

NONLOCAL MECHANISMS LEADING TO INSTABILITY IN DRY-LAND VEGETATION MODELS

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ABSTRACT. We consider a system of partial differential equations arising in vegetation dynamics. Roots interactions are modeled through nonlocal reaction terms which, in the limit of negligible roots extension, collapse to local reaction terms. We study, by means of linear stability analysis and computational simulation, the emergence of pattern formation due to both Turing instability and nonlocal effects. We show that, in the physical reasonable parameter range, interchangeability of local and nonlocal models is not justified, since their corresponding patterns differ substantially.

1. INTRODUCTION

Nonlocal reaction-diffusion equations has been an active area of mathematical research in the previous decades [2, 8, 13, 29]. Specially in areas like mathematical biology, the modeling process leads in a natural manner to the consideration of nonlocal terms, which may appear in both the diffusion and the reaction parts of the equations [3, 5, 17, 18, 20, 32, 34]. Typically, this type of terms arise when interactions among individuals or substances may be produced at a distance.

The usual representation of nonlocal terms is through integration of the variables of interest against a given *kernel*. And normally, upon rescaling, the kernel may be identified with a Dirac delta when the radius of interaction tends to zero, recovering in this limit a local reaction-diffusion equation.

Since, in general, the mathematical analysis and numerical discretization of nonlocal models is more complex than that of their local counterparts, the question of whether the replacement of the former by the latter gives similar predictions -at least qualitatively- or not, arises.

In this article, we study an instance of this general situation, a model for vegetation and soil water distribution in semi-arid lands. The nonlocality is caused by plants roots, which may considerably extend laterally away from the plant shoot. In the limit when roots and shoot may be considered at the same location, the nonlocal model collapses to a local one.

The main contribution of this article is to show that replacing nonlocal source terms by their local counterparts may lead to qualitatively different conclusions in the linear stability analysis. Comparisons between local and nonlocal formulations have already been considered in vegetation models. For instance, Eigentler and Sherratt [15] analyze the linear stability of local and nonlocal versions of a vegetation model and investigate how the nonlocal interaction affects the onset of pattern formation. Their analysis, however, focuses on the relation between the two formulations through the local limit and does not address the possibility of a qualitative change in the stability properties for small but finite interaction ranges. Similarly, Ruiz-Reynés et al. [36] compare local and nonlocal formulations of a model for clonal vegetation and find that the two models exhibit essentially the same qualitative behavior, with the nonlocal interaction mainly producing quantitative changes in the resulting patterns. Other studies, such as [40], have investigated the effects of nonlocal root interactions on pattern formation and Turing-type instabilities, but without

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addressing the specific question of whether the local approximation preserves the linear stability properties of the corresponding nonlocal model in the regime of short-range root interactions.

In contrast, our main interest is in identifying and characterizing parameter regimes in which the two formulations yield substantially different stability predictions. In particular, we show that the nonlocal model may be linearly unstable while the corresponding local model remains linearly stable, even when the spatial extension of the root system is small. Thus, our aim is not simply to compare local and nonlocal vegetation models, but to determine when the local approximation fails to reproduce the linear stability properties of the underlying nonlocal model. To this end, we combine the mathematical analysis of both formulations with computational experiments specifically guided by the analytical results.

We first introduce the models and the general assumptions that allow for a rigorous mathematical treatment. In Section 2, we state our main analytical results and discuss their interpretation in terms of the differences between the local and nonlocal models. Section 3 is devoted to numerical experiments designed to illustrate these differences and, in particular, to identify parameter regimes in which the two models lead to substantially different predictions. In Section 4, we provide the proofs of the analytical results. Finally, in Section 5, we summarize our findings and discuss their implications for the use of local approximations of nonlocal models.

1.1. Models. A defining feature of arid and semi-arid landscapes is the occurrence of vegetation patches forming spatially organized patterns of plants with a wide range of scales and morphologies [12, 26]. These geometrically structured vegetation patterns are the result of plants interactions with environmental and climatic factors which shape the spatial organization of biomass.

A variety of mathematical models have been proposed to elucidate the mechanisms responsible for the formation of these vegetation patterns [14, 26, 30, 31, 35, 37, 39]. In this work, we focus on the model introduced by Gilad et al. [22, 23], see also the monograph by Meron [33].

Gilad's et al. framework is based on several positive feedback processes linking vegetation biomass with water dynamics. Collectively, these feedbacks are incorporated into a system of three coupled nonlinear nonlocal parabolic partial differential equations governing the evolution of biomass, soil water, and surface water.

As pointed out by Meron [33], there are several situations in which the modeling equations are mathematically simplified. For instance, if we consider the spatial domain as a flat landscape, and assume a time scale of the water infiltration rate much shorter than that of the diffusive processes associated to seed dispersion and soil water diffusion, then the surface water equation can be neglected, reducing the problem to a two-component system of equations for biomass, b , and soil water content, w .

In dimensionless form, the resulting nonlocal reaction-diffusion (NLRD) model takes the form

$$\partial_t b - \delta_b \Delta b = \mathcal{F}_b[b, w] \quad \text{in } (0, T) \times \Omega, \quad (1.1)$$

$$\partial_t w - \delta_w \Delta w = \mathcal{F}_w[b, w] \quad \text{in } (0, T) \times \Omega, \quad (1.2)$$

where $T > 0$, $\Omega \subset \mathbb{R}^d$, for $d = 1, 2$ is, typically, a bounded domain, and δ_b, δ_w are the diffusion constants. The nonlocal reaction term $\mathcal{F} = (\mathcal{F}_b, \mathcal{F}_w)$ is defined by

$$\mathcal{F}_b[b, w] = b(\nu \mathcal{G}_b[b, w](1 - b) - 1), \quad (1.3)$$

$$\mathcal{F}_w[b, w] = p - w(\ell(b) + \gamma \mathcal{G}_w[b]). \quad (1.4)$$

Here, the constant $p > 0$ represents the precipitation rate, the function

$$\ell(b) = \frac{\nu}{1 + \rho b}$$

is the water loss rate due to evapo-transpiration, the nonlocal term $\mathcal{G}_b[b, w]$ represents the biomass growth rate coefficient for a logistic function, and $\mathcal{G}_w[b]$ captures the nonlocal water uptake by vegetation roots. The positive constants ν, γ and ρ stand for the soil water evaporation rate, the soil water consumption rate, and the evaporation reduction due to shading, respectively.

The nonlocal terms $\mathcal{G}_b, \mathcal{G}_w$ are specified through a Dirac approximating kernel, $J_\sigma(x)$, by the expressions

$$\begin{aligned}\mathcal{G}_b[\varphi, \psi](x) &= \tilde{s}(\varphi(x)) \int_{\Omega} J_{s(\varphi(x))}(x-y) \psi(y) dy, \\ \mathcal{G}_w[\varphi](x) &= \int_{\Omega} J_{s(\varphi(y))}(x-y) \varphi(y) \tilde{s}(\varphi(y)) dy.\end{aligned}\tag{1.5}$$

The root augmentation in response to biomass growth is modeled by the variable kernel variance

$$s(b) = s_0 \tilde{s}(b), \quad \text{with} \quad \tilde{s}(b) = 1 + \eta b,\tag{1.6}$$

providing a measure of the root-system lateral size. The positive parameter η quantifies the root augmentation per unit biomass, beyond the (minimal) seedling root-system size s_0 .

It is worth noticing that the kernel variance in the growth rate coefficient (1.5) only depends on the point being evaluated (x), while for the root water uptake it varies along the domain (y). This expresses the fact that the vegetation growth coefficient at x only depends on how far the roots at this point develop horizontally, while the water uptake at x may be due to plants growing at other locations y whose roots reach point x .

An additional simplifying situation pointed out by Meron [33], transforming the nonlocal problem to a local problem, is that in which the root expansion is very limited, mathematically, to just below the shoot location ($s_0 = 0$), leading to the local reaction-diffusion (LRD)-model

$$\partial_t b - \delta_b \Delta b = f_b(b, w) \quad \text{in } (0, T) \times \Omega,\tag{1.7}$$

$$\partial_t w - \delta_w \Delta w = f_w(b, w) \quad \text{in } (0, T) \times \Omega,\tag{1.8}$$

where the local reaction terms $f = (f_b, f_w)$ are given by the limit expressions of (1.3)-(1.4) as $s_0 \rightarrow 0$, i.e.

$$f_b(b, w) = b(\nu g_b(b, w)(1 - b) - 1),\tag{1.9}$$

$$f_w(b, w) = p - w(\ell(b) + \gamma g_w(b)),\tag{1.10}$$

being the growth rate $g_b(b, w)$ and the water uptake rate $g_w(b)$ defined as

$$g_b(b, w) = w \tilde{s}(b)^d, \quad g_w(b) = b \tilde{s}(b)^d.$$

We provide some details justifying this passage to the limit in Remark 1.2.

1.2. Uniform equilibria and spatial domain. Since the purpose of this article is to give a detailed account and comparison of the stability properties of uniform equilibria of the local and nonlocal models, it is convenient that both models share the same uniform equilibria.

Uniform equilibria of our local and nonlocal problems are constant points (b_*, w_*) satisfying both the differential equations and the boundary conditions, that we will fix in a moment. Thus, equilibria of the local reaction model must satisfy

$$f_b(b_*, w_*) = f_w(b_*, w_*) = 0,$$

whereas, in the nonlocal case, they must satisfy $\mathcal{F}_b[b_*, w_*] = \mathcal{F}_w[b_*, w_*] = 0$. Since

$$\begin{aligned}\mathcal{G}_b[b_*, w_*](x) &= g_b(b_*, w_*) \int_{\Omega} J_\sigma(x-y) dy, \\ \mathcal{G}_w[b_*](x) &= g_w(b_*) \int_{\Omega} J_\sigma(x-y) dy,\end{aligned}$$

for (b_*, w_*) to be an equilibrium of both the local and nonlocal models it must be verified

$$\int_{\Omega} J_\sigma(x-y) dy = 1 \quad \text{for all } x \in \Omega,$$

which is not true for general Dirac approximations, unless $\Omega = \mathbb{R}^d$.

To overcome this technical difficulty, we resort to periodic boundary conditions. From the mathematical point of view, this is a standard choice which does not introduce particular difficulties when the spatial domain can be expressed as a Cartesian product of intervals. From the biological

point of view, it can be interpreted as an idealization in which the overall domain is regarded as a repetitive landscape and the periodic domain represents a representative cell of that structure.

The use of periodic boundary conditions is also motivated by the preservation of spatially homogeneous equilibria, which facilitates the linear stability analysis. The main qualitative conclusion of our analysis, namely, that the local and nonlocal formulations may exhibit different linear stability properties, results from the different interaction mechanisms and is not expected to depend essentially on the periodic setting. We therefore expect this conclusion to remain relevant for sufficiently large bounded physical domains, although the precise stability thresholds and the resulting spatial patterns may be affected by the boundary conditions.

In particular, for standard non-flow boundary conditions, the boundary generally breaks the spatial homogeneity of the nonlocal interactions, so that the homogeneous equilibria considered here need not remain stationary solutions, see Remark 1.1. For kernels with small variance, these boundary effects are expected to be localized near the boundary, while the dynamics away from the boundary should be less affected. A detailed analysis of these boundary effects and of the corresponding nonuniform stationary states is beyond the scope of the present work.

Since, in addition, spatial dimensionality is not specially relevant in our study, we choose the spatial domain as $\Omega = \mathbb{T}$, the one-dimensional torus of length one, defining implicitly the periodic boundary conditions that complement the nonlocal (NLRD) and local (LRD) models (1.1)-(1.2) and (1.7)-(1.8). Notice that using the change of variables $(\tau, y) \rightarrow (t, x)$, with $\tau = L^2 t$ and $y = Lx$, a general torus of length L may be considered, at the cost of multiplying the reaction terms by the constant L^2 .

Remark 1.1 (Non-flow boundary conditions). We explain the technical reason why we choose the spatial domain as a torus, implying periodic boundary conditions for b and w , instead of the more common assumption of a bounded domain with solutions satisfying non-flow boundary conditions. Replacing $\mathbb{T}$ by any bounded $\Omega \subset \mathbb{R}$ and defining $J_\sigma = Q_\sigma$ (instead of its periodification), we have, for all $\sigma > 0$ and $x \in \Omega$,

$$\int_{\Omega} J_\sigma(x-y)dy = E_\sigma(x),$$

where $E_\sigma(x) \leq 1$ and $E_\sigma(x) \rightarrow 1$ as $\sigma \rightarrow 0$ uniformly in Ω . Then, $\mathcal{G}_b[b_*, w_*](x) = w_* \tilde{s}(b_*) E_{s(b_*)}(x)$, and $\mathcal{G}_w[b_*](x) = b_* \tilde{s}(b_*) E_{s(b_*)}(x)$, implying that there cannot exist a uniform equilibrium in this case.

1.3. Well-posedness. The existence of solutions of the nonlocal model (NLRD) has been studied for Dirichlet and Neumann boundary conditions by Goto et al. [24, 25] and Díaz et al. [10] in a more general framework than ours. The extension of their results to periodic boundary conditions is straightforward and therefore we omit the (lengthy) proof.

However, the kernel construction we consider for adapting to the periodic boundary conditions must be detailed. Let $Q : \mathbb{R} \rightarrow \mathbb{R}$ be such that

$$\begin{aligned} Q \in C^1(\mathbb{R}), \quad Q \geq 0, \quad Q(-x) = Q(x), \quad xQ'(x) \leq 0, \quad \text{supp}(Q) \subset (-1/2, 1/2), \\ \int_{\mathbb{R}} Q(x)dx = 1. \end{aligned} \tag{1.11}$$

For $\sigma \in (0, 1]$, we introduce the rescaling $Q_\sigma(x) = Q(x/\sigma)/\sigma$. Let $\tilde{Q}_\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be the 1-periodification of Q_σ , defined as

$$\tilde{Q}_\sigma(x) = \sum_{n \in \mathbb{Z}} Q_\sigma(x-n).$$

Then, we define $J_\sigma = \tilde{Q}_\sigma|_{\mathbb{T}}$. This is, J_σ is the 1-periodification of a compactly supported Dirac approximation defined on $\mathbb{R}$.

A direct adaptation of the existence and uniqueness results for weak solutions established in [10, 24, 25] to the present setting yields the following. Given initial biomass and soil-water distributions $b_0, w_0 : \mathbb{T} \rightarrow \mathbb{R}$ satisfying

$$b_0, w_0 \in C(\mathbb{T}), \quad 0 \leq b_0 \leq 1, \quad 0 \leq w_0 \leq M_w \quad \text{in } \mathbb{T},$$

for any constant $M_w > \max\{\|w_0\|_{L^\infty(\mathbb{T})}, p/\nu\}$, problem (NLRD) admits a weak solution with the regularity stated, for instance, in [25, Theorem 2.1], namely

$$b, w \in H^1(0, T; (H^1(\mathbb{T}))') \cap L^2(0, T; H^1(\mathbb{T})) \cap L^\infty(\Omega_T), \quad (1.12)$$

and satisfying the uniform bounds

$$0 \leq b \leq 1, \quad 0 \leq w \leq M_w, \quad \text{in } \Omega_T.$$

Moreover, for the simplified problem considered here, in which the porous medium type equation for surface water has been neglected (see the comment preceding (1.1)-(1.2)), one can establish higher regularity. In particular, under suitable additional regularity assumptions on the initial data, the corresponding weak solution is in fact a classical solution. Similar results are obtained for the simpler local reaction problem (LRD).

Remark 1.2. Under this regularity of data and solutions, the model simplification proposed by Meron when the root's lateral extension determined by s_0 is taken to the limit $s_0 \rightarrow 0$ can be rigorously justified. Setting $s_0 = 1/n$ and denoting by (b_n, w_n) the weak solution of (NLRD), the uniform bounds (1.3) allow to use a compactness argument implying the weak or weak* convergence of (b_n, w_n) to some function (b, w) in the norms stated in (1.12), together with an a.e. convergence in Ω_T and a strong convergence in, e.g., $L^1(\Omega_T)$. The main difficulty in the identification of the limit (b, w) as a weak solution of the local problem (LRD) lies in the nonlocal reaction terms, which is overcome by the following result.

Lemma 1.3. *Assume (1.11) and suppose that $u_n, v_n, z_n : \Omega_T \rightarrow \mathbb{R}$ are sequences such that $u_n \geq 0$, $\|u_n\|_{L^\infty(\Omega_T)} < K_u$, $\|z_n\|_{L^\infty(\Omega_T)} < K_z$, $v_n \rightarrow v$ and $z_n \rightarrow z$ strongly in $L^1(\Omega_T)$, for some positive constants K_u, K_z . Then*

$$\lim_{n \rightarrow \infty} \int_{\Omega_T} \left(\int_{\mathbb{T}} J_{s(u_n(t,x))}(x-y) v_n(t,y) dy \right) z_n(t,x) dx dt = \int_{\Omega_T} v(t,x) z(t,x) dx dt.$$

We first use this lemma with $u_n = b_n$, $v_n = w_n$ and $z_n = b_n(1 - b_n)\tilde{s}(b_n)\varphi$, where $\varphi \in L^2(0, T; H^1(\mathbb{T})) \cap L^\infty(\Omega_T)$ is a test function. The only condition of the lemma which is not trivially satisfied is $z_n \rightarrow z$ strongly in $L^1(\Omega_T)$. But this is a simple consequence of the $L^\infty(\Omega_T)$ uniform bound of b_n , its strong $L^1(\Omega_T)$ and a.e. convergences, and the dominated convergence theorem, together with the regularity $\varphi \in L^\infty(\Omega_T)$ of the test functions. Thus, we identify the limit of the nonlocal part of the reaction term corresponding to the equation for b as

$$\lim_{n \rightarrow \infty} \int_{\Omega_T} \mathcal{G}_b[b_n, w_n] b_n(1 - b_n)\varphi = \int_{\Omega_T} g_b(b, w) b(1 - b)\varphi.$$

Similarly, using Lemma 1.3 with $u_n = b_n$, $v_n = w_n\varphi$ and $z_n = b_n\tilde{s}(b_n)$, we identify the limit of the nonlocal part of the reaction term corresponding to the equation for w as

$$\lim_{n \rightarrow \infty} \int_{\Omega_T} w_n \mathcal{G}_w[b_n]\varphi = \int_{\Omega_T} w g_w(b)\varphi.$$

The proof of Lemma 1.3 is given in Section 4.2.

2. MAIN RESULTS

Uniform equilibria for problems (LRD) and (NLRD) are determined by the zeros of function f , which are also the equilibria of the *local reaction problem* (LR)

$$\begin{aligned} b' &= f_b(b, w) \quad \text{in } (0, T), \\ w' &= f_w(b, w) \quad \text{in } (0, T), \\ b(0) &= b_0, \quad w(0) = w_0, \end{aligned}$$

where prime denotes time differentiation and b_0, w_0 are non-negative constants. For studying instability emerging from the nonlocal terms, we first classify equilibria according to their stability with respect to problem (LR). It is trivial to check that for all $p > 0$, there exists the so-called *barren land* equilibrium, given by

$$(b_*^0, w_*^0) = (0, p/\nu). \quad (2.1)$$

For $p > 1$, the systems admit at least a second uniform equilibrium, the *fertile land* equilibrium (b_*^1, w_*^1) , with positive components. For ensuring that this is the unique additional equilibrium to the barren land equilibrium, we introduce some restrictions on the parameter values that, in any case, include the biologically inspired values considered by Gilad et al., see Remark 2.3.

Theorem 2.1. *Assume*

$$\eta > 0, \quad \rho > 0, \quad \nu > \frac{1}{2}, \quad \rho + \eta > \max(1, \rho\eta), \quad \gamma > \nu(\rho + \eta - 1). \quad (2.2)$$

- (1) *If $p < 1$, the barren land equilibrium (b_*^0, w_*^0) given by (2.1) is unique and stable for problem (LR).*
- (2) *If $p > 1$ there exist exactly two equilibria of (LR): the barren land equilibrium, which is unstable, and the fertile land equilibrium, (b_*^1, w_*^1) , with both components positive, which is stable. In addition, we have*

$$\frac{\nu p - \nu}{\nu p + \gamma} < b_*^1 < \frac{\nu p}{\nu p + \gamma}. \quad (2.3)$$

Regarding the linear stability of these equilibria for the reaction-diffusion models, the local problem (LRD) lies within a general structure for which the emergence of a Turing instability is well known to depend on the size of diffusivities ratio δ_w/δ_b , see e.g. [6, Section 3.1.2]. For the nonlocal problem (NLRD), we consider the function

$$H(\omega) = g'_w(b_*^1) \widehat{Q}_{s(b_*^1)}(\omega) + \frac{g_w(b_*^1) s'(b_*^1)}{s(b_*^1)} \omega \partial_\omega \widehat{Q}_{s(b_*^1)}(\omega), \quad (2.4)$$

for $\omega \in \mathbb{R}$. We also introduce the notation

$$R^j = Df(b_*^j, w_*^j), \quad D = \begin{pmatrix} \delta_b & 0 \\ 0 & \delta_w \end{pmatrix}, \quad A_k^j = R^j - 4\pi^2 k^2 D, \quad (2.5)$$

for $j = 0, 1$ and $k \in \mathbb{Z}$.

Theorem 2.2 (Stability of uniform equilibria). *Assume (1.11).*

- (1) *Let $p < 1$. Then the barren land equilibrium (b_*^0, w_*^0) is linearly stable for problems (LRD) and (NLRD).*
- (2) *Let $p > 1$ and assume*

$$\eta(1 - 2b_*^1) > 1, \quad (2.6)$$

$$\delta_w > r_+ \delta_b, \quad (2.7)$$

where r_+ is the largest (positive) root of (4.15). Then the fertile land equilibrium (b_*^1, w_*^1) is linearly unstable for problem (LRD).

- (3) *Let $p > 1$ and assume (2.6). Then, if*

$$\det(A_k^1) < w_*^1 R_{12}^1 \left(\ell'(b_*^1) (1 - \widehat{Q}_{s(b_*^1)}(k)) + \gamma (g'_w(b_*^1) - H(k) \widehat{Q}_{s(b_*^1)}(k)) \right) \quad (2.8)$$

for some $k \in \mathbb{Z}$ then (b_*^1, w_*^1) is linearly unstable for problem (NLRD).

Some comments are in order. Condition (2.6) is required to determine the sign of R_{11}^1 . It is satisfied if

$$1 < p < \frac{\gamma}{\nu} \frac{\eta - 1}{\eta + 1},$$

as it can be deduced replacing estimate (2.3) in (2.6). Notice that for the parameter values provided by Gilad et al., see Remark 2.3, the right hand side of this inequality is larger than one.

Condition (2.8) can be satisfied in two relevant situations. First, if the Turing condition (2.7) is satisfied, it can be shown that $\det(A_k^1) < 0$ for k in some interval $(k_1, k_2) \subset \mathbb{R}$. Moreover, in the proof of Theorem 2.2 for $p > 1$, we show that

$$0 > R_{21}^1 := -w_*^1 (\ell'(b_*^1) + \gamma g'_w(b_*^1)).$$

Since $\ell'(b_*^1) < 0$, the right-hand side of (2.8) is positive whenever $H(k) < 0$ for $k \in (k_1, k_2)$. Hence, under this sufficient condition, the Turing instability of the fertile land equilibrium in problem (LRD) is also present in problem (NLRD).

Second, if (2.7) is not satisfied, then $\det(A_k^1) > 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. In this case, $\det(A_k^1)$ can be shown to be an increasing function of diffusivities. Consequently, if $H(k) < 0$ for at least one nonzero $k \in \mathbb{Z}$ and the diffusivities are sufficiently small, condition (2.8) is satisfied. This gives the interesting situation in which the fertile land equilibrium is linearly stable for problem (LRD) but linearly unstable for problem (NLRD).

The condition $H(k) < 0$, however, cannot be expected to hold for arbitrary kernels satisfying the general assumptions (1.11). However, for the class of kernels satisfying

$$\widehat{Q}_\sigma(\omega) \geq 0 \quad \text{and} \quad \omega \partial_\omega \widehat{Q}_\sigma(\omega) \leq 0 \quad \text{for all } \omega \in \mathbb{R}, \quad (2.9)$$

the properties of the Fourier transform are compatible with the negativity of H . Kernels satisfying (2.9) do exist, as established by Tlas [38]. The Gaussian kernel is a familiar example satisfying these Fourier-space conditions, although it is not compactly supported and is therefore not included in the class considered here.

In conclusion, Theorem 2.2 provides sufficient conditions for the stability and instability of the homogeneous steady states of models (NLRD) and (LRD). These conditions are satisfied, in particular, for the class of kernels satisfying (2.9). However, if a kernel does not satisfy (2.9), the stability or instability conclusions of Theorem 2.2 may still hold, being necessary in this case a specific analysis of condition (2.8).

Remark 2.3. The dimensionless parameter values provided by Gilad et al. [23] are: $\eta = 3.5$, $\gamma = 16.6$, $\nu = 3.3$, $\rho = 0.95$. They satisfy (2.2).

Let us mention that, on one hand, an heuristic approach to linear stability analysis for model (NLRD) may be found in [23]. On the other hand, Díaz et al. [11, 9] follow a rigorous approach for the study of the multiplicity of solutions of the steady states corresponding to (NLRD).

TABLE 1. Parameter values (rounded)

η	γ	ν	ρ	δ_b	δ_w	r_+	s_0
3.35	16.6	3.3	0.95	5.2 e-4	5.2 e-2	9001	0.01,0.05,0.1

3. NUMERICAL EXPERIMENTS

The main purpose of this section is to illustrate the situation in which, for the same parameter data set (provided by Gilad et al. [23]), the fertile land uniform equilibrium (b_*^1, w_*^1) is unstable for problem (NLRD) while keeping its stability for the local problem (LRD).

This possibility is of practical importance since it reveals that replacing the nonlocal model (NLRD) by the simpler local model (LRD) may induce a severe change on the landscape shape outcome resulting from the modelling process.

Besides the shape, as shown in Figure 3, when instabilities arise and vegetation patches are formed, the total biomass contained by the ecological niche is smaller than that corresponding to the uniform state. Therefore, model election does not only affect to pattern formation but to land vegetation richness prediction.

The range of data values for which we find this behaviour is relevant. For the values proposed by Gilad et al. [23], see Table 1, the uniform fertile land equilibrium is stable for problem (LRD) for any precipitation rate, p . However, a contribution of nonlocality as mild as $s_0 = 0.05$ (Gilad et al. propose $s_0 = 0.125$) already drives this uniform equilibrium to instability for problem (NLRD).

Although model (LRD) is computationally much more efficient, in particular because it avoids the direct evaluation of the nonlocal terms, its use as a substitute for (NLRD) requires caution in certain parameter regimes. Our analysis shows that linear stability provides a useful way of identifying these regimes, by indicating the parameter values for which the homogeneous steady state loses stability and the predictions of the two models may differ.

3.1. Data and computational methodology. As commented in previous sections, we need to specify a kernel in order to accurately describe the unstable wave-numbers arising from nonlocal effects. We consider the Gaussian kernel

$$Q_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}.$$

Although the Gaussian kernel is not compactly supported, in the numerical implementation we use a truncated version of it, with the truncation taken sufficiently far into the Gaussian tails so that the discarded mass is numerically negligible. Thus, the kernel used in the computations have compact support, consistently with assumption (1.11).

For the non-truncated Gaussian kernel, we have $\hat{Q}_\sigma(\omega) = e^{-2\pi^2\sigma^2\omega^2} > 0$ for all $\omega \in \mathbb{R}$, and $\partial_\omega \hat{Q}_\sigma(\omega) = -4\pi^2\sigma^2\omega \hat{Q}_\sigma(\omega)$ so that condition (2.9) is satisfied. The truncation introduces small oscillations in the Fourier transform of the kernel, including possibly small negative values at sufficiently high frequencies. However, when the truncation is made sufficiently far into the Gaussian tails, these effects are numerically negligible in the range of frequencies relevant to the simulations.

Choosing the parameter values provided by Gilad et al. [23], we investigate the emergence of pattern formation in models (LRD) and (NLRD) for values of the precipitation rate in the range $p \in (1, 4)$, for several values of the parameter s_0 determining the roots extension, and thus the influence of the nonlocal terms, and for two kind of diffusivities constants, either satisfying or not satisfying the Turing condition (2.7), see Table 1.

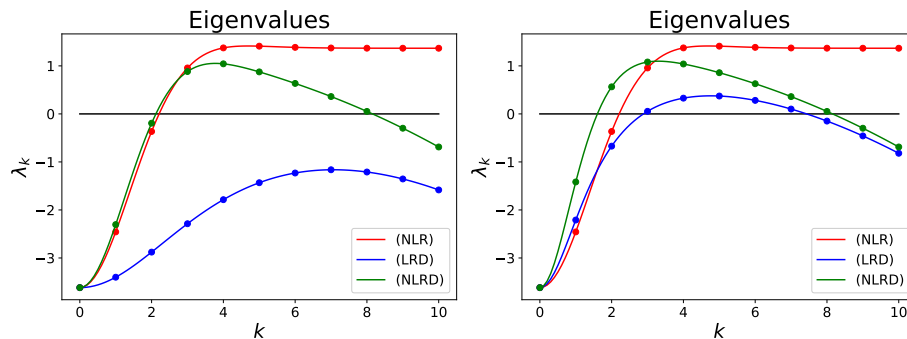


FIGURE 1. Eigenvalues. Left: Original data, for which Turing's condition is not satisfied. Right: Original data with δ_w modified so that Turing's condition is satisfied (factor $2r_+$). Data is fixed as in Table 1, with $p = 1.33$ and $s_0 = 0.05$.

Concerning the discretization numerical method for problems (LRD) and (NLRD), we use a semi-implicit spectral method for both problems, with a brute-force trapezoidal integration method for the nonlocal terms, to keep a good accuracy of their contribution, see [21, 16, 1] for faster although less accurate alternatives.

We consider a uniform mesh of the interval $\mathbb{T} = (0, 1)$, with identified borders, of $N = 200$ nodes. The time step is chosen to satisfy the usual CFL condition. In order to give space for high unstable wave-numbers to emerge, we multiply the reaction terms by $\gamma_r = 9$, which, under a suitable change of time-space scales, is equivalent to consider the spatial domain as a torus of length three. The equilibrium is assumed to be reached when the relative error between two iterations satisfies

$$\frac{\|b_{n+1} - b_n\|}{\|b_n\|} + \frac{\|w_{n+1} - w_n\|}{\|w_n\|} < \text{tol},$$

with $\text{tol} = 10^{-12}$ in the experiments. A further restriction of the final time, T , to $T < 1000$ is taken, although scarcely used. Typical running times for the (LRD) model in a standard i5 laptop are of the order of minutes, while for the (NLRD) model the order may increase to hours. In any case, optimization of execution time has not been pursued. The initial data is chosen as a small

perturbation of the uniform equilibria with the same wave number as that predicted by the linear stability analysis, e.g.

$$b_0(x) = b_*^1(1 + 0.001(\sin(2\pi k^*x) + \cos(2\pi k^*x))),$$

where k^* is the wave-number corresponding to the largest positive eigenvalue of the eigenvalue problem. If all the eigenvalues are negative, we just take b_0 as a random perturbation of b_*^1 with a normal distribution.

The codes for the (LRD) and (NLRD) models are run for a uniform mesh of $p \in [1, 4]$ of 90 values. For the nonlocal model, we experiment with the s_0 values 0.1, 0.05 and 0.01, to check the influence of roots extension in pattern emergence, and the convergence of the solution of (NLRD) to that of (LRD) as $s_0 \rightarrow 0$.

We also experiment with the Turing condition (2.7), setting in some experiments the values of δ_b, δ_w provided by Gilad et al. [23], which do not satisfy the Turing condition and therefore pattern formation is not expected for model (LRD), and in others modifying the values of δ_w by the rule $\delta_w = 2r_+\delta_b$, so that the condition is satisfied and patterns are expected in both models.

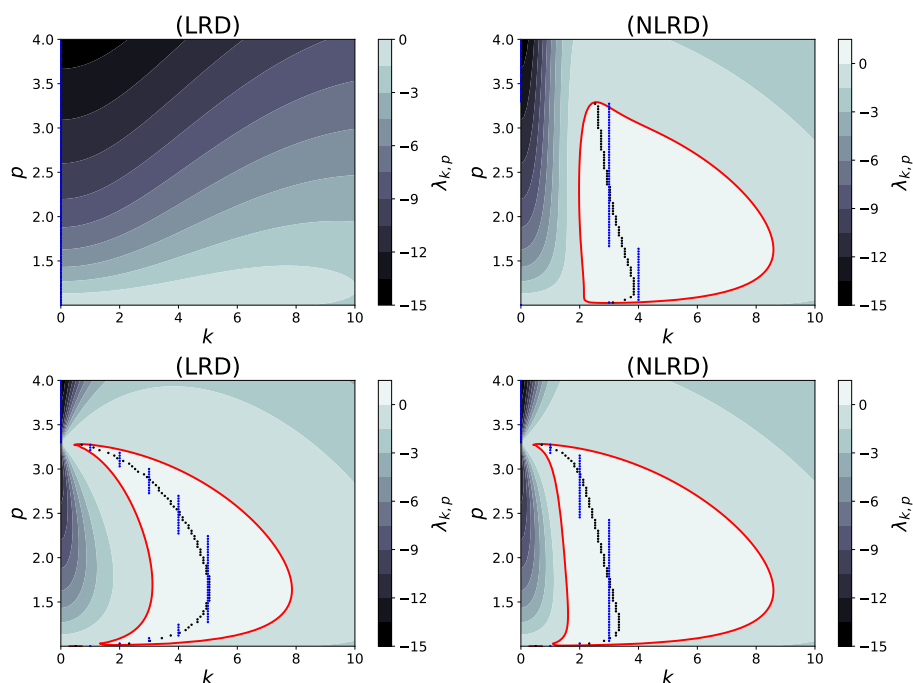


FIGURE 2. Eigenvalues as a function of (k, p) . Top row: original data, for which Turing's condition is not satisfied. Bottom row: data for which Turing's condition is satisfied (factor $2r_+$). The red line is the eigenvalue zero level, separating stable from unstable regions. The black line gives the $k \in \mathbb{R}$ where $\max \lambda_{k,p}$ is attained, for each p (if it is positive). The blue line is the $k \in \mathbb{Z}$ actually representing the dominant unstable mode. Data is fixed as in Table 1, with $s_0 = 0.05$.

3.2. Outcome. All the experiments are devoted to study the behavior of the fertile land equilibrium (b_*^1, w_*^1) , and are thus produced in the range $p > 1$. In Figure 1, we show the eigenvalues corresponding to the (LRD) and (NLRD) linearized problems, see (4.9)-(4.11), as a function of the wave-number for $p = 1.33$ and $s_0 = 0.05$. The left panel shows the case of Gilad's et al. original data for which Turing's condition (2.7) is not satisfied. The right panel shows the case with $\delta_w = 2r_+\delta_b$, for which Turing condition is satisfied. We also include the graphic corresponding to the nonlocal reaction problem (NLR), i.e. to problem (NLRD) without diffusion, to visualize how eigenvalues corresponding to problems (LRD) and (NLRD) are mediated by those of (NLR).

We see in this figure a general trend for values of s_0 large enough: For the pure nonlocal model (NLR) all but few wave-numbers are linearly unstable. Adding diffusion, i.e. considering problem (NLRD), reduces the unstable wave-numbers to a finite quantity. The largest eigenvalue corresponds to a small wave-number. If Turing's condition is satisfied, the largest eigenvalue for problem (LRD) is considerably larger than that of (NLRD), this is, the patterns differ notably.

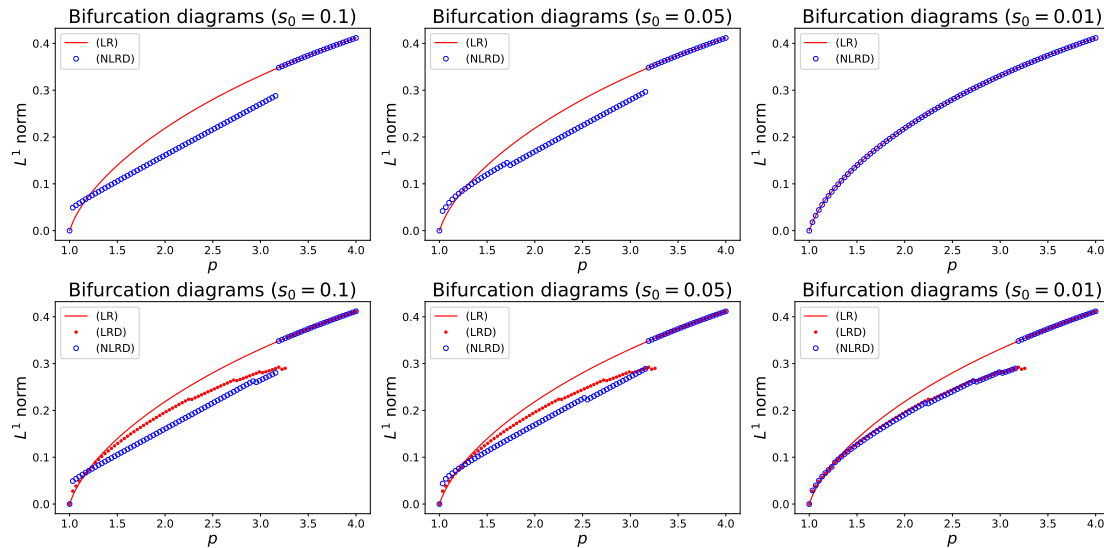


FIGURE 3. Bifurcation diagrams showing the background state defined by equilibria of problem (LR) and the total biomass, $\|b\|_{L^1(\mathbb{T})}$, of the solutions to (LRD) and (NLRD). Top row: Original data by Gilad et al., where the Turing's condition (2.7) is not satisfied, and thus the equilibria of (LRD) coincide with those of (LR). Bottom row: data for which Turing's condition is satisfied (factor $2r_+$).

In Figure 2, we show a broader view of the eigenvalues by projecting their values onto the (k, p) plane. The red lines delimit negative from positive values, and thus, stability from instability regions. The black line is the ridge of the surface, while the blue lines give the actual wave-numbers which are compatible with the periodic boundary conditions (integer numbers). Here, we see that instability only arises for a bounded interval of p .

In Figure 3, we show the numerically computed bifurcation diagrams corresponding to the original problems (LRD) and (NLRD). We plot the value of the $L^1(\mathbb{T})$ norm of the steady state solution $b(x)$ (the vegetation mass) as a function of the precipitation rate. As a background, we always plot the bifurcation diagram for the problem (LR) with local reaction and no diffusion. There are several properties to highlight:

- (1) The existence of intervals of p for which the total biomass is decreasing as a function of p .
- (2) The discontinuity of the total biomass $\|b\|_{L^1(\mathbb{T})}$ as a function of p , specially noticeable for model (NLRD).
- (3) The expectable convergence of (NLRD) to (LRD) as $s_0 \rightarrow 0$.

The loss of biomass when the precipitation rate increases is an unexpected behavior. To illustrate this phenomenon, in Figure 4, we plot the b -component of the steady state solution of (NLRD) for the Gilad's et al. original data and for $s_0 = 0.05$. The left panel shows the solutions for consecutive p values for which $\|b\|_{L^1(\mathbb{T})}$ is increasing, while the right shows the opposite situation. Apparently, there exists a monotonic relationship between mass and wave-number variation.

In Figure 5, we plot the b -component of the steady state solutions of (LRD) and (NLRD) for values of p for which the total biomass is approximately equal, i.e. crossing points for the bifurcation diagrams of (LRD) and (NLRD), for $s_0 = 0.05$ and satisfying Turing's condition. The question we want to ask is whether same total biomass for both models implies same pattern formation, which is answered negatively, in view of the left panel of the figure.

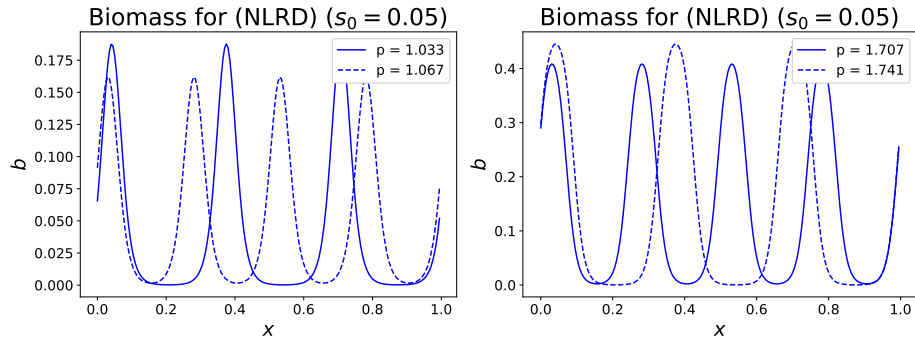


FIGURE 4. Original data by Gilad et al., where the Turing's condition (2.7) is not satisfied. Change in pattern wave-number may lead to both increase (left panel) or decrease (right panel) of total biomass.

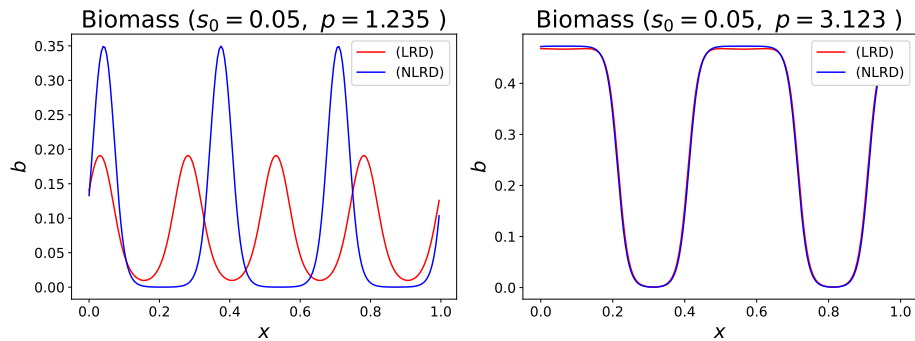


FIGURE 5. Data satisfying Turing's condition (factor $2r_+$). Values of p for which total biomass is approximately equal for both (NLR) and (NLRD) models.

4. PROOFS

Proof of Theorem 2.1. Existence of equilibria. We look for the existence of non-negative solutions of

$$f_b(b, w) = 0, \quad f_w(b, w) = 0, \quad (4.1)$$

with f_b, f_w defined by (1.9) and (1.10). Replacing $b = 0$ in (4.1), we easily deduce that $(b_*^0, w_*^0) = (0, p/\nu)$ is a solution for all $p > 0$.

Since $p > 0$, if another equilibrium (b_*, w_*) with $b_* > 0$ does exist, it must necessarily satisfy $w_* > 0$. Equating w_* in the second equation of (4.1) and replacing it in the first, we deduce that b_* must solve $\phi(b_*) = 0$, where

$$\phi(b) = (1 + \eta b)(1 + \rho b)(\nu p - (\nu p + \gamma)b) - \nu.$$

Notice that since $\nu > 0$, a necessary condition for the existence of a real positive root of ϕ is

$$b < \frac{\nu p}{\nu p + \gamma} =: b_m.$$

The roots of $\phi(b)$, being a cubic polynomial, can be expressed by an exact formula depending on the parameters. However, their large number make difficult to ascertain the sign of the roots. Thus, we resort to an analytical study of $\phi(b)$.

Differentiating $\phi(b)$ we deduce that $\phi'(b) < 0$ if $\Phi(b) > 0$, where $\Phi(b) = Ab^2 + Bb + C$, with

$$A = 3\rho\eta(\nu p + \gamma), \quad B = 2((\nu p + \gamma)(\rho + \eta) - \rho\eta\nu p), \quad C = \gamma - \nu p(\rho + \eta - 1).$$

We readily see that $A > 0$. The first condition in (2.2) implies that also $B > 0$ since it is equivalent to

$$p > \frac{-\gamma(\rho + \eta)}{\nu(\rho + \eta) - \rho\eta\nu},$$

which is true for all $p > 0$.

Let us denote by b_1, b_2 the roots of $\Phi(b)$. Then $\Phi(b) = A(b - b_1)(b - b_2)$ and it holds $A(b_1 + b_2) = -B < 0$ and $Ab_1b_2 = C$. If $p \leq 1$ we deduce, using (2.2), that $C > 0$ implying that b_1 and b_2 are negative. Therefore, $\Phi(b) > 0$ for $b \geq 0$, this is $\phi'(b) < 0$ for $b \geq 0$, and since in this case $\phi(0) = \nu(p - 1) \leq 0$ we deduce that there are no positive roots of $\phi(b)$ if $p \leq 1$. If $p > 1$ we first notice that $\phi(0) > 0$ and $\phi(b_m) < 0$, implying the existence of, at least, one root $b_*^1 \in (0, b_m)$ of $\phi(b)$. Since $A(b_1 + b_2) = -B < 0$, there exists at least a negative root of $\Phi(b)$ which, together with $\phi(0) > 0$, implies that b_*^1 is the unique positive root of $\phi(b)$. Finally, from the second equation of (4.1) we obtain the (positive) value of w_*^1 .

Stability of equilibria. We use the principle of linearized stability to deduce the local asymptotic stability or instability of hyperbolic equilibria from the corresponding properties of their linearizations. The elements of the Jacobian matrix $Df(b, w)$, are

$$\begin{aligned} \frac{\partial f_b}{\partial b}(b, w) &= \nu g_b(b, w)(1 - b) - 1 + \nu b((1 - b) \frac{\partial g_b}{\partial b}(b, w) - g_b(b, w)), \\ \frac{\partial f_b}{\partial w}(b, w) &= \nu b(1 - b) \frac{\partial g_b}{\partial w}(b, w), \\ \frac{\partial f_w}{\partial b}(b, w) &= -w(\ell'(b) + \gamma g'_w(b)), \\ \frac{\partial f_w}{\partial w}(b, w) &= -(\ell(b) + \gamma g_w(b)). \end{aligned}$$

Recall that (b_*^j, w_*^j) is stable if and only if $\det(R^j) > 0$ and $\text{tr}(R^j) < 0$.

For the barren land equilibrium $(b_*^0, w_*^0) = (0, p/\nu)$, we have

$$R^0 = \begin{pmatrix} p - 1 & 0 \\ p(\rho - \frac{\gamma}{\nu}) & -\nu \end{pmatrix}.$$

Therefore, if $p < 1$ then (b_*^0, w_*^0) is stable and if $p > 1$ it is unstable (saddle point). The fertile land equilibrium (b_*^1, w_*^1) does exist only if $p > 1$ and it satisfies

$$\nu g_b(b_*^1, w_*^1)(1 - b_*^1) = 1, \quad p = w_*^1(\ell(b_*^1) + \gamma g_w(b_*^1)). \quad (4.2)$$

Therefore, at this equilibrium, we obtain

$$R_{11}^1 = \nu b_*^1 w_*^1 (\eta - 1 - 2\eta b_*^1), \quad (4.3)$$

$$R_{12}^1 = \nu b_*^1 (1 - b_*^1) (1 + \eta b_*^1) = \frac{b_*^1}{w_*^1}, \quad (4.4)$$

$$R_{21}^1 = w_*^1 \left(\frac{\nu \rho}{(1 + \rho b_*^1)^2} - \gamma (1 + 2\eta b_*^1) \right), \quad (4.5)$$

$$R_{22}^1 = -\frac{p}{w_*^1}. \quad (4.6)$$

We have

$$\det(R^1) = p\nu b_*^1 (1 + 2\eta b_*^1) - p\nu b_*^1 \eta + \gamma b_*^1 (1 + 2\eta b_*^1) - b_*^1 \frac{\nu \rho}{(1 + \rho b_*^1)^2},$$

and then $\det(R^1) > 0$ if and only if

$$\left(p + \frac{\gamma}{\nu} \right) (1 + 2\eta b_*^1) > p\eta + \frac{\rho}{(1 + \rho b_*^1)^2}. \quad (4.7)$$

On one hand, we have $\rho > \rho/(1 + \rho b_*^1)^2$, and on the other hand, using that b_*^1 satisfies $\phi(b_*^1) = 0$, we deduce $\nu p - (\nu p + \gamma)b_*^1 < \nu$, this is $b_*^1 > \nu(p - 1)/(\nu p + \gamma)$. Therefore, for (4.7) to be true it suffices to prove that

$$\frac{\nu p + \gamma}{\nu} + 2\eta(p - 1) > p\eta + \rho,$$

which is equivalent to

$$p > \frac{\nu(2\eta + \rho) - \gamma}{\nu(1 + \eta)}. \quad (4.8)$$

The condition $\gamma > \nu(\rho + \eta - 1)$ assumed in (2.2) implies that the right hand side of (4.8) is smaller than one. Since we are in the case $p > 1$, inequality (4.8) is satisfied and we conclude that $\det(R^1) > 0$.

For the trace, we have that $\text{tr}(R^1) < 0$ is equivalent to $p > \nu b_*^1 (w_*^1)^2 (\eta - 1 - 2\eta b_*^1)$. Using the first equation of (4.2), we express this inequality as

$$\nu p(1 + \eta b_*^1)^2 (1 - b_*^1)^2 > b_*^1 (\eta - 1 - 2\eta b_*^1).$$

Expanding in terms of η , we obtain the condition

$$\nu p(b_*^1)^2 (1 - b_*^1)^2 \eta^2 + b_*^1 (2\nu p(1 - b_*^1)^2 + 2b_*^1 - 1) \eta + \nu p(1 - b_*^1)^2 + b_*^1 > 0.$$

The zero and second order coefficients are clearly positive, and since $\nu > 1/2$, see (2.2), and $p > 1$ we have $2\nu p > 1$, implying that also the first order coefficient is positive, and therefore the negativity of $\text{tr}(R^1)$. $\square$

4.1. Proof of Theorem 2.2. Since the diffusion terms for problems (LRD) and (NLRD) are linear, the only difference in their linearization around any equilibrium (b_*^j, w_*^j) resides in the linearization of the reaction terms f and $\mathcal{F}$.

Linearization of (NLRD) and eigenvalue problem. For linearizing the nonlocal problem (NLRD) we need to compute the Gateaux differentials of $\mathcal{F}$. In the next lemma, we use the symbol $\star$ to denote the convolution product in $\mathbb{T}$.

Lemma 4.1. *Let (b_*^j, w_*^j) , for $j = 0, 1$, be any of the two uniform equilibrium of (NLRD). We have*

$$\begin{aligned} \langle \partial_b \mathcal{F}_b[b_*^j, w_*^j], \varphi \rangle &= R_{11}^j \varphi, & \langle \partial_w \mathcal{F}_b[b_*^j, w_*^j], \psi \rangle &= R_{12}^j (J_{s(b_*^j)} \star \psi), \\ \langle \partial_b \mathcal{F}_w[b_*^j, w_*^j], \varphi \rangle &= R_{21}^j \varphi + \mathcal{N}^j[\varphi], & \langle \partial_w \mathcal{F}_w[b_*^j, w_*^j], \psi \rangle &= R_{22}^j \psi, \end{aligned}$$

where $\mathcal{N}$ is the linear operator given by

$$\mathcal{N}^j[\varphi] = w_*^j \gamma \left(g'_w(b_*^j) [\varphi - (J_{s(b_*^j)} \star \varphi)] - g_w(b_*^j) s'(b_*^j) (\partial_\sigma J_{s(b_*^j)} \star \varphi) \right).$$

The proof is given in Subsection 4.2.

The linearized problem corresponding to (NLRD) around (b_*^j, w_*^j) is then: Find $\varphi, \psi : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ such that

$$\partial_t \varphi - \delta_b \Delta \varphi = R_{11}^j \varphi + R_{12}^j (J_{s(b_*^j)} \star \psi) \quad \text{in } \Omega_T, \quad (4.9)$$

$$\partial_t \psi - \delta_w \Delta \psi = R_{21}^j \varphi + R_{22}^j \psi + \mathcal{N}^j[\varphi] \quad \text{in } \Omega_T, \quad (4.10)$$

$$\varphi(0, \cdot) = \varepsilon_b, \quad \psi(0, \cdot) = \varepsilon_w, \quad \text{in } \mathbb{T}, \quad (4.11)$$

where $\varepsilon_b, \varepsilon_w : \mathbb{T} \rightarrow \mathbb{R}$ are some functions with small $L^\infty(\mathbb{T})$ norm (perturbations of the equilibrium state). We solve the linear problem (4.9)-(4.11) by the method of separation of variables. Setting $\varphi(t, x) = e^{\lambda t} \Phi(x)$, $\psi(t, x) = e^{\lambda t} \Psi(x)$ and replacing in (4.9) and (4.10) we obtain the system

$$\lambda \Phi - \delta_b \Delta \Phi = R_{11}^j \Phi + R_{12}^j (J_{s(b_*^j)} \star \Psi) \quad \text{in } \mathbb{T},$$

$$\lambda \Psi - \delta_w \Delta \Psi = R_{21}^j \Phi + R_{22}^j \Psi + \mathcal{N}^j[\Phi] \quad \text{in } \mathbb{T}.$$

Multiplying the equations by $e^{-i2\pi kx}$, with $k \in \mathbb{Z}$, and integrating in $\mathbb{T}$ we obtain

$$\begin{aligned} \lambda \widehat{\Phi}[k] + (2\pi k)^2 \widehat{\Phi}[k] &= R_{11}^j \widehat{\Phi}[k] + R_{12}^j \mathcal{F}(J_{s(b_*^j)})[k] \widehat{\Psi}[k] \quad k \in \mathbb{Z}, \\ \lambda \widehat{\Psi}[k] + (2\pi k)^2 \widehat{\Psi}[k] &= R_{21}^j \widehat{\Phi}[k] + R_{22}^j \widehat{\Psi}[k] + \mathcal{L}^j[\widehat{\Phi}][k] \quad k \in \mathbb{Z}, \end{aligned} \quad (4.12)$$

where, using the convolution theorem for the Fourier transform, we define

$$\mathcal{L}^j[\widehat{\Phi}][k] = w_*^j \gamma \widehat{\Phi}[k] \left(g'_w(b_*^j) [1 - \mathcal{F}(J_{s(b_*^j)})[k]] - g_w(b_*^j) s'(b_*^j) \mathcal{F}(\partial_\sigma J_{s(b_*^j)})[k] \right).$$

In these expressions, both $\widehat{\xi}[k]$ and $\mathcal{F}(\xi)[k]$ denote the Fourier k -coefficient of ξ in $\mathbb{T}$.

Adding and subtracting the term $R_{12}^j \widehat{\Psi}[k]$ in (4.12), we obtain the eigenvalue problem $\lambda \widehat{\Theta} = \mathcal{A}^j[k] \widehat{\Theta}$, where $\Theta = (\Phi, \Psi)$,

$$\begin{aligned}\mathcal{A}^j[k] &= \mathcal{R}^j[k] - 4\pi^2 k^2 D, \\ \mathcal{R}^j[k] &= R^j + \begin{pmatrix} 0 & u^j[k] \\ v^j[k] & 0 \end{pmatrix}, \\ u^j[k] &= R_{12}^j (\mathcal{F}(J_{s(b_*^j)})[k] - 1), \\ v^j[k] &= w_*^j \gamma \left(g'_w(b_*^j) [1 - \mathcal{F}(J_{s(b_*^j)})[k]] - g_w(b_*^j) s'(b_*^j) \mathcal{F}(\partial_\sigma J_{s(b_*^j)})[k] \right).\end{aligned}$$

For each wave number $k \in \mathbb{Z}$, there exist two eigenvalues $\lambda_{k,1}^j$ and $\lambda_{k,2}^j$ determined as the roots of characteristic equation

$$\lambda^2 - \text{tr}(\mathcal{A}^j[k])\lambda + \det(\mathcal{A}^j[k]) = 0, \quad (4.13)$$

together with their corresponding eigenvectors $\widehat{\Theta}_{k,1}^j$ and $\widehat{\Theta}_{k,2}^j$. The linear subspace spanned by these eigenvectors is stable if $\text{Re}(\lambda_{k,i}^j) < 0$ for $i = 1, 2$, and unstable in other case. Therefore, the equilibrium (b_*^j, w_*^j) is linearly stable if and only if $\text{Re}(\lambda_{k,i}^j) < 0$ for $i = 1, 2$ and for all $k \in \mathbb{Z}$.

Linearization of (LRD) and eigenvalue problem. In this case, a similar approach as for problem (NLRD) leads to an eigenvalue problem in which the eigenvalues are determined by (4.13) with $\mathcal{A}^j[k]$ replaced by A_k^j , see (2.5). Notice that for any stable equilibrium for (LR) we have $\text{tr}(R^j) < 0$, implying that both $\text{tr}(\mathcal{A}^j[k])$ and $\text{tr}(A_k^j)$ are negative for all $k \in \mathbb{Z}$. Therefore, the uniform equilibria (b_*^j, w_*^j) may become linearly unstable (existence of an eigenvalue with positive real part) only if the corresponding determinants are negative for some $k \in \mathbb{Z} \setminus \{0\}$, see (4.13).

Case $p < 1$. The unique uniform stable equilibrium is the barren land equilibrium (b_*^0, w_*^0) , for which we have

$$\det(A_k^0) = (p - 1 - 4\pi^2 k^2 \delta_b)(-\nu - 4\pi^2 k^2 \delta_w) > 0,$$

for all $k \in \mathbb{Z}$. On noting that, for $j = 0, 1$,

$$\det(\mathcal{A}^j[k]) = \det(A_k^j) - N^j[k] \quad (4.14)$$

with

$$N^j[k] = R_{12}^j v^j[k] + R_{21}^j u^j[k] + u^j[k] v^j[k],$$

and taking into account that $R_{12}^0 = u^0[k] = 0$, we deduce $\det(\mathcal{A}^0[k]) = \det(A_k^0) > 0$. Therefore, the barren land equilibrium is linear stable for problems (LRD) and (NLRD) if $p < 1$.

Case $p > 1$. If $p > 1$, the unique stable uniform equilibrium for problem (LR) is the fertile land equilibrium (b_*^1, w_*^1) . Notice that for this equilibrium we have $R_{12}^1 > 0$, $R_{22}^1 < 0$, and $R_{11}^1 > 0$, the latter as a consequence of assumption (2.6), see (4.3)-(4.6). Then, the stability of (b_*^1, w_*^1) for problem (LR) also implies $R_{21}^1 < 0$.

According to [6, Section 3.1.2], this equilibrium becomes linearly unstable (Turing instability) for problem (LRD) under the conditions $R_{11}^1 > 0$, $R_{22}^1 < 0$ and (2.7), where

$$p(r) = (R_{11}^1)^2 r^2 + 2(2R_{12}^1 R_{21}^1 - R_{11}^1 R_{22}^1) r + (R_{22}^1)^2, \quad (4.15)$$

which are all satisfied for our problem.

For problem (NLRD), the identity (4.14) shows that the equilibrium (b_*^1, w_*^1) may become unstable only if

$$\det(A_k^1) < N^1[k] \quad \text{for some } k \in \mathbb{Z}.$$

For simplifying the expressions of $u^1[k]$ and $v^1[k]$, we use the following lemma relating the Fourier transforms of the kernels $Q_\sigma : \mathbb{R} \rightarrow \mathbb{R}$ and $J_\sigma : \mathbb{T} \rightarrow \mathbb{R}$. Abusing on notation, we use the same symbols $\widehat{\cdot}$ or $\mathcal{F}$ to denote the Fourier transforms in $\mathbb{R}$ and in $\mathbb{T}$.

Lemma 4.2. Assume (1.11) and let $\sigma > 0$. Then $\widehat{Q}_\sigma$ is real valued, $\widehat{Q}_\sigma(0) = 1$ and $|\widehat{Q}_\sigma(\omega)| \leq 1$ for $\omega \in \mathbb{R}$. In addition, $\mathcal{F}(J_\sigma)[k] = \widehat{Q}_\sigma(k)$ and $\mathcal{F}(\partial_\sigma J_\sigma)[k] = \frac{k}{\sigma} \partial_\omega \widehat{Q}_\sigma(k)$ for $k \in \mathbb{Z}$.

The proof is given in Subsection 4.2. Using Lema 4.2, we rewrite $u^1[k]$ and $v^1[k]$ as

$$\begin{aligned} u^1[k] &= R_{12}^1(\widehat{Q}_{s(b_*^1)}(k) - 1), \\ v^1[k] &= w_*^1 \gamma \left(g'_w(b_*^1)(1 - \widehat{Q}_{s(b_*^1)}(k)) - g_w(b_*^1) s'(b_*^1) \frac{k}{s(b_*^1)} \partial_\omega \widehat{Q}_{s(b_*^1)}(k) \right), \end{aligned}$$

resulting in

$$N^1[k] = R_{12}^1(R_{21}^1(\widehat{Q}_{s(b_*^1)}(k) - 1) + \widehat{Q}_{s(b_*^1)}(k)v^1[k]).$$

Using the explicit expressions $R_{21}^1 = -w_*^1(\ell'(b_*^1) + \gamma g'_w(b_*^1))$ and $v^1[k] = \gamma w_*^1(g'_w(b_*^1) - H(k))$, with $H(k)$ given in (2.4), we deduce that

$$N^1[k] = w_*^1 R_{12}^1 \left(\ell'(b_*^1)(1 - \widehat{Q}_{s(b_*^1)}(k)) + \gamma(g'_w(b_*^1) - H(k)) \widehat{Q}_{s(b_*^1)}(k) \right),$$

and the result follows from assumption (2.8).

4.2. Proofs of lemmas.

Proof of Lemma 1.3. We consider the decomposition

$$\int_{\Omega_T} \left(\int_{\mathbb{T}} J_{s(u_n(t,x))}(x-y) v_n(t,y) dy \right) z_n(t,x) dx dt = J_1(n) + J_2(n),$$

with

$$J_1(n) = \int_{\Omega_T} \tilde{V}_n(t,x) z_n(t,x) dx dt, \quad J_2(n) = \int_{\Omega_T} V_n(t,x) z_n(t,x) dx dt,$$

$$\tilde{V}_n(t,x) = \int_{\mathbb{T}} J_{s(u_n(t,x))}(x-y)(v_n(t,y) - v(t,y)) dy, \quad V_n(t,x) = \int_{\mathbb{T}} J_{s(u_n(t,x))}(x-y)v(t,y) dy.$$

We first show that $\lim_{n \rightarrow \infty} J_1(n) = 0$. Since z_n is uniformly bounded in $L^\infty(\Omega_T)$, we have

$$|J_1(n)| \leq K_z \int_{\Omega_T} |\tilde{V}_n(t,x)| dx dt. \quad (4.16)$$

For justifying the use of Fubini's theorem, we observe that

$$\begin{aligned} \int_{\mathbb{T}} J_{s(u_n(t,x))}(x-y) dx &= \int_{\mathbb{T}} \frac{n}{\tilde{s}(u_n(t,x))} J\left(\frac{n(x-y)}{\tilde{s}(u_n(t,x))}\right) dx \\ &\leq n \int_{\mathbb{T}} J\left(\frac{n(x-y)}{\tilde{s}(u_n(t,x))}\right) dx \\ &= \int_{\mathbb{T}} J\left(\frac{\bar{x} - ny}{\tilde{s}(u_n(t, \bar{x}/n))}\right) d\bar{x} \leq J(0)|\mathbb{T}|, \end{aligned}$$

since J is radially decreasing and $\tilde{s} \geq 1$, see (1.6). Therefore, from (4.16) we deduce that

$$\begin{aligned} |J_1(n)| &\leq K_z \int_{\Omega_T} |v_n(t,y) - v(t,y)| \left(\int_{\mathbb{T}} J_{s(u_n(t,x))}(x-y) dx \right) dy dt \\ &\leq K_z J(0)|\mathbb{T}| \int_{\Omega_T} |v_n(t,y) - v(t,y)| dy \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, in view of the strong convergence $v_n \rightarrow v$ in $L^1(\Omega_T)$.

For computing the limit of $J_2(n)$, notice that if we prove that $V_n \rightarrow v$ strongly in $L^1(\Omega_T)$ then, using the uniform $L^\infty(\Omega_T)$ boundedness of z_n , we obtain

$$\lim_{n \rightarrow \infty} J_2(n) = \int_{\Omega_T} v(t,x) z(t,x) dx dt,$$

which implies the assertion of the lemma.

The strong convergence $V_n \rightarrow v$ in $L^1(\Omega_T)$ is proven by adapting the arguments in [4, Prop. 4.21 and Th. 4.22] to our mollifier $J_{s(u_n(t,x))}$. Following [4, Prop. 4.21], we first assume that v is continuous in Ω_T , so that given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|v(t, x-y) - v(t, y)| < \varepsilon \quad \text{for all } x \in \mathbb{T}, y \in B_\delta(0).$$

Because of the uniform $L^\infty(\Omega_T)$ bound of u_n and to the definition of s , see (1.6), we have

$$s(u_n(t, x)) < \delta \quad \text{for } n > \frac{1 + \eta K_u}{\delta}, \quad \text{and for all } (t, x) \in \Omega_T.$$

Then, for $x \in \mathbb{T}$ fixed, since J_σ is radial and compactly supported, we obtain, for $n > (1 + \eta K_u)/\delta$,

$$\begin{aligned} |V_n(t, x) - v(t, x)| &\leq \int_{B_{s(u_n(t, x))}(x)} J_{s(u_n(t, x))}(x - y) |v(t, y) - v(t, x)| dy \\ &\leq \int_{B_\delta(0)} J_{s(u_n(t, x))}(z) |v(t, x - \bar{x}) - v(t, x)| d\bar{x} \leq \varepsilon, \end{aligned}$$

implying the uniform convergence $V_n \rightarrow v$ in Ω_T when v is continuous. The proof of the strong convergence $V_n \rightarrow v$ in $L^1(\Omega_T)$ when $v \in L^1(\Omega_T)$ is then deduced by an approximation argument like in [4, Th. 4.22]. $\square$

Proof of Lemma 4.1. Let (b_*, w_*) denote any of the equilibria. The Gateaux derivative coincides with the usual derivative for the local terms. For the nonlocal terms, we have

$$\mathcal{G}_b[b_* + h\varphi, w_*](t, x) = \tilde{s}(b_* + h\varphi(t, x)) \int_{\mathbb{T}} J_{s(b_* + h\varphi(t, x))}(x - y) w_* dy = w_* \tilde{s}(b_* + h\varphi(t, x)),$$

and thus

$$\partial_b \mathcal{G}_b[b_*, w_*] = \frac{d}{dh} \mathcal{G}_b[b_* + h\varphi, w_*] \Big|_{h=0} = w_* \tilde{s}'(b_*) \varphi.$$

Similarly,

$$\mathcal{G}_b[b_*, w_* + h\psi] = \tilde{s}(b_*) \int_{\mathbb{T}} J_{s(b_*)}(x - y) (w_* + h\psi(t, y)) dy$$

leading to

$$\partial_b \mathcal{G}_w[b_*, w_*] = \frac{d}{dh} \mathcal{G}_b[b_*, w_* + h\psi] \Big|_{h=0} = \tilde{s}(b_*) \int_{\mathbb{T}} J_{s(b_*)}(x - y) \psi(t, y) dy = \tilde{s}(b_*) (J_{s(b_*)} \star \psi).$$

For the Gateaux derivatives of $\mathcal{G}_w[b_*]$, we have

$$\mathcal{G}_w[b_* + h\varphi](t, x) = \int_{\mathbb{T}} J_{s(b_* + h\varphi(t, y))}(x - y) \tilde{s}(b_* + h\varphi(t, y)) (b_* + h\varphi(t, y)) dy,$$

and then

$$\begin{aligned} \frac{d}{dh} \mathcal{G}_w[b_* + h\varphi](t, x) &= \int_{\mathbb{T}} J_{s(b_* + h\varphi(t, y))}(x - y) \tilde{s}'(b_* + h\varphi(t, y)) \varphi(t, y) (b_* + h\varphi(t, y)) dy \\ &\quad + \int_{\mathbb{T}} J_{s(b_* + h\varphi(t, y))}(x - y) \tilde{s}(b_* + h\varphi(t, y)) \varphi(t, y) dy \\ &\quad + \int_{\mathbb{T}} \tilde{s}(b_* + h\varphi(t, y)) (b_* + h\varphi(t, y)) \frac{d}{dh} J_{s(b_* + h\varphi(t, y))}(x - y) dy. \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_b \mathcal{G}_w[b_*] &= \frac{d}{dh} \mathcal{G}_w[b_* + h\varphi](t, x) \Big|_{h=0} \\ &= g'_w(b_*) \int_{\mathbb{T}} J_{s(b_*)}(x - y) \varphi(t, y) dy + g_w(b_*) \int_{\mathbb{T}} \frac{d}{dh} J_{s(b_* + h\varphi(t, y))}(x - y) \Big|_{h=0} dy. \end{aligned}$$

The integral term can be expanded as

$$\frac{d}{dh} J_{s(b_* + h\varphi(t, y))}(x - y) = \partial_\sigma J_{s(b_* + h\varphi(t, y))}(x - y) s'(b_* + h\varphi(t, y)) \varphi(t, y),$$

leading to

$$\frac{d}{dh} J_{s(b_* + h\varphi(t, y))}(x - y) \Big|_{h=0} = \partial_\sigma J_{s(b_*)}(x - y) s'(b_*) \varphi(t, y).$$

Therefore

$$\partial_b \mathcal{G}_w[b_*] = g'_w(b_*) (J_{s(b_*)} \star \varphi) + g_w(b_*) s'(b_*) (\partial_\sigma J_{s(b_*)} \star \varphi).$$

We trivially have $\partial_w \mathcal{G}_w[b_*] = 0$.

Now, the Gateaux derivatives of $\mathcal{F}_b[b_*, w_*]$ can be deduced easily. For instance, on noting that $\mathcal{G}_b[b_*, w_*] = g_b(b_*, w_*)$, we obtain

$$\begin{aligned}\partial_b \mathcal{F}_b[b_*, w_*] &= \frac{d}{dh} \mathcal{F}_b[b_* + h\varphi, w_*] \Big|_{h=0} \\ &= \varphi(\nu \mathcal{G}_b[b_*, w_*](1 - b_*) - 1 - b_* \nu \mathcal{G}_b[b_*, w_*] + b_*(1 - b_*) \nu w_* \tilde{s}'(b_*)) \\ &= \partial_b f_b(b_*, w_*) \varphi.\end{aligned}$$

The other derivatives are computed similarly. $\square$

Proof of Lemma 4.2. We have $\widehat{Q}_\sigma(\omega) = \widehat{Q}(\sigma\omega)$. Since Q is even, we deduce $\widehat{Q}$ is real. Since Q is normalized in $L^1(\mathbb{R})$ we deduce $\widehat{Q}(0) = 1$ and $|\widehat{Q}(\omega)| \leq 1$ for all $\omega \in \mathbb{R}$. The identity $\mathcal{F}(J_\sigma)[k] = \widehat{Q}_\sigma(k)$ is the Poisson summation formula, which is valid for compactly supported $C(\mathbb{R})$ functions, see e.g. [28, Ch. IV, 1.15]. For proving the second identity, we notice that

$$\partial_\sigma Q_\sigma(x) = -\frac{1}{\sigma} \partial_x(x Q_\sigma(x)),$$

which is continuous and compactly supported. The formula for the Fourier transform of the derivative of a function gives

$$\mathcal{F}(\partial_\sigma Q_\sigma)(\omega) = -\frac{i2\pi\omega}{\sigma} \mathcal{F}(x Q_\sigma(x))(\omega) = \frac{\omega}{\sigma} \partial_\omega \mathcal{F}(Q_\sigma)(\omega).$$

Using again Poisson summation formula, we deduce $\mathcal{F}(\partial_\sigma J_\sigma)[k] = \frac{k}{\sigma} \partial_\omega \widehat{Q}_\sigma(k)$. $\square$

5. CONCLUSIONS

Interactions among biological matter are often at a distance, involving nonlocal terms in their mathematical description. Since the resulting models are normally more intricate than the local ones, and the latter are obtained through a well justified mathematical argument when the distance becomes negligible, the question of model interchangeability arises.

We have analyzed in this article a well-known model with these characteristics. The system of nonlocal reaction-diffusion equations (NLRD), where the nonlocal terms represent the roots water uptaking mechanism at a distance from their shoot. The corresponding limit model (LRD) has a classical (local) reaction-diffusion form, being its mathematical properties well described in the literature. In particular, the linear instability of its uniform equilibria for certain parameter choices.

We investigated model interchangeability through linear stability analysis and numerical simulations. Based on the physical parameter values reported in the biological literature, our results indicate that, for the parameter regimes and kernels considered in this study, pattern formation in the model is mainly driven by the nonlocal effects associated with the root terms, rather than by a Turing instability arising from the diffusion ratio.

We checked that the linear stability analysis is useful for determining pattern wave-numbers. However, it does not suffice to understand total biomass production, which is related to pattern amplitude, whose magnitude is not provided by the linear analysis. There exist several methods that provide the amplitude approximation, e.g. the weakly nonlinear analysis [27, 7, 19], whose extension to nonlocal equations could be of general and independent interest.

Our most intriguing finding is that there exist situations in which the models predict a loss of total biomass for increasing precipitation rates. We do not understand this counter-intuitive outcome. The instability wave-number change might be a factor, but other nonlinear effects affecting the pattern amplitude could be involved. Thus, further research will be devoted to clarify this phenomenon.

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