2$. In the next step, consider $x \in \mathbb{R}^n$ such that $\|x\| = R$. Since $v_S(x) = z_S(1 - \|x\|^{2-n})$ in Ω_1 , in particular in Ω_R , the additional assumption for δ_S implies the following for each x such that $\|x\| = R$

$$v_S(x) - w_S(x) = z_S(1 - \|x\|^{2-n}) - \delta_S\|x\|^{-p} = z_S(1 - R^{2-n}) - \delta_S R^{-p} = 0.$$

Moreover, by the properties of v_S and w_S , we have

$$\lim_{\|x\| \rightarrow \infty} [v_S(x) - w_S(x)] = 0.$$

Therefore, the Maximum Principle leads to the conclusion that $v_S - w_S \geq 0$ in Ω_R . In particular $w_S = v_S = S$ when $\|x\| = R$. $\square$

Having the supersolution and subsolution of the equation in (2.1) constructed and relation between them established, we can formulate main theorem of this part.

Theorem 2.8. *Assume that (A1) and (A2) hold and let A and B be defined as in (2.5) and (2.6). Then, for every $S \in (A, B]$, problem (2.1) possesses at least one solution.*

Proof. Fix $S \in (A, B]$ and take v_S and w_S as given in Propositions 2.6 and 2.7 respectively. The Noussair-Swanson Lemma yields us the existence of solution u_S to the equation in (2.1), such that

$$\delta_S\|x\|^{-p} \leq u_S \leq \frac{1}{(n-2)^2} \int_{1-\|x\|^{2-n}}^1 \int_0^q (1-\xi)^{\frac{2n-2}{2-n}} \tilde{f}_S((1-\xi)^{\frac{1}{2-n}}) d\xi dq.$$

Moreover, by the last assertion of that Lemma, we obtain $u_S = v_S$ when $\|x\| = R$, thus $u_S(x) = S$ on the boundary of the ball. Therefore u_S is the solution of (2.1), which proves Theorem 2.8. $\square$

The above Theorem yield us the location of solution u_S , which will be visualized in the figures below with parameters $n = 3$, $R = 1.2$, $p = 3$, $a = 0.4$, $c = 0.8$, $d = 1$ and $D = 1$ which in turn yields $A = \frac{7}{8}$ and $B = \frac{7}{6}$. Therefore we take $S = 1$. Note that the choice of $S = 1$ may still be available by taking different values of the above parameters.

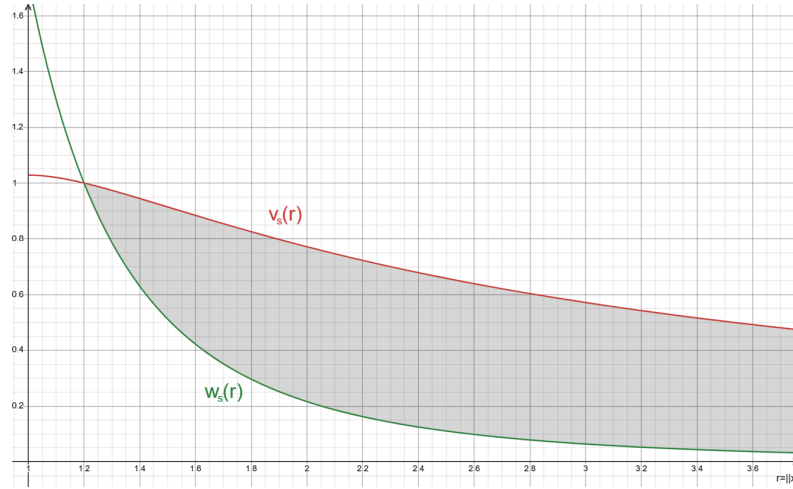


FIGURE 1. Location of the solution. $u_S(x)$ exists in the shaded area. Note that $v_S(1.2) = w_S(1.2) = S = 1$ which follows from taking $R = 1.2$. $w_S(r) > v_S(r)$ for $r < 1.2$.



FIGURE 2. Horizontal axis is shown in logarithmical scale.

2.2. Implications for Constantin's problem. As a consequence of the previous subsection, we acquire quantitative results for the problem without additional boundary condition, that is

$$\begin{aligned} \Delta u(x) + f(x, u(x)) + k(x, x \cdot \nabla u(x)) &= 0 \quad \text{for } x \in \Omega_R, \\ \lim_{\|x\| \rightarrow \infty} u(x) &= 0, \end{aligned} \tag{2.19}$$

with assumptions (A1) and (A2) held.

The first step is to describe auxiliary linear problems analogous to (2.10) that will generate supersolutions of (2.19). To this effect we can now consider c_i with formula $c_i := \frac{[2(n-2)d-c]i}{2j}$,

where $i \in \{1, 2, \dots, j\}$, $j \in \mathbb{N}$, $\beta \in (0, 1]$ and c and d given in (A1). We follow ideas presented in [22].

Remark 2.9. Let j be a fixed element of $\mathbb{N}$ and $\beta \in (0, 1]$. There exist an uncountable set of finite sequences consist of continuous functions $\tilde{f}_i^\beta : [1, +\infty) \rightarrow (0, +\infty)$, $i = 1, 2, \dots, j$ such that in each of sequences the following inequalities hold

$$cr^{-n-1} < \tilde{f}_i^\beta(r) < \tilde{f}_{i+1}^\beta(r) \quad \text{for } i = 1, 2, \dots, j-1 \quad (2.20)$$

and

$$\int_1^\infty r^{n-1} \tilde{f}_i^\beta(r) dr < 2d(n-2) + D, \quad \text{for } i = 1, 2, \dots, j. \quad (2.21)$$

Proof. Fix $\beta \in (0, 1]$. It is sufficient to consider $\tilde{f}_i^\beta(r) = (c + D + c_i\beta)r^{-n-1}$ with $c_i = \frac{[2(n-2)d-c]i}{2j}$ for any $i \in \{1, 2, \dots, j\}$ and D given in (A2). The arbitrariness of β guarantees an uncountable set of such sequences. In [15], the process with one finite sequence $\tilde{f}_i^\beta(r) = (c + D + c_i)r^{-n-1}$, that is where $\beta = 1$ was presented. $\square$

For $\beta \in (0, 1]$, we look for supersolutions v_i^β given analogously to v_S , as radial solutions of j auxiliary linear elliptic equations:

$$\begin{aligned} -\Delta v(x) &= \tilde{f}_i^\beta(\|x\|) \quad \text{for } x \in \Omega_1, \\ \lim_{\|x\| \rightarrow \infty} v(x) &= 0. \end{aligned} \quad (2.22)$$

Proposition 2.10. Fix $\beta \in (0, 1]$. Then

$$v_i^\beta(x) = \frac{1}{(n-2)^2} \int_{1-\|x\|^{2-n}}^1 \int_0^q (1-\xi)^{\frac{2n-2}{2-n}} \tilde{f}_i^\beta((1-\xi)^{\frac{1}{2-n}}) d\xi dq$$

is a radial solution of (2.22) for each $i = 1, 2, \dots, j$ that satisfies

$$0 < v_i^\beta(x) < 2d + \frac{D}{(n-2)} \quad \text{for } \|x\| > 1, \quad (2.23)$$

$$-(2d(n-2) + D)\|x\|^{2-n} \leq x \cdot \nabla v_i^\beta(x) \leq 0 \quad \text{for } \|x\| > 1, \quad (2.24)$$

$$0 < v_1^\beta(t) < \dots < v_i^\beta(t) < v_{i+1}^\beta(t) < \dots < v_{j-1}^\beta(t) < v_j^\beta(t). \quad (2.25)$$

Moreover, v_i^β is a solution to (2.19).

Proof. Fix $\beta \in (0, 1]$ and $i \in \{1, 2, \dots, j\}$. That $-\Delta v(x) = \tilde{f}_i^\beta(\|x\|)$ follows analogously to the proof of Proposition 2.6. Indeed, by substitution $t = 1 - \|x\|^{2-n}$, we obtain

$$\begin{aligned} h_i^\beta(t) &:= \frac{1}{(n-2)^2} (1-t)^{\frac{2n-2}{2-n}} \tilde{f}_i^\beta((1-t)^{\frac{1}{2-n}}), \\ z_i^\beta(t) &:= \int_t^1 \left[\int_0^q h_i^\beta(\xi) d\xi \right] dq, \\ v_i^\beta(x) &= \int_{1-\|x\|^{2-n}}^1 \left[\int_0^q h_i^\beta(\xi) d\xi \right] dq = z_i^\beta(1 - \|x\|^{2-n}) \quad \text{in } \Omega_1. \end{aligned} \quad (2.26)$$

Thus, properties of h_i^β and z_i^β follow directly from properties of h_S and z_S , in particular

$$0 < z_i^\beta(t) \text{ in } [0, 1) \quad \text{and} \quad z_i^\beta(t) < 2d + \frac{D}{(n-2)} \text{ in } [0, 1], \quad (2.27)$$

$$-\left(2d + \frac{D}{(n-2)}\right) < (z_i^\beta)'(t) \text{ in } (0, 1) \quad \text{and} \quad (z_i^\beta)'(t) < 0 \text{ in } (0, 1]. \quad (2.28)$$

To prove that for each $t \in (0, 1)$, we have

$$z_1^\beta(t) < \dots < z_i^\beta(t) < z_{i+1}^\beta(t) < \dots < z_{j-1}^\beta(t) < z_j^\beta(t), \quad (2.29)$$

we notice that $(z_i^\beta)(t) = \int_t^1 [\int_0^q h_i^\beta(\xi) d\xi] dq$ implies $z_i^\beta(1) = 0$, $(z_i^\beta)'(t) = -\int_0^t h_i^\beta(\xi) d\xi$ with $(z_i^\beta)'(0) = 0$ and $(z_i^\beta)''(t) = -h_i^\beta(t)$ for any $t \in [0, 1]$. Moreover, the formula for $h_i^\beta(t)$ leads to the conclusion that $h_1^\beta(t) < h_2^\beta(t) < \dots < h_j^\beta(t)$ for all $t \in (0, 1)$. Therefore, for each $i \in \{1, 2, \dots, j-1\}$, we have

$$-(z_{i+1}^\beta(t) - z_i^\beta(t))'' = h_{i+1}^\beta(t) - h_i^\beta(t) > 0,$$

which means that $(z_{i+1}^\beta(\cdot) - z_i^\beta(\cdot))'$ is a strictly decreasing function. By $((z_{i+1}^\beta(0) - z_i^\beta(0))' = 0$ and the continuity of $z(\cdot)$, we obtain that $(z_{i+1}^\beta(\cdot) - z_i^\beta(\cdot))' < 0$ for any $t \in (0, 1)$, thus $(z_{i+1}^\beta(\cdot) - z_i^\beta(\cdot))$ is strictly decreasing as well. Combining this with the fact that $((z_{i+1}^\beta(1) - z_i^\beta(1)) = 0$ we obtain $z_{i+1}^\beta(t) - z_i^\beta(t) > 0$ in $(0, 1)$ thus proving (2.29).

From $v_i^\beta(x) = z_i^\beta(1 - \|x\|^{2-n})$ in Ω_1 , estimates (2.23) and (2.25) follow directly from (2.27) and (2.29) respectively. While (2.24) can be proven analogously to (2.13) using the estimates in (2.28). Combining this fact with conditions in the assumption (A2), we conclude that $k(x, x \cdot \nabla v_i^\beta(x)) \leq D\|x\|^{-1-n}$ and further, by analogous reasoning to the one in previous subsection, we obtain

$$\begin{aligned} \Delta v_i^\beta(x) + f(x, v_i^\beta(x)) + k(x, x \cdot \nabla v_i^\beta(x)) &\leq \Delta v_i^\beta(x) + c\|x\|^{-n-1} + D\|x\|^{-1-n} \\ &\leq \Delta v_i^\beta(x) + \tilde{f}_i^\beta(\|x\|) = 0. \end{aligned} \quad (2.30)$$

In consequence, we derive that v_i^β satisfies the definition of super solution to (2.19). $\square$

The main result in this part is the following theorem.

Theorem 2.11. *Let $i \in \{1, 2, \dots, j\}$ and $\beta \in (0, 1]$. If k and f are functions that satisfy (A1) and (A2), then the problem (2.19) possesses j non-trivial solutions u_i^β in Ω_R such that $u_i^\beta(x) \leq v_i^\beta(x)$ for $x \in \overline{\Omega}_R$. In particular $u_i^\beta(x) = v_i^\beta(x)$ when $\|x\| = R$. Furthermore, for any $i = 1, 2, \dots, j-1$, $u_i^\beta(x) \leq u_{i+1}^\beta(x)$ in $\overline{\Omega}_R$ and $u_i^\beta(x) < u_{i+1}^\beta(x)$ on $\partial\overline{\Omega}_R$.*

Proof. Fix $\beta \in (0, 1]$. We can construct the first solution by taking the positive supersolution v_1^β and the trivial subsolution $w_0 \equiv 0$ in $\overline{\Omega}_R$. It is obvious that $w_0 < v_1^\beta$, thus by the Noussair-Swanson Lemma, we obtain the solution u_1^β to the problem (2.19) in Ω_R that satisfies $0 \leq u_1^\beta(x) \leq v_1^\beta(x)$ for any $x \in \overline{\Omega}_R$. By the last assertion of this Lemma, we obtain $u_1^\beta(x) = v_1^\beta(x)$ for $x \in \partial\overline{\Omega}_R$, a crucial equality in construction of strictly increasing sequence of solutions. Moreover the continuity of u_1^β in $\overline{\Omega}_R$ implies that for some $\varepsilon > 0$ the inequality $u_1^\beta(x) > 0$ is satisfied for $R < \|x\| < R + \varepsilon$. Thus u_1^β itself is not a trivial solution.

Now, we can treat u_1^β as a non-negative subsolution to (2.19). Taking v_2^β , that is, another positive supersolution of (2.19), we note that $u_1^\beta \leq v_1^\beta < v_2^\beta$ for any $x \in \overline{\Omega}_R$. Thus we may apply Noussair-Swanson lemma again to get another solution u_2^β to (2.19) in Ω_R such that $u_1^\beta(x) \leq u_2^\beta(x) \leq v_2^\beta(x)$ for $x \in \overline{\Omega}_R$ and $u_2^\beta(x) = v_2^\beta(x)$ for all $x \in \partial\overline{\Omega}_R$. Analogously to u_1^β , u_2^β is not a trivial solution. Moreover, $u_1^\beta(x) = v_1^\beta(x) < v_2^\beta(x) = u_2^\beta(x)$ for all $x \in \partial\overline{\Omega}_R$ implies $u_1^\beta \neq u_2^\beta$. Let us note that we may continue this process by taking u_2^β as a non-negative subsolution of (2.19) and v_3^β as positive supersolution. Moreover $u_2^\beta \leq v_2^\beta < v_3^\beta$ in Ω_R , thus another application of Noussair-Swanson lemma yields us a non-trivial solution u_3^β such that $u_2^\beta(x) \leq u_3^\beta(x) \leq v_3^\beta(x)$ for $x \in \overline{\Omega}_R$. In particular $u_1^\beta(x) < u_2^\beta(x) = v_2^\beta < v_3^\beta = u_3^\beta$ for all $x \in \partial\overline{\Omega}_R$.

Let us note that we can repeat this process analogously to show the following pattern for $i = 2, 3, \dots, j-1$. Having a non-negative, non-trivial solution u_i^β such that $u_i^\beta \leq v_i^\beta$ in $\overline{\Omega}_R$ and $u_i^\beta = v_i^\beta$ for $x \in \partial\overline{\Omega}_R$ constructed, we may treat it as a subsolution of (2.19). Next, we can compare u_i^β with the positive supersolution v_{i+1}^β . Due to the fact that supersolutions create a strictly increasing sequence for our fixed β , we obtain $u_i^\beta \leq v_i^\beta < v_{i+1}^\beta$ for any $x \in \overline{\Omega}_R$. Applying the Noussair-Swanson lemma yields us the existence of a non-trivial solution u_{i+1}^β to the problem (2.19) in Ω_R , such that $u_i^\beta(x) \leq u_{i+1}^\beta(x) \leq v_{i+1}^\beta(x)$ for any $x \in \overline{\Omega}_R$ and $u_{i+1}^\beta(x) = v_{i+1}^\beta(x) > v_i^\beta(x) = u_i^\beta(x)$, for $x \in \partial\overline{\Omega}_R$, thus solutions themselves create strictly increasing finite sequence.

We can choose any other $\beta \in (0, 1]$ and start over to generate finite sequence of solutions u_i^β such that for any $i = 2, 3, \dots, j-1$ we have $u_i^\beta(x) \leq u_{i+1}^\beta(x) \leq v_{i+1}^\beta(x)$ for every $x \in \overline{\Omega_R}$ and $u_i^\beta(x) = v_i^\beta(x) < v_{i+1}^\beta(x) = u_{i+1}^\beta(x)$ for every $x \in \partial\overline{\Omega_R}$. $\square$

3. PROBLEM WITH SINGULAR PART

3.1. Constantin's problem with a singularity - formulation. Now we analyze the problem

$$\begin{aligned} \Delta u(x) + f(x, u(x)) - b(x)(u(x))^{-1} \|\nabla u(x)\|^2 + k(x, x \cdot \nabla u(x)) &= 0 \quad \text{for } x \in \Omega_R \\ \lim_{\|x\| \rightarrow +\infty} u(x) &= 0, \end{aligned} \quad (3.1)$$

where $b > 0$. By the solution of (3.1) we understand a function u such that for every bounded M , $\overline{M} \subset \Omega_R$, $u \in C^{2+\gamma}(\overline{M})$, $\gamma \in (0, 1)$, and u satisfies (3.1). In this part we consider f , b , k and g satisfying the following conditions:

(A1') $f : \Omega_1 \times \mathbb{R} \rightarrow [0, +\infty)$ is a locally Hölder-continuous function with an exponent $\gamma \in (0, 1)$, where $\Omega_1 = \{x \in \mathbb{R}^n, \|x\| > 1\}$, $n > 2$, $f(x, \cdot)$ is of C^1 class and there exist constants $d > 0$ and $0 < c < 2d(n-2)$ such that for all $r \in (1, +\infty)$,

$$\sup_{u \in [0, 4d]\|x\|=r} \sup f(x, u) \leq cr^{-n-1}. \quad (3.2)$$

(A2') $k : \Omega_1 \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally Hölder-continuous function with an exponent $\gamma \in (0, 1)$, $k(x, \cdot)$ is of C^1 class, $k(x, 0) = 0$ on Ω_1 and

- (1) for any bounded subset $M \subset \Omega_1$ there exists a constant $C > 0$ such that for all $x \in \overline{M}$ and $\tau \in \mathbb{R}$, $|k(x, \tau)| \leq C(1 + \tau^2)$,
- (2) there exist $g : \Omega_1 \rightarrow \mathbb{R}$ of C^1 class, constants $D \in (0, 2d(n-2))$, $a \in (0, \frac{1}{2})$ and $p > 2n-4$ such that for $\tau \in [-(2d(n-2) + D)\|x\|^{2-n}, 0]$ and $x \in \Omega_1$, we have

$$\begin{aligned} -a^2\|x\|^{-3} \leq g(x) &\leq \frac{\frac{p}{2} + 2 - n}{\|x\|^2}, \\ k(x, \tau) &\geq g(x)\tau, \\ k(x, \tau) &\leq D\|x\|^{-1-n}. \end{aligned}$$

(A3) $b : \Omega_1 \rightarrow (0, +\infty)$ is a locally Hölder-continuous function with an exponent $\gamma \in (0, 1)$.

3.2. Solutions on bounded domains. In this part, we consider the equation

$$\Delta u(x) + f(x, u(x)) - b(x)(u(x))^{-1} \|\nabla u(x)\|^2 + k(x, x \cdot \nabla u(x)) = 0 \quad (3.3)$$

in some arbitrary annulus in Ω_R . We apply the results obtained by Cui [10, lemma 3].

Lemma 3.1. *Let (A1'), (A2'), (A3) hold and consider a bounded set $\Omega \subset \Omega_R$ with smooth boundary. If there exist a positive supersolution $v \in C(\overline{\Omega}) \cap C^2(\Omega)$ and a positive subsolution $w \in C(\overline{\Omega}) \cap C^2(\Omega)$ of (3.3) such that $w(x) \leq v(x)$ for $x \in \overline{\Omega}$, then (3.3) has a solution $u \in C^{2+\gamma}(\Omega)$ in Ω such that $w(x) \leq u(x) \leq v(x)$ for $x \in \Omega$.*

Starting from now, for each $L \in \mathbb{N}$, let us define $A_L := \{x \in \mathbb{R}^n, R + \frac{1}{L+1} < \|x\| < R + L\}$.

Lemma 3.2. *Let (A1'), (A2'), (A3) hold, and assume $b(x) \leq \frac{1}{2}$. For each $L \in \mathbb{N}$ and each $\beta \in (0, 1]$ there exists a solution u_L^β of (3.3) in $A_L \subset \Omega_R$, such that $\delta_0^\beta \|x\|^{-p} \leq u_L^\beta \leq v_0^\beta$ for all $x \in \overline{A_L}$ with $0 < \delta_0^\beta = \frac{z_1^\beta(1-R^{2-n})}{R^{-p}}$ and z_1^β given in (2.14).*

Proof. Fix $L \in \mathbb{N}$ and $\beta \in (0, 1]$. Let us consider the first super solution v_1^β of the problem without the singular part given in the Proposition 2.10. Taking (2.30) and the assumption $b(x) > 0$ into account, we obtain for $v_0^\beta = v_1^\beta$, the inequality

$$\begin{aligned} \Delta v_0^\beta(x) + f(x, v_0^\beta(x)) - b(x)(v_0^\beta(x))^{-1} \|\nabla v_0^\beta(x)\|^2 + k(x, x \cdot \nabla v_0^\beta(x)) \\ \leq \Delta v_1^\beta(x) + f(x, v_1^\beta(x)) + k(x, x \cdot \nabla v_1^\beta(x)) \leq 0. \end{aligned} \quad (3.4)$$

Thus v_0^β is a positive supersolution of the equation given in (3.3) in each subset of Ω_R and $\lim_{\|x\| \rightarrow +\infty} v_0^\beta(x) = 0$.

Let us also consider $w_0^\beta := \delta_0^\beta \|x\|^{-p}$ given analogously to the subsolution w described in Proposition 2.7 and additionally assume $b(x) \leq \frac{1}{2}$. We recall that $p > 2n - 4$, $x \cdot \nabla w_0^\beta(x) = -\delta_0^\beta p \|x\|^{-p}$, $\Delta w_0^\beta(x) = \delta_0^\beta p(p+2-n) \|x\|^{-p-2}$ and we calculate $\|\nabla w_0^\beta(x)\|^2 = (\delta_0^\beta)^2 p^2 \|x\|^{-2p-2}$. Taking $\delta_0^\beta > 0$, (A3) and all of above inequalities and formulas into account, we estimate

$$\begin{aligned} & \Delta w_0^\beta(x) + k(x, x \cdot \nabla w_0^\beta(x)) + f(x, w_0^\beta(x)) - b(x)(w_0^\beta(x))^{-1} \|\nabla w_0^\beta(x)\|^2 \\ & \geq \Delta w_0^\beta(x) + g(x)x \cdot \nabla w_0^\beta(x) - b(x)(w_0^\beta(x))^{-1} \|\nabla w_0^\beta(x)\|^2 \\ & = \delta_0^\beta p(p+2-n) \|x\|^{-p-2} + [-g(x)]\delta_0^\beta p \|x\|^{-p} + [-b(x)]((\delta_0^\beta)^{-1} \|x\|^p)[(\delta_0^\beta)^2 p^2 \|x\|^{-2p-2}] \\ & \geq \delta_0^\beta p(p+2-n) \|x\|^{-p-2} - \frac{\frac{p}{2} + 2 - n}{\|x\|^2} \delta_0^\beta p \|x\|^{-p} - \frac{1}{2} \delta_0^\beta p^2 \|x\|^{-p-2} \\ & = \delta_0^\beta p \|x\|^{-p-2} [p+2-n - \frac{p}{2} - 2 + n - \frac{p}{2}] = 0. \end{aligned}$$

Thus, w_0^β provides us a positive subsolution of the equation given in (3.3) in each subset of Ω_R and $\lim_{\|x\| \rightarrow +\infty} w_0^\beta(x) = 0$. Analogously to the proof in the part without singularity, the Maximum Principle leads to the conclusion that $v_0^\beta - w_0^\beta \geq 0$ in Ω_R . In particular, $v_0^\beta(x) \geq w_0^\beta(x)$ in any $\overline{A}_L \subset \Omega_R$. Bearing Lemma 3.1 in mind, we obtain the existence of a solution u_L^β of the equation (3.3) in A_L , such that $w_0^\beta \leq u_L^\beta \leq v_0^\beta$ for all $x \in \overline{A}_L$. In particular $w_0^\beta = u_L^\beta = v_0^\beta$ when $\|x\| = R$. $\square$

Remark 3.3. Note that the estimates $g(x) \leq \frac{\frac{p}{2}+2-n}{\|x\|^2}$ and $b \leq \frac{1}{2}$ were chosen in such a way that during the calculation whether w_0^β satisfies the definition of subsolution, we obtained the expression $p+2-n - \frac{p}{2} - 2 + n - \frac{p}{2} = 0$. Now, we can give more general estimates for both $g(x)$ and $b(x)$. In particular, let us introduce constant $Q > 0$, such that $g(x) \leq \frac{Qb(x)}{\|x\|^2}$. If $p > 2n - 4$ and $b(x) \leq \frac{p+2-n}{Q+p}$, then for any $\beta \in (0, 1]$, w_0^β satisfies the definition of subsolution of the equation given in (3.3). Note that if $Q \in (0, p+2n-4)$, then $\frac{p+2-n}{Q+p} \in (\frac{1}{2}, 1)$, which expands the upper estimate for b in comparison to Lemma 3.2.

Proof. Indeed, by taking $g(x) \leq \frac{Qb(x)}{\|x\|^2}$, $b(x) \leq \frac{p+2-n}{Q+p}$ and following the calculations in the proof of Lemma 3.2, we obtain

$$\begin{aligned} & \delta_0^\beta p(p+2-n) \|x\|^{-p-2} + [-g(x)]\delta_0^\beta p \|x\|^{-p} + [-b(x)]((\delta_0^\beta)^{-1} \|x\|^p)[(\delta_0^\beta)^2 p^2 \|x\|^{-2p-2}] \\ & \geq \delta_0^\beta p(p+2-n) \|x\|^{-p-2} + [-Qb(x)]\delta_0^\beta p \|x\|^{-p-2} + [-b(x)p]\delta_0^\beta p \|x\|^{-p-2} \\ & = \delta_0^\beta p \|x\|^{-p-2} [(p+2-n) - (Q+p)b(x)] \geq 0. \end{aligned} \quad \square$$

3.3. Solutions on exterior domains. Having a solution in the bounded domain constructed, we can start the application of classical procedure that is based on the domain approximation method and estimates of solutions for elliptic equations to obtain existence results in unbounded domain (see e.g. [10, 19, 21]). Here, the unbounded domain is exterior of the ball and we investigate the case when we established lower and upper bounds for solutions.

Theorem 3.4. *Let (A1'), (A2'), (A3) hold. Assume additionally that for all $x \in \Omega_R$, $b(x) \leq \frac{1}{2}$ and $g(x) \leq 0$. Then for any $\beta \in (0, 1]$, there exists a solution u_0^β of (3.1) such that*

$$\begin{aligned} & \delta_0^\beta \|x\|^{-p} \leq u_0^\beta(x) \leq v_0^\beta(x) \quad \text{in } x \in \Omega_R, \\ & u_0^\beta(x) = O(\|x\|^{2-n}) \quad \text{when } \|x\| \rightarrow +\infty. \end{aligned} \quad (3.5)$$

Proof. Fix $\beta \in (0, 1]$. For any $i \in \mathbb{N}$, let us consider the following annulus $M_i := \{x \in \mathbb{R}^n, R + \frac{1}{i+1} < \|x\| < R + i\}$. Lemma 3.2 yields us the existence of at least one positive solution u_i^β in annulus M_i , such that $0 < w_0^\beta(x) \leq u_i^\beta(x) \leq v_0^\beta(x)$ for any $x \in \overline{M}_i$. Apart from being supersolution of the

problem in each M_i , v_0^β plays the other crucial role. It provides the upper bound for the sequence $\{u_j\}_{j \geq i}$ in each M_i , thus we can apply the classical procedure (see, among others, [7, Lemma 2.3], [19, Theorem 3.3], [21, Theorem 4.1]), to get that for a given $m \geq 1$, there exists a positive constant K such that for all $i \geq m$

$$\|u_i^\beta\|_{C^{2+\gamma}(\overline{M}_m)} \leq K. \quad (3.6)$$

In the next step, let us take $m = 1$. By (3.6) we see that $\|u_i^\beta\|_{C^{2+\gamma}(\overline{M}_1)} \leq K_1$ for all $i \geq 1$. Therefore, by compactness of injection $C^{2+\gamma}(\overline{M}_1)$ into $C^2(\overline{M}_1)$, we obtain a subsequence $\{u_{i_p}\}_{p \in \mathbb{N}}$ tending to some u^1 in $C^2(M_1)$. Thus u^1 satisfies the equation from (3.1) in M_1 . Further, we consider annulus M_2 and notice that estimation (3.6) holds for this subsequence. By analogous reasoning to the one for $m = 1$, we obtain the existence of $u^2 \in C^2(M_2)$ satisfying the equation in M_2 . Repeated application of this iterative procedure finally yields us the sequence $\{u^m\}_{m \in \mathbb{N}}$, where u^m satisfies the equation from (3.1) in M_m for $m = 1, 2, \dots$, and

$$u^{m+1} = u^m \quad \text{in } M_m. \quad (3.7)$$

Now, let $u_0^\beta(x) := u^m(x)$ for any $x \in M_m$. Existence of the assertion (3.7) implies that u_0^β must be well-defined and shows us that u_0^β satisfies the equation in (3.1) at each point of Ω_R . Moreover, for any bounded set $\overline{M} \subset \Omega_R$, there exists $m \in \mathbb{N}$, such that $\overline{M} \subset M_m$, and further, $u_0^\beta|_{\overline{M}} = u^m$. By applying standard regularity argument based on the Schauder-type estimates, we obtain that $u_0^\beta \in C^{2+\gamma}(\overline{M})$ and finally, by the estimates for each u_i^β , $i \in \mathbb{N}$, we obtain the required inequalities for the solution u_0^β . $\square$

3.4. Problem with singularity and additional boundary condition. With the results from the previous subsection established, we can present the following corollary concerning the singular boundary value problem.

Corollary 3.5. *Let $S \in (A, B]$ and (A1'), (A2'), (A3) hold. Then, there exists $u_{0_S}(x)$ that is the solution to (1.4), namely the problem*

$$\begin{aligned} \Delta u(x) + f(x, u(x)) - b(x)(u(x))^{-1} \|\nabla u(x)\|^2 + k(x, x \cdot \nabla u(x)) &= 0, \quad x \in \Omega_R, \\ u(x) &= S \quad \text{when } \|x\| = R, \\ \lim_{\|x\| \rightarrow \infty} u(x) &= 0. \end{aligned}$$

Proof. Fix $S \in (A, B]$. We can now choose $v_0 := v_S$ and $w_0 = w_S$, described in Propositions 2.6 and 2.7 respectively, as the super and subsolution of (3.3) and repeat all of procedures from subsections 3.2 and 3.3. Lemma 3.2 and the selection of $v_0 = v_S$ imply that $u_{0_S} = v_0 = v_S$ when $\|x\| = R$, which yields us $u_{0_S}(x) = S$ for $\|x\| = R$. Thus $u_{0_S}(x)$ is the solution to (1.4). $\square$

Example 3.6. Let $S \in (A, B]$. We can apply our results to the problem

$$\begin{aligned} \Delta u(x) + \frac{\ln(u+2)(x_1+x_3)^4}{11\|x\|^8} + \frac{u^3(x_2+x_3)^2}{3(u+2)\|x\|^6} \\ - (u(x))^{-1} \|\nabla u(x)\|^2 + \left[\frac{(\sin(x_2))^2}{(2+x_1^2)\|x\|^2} - \frac{a^2}{\|x\|^4} \right] x \cdot \nabla u(x) &= 0 \quad \text{in } \Omega_R, \\ u(x) &= S, \quad \text{for } \|x\| = R, \\ \lim_{\|x\| \rightarrow \infty} u(x) &= 0, \end{aligned}$$

where $\Omega_R := \{x \in \mathbb{R}^3, \|x\| > R > 1\}$, $n = 3$, $c = 0.8$, $a \in (0, \frac{1}{2})$.

Proof. First, take $d = 1$, $c = 0.8$, $n = 3$, $D = 2ad(n-2) = 2a$ and $p \geq 3$. In this case we calculate

$$\begin{aligned} f(x, u) &= \frac{\ln(u+2)(x_1+x_3)^4}{11\|x\|^8} + \frac{u^3(x_2+x_3)^2}{3(u+2)\|x\|^6}, \\ A &= (c+D) \frac{R^{1-n}[R(n-1) - (n-2)]}{(n-1)(n-2)} = (2a+0.8) \frac{2R-1}{2R^2}, \end{aligned}$$

$$B = (D + \frac{2(n-2)d+c}{2}) \frac{R^{1-n}[R(n-1)-(n-2)]}{(n-1)(n-2)} = (2a+1.4) \frac{2R-1}{2R^2} \leq 1.2.$$

To show that (A1') holds, we claim that $f(x, u(x)) \leq c\|x\|^{-n-1} = 0.8\|x\|^{-4}$. We estimate

$$f(x, u(x)) \leq \frac{\ln(3.2)(x_1 + x_3)^4}{11\|x\|^8} + \frac{1.2^3(x_2 + x_3)^2}{9.6\|x\|^6} \leq 0.8\|x\|^{-4}.$$

Now, take

$$k(x, x \cdot \nabla u(x)) = \left[\frac{(\sin(x_2))^2}{(2+x_1^2)\|x\|^2} - \frac{a^2}{\|x\|^4} \right] x \cdot \nabla u(x).$$

We can consider such function as a case where $k(x, x \cdot \nabla u(x)) = g(x)x \cdot \nabla u(x)$. Finally, it is obvious that for $p \geq 3$, $g(x)$ given as

$$g(x) = \frac{(\sin(x_2))^2}{(2+x_1^2)\|x\|^2} - \frac{a^2}{\|x\|^4}$$

satisfies

$$-\frac{a^2}{\|x\|^3} \leq g(x) \leq \frac{\frac{p}{2}-1}{\|x\|^2},$$

which completes the proof that k satisfies (A2'). $\square$

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