

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR NONLINEAR POISSON-POISSON ELLIPTIC SYSTEMS

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ABSTRACT. We study a nonlinear Poisson-Poisson system in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, whose reduced form is the nonlocal elliptic equation $-\Delta u + \lambda(-\Delta)^{-1}u = |u|^{p-2}u$. We assume that $\lambda \in (-\lambda_1(\Omega)^2, \infty)$, where $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω , and that the exponent is subcritical, $2 < p < 2^* = 2N/(N-2)$. We first show that, throughout this coercive range, the quadratic part of the associated energy functional defines a norm equivalent to the standard norm of $H_0^1(\Omega)$. We then establish the Palais-Smale condition on the Nehari manifold and use the Krasnosel'skii genus to prove the existence of infinitely many pairs of nontrivial weak solutions. More precisely, there exists a sequence of solutions (u_k, ϕ_{u_k}) such that $\|u_k\|_\lambda^2 \rightarrow \infty$. A three-dimensional finite element experiment illustrates two distinct discrete solutions, one of which changes sign.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain. In this work we study the coupled elliptic system

$$\begin{aligned} -\Delta u + \lambda \phi &= |u|^{p-2}u, & \text{in } \Omega, \\ -\Delta \phi &= u, & \text{in } \Omega, \\ u &= 0, \quad \phi = 0, & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$\lambda \in (-\lambda_1(\Omega)^2, \infty), \quad 2 < p < 2^* := \frac{2N}{N-2},$$

and $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω . Systems of this type arise in models of electrostatic interactions, ionic transport, and semiconductor theory. They describe the coupling between electric fields and charge densities in a medium and have attracted considerable attention because of their applications to electrolytes and semiconductor devices; see, for instance, [14]. From the analytical point of view, the system admits an interpretation as a semilinear elliptic equation with a nonlocal term, since from the second equation one formally obtains

$$\phi = (-\Delta)^{-1}u,$$

and, therefore, the problem can be written as

$$-\Delta u + \lambda(-\Delta)^{-1}u = |u|^{p-2}u \quad \text{in } \Omega. \tag{1.2}$$

The variational framework for nonlinear elliptic problems is rooted in the critical point methods developed by Ambrosetti and Rabinowitz [4]. These methods provide the natural tools for combining compactness and minimax arguments in the search for nontrivial solutions.

A classical local counterpart is obtained by replacing the inverse Laplacian term with a linear perturbation. This leads to the Brezis-Nirenberg problem [7], in which the authors studied elliptic equations with critical Sobolev growth and proved, for $N \geq 4$, the existence of positive solutions

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in an appropriate range of the linear parameter. This result showed that a lower-order linear perturbation can decisively modify the existence of solutions in the critical regime, where the Sobolev embedding ceases to be compact. Later, Lions [12] introduced the concentration-compactness principle, which became a central tool for treating the loss of compactness in critical problems.

In the nonlocal setting, Alvarez-Caudevilla, Colorado and Ortega [2] studied the related problem

$$-\Delta u = \gamma(-\Delta)^{-1}u + |u|^{q-1}u, \quad \gamma > 0,$$

where q is used here to distinguish their exponent notation from the one adopted in the present paper. Equivalently, their equation can be written as

$$-\Delta u - \gamma(-\Delta)^{-1}u = |u|^{q-1}u.$$

Thus, in the notation of (1.1), their nonlocal coefficient corresponds to $\lambda = -\gamma < 0$. Their problem lies in the coercive range considered here whenever $0 < \gamma < \lambda_1(\Omega)^2$. The authors proved the existence of positive solutions both in the subcritical case and in the critical case. In the critical case, they showed that the nonlocal term $(-\Delta)^{-1}u$ modifies the dimensional range of existence: while in the classical Brezis-Nirenberg problem positive solutions are obtained for $N \geq 4$, in the problem with inverse operator the method leads to the range $N \geq 7$. Moreover, they interpreted the model as a fourth-order equation with Navier conditions, equivalently formulated as a coupled elliptic system of second order.

The connection with fourth-order equations must be understood in the context of the theory of polyharmonic problems. Gazzola, Grunau and Sweers [11] show that, in this type of problems, boundary conditions and the loss of maximum principles play a decisive role. In particular, biharmonic problems with Navier conditions can be reformulated as coupled second-order systems, which provides a natural framework for comparing fourth-order equations with Poisson-Poisson formulations.

Alvarez-Caudevilla and Galaktionov [3] studied fourth-order semilinear parabolic equations of Cahn-Hilliard type. At the stationary level, their equations are related to inverse-Laplacian problems through a formal fourth-order reformulation. They obtained results on stationary states, global existence, and finite-time blow-up, depending on the nonlinear exponent.

By formally eliminating the auxiliary variable in problems with inverse Laplacian operator, one obtains fourth-order equations with Navier-type conditions. In this local fourth-order framework, Feng and Lu [9] studied semilinear biharmonic problems of the form $\Delta^2 u = f(x, u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, with $N > 4$, considering both Navier conditions $u = \Delta u = 0$ on $\partial\Omega$, and Dirichlet conditions $u = \frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. By means of critical point theory for even functionals and strongly monotone operators, they obtained results of existence, uniqueness and multiplicity of nontrivial weak solutions, including results of infinitely many solutions.

Works on critical polyharmonic problems provide a different antecedent, mainly related to the influence of the critical exponent and of the dimension in higher-order problems. Bernis and Grunau [5] studied Dirichlet problems for polyharmonic operators in the ball, of the form $(-\Delta)^K u = \lambda u + |u|^{q-1}u$, where $q = \frac{N+2K}{N-2K}$ is the critical Sobolev exponent, and analyzed multiple critical dimensions for the existence of nontrivial radial solutions.

We also mention the recent work of Afonso and Mascaro [1], who established infinitely many solutions for a perturbed Schrödinger-Bopp-Podolsky system in a bounded domain by means of symmetric variational methods. Their higher-order coupling and prescribed-frequency framework differ from the Poisson-Poisson system considered here. In the present problem, the inverse Dirichlet Laplacian appears as a linear nonlocal perturbation, and the exact coercive range $\lambda > -\lambda_1(\Omega)^2$ is considered.

The purpose of the present paper is to establish an infinite multiplicity result for the subcritical Poisson-Poisson system (1.1). More precisely, for every $\lambda \in (-\lambda_1(\Omega)^2, \infty)$ and every $2 < p < 2^*$, we prove the existence of infinitely many pairs of nontrivial weak solutions. The argument relies on the coercivity of the quadratic part of the energy functional, the geometry of the Nehari manifold, the Palais-Smale condition, and the Krasnosel'skii genus.

In fact, we obtain a sequence of weak solutions (u_k, ϕ_{u_k}) whose associated energy levels are unbounded. Equivalently, with respect to the norm induced by the quadratic part of the functional,

the sequence satisfies $\|u_k\|_\lambda^2 \rightarrow \infty$. This complements previous results on inverse-Laplacian and higher-order problems by showing that the Poisson-Poisson coupling admits a sufficiently rich variational structure to generate infinitely many solutions.

The paper is organized as follows. Section 2 recalls the functional and variational tools used throughout the paper. In Section 3 we introduce the energy functional and establish the coercivity and equivalence properties of the associated norm. Section 4 is devoted to the Nehari manifold, while Section 5 establishes the Palais-Smale condition. The infinite multiplicity result is proved in Section 6. Finally, Section 7 presents a three-dimensional finite element experiment illustrating two distinct discrete solutions.

2. PRELIMINARIES

We use the standard notation $L^p(\Omega)$ and $W^{m,p}(\Omega)$ for Lebesgue and Sobolev spaces. We set $H^m(\Omega) = W^{m,2}(\Omega)$ and denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

Definition 2.1. A pair $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is called a weak solution of (1.1) if

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla \phi \, dx + \lambda \int_{\Omega} \phi \phi \, dx &= \int_{\Omega} |u|^{p-2} u \phi \, dx \quad \text{in } \Omega, \\ \int_{\Omega} \nabla \phi \cdot \nabla \psi \, dx &= \int_{\Omega} u \psi \, dx. \end{aligned} \quad (2.1)$$

for all $(\phi, \psi) \in H_0^1(\Omega)$.

Such a weak solution will simply be called a solution of (1.1). The following results can be found in [6, 8].

Theorem 2.2 (Rellich-Kondrashov). *Let $\Omega \subset \mathbb{R}^N$ a bounded domain and $p \in [1, 2^*)$. Then each bounded sequence in $H_0^1(\Omega)$ has a strongly convergent subsequence in $L^p(\Omega)$.*

Let H be a Hilbert space.

Theorem 2.3. (1) *Each bounded sequence in H has a weakly convergent subsequence.*

(2) *If $u_k \rightharpoonup u$ weak in H then*

$$\|u\| \leq \liminf_{k \rightarrow \infty} \|u_k\|.$$

(3) *If $u_k \rightharpoonup u$ weak in H and $\lim_{k \rightarrow \infty} \|u_k\| = \|u\|$, then $u_k \rightarrow u$ strongly in H .*

(4) *If $u_k \rightharpoonup u$ weakly in H , then (u_k) is bounded in H .*

Theorem 2.4 (Fundamental property of weak convergence). *Every bounded sequence in a Hilbert space H has a weakly convergent subsequence in H .*

Proposition 2.5. *Let $J : H \rightarrow \mathbb{R}$ be a C^1 function, let $\mathcal{M}$ be a C^1 -submanifold of H , and let u be a minimum (or a maximum) point of J on $\mathcal{M}$. Then*

$$\langle \nabla J(u), v \rangle = 0 \quad \text{for all } v \in T_u \mathcal{M}.$$

Definition 2.6. Let $J : H \rightarrow \mathbb{R}$ be a C^1 -functional and let $\mathcal{M}$ be a C^1 -submanifold of H . A point $u \in \mathcal{M}$ is called a *critical point* of J on $\mathcal{M}$ if

$$\langle \nabla J(u), v \rangle = 0 \quad \forall v \in T_u \mathcal{M}. \quad (2.2)$$

If $\mathcal{M} = H$, one simply says that u is a critical point of J , and condition (2.2) is equivalent to $\nabla J(u) = 0$.

A number $c \in \mathbb{R}$ is called a *critical value* of J on $\mathcal{M}$ if $c = J(u)$ for some critical point u of J on $\mathcal{M}$. Otherwise, c is called a *regular value* of J on $\mathcal{M}$.

Suppose now that $\mathcal{M}$ is a C^2 -Hilbert submanifold of H , and that $J : H \rightarrow \mathbb{R}$ is a C^2 -functional. Let us fix a C^2 -function $\Psi : \mathcal{O} \rightarrow \mathbb{R}$, defined on a neighborhood $\mathcal{O}$ of $\mathcal{M}$ in H , such that $\mathcal{M} = \Psi^{-1}(a_0)$, where a_0 is a regular value of Ψ .

Definition 2.7. The *gradient field of J on $\mathcal{M}$* is obtained by orthogonally projecting $\nabla J(u)$ onto $T_u\mathcal{M}$ for each $u \in \mathcal{M}$; that is,

$$\nabla_{\mathcal{M}}J(u) := \nabla J(u) - \frac{\langle \nabla J(u), \nabla \Psi(u) \rangle}{\|\nabla \Psi(u)\|^2} \nabla \Psi(u).$$

By shrinking $\mathcal{O}$ if necessary, we may assume without loss of generality that $\nabla \Psi(u) \neq 0$ for all $u \in \mathcal{O}$. Henceforth, $\mathcal{M}$ will denote a C^2 -Hilbert submanifold of a Hilbert space H , $J : H \rightarrow \mathbb{R}$ a C^2 -function, and $\nabla_{\mathcal{M}}J$ the gradient of J restricted to $\mathcal{M}$.

Definition 2.8. A sequence (u_k) satisfying

$$u_k \in \mathcal{M}, \quad J(u_k) \rightarrow c, \quad \nabla_{\mathcal{M}}J(u_k) \rightarrow 0,$$

is called a *Palais-Smale sequence* for J on $\mathcal{M}$ at c . We say that J satisfies the *Palais-Smale condition* $(PS)_{\mathcal{M},c}$ if every such sequence has a convergent subsequence in H . We simply say that J satisfies the *Palais-Smale condition on $\mathcal{M}$* if it satisfies $(PS)_{\mathcal{M},c}$ for every $c \in \mathbb{R}$.

Definition 2.9. A subset S of a vector space is said to be *symmetric* if

$$u \in S \iff -u \in S.$$

A function $f : S_1 \rightarrow S_2$ between two symmetric subsets S_1 and S_2 is said to be *odd* if

$$f(-u) = -f(u) \quad \text{for all } u \in S_1.$$

Definition 2.10 (Krasnoselskii genus). Let H be a real Banach (or Hilbert) space and let $S \subset H$ be a closed symmetric set such that

$$S = -S, \quad 0 \notin S.$$

The *Krasnoselskii genus* of S , denoted by $\text{gen}(S)$, is defined as the least integer $k \in \mathbb{N}$ such that there exists a continuous odd map

$$f : S \rightarrow \mathbb{R}^k \setminus \{0\}.$$

If no such integer exists, we set

$$\text{gen}(S) = +\infty.$$

Proposition 2.11. Let $\mathcal{S}$ and $\mathcal{R}$ be symmetric subsets of H . The Krasnosel'skii genus has the following properties:

- (1) *Nontriviality:* $\text{gen}(S) = 0$ if and only if $S = \emptyset$.
- (2) *Subadditivity:* If $\mathcal{S}$ and $\mathcal{R}$ are closed, then $\text{gen}(\mathcal{S} \cup \mathcal{R}) \leq \text{gen}(\mathcal{S}) + \text{gen}(\mathcal{R})$.
- (3) *Monotonicity:* If there exists a continuous odd function $f : \mathcal{S} \rightarrow \mathcal{R}$, then $\text{gen}(\mathcal{S}) \leq \text{gen}(\mathcal{R})$.
- (4) *Continuity:* If $\mathcal{S}$ is closed, then there exists a symmetric neighborhood $\mathcal{U}$ of $\mathcal{S}$ in H such that $\text{gen}(\mathcal{U}) = \text{gen}(\mathcal{S})$.
- (5) *Finiteness:* If $\mathcal{S}$ is compact and does not contain the origin, then $\text{gen}(\mathcal{S}) < \infty$. If $\mathcal{S}$ is finite and does not contain the origin, then $\text{gen}(\mathcal{S}) = 1$.

Theorem 2.12. Let $\mathcal{M}$ be a symmetric set not containing the origin. Assume that J is even, bounded from below on $\mathcal{M}$, and satisfies the Palais-Smale condition on $\mathcal{M}$. Then

$$\text{gen}(\mathcal{M}) \leq \sum_{c \in \mathbb{R}} \text{gen}(K_c),$$

where $K_c := \{u \in \mathcal{M} : J(u) = c, \nabla_{\mathcal{M}}J(u) = 0\}$. In particular, J possesses at least $\text{gen}(\mathcal{M})$ pairs of critical points $\pm u$ on $\mathcal{M}$.

Moreover, if $\text{gen}(\mathcal{M}) = \infty$, then the set of critical values of J on $\mathcal{M}$ is unbounded.

For a proof of the above theorem, see [8].

Corollary 2.13. (a) $\text{gen}(S^{m-1}) = m$.

(b) Let H be an infinite-dimensional Hilbert space and let $S_H := \{u \in H : \|u\| = 1\}$ denote its unit sphere. Then $\text{gen}(S_H) = \infty$.

3. FUNCTIONAL J

For each $u \in H_0^1(\Omega)$, consider $\phi_u \in H_0^1(\Omega)$ solution of

$$\begin{aligned} -\Delta\phi_u &= u \quad \text{in } \Omega, \\ \phi_u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \quad (3.1)$$

By the Lax-Milgram theorem, this problem has a unique solution and

$$\|\phi_u\|_{H_0^1(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

We define the energy functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\Omega} u\phi_u dx - \frac{1}{p} \int_{\Omega} |u|^p dx \quad (3.2)$$

Proposition 3.1. *The functional J is of class C^2 , and*

$$J'(u)v = \int_{\Omega} \nabla u \cdot \nabla v dx + \lambda \int_{\Omega} \phi_u v dx - \int_{\Omega} |u|^{p-2} uv dx, \quad \forall u, v \in H_0^1(\Omega).$$

Consequently, (u, ϕ_u) is a weak solution of (1.1) if and only if u is a critical point of J .

Note that

$$\int_{\Omega} u\phi_u dx = - \int_{\Omega} \Delta\phi_u \phi_u dx = \int_{\Omega} \nabla\phi_u \cdot \nabla\phi_u dx = \int_{\Omega} |\nabla\phi_u|^2 dx \geq 0. \quad (3.3)$$

From (3.2) and (3.3), we have

$$J(u) \geq \frac{c\lambda}{2} \|\nabla u\|_2^2 - \frac{1}{p} \|u\|_p^p,$$

where

$$c_\lambda = \begin{cases} 1, & \lambda \geq 0, \\ 1 + \frac{\lambda}{\lambda_1(\Omega)^2}, & -\lambda_1(\Omega)^2 < \lambda < 0. \end{cases}$$

Let $\lambda > -\lambda_1(\Omega)^2$, define

$$\langle u, v \rangle_\lambda := \int_{\Omega} \nabla u \cdot \nabla v dx + \lambda \int_{\Omega} \phi_u v dx,$$

and

$$\langle u, v \rangle := \langle u, v \rangle_0,$$

for all $u, v \in H_0^1(\Omega)$.

Lemma 3.2. $\langle \cdot, \cdot \rangle_\lambda$ defines an inner product in $H_0^1(\Omega)$.

Proof. (1) Bilinearity. The map $u \mapsto \phi_u$ is linear because the Dirichlet problem is linear. Indeed, if $-\Delta\phi_u = u$ and $-\Delta\phi_w = w$, then for $\alpha, \beta \in \mathbb{R}$, we have

$$-\Delta(\alpha\phi_u + \beta\phi_w) = \alpha u + \beta w,$$

hence $\phi_{\alpha u + \beta w} = \alpha\phi_u + \beta\phi_w$. Therefore,

$$\langle \alpha u + \beta w, v \rangle_\lambda = \int_{\Omega} \nabla(\alpha u + \beta w) \cdot \nabla v dx + \lambda \int_{\Omega} \phi_{\alpha u + \beta w} v dx,$$

so we obtain

$$\langle \alpha u + \beta w, v \rangle_\lambda = \alpha \langle u, v \rangle_\lambda + \beta \langle w, v \rangle_\lambda.$$

Similarly, linearity holds in the second variable.

(2) Symmetry. The first term is symmetric:

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} \nabla v \cdot \nabla u dx.$$

For the second term, since $-\Delta\phi_u = u$ and $-\Delta\phi_v = v$, integration by parts yields

$$\int_{\Omega} \phi_u v dx = \int_{\Omega} \phi_u (-\Delta\phi_v) dx = \int_{\Omega} \nabla\phi_u \cdot \nabla\phi_v dx.$$

Interchanging u and v ,

$$\int_{\Omega} \phi_v u \, dx = \int_{\Omega} \nabla \phi_v \cdot \nabla \phi_u \, dx.$$

Hence

$$\int_{\Omega} \phi_u v \, dx = \int_{\Omega} \phi_v u \, dx,$$

and therefore $\langle u, v \rangle_{\lambda} = \langle v, u \rangle_{\lambda}$.

(3) Positive definiteness. We compute

$$\begin{aligned} \langle u, u \rangle_{\lambda} &= \int_{\Omega} |\nabla u|^2 \, dx + \lambda \int_{\Omega} \phi_u u \, dx \\ &= \int_{\Omega} |\nabla u|^2 \, dx + \lambda \int_{\Omega} \phi_u (-\Delta \phi_u) \, dx \\ &= \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|\nabla \phi_u\|_{L^2}^2. \end{aligned}$$

If $\lambda \geq 0$, positivity is immediate. Assume now $\lambda < 0$. Since $-\Delta \phi_u = u$, by the variational characterization of the first eigenvalue,

$$\|\phi_u\|_2 \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|\nabla \phi_u\|_2. \quad (3.4)$$

Moreover, by Hölder inequality and (3.4), we have

$$\|\nabla \phi_u\|_2^2 = \int_{\Omega} |\nabla \phi_u|^2 \, dx = \int_{\Omega} \phi_u u \, dx \leq \|u\|_2 \|\phi_u\|_2 \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|u\|_2 \|\nabla \phi_u\|_2.$$

Then

$$\|\nabla \phi_u\|_2 \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|u\|_2$$

implies that

$$\int_{\Omega} |\nabla \phi_u|^2 \, dx \leq \frac{1}{\lambda_1(\Omega)} \|u\|_2^2. \quad (3.5)$$

On the other hand, by Poincaré inequality for u , we have

$$\|u\|_2 \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|\nabla u\|_2. \quad (3.6)$$

From (3.5) and (3.6), we obtain

$$\int_{\Omega} u \phi_u \, dx = \int_{\Omega} |\nabla \phi_u|^2 \, dx \leq \frac{1}{\lambda_1(\Omega)} \left(\frac{1}{\lambda_1(\Omega)} \|\nabla u\|_2^2 \right) = \frac{1}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2.$$

Since $\lambda < 0$, it follows that

$$\lambda \int_{\Omega} u \phi_u \, dx \geq \frac{\lambda}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2.$$

So,

$$\|\nabla u\|_2^2 + \lambda \int_{\Omega} u \phi_u \, dx \geq \|\nabla u\|_2^2 + \frac{\lambda}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2 = \left(1 + \frac{\lambda}{\lambda_1(\Omega)^2} \right) \|\nabla u\|_2^2.$$

Since $\lambda > -\lambda_1(\Omega)^2$, we have $1 + \frac{\lambda}{\lambda_1(\Omega)^2} > 0$, which implies

$$\|u\|_{\lambda}^2 = \langle u, u \rangle_{\lambda} \geq \left(1 + \frac{\lambda}{\lambda_1(\Omega)^2} \right) \|\nabla u\|_2^2 > 0.$$

So, $\langle u, u \rangle_{\lambda} > 0$ if $u \neq 0$, and if $\langle u, u \rangle_{\lambda} = 0$ then $\|\nabla u\|_2 = 0$ implies $u = 0$ in $H_0^1(\Omega)$. Therefore $\langle \cdot, \cdot \rangle_{\lambda}$ is positive definite. $\square$

We denote by $\|\cdot\|_{\lambda}$ induced norm, this is

$$\|u\|_{\lambda}^2 = \int_{\Omega} |\nabla u|^2 \, dx + \lambda \int_{\Omega} \phi_u u \, dx = \|\nabla u\|_2^2 + \lambda \int_{\Omega} \phi_u u \, dx,$$

where ϕ_u is solution Poisson equation and $\|u\| = \|\nabla u\|_2$ for all $u \in H_0^1(\Omega)$.

Lemma 3.3. $\|\cdot\|_\lambda$ is equivalent to $\|\cdot\|$.

Proof. Using Poincaré inequality,

$$\lambda_1(\Omega) \int_{\Omega} |\phi_u|^2 dx \leq \int_{\Omega} |\nabla \phi_u|^2 dx.$$

Moreover, by Hölder's inequality,

$$\int_{\Omega} \phi_u u dx \leq \|\phi_u\|_2 \|u\|_2.$$

Applying again the Poincaré inequality,

$$\|\phi_u\|_2 \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|\nabla \phi_u\|_2, \quad \|u\|_2 \leq \frac{1}{\sqrt{\lambda_1(\Omega)}} \|\nabla u\|_2.$$

Hence,

$$\int_{\Omega} \phi_u u dx \leq \frac{1}{\lambda_1(\Omega)} \|\nabla \phi_u\|_2 \|\nabla u\|_2.$$

Since $\|\nabla \phi_u\|_2^2 = \int_{\Omega} \phi_u u dx$, we deduce that

$$\|\nabla \phi_u\|_2^2 \leq \frac{1}{\lambda_1(\Omega)} \|\nabla \phi_u\|_2 \|\nabla u\|_2.$$

Therefore,

$$\|\nabla \phi_u\|_2 \leq \frac{1}{\lambda_1(\Omega)} \|\nabla u\|_2,$$

and consequently,

$$\int_{\Omega} \phi_u u dx = \|\nabla \phi_u\|_2^2 \leq \frac{1}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2. \quad (3.7)$$

Now,

$$\|u\|_\lambda^2 = \|\nabla u\|_2^2 + \lambda \int_{\Omega} \phi_u u dx,$$

then, of (3.7) we have

$$\|u\|_\lambda^2 \leq \|\nabla u\|_2^2 + |\lambda| \int_{\Omega} \phi_u u dx \leq \|\nabla u\|_2^2 + \frac{|\lambda|}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2 \leq \left(1 + \frac{|\lambda|}{\lambda_1(\Omega)^2}\right) \|\nabla u\|_2^2. \quad (3.8)$$

On the other hand, if $\lambda \geq 0$, we have $\|u\|_\lambda^2 \geq \|\nabla u\|_2^2$, this is

$$\|u\|_\lambda \geq \|\nabla u\|_2. \quad (3.9)$$

If $-\lambda_1(\Omega)^2 < \lambda < 0$, then of (3.7), we obtain $\lambda \int_{\Omega} \phi_u u dx \geq \frac{\lambda}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2$ implies $\|\nabla u\|_2^2 + \lambda \int_{\Omega} \phi_u u dx \geq \|\nabla u\|_2^2 + \frac{\lambda}{\lambda_1(\Omega)^2} \|\nabla u\|_2^2$, this is

$$\|u\|_\lambda^2 \geq \left(1 + \frac{\lambda}{\lambda_1(\Omega)^2}\right) \|\nabla u\|_2^2. \quad (3.10)$$

Since $\lambda > -\lambda_1(\Omega)^2$, we have

$$1 + \frac{\lambda}{\lambda_1(\Omega)^2} > 0.$$

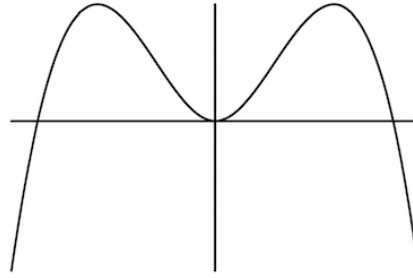
By (3.8), (3.9) and (3.10), there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|u\| \leq \|u\|_\lambda \leq C_2 \|u\|,$$

for all $u \in H_0^1(\Omega)$. Hence the norms are equivalent. $\square$

Clearly $(H_0^1(\Omega), \|\cdot\|_\lambda)$ is Hilbert space. To study the critical points of J , it is convenient to analyze its graph. To this end, let us fix a direction $u \in H_0^1(\Omega)$, $u \neq 0$, and examine the graph of J along the line generated by u . That is, consider the function $J_u : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$J_u(t) := J(tu) = \left(\frac{1}{2} \|u\|_\lambda^2\right) t^2 - \left(\frac{1}{p} \|u\|_p^p\right) |t|^p.$$

FIGURE 1. Graph of $J_u(t)$

Since we have assumed that $p > 2$, the graph of this function has the shape in Figure 1 and satisfies $\lim_{t \rightarrow -\infty} J(tu) = -\infty$.

Observe that J is not bounded from below and that it has a local minimum at 0. Obviously, 0 is a solution of problem (1.1). We are interested in determining whether this problem admits a nontrivial solution.

Direct calculations show that $J'_u(t) = t\|u\|_\lambda^2 - t^{p-1}\|u\|_p^p$, and if $J'_u(t) = 0$ with $t > 0$ implies that

$$t = \left(\frac{\|u\|_\lambda^2}{\|u\|_p^p} \right)^{\frac{1}{p-2}}$$

is critical point of J_u , moreover $J''_u(t) = (2-p)\|u\|_\lambda^2 < 0$, then $t_u = t$ is the unique maximum point of J_u on $(0, \infty)$. The nontrivial critical points of J are contained in the set $\{t_u u : u \in H_0^1(\Omega), \|u\|_\lambda = 1\}$. In fact, if u is critical point of J , then $J'(u) = 0 = \nabla J(u)$, in particular $\langle J'(u), u \rangle_\lambda = J'(u)(u) = 0$, since $u = \|u\|_\lambda v = tv$ with $\|v\|_\lambda = 1$, then $J'(tv)(tv) = tJ'(tv)(v) = 0$ implies $J'(tv)(v) = 0$, but $\frac{d}{dt}J(tv) = J'(tv)(v)$, so $t = \|u\|_\lambda$ it is precisely the value where the function J_v attains its maximum, is it $t = t_v$, so $u = tv = t_v v$ with $\|v\|_\lambda = 1$. This set is called the *Nehari manifold* and denoted

$$\mathcal{N} = \{u \in H_0^1(\Omega)/u \neq 0, J'(u)u = 0\} = \{u \in H_0^1(\Omega)/u \neq 0, \|u\|_\lambda^2 = \|u\|_p^p\}. \quad (3.11)$$

Obviously it contains all the non-trivial critical points of J , and has the following properties.

4. NEHARI MANIFOLD

Proposition 4.1. (1) *There exists $d_0 > 0$ such that $\|u\|_\lambda \geq d_0$ for all $u \in \mathcal{N}$. Consequently, $\mathcal{N}$ is a closed subset of $H_0^1(\Omega)$.*

(2) *$\mathcal{N}$ is a C^2 Hilbert submanifold of $H_0^1(\Omega)$ and is called the Nehari manifold.*

(3) *$u \notin T_u \mathcal{N}$ for every $u \in \mathcal{N}$.*

(4) *For each $u \in H_0^1(\Omega)$, $u \neq 0$, there exists a unique $t_u \in (0, \infty)$ such that $t_u u \in \mathcal{N}$. Moreover, t_u is the unique point in $(0, \infty)$ satisfying $\max_{t \geq 0} J(tu) = J(t_u u)$.*

Proof. (1) By Poincare inequality

$$\|u\|_p \leq C\|\nabla u\|_2$$

where $C > 0$ depending on Ω and p , $\|\nabla u\|_2$ is a norm in $H_0^1(\Omega)$. Then by the Lemma 3.3 implies $\|u\|_p \leq C\|u\|_\lambda$, therefore $\|u\|_p^p \leq C^p\|u\|_\lambda^p$ hence

$$C^{-p} \leq \frac{\|u\|_\lambda^p}{\|u\|_p^p} = \frac{\|u\|_\lambda^p}{\|u\|_\lambda^2} = \|u\|_\lambda^{p-2}.$$

this is,

$$0 < d_0 := C^{-p/(p-2)} \leq \|u\|_\lambda, \quad \forall u \in \mathcal{N}.$$

Consequently,

$$\mathcal{N} = \{u \in H_0^1(\Omega) : \|u\|_\lambda \geq d_0 \text{ and } \|u\|_\lambda^2 - \|u\|_p^p = 0\},$$

which is clearly a closed subset of $H_0^1(\Omega)$.

(2) Consider the function $\Psi : H_0^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ defined by

$$\Psi(u) = \|u\|_\lambda^2 - \|u\|_p^p.$$

Then $\mathcal{N} = \Psi^{-1}(0)$, Ψ is of class C^2 , and its derivative is

$$\Psi'(u)v = 2\langle u, v \rangle_\lambda - p \int_\Omega |u|^{p-2}uv \, dx, \quad \forall u, v \in H_0^1(\Omega).$$

Moreover, 0 is a regular value of Ψ since

$$\Psi'(u)u = 2\|u\|_\lambda^2 - p\|u\|_p^p = (2-p)\|u\|_\lambda^2 \neq 0, \quad \forall u \in \mathcal{N}. \quad (4.1)$$

This proves that $\mathcal{N}$ is a C^2 manifold.

(3) From (4.1), it follows that $u \notin \ker \Psi'(u) =: T_u\mathcal{N}$.

(4) Let $0 \neq u \in H_0^1(\Omega)$ and define $J_u : (0, \infty) \rightarrow \mathbb{R}$ by

$$J_u(t) = J(tu) = \left(\frac{1}{2}\|u\|_\lambda^2\right)t^2 - \left(\frac{1}{p}\|u\|_p^p\right)t^p.$$

This function has a unique critical point in $(0, \infty)$, and this point is a maximum. Furthermore, for $t \in (0, \infty)$,

$$J'_u(t) = J'(tu)u = 0 \iff J'(tu)(tu) = 0 \iff tu \in \mathcal{N}.$$

Hence, J_u attains its maximum at t if and only if $tu \in \mathcal{N}$. □

Lemma 4.2. *If $u \in \mathcal{N}$, then $J(tu) \leq J(u)$ for all $t \geq 0$.*

Proof. We have

$$J(tu) = \frac{1}{2}\|u\|_\lambda^2 t^2 - \frac{1}{p}\|u\|_p^p |t|^p = \frac{1}{2}\|u\|_\lambda^2 t^2 - \frac{1}{p}\|u\|_\lambda^2 |t|^p = \|u\|_\lambda^2 \left(\frac{pt^2 - 2|t|^p}{2p}\right) \leq \|u\|_\lambda^2 \frac{p-2}{2p} = J(u). \quad \square$$

Corollary 4.3. (1) $\inf_{u \in \mathcal{N}} J(u) > 0$.

(2) *If $u \in \mathcal{N}$ is a critical point of J on $\mathcal{N}$, then u is a nontrivial critical point of $J : H_0^1(\Omega) \rightarrow \mathbb{R}$, and consequently a nontrivial solution of system (1.1).*

Proof. (1) Observe that

$$J(u) = \frac{p-2}{2p} \|u\|_\lambda^2 = \frac{p-2}{2p} \|u\|_p^p, \quad \forall u \in \mathcal{N}. \quad (4.2)$$

The result follows From Proposition 4.1.

(2) Since u is a critical point of J restricted to $\mathcal{N}$ and 0 is a regular value of Ψ , there exists $\mu \in \mathbb{R}$ such that

$$J'(u) = \mu \Psi'(u).$$

Evaluating this identity at u and using $u \in \mathcal{N}$, we obtain

$$0 = J'(u)u = \mu \Psi'(u)u = \mu(2-p)\|u\|_\lambda^2.$$

Since $p > 2$ and $u \neq 0$, it follows that $\mu = 0$. Consequently, $J'(u) = 0$. Therefore, u is a nontrivial critical point of $J : H_0^1(\Omega) \rightarrow \mathbb{R}$, and hence a nontrivial weak solution of problem (1.1). □

Proposition 4.4. *If $p \in (2, 2^*)$, then J attains its minimum on $\mathcal{N}$.*

Proof. Let $c := \inf_{\mathcal{N}} J$, then there exists $(u_k) \subset \mathcal{N}$ such that $J(u_k) \rightarrow c$. Since $(u_k) \subset \mathcal{N}$ we have

$$J(u_k) = \frac{1}{2}\|u_k\|_\lambda^2 - \frac{1}{p}\|u_k\|_p^p = \frac{1}{2}\|u_k\|_\lambda^2 - \frac{1}{p}\|u_k\|_\lambda^2 = \frac{p-2}{2p}\|u_k\|_\lambda^2$$

then $(\|u_k\|_\lambda) \subset \mathbb{R}$ converge, hence $(u_k) \subset \mathcal{N} \subset H_0^1(\Omega)$ is bounded in $H_0^1(\Omega)$, and therefore has a subsequence which, for simplicity, we will denote in the same way such that

$$u_k \rightharpoonup u \quad \text{weak in } H_0^1(\Omega).$$

Moreover $p < 2^*$, by Rellich Kondrashov there exists a subsequence of (u_k) , which we also denote by (u_k) , that converges in $L^p(\Omega)$, that is, there exists $u \in L^p(\Omega)$ such that $u_k \rightarrow u$ strongly in $L^p(\Omega)$, then $\frac{p-2}{2p}\|u_k\|_p^p \rightarrow \frac{p-2}{2p}\|u\|_p^p = c > 0$, then $u \neq 0$.

Proposition 4.1, shows that there exists $t_u \in (0, \infty)$ such that $t_u u \in \mathcal{N}$. Since $u_k \in \mathcal{N}$ and $J(t_u u_k) \leq J(u_k)$, we have

$$\begin{aligned} c &\leq J(t_u u) = \frac{1}{2} \|t_u u\|_\lambda^2 - \frac{1}{p} \|t_u u\|_p^p \\ &\leq \frac{1}{2} \liminf_{k \rightarrow \infty} \|t_u u_k\|_\lambda^2 - \lim_{k \rightarrow \infty} \frac{1}{p} \|t_u u_k\|_p^p \\ &= \liminf_{k \rightarrow \infty} \left(\frac{1}{2} \|t_u u_k\|_\lambda^2 - \frac{1}{p} \|t_u u_k\|_p^p \right) \\ &= \liminf_{k \rightarrow \infty} J(t_u u_k) \\ &\leq \liminf_{k \rightarrow \infty} J(u_k) = c. \end{aligned}$$

Then $J(t_u u) = c$, this is, J attains its minimum at $\mathcal{N}$. Moreover, since

$$0 < c = \frac{p-2}{2p} \|t_u u\|_p^p = \lim_{k \rightarrow \infty} \frac{p-2}{2p} \|t_u u_k\|_p^p = |t_u|^p c,$$

we conclude that $t_u = 1$. Furthermore, since

$$c = \frac{p-2}{2p} \|u\|_\lambda^2 \leq \liminf_{k \rightarrow \infty} \frac{p-2}{2p} \|u_k\|_\lambda^2 = \lim_{k \rightarrow \infty} \frac{p-2}{2p} \|u_k\|_\lambda^2 = c,$$

it follows that $\lim_{k \rightarrow \infty} \|u_k\|_\lambda^2 = \|u\|_\lambda^2$. Therefore, $u_k \rightarrow u$ strongly in $H_0^1(\Omega)$, and $J(u) = c$. $\square$

5. INFINITELY MANY SOLUTIONS

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, let $\lambda \in (-\lambda_1(\Omega)^2, \infty)$ and let $p \in (2, 2^*)$. We shall prove that problem (1.1) has infinitely many weak solutions.

5.1. Palais-Smale condition. On $H_0^1(\Omega)$, we consider inner product

$$\langle u, v \rangle_\lambda = \int_\Omega (\nabla u \cdot \nabla v + \lambda \phi_u v) \, dx, \quad (5.1)$$

and we denote by $\|\cdot\|_\lambda$ the induced norm. Recall that the nontrivial solutions (weak solutions) of (1.1) are precisely the critical points of the functional

$$J : H_0^1(\Omega) \rightarrow \mathbb{R},$$

defined by

$$J(u) = \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{p} \|u\|_p^p,$$

on the Nehari manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega) : u \neq 0, \|u\|_\lambda^2 = \|u\|_p^p\} = \Psi^{-1}(0),$$

where $\Psi : H_0^1(\Omega) \setminus \{0\} \rightarrow \mathbb{R}$ is the function

$$\Psi(u) = \|u\|_\lambda^2 - \|u\|_p^p.$$

Let ∇J and $\nabla \Psi$ denote the gradients of J and Ψ with respect to the inner product $\langle \cdot, \cdot \rangle_\lambda$, that is,

$$\nabla J(u) = u - \nabla \Phi(u), \quad \nabla \Psi(u) = 2u - p \nabla \Phi(u),$$

where $\Phi(u) := \frac{1}{p} \|u\|_p^p$. Then

$$\Phi'(u)v = \langle \nabla \Phi(u), v \rangle_\lambda = \int_\Omega |u|^{p-2} uv \, dx, \quad \forall u, v \in H_0^1(\Omega).$$

Using Hölder's and Sobolev's inequalities, we obtain

$$|\langle \nabla \Phi(u), v \rangle_\lambda| \leq \int_\Omega |u|^{p-1} |v| \, dx \leq \| |u|^{p-1} \|_{\frac{p}{p-1}} \|v\|_p = \|u\|_p^{p-1} \|v\|_p \leq C \|u\|_p^{p-1} \|v\|_\lambda, \quad (5.2)$$

for all $u, v \in H_0^1(\Omega)$.

Replacing v by $\nabla \Phi(u)$, we conclude that

$$\|\nabla \Phi(u)\|_\lambda \leq C \|u\|_p^{p-1}, \quad \forall u \in H_0^1(\Omega). \quad (5.3)$$

Proposition 5.1. *The functional J satisfies the Palais-Smale condition on $\mathcal{N}$.*

Proof. Let (u_k) be a sequence in $\mathcal{N}$ such that $J(u_k) \rightarrow c$ and $\nabla_{\mathcal{N}} J(u_k) \rightarrow 0$. Then

$$J(u_k) = \frac{1}{2}\|u_k\|_{\lambda}^2 - \frac{1}{p}\|u_k\|_p^p = \frac{1}{2}\|u_k\|_{\lambda}^2 - \frac{1}{p}\|u_k\|_{\lambda}^2 = \frac{p-2}{2p}\|u_k\|_{\lambda}^2 = \frac{p-2}{2p}\|u_k\|_p^p \rightarrow c > 0,$$

then $(u_k) \subset H_0^1(\Omega)$ is bounded.

Since $\nabla_{\mathcal{N}} J(u_k) \rightarrow 0$ and (u_k) is bounded in $H_0^1(\Omega)$, it follows that

$$\langle \nabla_{\mathcal{N}} J(u_k), u_k \rangle_{\lambda} \rightarrow 0. \quad (5.4)$$

We first prove that $\nabla J(u_k) \rightarrow 0$. Since

$$\nabla_{\mathcal{N}} J(u_k) = \nabla J(u_k) - \frac{\langle \nabla J(u_k), \nabla \Psi(u_k) \rangle_{\lambda}}{\|\nabla \Psi(u_k)\|_{\lambda}^2} \nabla \Psi(u_k).$$

Let

$$t_k = \frac{\langle \nabla J(u_k), \nabla \Psi(u_k) \rangle_{\lambda}}{\|\nabla \Psi(u_k)\|_{\lambda}^2}.$$

Then

$$\nabla J(u_k) = \nabla_{\mathcal{N}} J(u_k) + t_k \nabla \Psi(u_k), \quad t_k \in \mathbb{R}. \quad (5.5)$$

Taking the inner product with u_k and using Proposition 4.1, we conclude that there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \langle \nabla_{\mathcal{N}} J(u_k), u_k \rangle_{\lambda} &= \langle \nabla J(u_k), u_k \rangle_{\lambda} - t_k \langle \nabla \Psi(u_k), u_k \rangle_{\lambda} \\ &= \langle u_k - \nabla \Phi(u_k), u_k \rangle_{\lambda} - t_k \langle 2u_k - p \nabla \Phi(u_k), u_k \rangle_{\lambda} \\ &= \|u_k\|_{\lambda}^2 - \langle \nabla \Phi(u_k), u_k \rangle_{\lambda} - 2t_k \|u_k\|_{\lambda}^2 + t_k p \langle \nabla \Phi(u_k), u_k \rangle_{\lambda} \\ &= (1 - 2t_k) \|u_k\|_{\lambda}^2 - \int_{\Omega} |u_k|^{p-2} u_k u_k \, dx + t_k p \int_{\Omega} |u_k|^{p-2} u_k u_k \, dx \\ &= (1 - 2t_k) \|u_k\|_{\lambda}^2 + (t_k p - 1) \int_{\Omega} |u_k|^p \, dx \\ &= (1 - 2t_k) \|u_k\|_{\lambda}^2 + (t_k p - 1) \|u_k\|_p^p \\ &= t_k (p - 2) \|u_k\|_{\lambda}^2, \end{aligned}$$

hence

$$|\langle \nabla_{\mathcal{N}} J(u_k), u_k \rangle_{\lambda}| = |t_k| (p - 2) \|u_k\|_{\lambda}^2 \geq c_1 |t_k|.$$

From (5.4), it follows that $t_k \rightarrow 0$. On the other hand, using inequality (5.3) we conclude that there exists a constant $c_2 > 0$ such that

$$\|\nabla \Psi(u_k)\|_{\lambda} \leq 2\|u_k\|_{\lambda} + p\|\nabla \Phi(u_k)\|_{\lambda} \leq 2\|u_k\|_{\lambda} + pC\|u_k\|_p^{p-1} \leq c_2.$$

Thus, since $(\nabla \Psi(u_k))$ is bounded, $\nabla_{\mathcal{N}} J(u_k) \rightarrow 0$, and $t_k \rightarrow 0$, identity (5.5) implies that $\nabla J(u_k) \rightarrow 0$.

We now prove that (u_k) has a convergent subsequence. Theorems 2.4 and 2.2 ensure that (u_k) has a subsequence, still denoted by (u_k) , such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_k &\rightarrow u \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

It follows from (5.2) that

$$|\langle \nabla \Phi(u_k), u_k - u \rangle_{\lambda}| \leq \|u_k\|_p^{p-1} \|u_k - u\|_p \leq c_3 \|u_k - u\|_p \rightarrow 0.$$

Therefore,

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \langle \nabla J(u_k), u_k - u \rangle_{\lambda} \\ &= \lim_{k \rightarrow \infty} \langle u_k, u_k - u \rangle_{\lambda} - \lim_{k \rightarrow \infty} \langle \nabla \Phi(u_k), u_k - u \rangle_{\lambda} \\ &= \lim_{k \rightarrow \infty} \langle u_k, u_k - u \rangle_{\lambda} \end{aligned}$$

$$\begin{aligned}
&= \lim_{k \rightarrow \infty} \|u_k\|_\lambda^2 - \lim_{k \rightarrow \infty} \langle u_k, u \rangle_\lambda \\
&= \lim_{k \rightarrow \infty} \|u_k\|_\lambda^2 - \|u\|_\lambda^2.
\end{aligned}$$

Proposition 2.3 then implies that $u_k \rightarrow u$ strongly in $H_0^1(\Omega)$, which proves that J satisfies the Palais-Smale condition on $\mathcal{N}$. $\square$

6. MAIN RESULTS

Theorem 6.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, let $\lambda \in (-\lambda_1(\Omega)^2, \infty)$, and let $p \in (2, 2^*)$. Then the system (1.1) has at least one nontrivial solution.*

The above result is an immediate consequence of Propositions 4.4, 2.5, and Corollary 4.3.

Theorem 6.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, let $\lambda \in (-\lambda_1(\Omega)^2, \infty)$, and let $p \in (2, 2^*)$. Then system (1.1) possesses a sequence of solutions (u_k) satisfying $\lim_{k \rightarrow \infty} \|u_k\|_\lambda^2 = \infty$.*

Proof. We shall apply Theorem (2.12). Observe that $0 \notin \mathcal{N}$ and that

$$u \in \mathcal{N} \iff -u \in \mathcal{N} \quad \text{and} \quad J(u) = J(-u), \quad \forall u \in H_0^1(\Omega).$$

Propositions (4.1) and (5.1) ensure that J is bounded from below on $\mathcal{N}$ and satisfies the Palais-Smale condition on $\mathcal{N}$. Let

$$S_\lambda := \{u \in H_0^1(\Omega) : \|u\|_\lambda = 1\}$$

be the unit sphere. The map $h : S_\lambda \rightarrow \mathcal{N}$ defined by

$$h(u) := \frac{u}{\|u\|_p^{\frac{p}{p-2}}}$$

is odd and continuous. Using Corollary 2.13 and the monotonicity of the genus, we conclude that

$$\infty = \text{gen}(S_\lambda) \leq \text{gen}(\mathcal{N}).$$

Theorem 2.12 therefore guarantees that J has an unbounded sequence of critical values on $\mathcal{N}$; that is, there exists a sequence of critical points (u_k) of J on $\mathcal{N}$ such that

$$J(u_k) = \frac{p-2}{2p} \|u_k\|_\lambda^2 \rightarrow \infty.$$

This completes the proof. $\square$

Remark 6.3. The previous result suggests that the variational approach developed in this paper may also be relevant for higher-order elliptic problems. As discussed in the Introduction, problems involving the inverse Laplacian are naturally related, at least formally, to higher-order equations. Recent works on triharmonic and polyharmonic equations provide further motivation in this direction. In particular, Meng and Feng [13] studied the asymptotic behavior of large radial solutions for triharmonic equations, whereas Feng, Lu and Xu [10] obtained multiplicity results for nonlinear polyharmonic equations under different boundary conditions. It would be interesting to investigate whether the Nehari manifold and Krasnosel'skii genus methods used here can be adapted to suitable higher-order counterparts of the present problem. Such an extension would require a separate analysis of the associated functional setting, boundary conditions, and coercivity properties.

7. NUMERICAL EXPERIMENTS

In this section, we present a three-dimensional numerical experiment for the nonlinear Poisson-Poisson system. The aim is to illustrate the existence of different non-trivial discrete solutions and the associated variational structure. We first compute a solution that is nonnegative up to numerical precision and compare it on three meshes. We then use a different initial profile to obtain a second solution that changes sign. We do not aim to prove the infinite multiplicity result numerically, nor to derive convergence rates or a priori error estimates. We consider

$$\Omega = (0, 1)^3, \quad p = 3, \quad \lambda = 1,$$

with homogeneous Dirichlet boundary conditions. Since $N = 3$, one has $2^* = 6$, and therefore

$$2 < p = 3 < 2^*.$$

Let $\mathcal{T}_h$ be a tetrahedral partition of Ω , and let $V_h \subset H_0^1(\Omega)$ be the space of continuous piecewise linear finite elements, namely

$$V_h = \{v_h \in C(\overline{\Omega}) : v_h|_K \in P_1(K), \forall K \in \mathcal{T}_h, v_h = 0 \text{ on } \partial\Omega\}.$$

We look for $(u_h, \phi_h) \in V_h \times V_h$ such that

$$\begin{aligned} \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx + \lambda \int_{\Omega} \phi_h v_h \, dx &= \int_{\Omega} |u_h|^{p-2} u_h v_h \, dx, \\ \int_{\Omega} \nabla \phi_h \cdot \nabla \psi_h \, dx &= \int_{\Omega} u_h \psi_h \, dx, \end{aligned}$$

for every $v_h, \psi_h \in V_h$.

The computations were performed in FreeFem++ using continuous P_1 finite elements on tetrahedral meshes of the unit cube. The parameter n denotes the number of subdivisions in each coordinate direction. The mesh $n = 18$ is used for the detailed numerical results and graphical illustrations.

To obtain two different discrete solutions, we use the initial profiles

$$u_{0,h}^{(1)} = I_h(\sin(\pi x) \sin(\pi y) \sin(\pi z))$$

and

$$u_{0,h}^{(2)} = I_h(\sin(2\pi x) \sin(\pi y) \sin(\pi z)),$$

where I_h denotes the nodal interpolation operator. The first profile is positive in the interior of the cube, whereas the second one changes sign across the plane $x = 1/2$. The latter is used to guide Newton's method toward a sign-changing solution.

For $j = 1, 2$, the corresponding initial potential $\phi_{0,h}^{(j)} \in V_h$ is obtained from

$$\int_{\Omega} \nabla \phi_{0,h}^{(j)} \cdot \nabla \psi_h \, dx = \int_{\Omega} u_{0,h}^{(j)} \psi_h \, dx, \quad \forall \psi_h \in V_h.$$

Before starting Newton's method, each initial pair is rescaled using the discrete Nehari identity. We define

$$A_h^{(j)} = \int_{\Omega} |\nabla u_{0,h}^{(j)}|^2 \, dx + \lambda \int_{\Omega} u_{0,h}^{(j)} \phi_{0,h}^{(j)} \, dx \quad \text{and} \quad B_h^{(j)} = \int_{\Omega} |u_{0,h}^{(j)}|^p \, dx.$$

Since the potential associated with $tu_{0,h}^{(j)}$ is $t\phi_{0,h}^{(j)}$, we take

$$t_0^{(j)} = \left(\frac{A_h^{(j)}}{B_h^{(j)}} \right)^{\frac{1}{p-2}},$$

and replace

$$u_{0,h}^{(j)} \leftarrow t_0^{(j)} u_{0,h}^{(j)}, \quad \phi_{0,h}^{(j)} \leftarrow t_0^{(j)} \phi_{0,h}^{(j)}.$$

For $p = 3$, this reduces to

$$t_0^{(j)} = \frac{A_h^{(j)}}{B_h^{(j)}}.$$

The nonlinear discrete system is solved separately for the two initial pairs by Newton's method with the fixed damping factor $\theta = 0.7$. Both computations are carried out directly on the full cube. In particular, no symmetry condition is imposed in the computation of the second solution.

For an iterate (u_h^k, ϕ_h^k) , we define

$$R_{1,h}^k(v_h) = \int_{\Omega} \nabla u_h^k \cdot \nabla v_h \, dx + \lambda \int_{\Omega} \phi_h^k v_h \, dx - \int_{\Omega} |u_h^k|^{p-2} u_h^k v_h \, dx,$$

and

$$R_{2,h}^k(\psi_h) = \int_{\Omega} \nabla \phi_h^k \cdot \nabla \psi_h \, dx - \int_{\Omega} u_h^k \psi_h \, dx.$$

The Newton correction

$$(\delta u_h^k, \delta \phi_h^k) \in V_h \times V_h$$

is computed from

$$\begin{aligned} \int_{\Omega} \nabla \delta u_h^k \cdot \nabla v_h \, dx + \lambda \int_{\Omega} \delta \phi_h^k v_h \, dx - (p-1) \int_{\Omega} |u_h^k|^{p-2} \delta u_h^k v_h \, dx &= -R_{1,h}^k(v_h), \\ \int_{\Omega} \nabla \delta \phi_h^k \cdot \nabla \psi_h \, dx - \int_{\Omega} \delta u_h^k \psi_h \, dx &= -R_{2,h}^k(\psi_h). \end{aligned}$$

The damped update is

$$u_h^{k+1} = u_h^k + \theta \delta u_h^k, \quad \phi_h^{k+1} = \phi_h^k + \theta \delta \phi_h^k, \quad \theta = 0.7.$$

The actual increment is measured by

$$\Delta_k = \left(\|u_h^{k+1} - u_h^k\|_{H^1(\Omega)}^2 + \|\phi_h^{k+1} - \phi_h^k\|_{H^1(\Omega)}^2 \right)^{1/2}.$$

To monitor the two discrete equations, let $r_{1,h}, r_{2,h} \in V_h$ satisfy

$$\begin{aligned} \int_{\Omega} \nabla r_{1,h} \cdot \nabla v_h \, dx &= R_{1,h}(v_h), \quad \forall v_h \in V_h, \\ \int_{\Omega} \nabla r_{2,h} \cdot \nabla \psi_h \, dx &= R_{2,h}(\psi_h), \quad \forall \psi_h \in V_h. \end{aligned}$$

The normalized residual indicators are defined by

$$\rho_{i,h} = \frac{\|\nabla r_{i,h}\|_{L^2(\Omega)}}{1 + \|u_h\|_{H^1(\Omega)} + \|\phi_h\|_{H^1(\Omega)}}, \quad i = 1, 2.$$

The iteration is stopped when

$$\Delta_k < 10^{-8}, \quad \max\{\rho_{1,h}, \rho_{2,h}\} < 10^{-8}.$$

For each computed solution, we evaluate the discrete energy

$$J_h(u_h) = \frac{1}{2} \int_{\Omega} |\nabla u_h|^2 \, dx + \frac{\lambda}{2} \int_{\Omega} u_h \phi_h \, dx - \frac{1}{p} \int_{\Omega} |u_h|^p \, dx,$$

with

$$\|u_h\|_{\lambda,h}^2 = \int_{\Omega} |\nabla u_h|^2 \, dx + \lambda \int_{\Omega} u_h \phi_h \, dx, \quad \|u_h\|_{p,h}^p = \int_{\Omega} |u_h|^p \, dx,$$

and the Nehari discrepancy

$$E_{N,h} = \left| \|u_h\|_{\lambda,h}^2 - \|u_h\|_{p,h}^p \right|.$$

As a simple check of the effect of mesh refinement, the first computation was repeated for $n \in \{10, 14, 18\}$. The same initial profile, damping parameter, stopping criteria, and numerical settings were used in the three cases.

Besides comparing the energy and maximum values, we also compare the discrete functions obtained on successive meshes. Since these solutions are computed on different meshes, the solution on the coarser mesh is first evaluated at the nodes of the finer mesh.

Let $\mathcal{N}_{h_n}$ denote the set of nodes of the mesh with n subdivisions, and define the discrete nodal ℓ^2 -norm by

$$\|v_h\|_{\ell^2(\mathcal{N}_{h_n})} = \left(\sum_{x_i \in \mathcal{N}_{h_n}} |v_h(x_i)|^2 \right)^{1/2}.$$

We then set

$$D_{n,n_{\text{prev}}}^{(1)} = \frac{\|u_{h_n}^{(1)} - \mathcal{I}_{n \leftarrow n_{\text{prev}}} u_{h_{n_{\text{prev}}}}^{(1)}\|_{\ell^2(\mathcal{N}_{h_n})}}{\|u_{h_n}^{(1)}\|_{\ell^2(\mathcal{N}_{h_n})}},$$

where n_{prev} denotes the subdivision parameter of the previous (coarser) mesh, and $\mathcal{I}_{n \leftarrow n_{\text{prev}}} u_{h_{n_{\text{prev}}}}^{(1)}$ denotes the solution computed on the coarser mesh and evaluated at the nodes of the finer mesh. The results for the first solution are collected in Table 1.

TABLE 1. Comparison of the first discrete solution on successive meshes.

n	DOF	$J_h(u_h^{(1)})$	$E_{N,h}^{(1)}$	$\max\{\rho_{1,h}, \rho_{2,h}\}^{(1)}$	$\max u_h^{(1)} $	$D_{n,n_{\text{prev}}}^{(1)}$
10	1458	1413.96	9.16698×10^{-8}	2.02532×10^{-11}	61.0098	—
14	4394	1321.66	9.32887×10^{-8}	2.07359×10^{-11}	59.3043	2.35114×10^{-2}
18	9826	1284.53	9.38417×10^{-8}	2.09670×10^{-11}	58.6294	9.99541×10^{-3}

For the first solution, both the discrete energy and the maximum value decrease as n increases. More importantly, the relative nodal difference between successive solutions also decreases. This gives a simple consistency check under mesh refinement. We do not use these values to claim a convergence rate or a finite element error estimate.

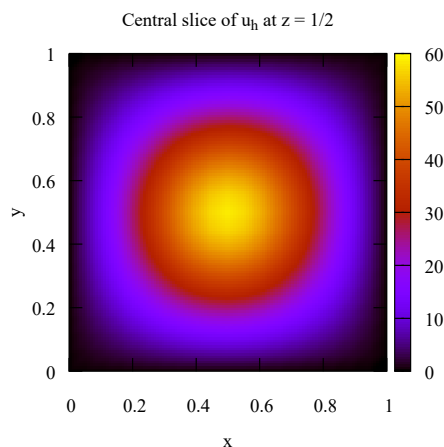
The second solution was computed on the representative mesh $n = 18$. The main numerical values for both solutions on this mesh are given in Table 2.

TABLE 2. Main numerical values for the two solutions computed on the mesh $n = 18$.

solution	steps	final Δ_k	$J_h(u_h)$	$\ u_h\ _{\lambda,h}^2$	$E_{N,h}$	$\max\{\rho_{1,h}, \rho_{2,h}\}$	$\min u_h$	$\max u_h$
$u_h^{(1)}$	19	6.51079×10^{-9}	1284.53	7707.17	9.38417×10^{-8}	2.09670×10^{-11}	-1.27588×10^{-31}	58.6294
$u_h^{(2)}$	20	9.29682×10^{-9}	9570.74	57424.4	5.47880×10^{-7}	1.00378×10^{-11}	-126.38	126.38

The final increments and residual indicators in Table 2 satisfy the prescribed stopping criteria. The Nehari discrepancies are also very small compared with the size of the corresponding quantities. Thus, both computed pairs are consistent with the discrete system.

The minimum value of $u_h^{(1)}$ is of the order of floating-point round-off. Therefore, the first solution can be regarded as nonnegative up to numerical precision. Figures 2 and 3 show the central slices of $u_h^{(1)}$ and its associated potential $\phi_h^{(1)}$, respectively, in the plane $z = 1/2$.

FIGURE 2. Central slice of $u_h^{(1)}$ in the plane $z = 1/2$.

The first solution is concentrated near the center of the cube. The potential has a smoother spatial profile, consistently with the elliptic smoothing effect of the discrete Poisson equation. In contrast, the second solution satisfies

$$\min u_h^{(2)} = -126.38 < 0 < 126.38 = \max u_h^{(2)},$$

and therefore changes sign. Since the first solution is nonnegative up to numerical precision, $u_h^{(2)}$ is different from both $u_h^{(1)}$ and $-u_h^{(1)}$. Their different discrete energies give an additional numerical distinction between the two solutions.

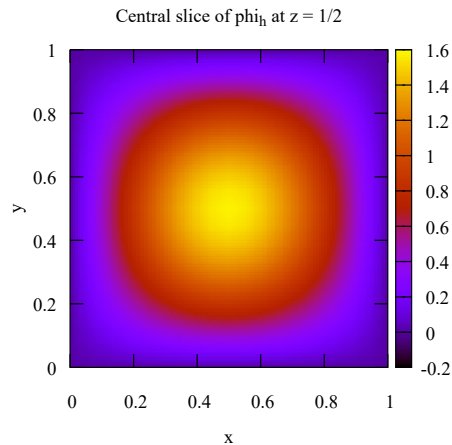
FIGURE 3. Central slice of $\phi_h^{(1)}$ in the plane $z = 1/2$.

Figure 4 shows the second solution in the plane $z = 1/2$.

FIGURE 4. Central slice of the sign-changing solution $u_h^{(2)}$ in the plane $z = 1/2$.

The positive and negative regions displayed in Figure 4 provide a visual illustration of the nodal structure of the second solution.

Finally, for $p = 3$, an exact discrete solution satisfying the Nehari identity obeys

$$J_h(u_h) = \frac{1}{6} \|u_h\|_{\lambda, h}^2.$$

The values in Table 2 are consistent with the discrete Nehari identity. Indeed, for the two computed solutions, we obtain

$$\frac{7707.17}{6} \approx 1284.53, \quad \frac{57424.4}{6} \approx 9570.74.$$

In summary, the experiment gives two distinct nontrivial discrete solutions of the nonlinear Poisson-Poisson system. The first one is nonnegative up to numerical precision, while the second one changes sign. The comparison on the meshes $n = 10, 14, 18$ also shows that the relative difference between successive approximations decreases under mesh refinement. These computations do not prove the infinite multiplicity result, but they provide a numerical illustration that is consistent with the analytical results of the paper.

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