

CRITICAL GRUSHIN-CHOQUARD PROBLEMS WITH CONCAVE AND SUPERLINEAR PERTURBATIONS

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ABSTRACT. Let $m, n \geq 1$, $\gamma > 0$, and $N_\gamma = m + (1 + \gamma)n > 2$. Let $\Omega \subset \mathbb{R}^m \times \mathbb{R}^n$ be a bounded smooth domain intersecting $\{0\} \times \mathbb{R}^n$. We study the boundary-value problem

$$\begin{aligned} -\Delta_\gamma u &= \lambda g(z, u) + \left(\int_\Omega \frac{|u(w)|^p}{d(z-w)^\mu} dw \right) |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $z = (x, y)$, $\Delta_\gamma = \Delta_x + (1 + \gamma)^2 |x|^{2\gamma} \Delta_y$, $d(z) = (|x|^{2(1+\gamma)} + |y|^2)^{1/[2(1+\gamma)]}$, $0 < \mu < N_\gamma$, and $p = (2N_\gamma - \mu)/(N_\gamma - 2)$. We use a nonlocal Brézis-Lieb splitting and variational methods to prove compactness below the critical energy. For $1 < q < 2$ and small $\lambda > 0$, the problem with $g(z, u) = |u|^{q-2}u$ has a nontrivial nonnegative local minimizer of negative energy. A second nontrivial nonnegative solution is obtained under a quantitative concentration condition. We verify this condition in an explicit range of q under a whole-space extremal and decay assumption. For $2 < q < 2_\gamma^* = 2N_\gamma/(N_\gamma - 2)$ and large λ , there is a nontrivial nonnegative mountain-pass solution whose energy and norm tend to zero as $\lambda \rightarrow \infty$. The large-parameter existence result extends to general subcritical Carathéodory nonlinearities satisfying the stated growth and superlinearity assumptions.

1. INTRODUCTION

Let $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, $m, n \geq 1$, and $N = m + n$. For $\gamma > 0$, define

$$\Delta_\gamma = \Delta_x + (1 + \gamma)^2 |x|^{2\gamma} \Delta_y, \quad d(z) = \left(|x|^{2(1+\gamma)} + |y|^2 \right)^{\frac{1}{2(1+\gamma)}}.$$

The operator degenerates on $\{0\} \times \mathbb{R}^n$. Its dilations are $\delta_t(x, y) = (tx, t^{1+\gamma}y)$, and its homogeneous dimension is $N_\gamma = m + (1 + \gamma)n > 2$. Set

$$2_\gamma^* = \frac{2N_\gamma}{N_\gamma - 2}, \quad p = 2_{\gamma, \mu}^* = \frac{2N_\gamma - \mu}{N_\gamma - 2}, \quad 0 < \mu < N_\gamma.$$

Throughout this paper, we assume

$$\begin{aligned} m, n &\geq 1, \quad \gamma > 0, \\ \Omega &\subset \mathbb{R}^m \times \mathbb{R}^n \text{ is bounded and smooth, } \Omega \cap (\{0\} \times \mathbb{R}^n) \neq \emptyset. \end{aligned} \tag{1.1}$$

The intersection condition permits anisotropic concentration at an interior point $(0, y_0)$. Such concentration preserves the Dirichlet integral and the critical Choquard energy; see Proposition 3.1.

For $\lambda > 0$ and $q \in (1, 2_\gamma^*) \setminus \{2\}$, we consider

$$\begin{aligned} -\Delta_\gamma u &= \lambda |u|^{q-2}u + \left(\int_\Omega \frac{|u(w)|^p}{d(z-w)^\mu} dw \right) |u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

2020 *Mathematics Subject Classification*. 35J70, 35H20, 35A15, 35B33, 35B38.

Key words and phrases. Grushin operator; Choquard equation; concave perturbation; Hardy-Littlewood-Sobolev critical exponent; superlinear perturbation.

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Submitted September 21, 2026. Published October 23, 2026.

We also replace the power perturbation by a subcritical Carathéodory function and study

$$\begin{aligned} -\Delta_\gamma u &= \lambda f(z, u) + \left(\int_\Omega \frac{|u(w)|^p}{d(z-w)^\mu} dw \right) |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.3)$$

The interaction between a lower-order perturbation and critical growth was studied by Brézis and Nirenberg [9]. Ambrosetti, Brézis and Cerami [4] treated concave and convex powers, while Cerami, Fortunato and Struwe [10] studied bifurcation and multiplicity. The variational arguments use mountain-pass theory [5, 39] and critical compactness methods [24, 25].

Choquard equations originate in the models of Pekar [36] and the analysis of Lieb [21]. Their nonlocal variational structure was studied in [1, 23, 32, 33, 34]. Gao and Yang [17] considered the critical Choquard analogue of the Brézis–Nirenberg problem. Related fractional critical problems were treated in [35, 13].

The Grushin operator belongs to the degenerate elliptic models developed in [38, 18, 6]. Embedding and regularity results for the associated vector fields appear in [11, 12, 15, 20, 26, 27, 30, 31]. The corresponding potential estimates extend the classical theory in [14, 22, 37]. We use the Grushin Hardy–Littlewood–Sobolev inequality proved in [19].

Alves and de Holanda [2] studied subcritical Grushin equations. Alves, Gandal, Loiudice and Tyagi [3] treated local critical problems with power perturbations. Gandal, Loiudice and Tyagi [16] proved attainment and sharp two-sided decay for local Grushin Sobolev extremals. Their Euler–Lagrange equation is local. Those results therefore do not supply an extremal for the critical Choquard quotient considered here. Bifurcation and multiplicity for local critical Grushin problems were obtained in [28, 29].

For the critical Grushin–Choquard problem with perturbation λu , Kanungo, Mishra and Molica Bisci [19] proved bifurcation and multiplicity results. Their Hardy–Littlewood–Sobolev inequality bounds the nonlocal form. It does not establish attainment or decay for its sharp whole-space quotient. Bernini and Malanchini [7] studied an entire Grushin–Choquard equation with a positive mass term. The present concentration problem instead leads to a massless critical equation. We study the effect of concave powers, superlinear powers and general subcritical perturbations on existence and compactness.

Let $H_\gamma(\Omega)$ be the completion of $C_c^\infty(\Omega)$ under the norm $\|u\|_\gamma^2 = \int_\Omega |\nabla_\gamma u|^2 dz$, where $\nabla_\gamma u = (\nabla_x u, (1+\gamma)|x|^\gamma \nabla_y u)$. Define

$$\mathcal{B}(u) = \int_\Omega \int_\Omega \frac{|u(z)|^p |u(w)|^p}{d(z-w)^\mu} dz dw, \quad S_{\gamma,\mu}(\Omega) = \inf_{u \in H_\gamma(\Omega) \setminus \{0\}} \frac{\|u\|_\gamma^2}{\mathcal{B}(u)^{1/p}}.$$

The embedding into $L^{2^*_\gamma}(\Omega)$ is not compact. We prove compactness for the perturbed functional below

$$c^* = \frac{p-1}{2p} S_{\gamma,\mu}(\Omega)^{\frac{p}{p-1}}. \quad (1.4)$$

For $1 < q < 2$, a weak limit may have negative energy. The compactness threshold becomes $c^* - D_q \lambda^{2/(2-q)}$. A second solution requires a path below this level.

We say that $(\mathcal{C}_q)$ holds if there exist $a, b, c_0, C_0, \varepsilon_0 > 0$ and nonnegative $\omega_\varepsilon \in H_\gamma(\Omega)$, $0 < \varepsilon < \varepsilon_0$, such that

$$\mathcal{B}(\omega_\varepsilon) = 1, \quad \|\omega_\varepsilon\|_\gamma^2 \leq S_{\gamma,\mu}(\Omega) + C_0 \varepsilon^a, \quad |\omega_\varepsilon|_q^q \geq c_0 \varepsilon^b, \quad aq > 2b.$$

Section 3 verifies this condition under a whole-space extremal and decay assumption. The nonlocal truncation error is $O(\varepsilon^{N_\gamma - \mu/2})$, whereas the Dirichlet error is $O(\varepsilon^{N_\gamma - 2})$. Both rates enter the admissible range of q . For $2 < q < 2^*_\gamma$, a fixed test function instead puts the mountain-pass level below c^* when λ is large.

We first treat the concave range. The negative-energy solution needs no whole-space profile. The second solution requires $(\mathcal{C}_q)$.

Theorem 1.1. *Assume (1.1), $N_\gamma > 2$, $0 < \mu < N_\gamma$ and $1 < q < 2$. Then there exists $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0)$, problem (1.2) admits a nontrivial nonnegative weak solution*

$u_\lambda \in H_\gamma(\Omega)$. It is a local minimizer and

$$J_{\lambda,q}(u_\lambda) < 0, \quad \|u_\lambda\|_\gamma \leq C\lambda^{\frac{1}{2-q}}$$

for a constant $C > 0$ independent of λ . In particular, $\|u_\lambda\|_\gamma \rightarrow 0$ as $\lambda \rightarrow 0^+$.

If, in addition, (C_q) holds, then λ_0 can be chosen so that (1.2) also has a nontrivial nonnegative weak solution $v_\lambda \in H_\gamma(\Omega)$ for every $\lambda \in (0, \lambda_0)$. This solution is obtained at a mountain-pass level and satisfies

$$J_{\lambda,q}(u_\lambda) < 0 < J_{\lambda,q}(v_\lambda).$$

Hence u_λ and v_λ are distinct.

Remark 1.2. Condition (C_q) is stronger than $(\|\omega_\varepsilon\|_\gamma^2 - S_{\gamma,\mu}(\Omega))/|\omega_\varepsilon|_q^q \rightarrow 0$. The latter limit alone does not permit the two simultaneous choices required in the mountain-pass estimate. Proposition 3.2 below shows that a normalized whole-space extremal with the sharp decay $d(z)^{2-N_\gamma}$ gives

$$\|\omega_\varepsilon\|_\gamma^2 - S_{\gamma,\mu}(\Omega) = O(\varepsilon^a), \quad |\omega_\varepsilon|_q^q \geq c\varepsilon^b,$$

where

$$a = \min\{N_\gamma - 2, N_\gamma - \mu/2\}, \quad b = N_\gamma - q(N_\gamma - 2)/2.$$

Thus (C_q) follows when $q > 2N_\gamma/(N_\gamma - 2 + a)$. If $0 < \mu \leq 4$, the admissible interval is $N_\gamma/(N_\gamma - 2) < q < 2$. It is nonempty exactly when $N_\gamma > 4$. These inequalities do not establish attainment or decay of the whole-space Grushin-Choquard extremal.

Corollary 1.3. Assume the hypotheses of Theorem 1.1 and the whole-space extremal property stated in (3.2). If

$$q > \frac{2N_\gamma}{N_\gamma - 2 + \min\{N_\gamma - 2, N_\gamma - \mu/2\}},$$

then problem (1.2) has the two distinct nonnegative solutions described in Theorem 1.1 for every sufficiently small $\lambda > 0$. In particular, this conclusion holds under the sharp-profile property when

$$0 < \mu \leq 4, \quad N_\gamma > 4, \quad \frac{N_\gamma}{N_\gamma - 2} < q < 2.$$

The second main result treats the superlinear power range.

Theorem 1.4. Assume (1.1), $N_\gamma > 2$, $0 < \mu < N_\gamma$ and $2 < q < 2_\gamma^*$. Then there exists $\Lambda_* > 0$ such that, for every $\lambda \geq \Lambda_*$, problem (1.2) admits a nontrivial nonnegative weak solution $u_\lambda \in H_\gamma(\Omega)$. This solution is obtained at a mountain-pass level $c_{\lambda,q}$ satisfying

$$0 < c_{\lambda,q} < c^*, \quad c_{\lambda,q} \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Moreover, $\|u_\lambda\|_\gamma \rightarrow 0$ as $\lambda \rightarrow +\infty$.

Remark 1.5. The large parameter assumption in Theorem 1.4 is used only to force the mountain-pass level below the critical compactness threshold. Removing this assumption would require sharp domain-dependent test-function expansions, which are not used here.

Next we consider a parameterized general perturbation. Set

$$F(z, t) = \int_0^t f(z, s) ds.$$

Theorem 1.6. Assume (1.1), $N_\gamma > 2$ and $0 < \mu < N_\gamma$. Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying the following conditions

- (A1) $f(z, t) = 0$ for $t \leq 0$, and $f(z, t) = o(t)$ as $t \rightarrow 0^+$ uniformly for a.e. $z \in \Omega$;
- (A2) there exist $a_1, a_2 > 0$ and $r \in (2, 2_\gamma^*)$ such that $0 \leq f(z, t) \leq a_1 t + a_2 t^{r-1}$ for a.e. $z \in \Omega$ and all $t \geq 0$;
- (A3) there exists $\theta \in (2, r]$ such that $0 < \theta F(z, t) \leq t f(z, t)$ for a.e. $z \in \Omega$ and all $t > 0$;
- (A4) there exist a nonempty open set U compactly contained in Ω and constants $b > 0$ and $\sigma \in (2, r]$ such that $F(z, t) \geq b t^\sigma$ for a.e. $z \in U$ and all $t \geq 0$.

Then there exists $\Lambda_f > 0$ such that, for every $\lambda \geq \Lambda_f$, problem (1.3) admits a nontrivial nonnegative weak solution.

The inequalities $\theta, \sigma \leq r$ make the lower growth conditions compatible with (A2). Condition (A4) provides a test function supported in U and a mountain-pass bound of order $\lambda^{-2/(\sigma-2)}$. The compactness and cut-off estimates above yield the three existence results.

The paper is organized as follows. Section 2 recalls the functional setting, proves the Choquard Brézis-Lieb splitting and establishes compactness below the critical level. Section 3 compares the bounded- and whole-space constants and gives quantitative cut-off estimates. Section 4 proves Theorems 1.1 and 1.4. Section 5 proves Theorem 1.6.

2. FUNCTIONAL SETTING AND COMPACTNESS BELOW THE CRITICAL LEVEL

Let $\Omega \subset \mathbb{R}^m \times \mathbb{R}^n$ be a bounded smooth domain satisfying (1.1). The Grushin gradient is

$$\nabla_\gamma u = (\nabla_x u, (1 + \gamma)|x|^\gamma \nabla_y u).$$

We denote by $H_\gamma(\Omega)$, equivalently the zero-boundary Grushin Sobolev space $H_{\gamma,0}^1(\Omega)$, the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_\gamma = \left(\int_\Omega |\nabla_\gamma u|^2 dz \right)^{1/2}.$$

The scalar product is

$$\langle u, v \rangle_\gamma = \int_\Omega \nabla_\gamma u \cdot \nabla_\gamma v dz.$$

The embedding $H_\gamma(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $1 \leq s \leq 2_\gamma^*$ and compact for $1 \leq s < 2_\gamma^*$; see [15, 30, 3]. We write $|u|_s$ for the norm of u in $L^s(\Omega)$. For $1 \leq s \leq 2_\gamma^*$, let $C_s > 0$ be such that

$$|u|_s \leq C_s \|u\|_\gamma \quad \text{for all } u \in H_\gamma(\Omega). \quad (2.1)$$

For $u \in H_\gamma(\Omega)$ define the critical Choquard quantity

$$\mathcal{B}(u) = \int_\Omega \int_\Omega \frac{|u(z)|^p |u(w)|^p}{d(z-w)^\mu} dz dw. \quad (2.2)$$

The Grushin Hardy-Littlewood-Sobolev inequality [19] gives

$$\left| \int_\Omega \int_\Omega \frac{h(z)k(w)}{d(z-w)^\mu} dz dw \right| \leq C |h|_t |k|_t, \quad t = \frac{2N_\gamma}{2N_\gamma - \mu},$$

for every $h, k \in L^t(\Omega)$. Since $pt = 2_\gamma^*$, the quantity $\mathcal{B}(u)$ is finite for every $u \in H_\gamma(\Omega)$. Define

$$S_{\gamma,\mu}(\Omega) = \inf_{u \in H_\gamma(\Omega) \setminus \{0\}} \frac{\|u\|_\gamma^2}{\mathcal{B}(u)^{1/p}}. \quad (2.3)$$

The Hardy-Littlewood-Sobolev and Sobolev inequalities give $S_{\gamma,\mu}(\Omega) > 0$. A nonzero test function gives $S_{\gamma,\mu}(\Omega) < \infty$. Hence

$$\mathcal{B}(u) \leq S_{\gamma,\mu}(\Omega)^{-p} \|u\|_\gamma^{2p} \quad \text{for all } u \in H_\gamma(\Omega). \quad (2.4)$$

Definition 2.1. A function $u \in H_\gamma(\Omega)$ is a weak solution of (1.2) if

$$\int_\Omega \nabla_\gamma u \cdot \nabla_\gamma \varphi dz = \lambda \int_\Omega |u|^{q-2} u \varphi dz + \int_\Omega \left(\int_\Omega \frac{|u(w)|^p}{d(z-w)^\mu} dw \right) |u|^{p-2} u \varphi dz$$

for all $\varphi \in H_\gamma(\Omega)$.

Definition 2.2. A function $u \in H_\gamma(\Omega)$ is a weak solution of (1.3) if

$$\int_\Omega \nabla_\gamma u \cdot \nabla_\gamma \varphi dz = \lambda \int_\Omega f(z, u) \varphi dz + \int_\Omega \left(\int_\Omega \frac{|u(w)|^p}{d(z-w)^\mu} dw \right) |u|^{p-2} u \varphi dz$$

for all $\varphi \in H_\gamma(\Omega)$.

For $u \in H_\gamma(\Omega)$, let $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. For $\lambda > 0$ and $1 < q < 2_\gamma^*$ set

$$J_{\lambda,q}(u) = \frac{1}{2}\|u\|_\gamma^2 - \frac{\lambda}{q} \int_\Omega (u^+)^q dz - \frac{1}{2p} \mathcal{B}(u^+).$$

The functional $J_{\lambda,q}$ is of class C^1 on $H_\gamma(\Omega)$ and

$$\langle J'_{\lambda,q}(u), \varphi \rangle = \int_\Omega \nabla_\gamma u \cdot \nabla_\gamma \varphi dz - \lambda \int_\Omega (u^+)^{q-1} \varphi dz - \int_\Omega \left(\int_\Omega \frac{(u^+(w))^p}{d(z-w)^\mu} dw \right) (u^+)^{p-1} \varphi dz.$$

If u is a critical point of $J_{\lambda,q}$, testing the equation by u^- gives

$$0 = \langle J'_{\lambda,q}(u), u^- \rangle = -\|u^-\|_\gamma^2,$$

because $u^+ u^- = 0$ a.e. and $\nabla_\gamma u^+ \cdot \nabla_\gamma u^- = 0$ a.e. Hence every nonzero critical point of $J_{\lambda,q}$ is a nonnegative weak solution of (1.2).

Recall that the critical compactness level c^* is defined in (1.4).

Lemma 2.3 (Nonlocal splitting and derivative convergence). *Let $\{u_j\} \subset H_\gamma(\Omega)$ be bounded and suppose that $u_j \rightarrow u$ a.e. in Ω . Then*

$$\mathcal{B}(u_j) = \mathcal{B}(u_j - u) + \mathcal{B}(u) + o(1). \quad (2.5)$$

If, in addition, $u_j \rightarrow u$ in $H_\gamma(\Omega)$, then for every $\varphi \in H_\gamma(\Omega)$,

$$\int_\Omega \left(\int_\Omega \frac{|u_j(w)|^p}{d(z-w)^\mu} dw \right) |u_j|^{p-2} u_j \varphi dz \rightarrow \int_\Omega \left(\int_\Omega \frac{|u(w)|^p}{d(z-w)^\mu} dw \right) |u|^{p-2} u \varphi dz. \quad (2.6)$$

Proof. Put $t = \frac{2N_\gamma}{2N_\gamma - \mu}$, $t' = \frac{2N_\gamma}{\mu}$. Then $pt = 2_\gamma^*$ and $1/t + 1/t' = 1$. Since $\{u_j\}$ is bounded in $L^{2_\gamma^*}(\Omega)$, the sequences $|u_j|^p$, $|u_j - u|^p$ and $|u|^p$ are bounded in $L^t(\Omega)$. The power form of the Brézis-Lieb lemma [8] gives

$$h_j := |u_j|^p - |u_j - u|^p - |u|^p \rightarrow 0 \quad \text{in } L^t(\Omega). \quad (2.7)$$

Indeed, for every $\delta > 0$, the power inequality yields

$$|h_j| \leq \delta |u_j - u|^p + C_\delta |u|^p.$$

The functions $(|h_j| - \delta |u_j - u|^p)^+$ tend to zero a.e. and are bounded by $C_\delta |u|^p \in L^t(\Omega)$. Dominated convergence therefore gives

$$\limsup_{j \rightarrow \infty} |h_j|_t \leq \delta \sup_j |u_j - u|_{2_\gamma^*}^p.$$

Letting $\delta \rightarrow 0$ proves (2.7). Let

$$T(h)(z) = \int_\Omega \frac{h(w)}{d(z-w)^\mu} dw.$$

The Grushin Hardy-Littlewood-Sobolev inequality implies that $T : L^t(\Omega) \rightarrow L^{t'}(\Omega)$ is continuous. Hence

$$T(h_j) \rightarrow 0 \quad \text{in } L^{t'}(\Omega). \quad (2.8)$$

We first prove the energy splitting. We denote $a_j = |u_j - u|^p$ and $b = |u|^p$. Then $|u_j|^p = a_j + b + h_j$. Using the symmetry of the kernel, we obtain the exact identity

$$\mathcal{B}(u_j) - \mathcal{B}(u_j - u) - \mathcal{B}(u) = 2 \int_\Omega T(a_j) b dz + 2 \int_\Omega T(a_j) h_j dz + 2 \int_\Omega T(b) h_j dz + \int_\Omega T(h_j) h_j dz.$$

The second, third and fourth terms tend to zero by (2.7), (2.8), the boundedness of a_j in $L^t(\Omega)$ and Hölder's inequality. For the first term, note that $a_j \rightarrow 0$ in $L^t(\Omega)$. Indeed, $u_j - u \rightarrow 0$ a.e. and $\{a_j\}$ is bounded in $L^t(\Omega)$. Since $T(b) \in L^{t'}(\Omega)$, we have

$$\int_\Omega T(a_j) b dz = \int_\Omega a_j T(b) dz \rightarrow 0.$$

This proves (2.5).

We now prove the convergence of the derivative. Let

$$G_j = T(|u_j|^p) |u_j|^{p-2} u_j, \quad G = T(|u|^p) |u|^{p-2} u.$$

By Hölder's inequality and

$$\frac{1}{(2_\gamma^*)'} = \frac{1}{t'} + \frac{p-1}{2_\gamma^*},$$

the sequence $\{G_j\}$ is bounded in $L^{(2_\gamma^*)'}(\Omega)$.

We first test against a bounded function $\psi \in L^\infty(\Omega)$. Put $s_0 = (p-1)t$. We have

$$0 < s_0 = (p-1)t = 2_\gamma^* - t < 2_\gamma^*.$$

If $s_0 \geq 1$, the compact embedding gives $u_j \rightarrow u$ in $L^{s_0}(\Omega)$. If $s_0 < 1$, the convergence in $L^1(\Omega)$ and the finiteness of $|\Omega|$ give

$$\int_\Omega |u_j - u|^{s_0} dz \leq |\Omega|^{1-s_0} |u_j - u|_1^{s_0} \rightarrow 0.$$

In the latter case $p < 2$, and the map $s \mapsto |s|^{p-2}s$ is $(p-1)$ -Hölder continuous. In both cases the standard power estimate gives

$$|u_j|^{p-2}u_j\psi \rightarrow |u|^{p-2}u\psi \quad \text{in } L^t(\Omega).$$

Moreover, $|u_j|^p \rightharpoonup |u|^p$ in $L^t(\Omega)$, and therefore

$$T(|u_j|^p) \rightharpoonup T(|u|^p) \quad \text{in } L^{t'}(\Omega).$$

Consequently,

$$\int_\Omega G_j \psi dz \rightarrow \int_\Omega G \psi dz.$$

Finally, let $\varphi \in H_\gamma(\Omega)$. Since $H_\gamma(\Omega) \hookrightarrow L^{2_\gamma^*}(\Omega)$ and $L^\infty(\Omega)$ is dense in $L^{2_\gamma^*}(\Omega)$, choose $\psi_k \in L^\infty(\Omega)$ such that $\psi_k \rightarrow \varphi$ in $L^{2_\gamma^*}(\Omega)$. The uniform boundedness of $\{G_j\}$ in $L^{(2_\gamma^*)'}(\Omega)$ yields

$$\left| \int_\Omega G_j(\varphi - \psi_k) dz \right| \leq C|\varphi - \psi_k|_{2_\gamma^*},$$

and the same estimate holds with G in place of G_j . Letting first $j \rightarrow \infty$ and then $k \rightarrow \infty$ gives (2.6). $\square$

Lemma 2.4 (Compactness below the critical level). *Let $1 < q < 2_\gamma^*$ and let $\{u_j\} \subset H_\gamma(\Omega)$ be a Palais-Smale sequence for $J_{\lambda,q}$ at level c .*

- (i) *If $2 < q < 2_\gamma^*$ and $c < c^*$, then $\{u_j\}$ has a strongly convergent subsequence in $H_\gamma(\Omega)$.*
- (ii) *If $1 < q < 2$, then there is a constant $D_q > 0$, independent of λ , such that $\{u_j\}$ has a strongly convergent subsequence whenever*

$$c < c^* - D_q \lambda^{\frac{2}{2-q}}.$$

Proof. We first prove boundedness. Put

$$A_j = \|u_j\|_\gamma^2, \quad C_j = |u_j^+|_q^q, \quad B_j = \mathcal{B}(u_j^+).$$

If $q > 2$, then

$$c + o(1) + o(1)\|u_j\|_\gamma = J_{\lambda,q}(u_j) - \frac{1}{q} \langle J'_{\lambda,q}(u_j), u_j \rangle = \left(\frac{1}{2} - \frac{1}{q}\right)A_j + \left(\frac{1}{q} - \frac{1}{2p}\right)B_j.$$

Since $2 < q < 2_\gamma^* < 2p$, both coefficients are positive. Hence $\{u_j\}$ is bounded.

If $1 < q < 2$, it follows that

$$\begin{aligned} c + o(1) + o(1)\|u_j\|_\gamma &= J_{\lambda,q}(u_j) - \frac{1}{2p} \langle J'_{\lambda,q}(u_j), u_j \rangle \\ &= \frac{p-1}{2p}A_j - \lambda \left(\frac{1}{q} - \frac{1}{2p}\right)C_j \\ &\geq \frac{p-1}{2p}\|u_j\|_\gamma^2 - C\lambda\|u_j\|_\gamma^q, \end{aligned}$$

where we have used (2.1). Since $q < 2$, boundedness follows.

Next, the negative part of a Palais-Smale sequence is small. Indeed,

$$\langle J'_{\lambda,q}(u_j), u_j^- \rangle = -\|u_j^-\|_\gamma^2 = o(1)\|u_j^-\|_\gamma,$$

and therefore

$$\|u_j^-\|_\gamma \rightarrow 0. \quad (2.9)$$

Set $w_j = u_j^+$. By (2.9), $\{w_j\}$ is bounded in $H_\gamma(\Omega)$ and, up to a subsequence,

$$w_j \rightharpoonup u \text{ in } H_\gamma(\Omega), \quad w_j \rightarrow u \text{ in } L^s(\Omega), \quad 1 \leq s < 2_\gamma^*, \quad w_j \rightarrow u \text{ a.e. in } \Omega,$$

with $u \geq 0$. Since $u_j - w_j = -u_j^-$ tends to zero strongly in $H_\gamma(\Omega)$, the weak limit of u_j is also u .

We claim that u is a critical point of $J_{\lambda,q}$. Let $\varphi \in H_\gamma(\Omega)$. The gradient part passes to the limit by weak convergence. The subcritical term passes to the limit because $w_j \rightarrow u$ in $L^q(\Omega)$. The critical Choquard term passes to the limit by Lemma 2.3 applied to w_j . Since $J'_{\lambda,q}(u_j) \rightarrow 0$ and $u_j - w_j \rightarrow 0$, we obtain

$$\langle J'_{\lambda,q}(u), \varphi \rangle = 0.$$

Put $v_j = w_j - u$. Here $\mathcal{B}(v_j)$ is always understood in the sense of (2.2), with absolute values, so no sign condition on v_j is needed. Then

$$\|u_j\|_\gamma^2 = \|w_j\|_\gamma^2 + o(1) = \|v_j\|_\gamma^2 + \|u\|_\gamma^2 + o(1), \quad (2.10)$$

$$\mathcal{B}(w_j) = \mathcal{B}(v_j) + \mathcal{B}(u) + o(1), \quad (2.11)$$

and, by compactness of the embedding into $L^q(\Omega)$,

$$|w_j|_q^q = |u|_q^q + o(1). \quad (2.12)$$

Using (2.10)–(2.12), the identity $J_{\lambda,q}(u_j) = c + o(1)$ and the fact that u is critical, we obtain

$$c = J_{\lambda,q}(u) + \frac{1}{2}\|v_j\|_\gamma^2 - \frac{1}{2p}\mathcal{B}(v_j) + o(1). \quad (2.13)$$

Similarly, from $\langle J'_{\lambda,q}(u_j), u_j \rangle = o(1)$ and $\langle J'_{\lambda,q}(u), u \rangle = 0$, we obtain

$$\|v_j\|_\gamma^2 - \mathcal{B}(v_j) = o(1). \quad (2.14)$$

Passing to a further subsequence, we can assume $\|v_j\|_\gamma^2 \rightarrow \ell$ and $\mathcal{B}(v_j) \rightarrow \ell$, with $\ell \geq 0$. By (2.3),

$$S_{\gamma,\mu}(\Omega)\mathcal{B}(v_j)^{1/p} \leq \|v_j\|_\gamma^2.$$

Thus either $\ell = 0$ or

$$\ell \geq S_{\gamma,\mu}(\Omega)^{\frac{p}{p-1}}. \quad (2.15)$$

If $\ell = 0$, then $v_j \rightarrow 0$ in $H_\gamma(\Omega)$, and hence $u_j \rightarrow u$ strongly in $H_\gamma(\Omega)$. It remains to rule out $\ell > 0$.

Assume first $q > 2$. Since u is a critical point and $u \geq 0$,

$$\|u\|_\gamma^2 = \lambda|u|_q^q + \mathcal{B}(u),$$

and therefore

$$J_{\lambda,q}(u) = \lambda\left(\frac{1}{2} - \frac{1}{q}\right)|u|_q^q + \frac{p-1}{2p}\mathcal{B}(u) \geq 0.$$

Using (2.13), (2.14) and (2.15), if $\ell > 0$ then

$$c = J_{\lambda,q}(u) + \frac{p-1}{2p}\ell \geq c^*,$$

which contradicts $c < c^*$.

Now assume $1 < q < 2$. Let $a = (p-1)/(2p)$ and

$$b = \left(\frac{1}{q} - \frac{1}{2p}\right)C_q^q.$$

For every critical point w of $J_{\lambda,q}$,

$$J_{\lambda,q}(w) = J_{\lambda,q}(w) - \frac{1}{2p}\langle J'_{\lambda,q}(w), w \rangle = a\|w\|_\gamma^2 - \lambda\left(\frac{1}{q} - \frac{1}{2p}\right)|w^+|_q^q \geq a\|w\|_\gamma^2 - \lambda b\|w\|_\gamma^q.$$

The minimizing point of $s \mapsto as^2 - \lambda bs^q$ is $s = (\lambda bq/(2a))^{1/(2-q)}$. Its minimum is $-D_q \lambda^{2/(2-q)}$, where

$$D_q = \frac{a(2-q)}{q} \left(\frac{bq}{2a} \right)^{\frac{2}{2-q}}.$$

Therefore $J_{\lambda,q}(u) \geq -D_q \lambda^{2/(2-q)}$. If $\ell > 0$, then

$$c \geq -D_q \lambda^{\frac{2}{2-q}} + \frac{p-1}{2p} S_{\gamma,\mu}(\Omega)^{\frac{p}{p-1}} = c^* - D_q \lambda^{\frac{2}{2-q}},$$

contradicting the assumed bound on c . Hence $\ell = 0$ also in the concave case. The proof is complete. $\square$

3. WHOLE-SPACE PROFILES AND QUANTITATIVE CUT-OFF ESTIMATES

We compare the sharp constants and estimate the truncation errors under a whole-space profile assumption. Let $\mathcal{D}_\gamma^{1,2}(\mathbb{R}^N)$ be the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to

$$\|u\|_{\gamma,\mathbb{R}^N}^2 = \int_{\mathbb{R}^N} |\nabla_\gamma u|^2 dz.$$

For $u \in \mathcal{D}_\gamma^{1,2}(\mathbb{R}^N)$ we set

$$\mathcal{B}_\infty(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(z)|^p |u(w)|^p}{d(z-w)^\mu} dz dw, S_{\gamma,\mu}^\infty = \inf_{u \in \mathcal{D}_\gamma^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|_{\gamma,\mathbb{R}^N}^2}{\mathcal{B}_\infty(u)^{1/p}}. \quad (3.1)$$

Proposition 3.1. *For every bounded domain satisfying (1.1),*

$$S_{\gamma,\mu}(\Omega) = S_{\gamma,\mu}^\infty.$$

Proof. Extension by zero gives $S_{\gamma,\mu}^\infty \leq S_{\gamma,\mu}(\Omega)$. For the reverse inequality choose $z_0 = (0, y_0) \in \Omega$ and $r > 0$ so that a Grushin ball of radius $2r$ centered at z_0 is contained in Ω . Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be nonzero and define

$$\phi_{\varepsilon,z_0}(x, y) = \varepsilon^{-\frac{N_\gamma-2}{2}} \phi\left(\frac{x}{\varepsilon}, \frac{y-y_0}{\varepsilon^{1+\gamma}}\right).$$

For sufficiently small ε , this function belongs to $C_c^\infty(\Omega)$. The change of variables associated with the dilation δ_ε gives

$$\|\phi_{\varepsilon,z_0}\|_\gamma = \|\phi\|_{\gamma,\mathbb{R}^N}, \quad \mathcal{B}(\phi_{\varepsilon,z_0}) = \mathcal{B}_\infty(\phi).$$

Taking first the infimum over ϕ and then using density in (3.1) proves the reverse inequality. $\square$

The sharp-profile property required below is the following statement about quotient (3.1): there is a nonnegative extremal $U \in \mathcal{D}_\gamma^{1,2}(\mathbb{R}^N)$ and constants $c, C > 0$ such that

$$\begin{aligned} \mathcal{B}_\infty(U) &= 1, \quad \|U\|_{\gamma,\mathbb{R}^N}^2 = S_{\gamma,\mu}^\infty, \\ \frac{c}{1+d(z)^{N_\gamma-2}} &\leq U(z) \leq \frac{C}{1+d(z)^{N_\gamma-2}}, \\ |\nabla_\gamma U(z)| &\leq C d(z)^{1-N_\gamma} \quad \text{for } d(z) \geq 1. \end{aligned} \quad (3.2)$$

If the infimum is attained, its Euler-Lagrange equation, with this normalization, is

$$-\Delta_\gamma U = S_{\gamma,\mu}^\infty \left(\int_{\mathbb{R}^N} \frac{U(w)^p}{d(z-w)^\mu} dw \right) U^{p-1} \quad \text{in } \mathbb{R}^N. \quad (3.3)$$

Thus (3.2) asks not only for attainment but also for the sharp harmonic-order decay and the corresponding gradient bound.

Proposition 3.2 (Quantitative cut-off estimates). *Let $1 < q < 2$ and assume (3.2). Put*

$$a = \min\{N_\gamma - 2, N_\gamma - \mu/2\}, \quad b = N_\gamma - \frac{q(N_\gamma - 2)}{2}.$$

Then there are nonnegative functions $\omega_\varepsilon \in H_\gamma(\Omega)$ and constants $c_0, C_0, \varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$,

$$\mathcal{B}(\omega_\varepsilon) = 1, \quad \|\omega_\varepsilon\|_\gamma^2 \leq S_{\gamma,\mu}(\Omega) + C_0 \varepsilon^a, \quad |\omega_\varepsilon|_q^q \geq c_0 \varepsilon^b. \quad (3.4)$$

Consequently, $(\mathcal{C}_q)$ holds whenever

$$q > \frac{2N_\gamma}{N_\gamma - 2 + a}. \quad (3.5)$$

In particular, if $0 < \mu \leq 4$, then $a = N_\gamma - 2$ and (3.5) is precisely

$$q > \frac{N_\gamma}{N_\gamma - 2}.$$

Proof. Fix $z_0 = (0, y_0) \in \Omega$. Choose $r > 0$ such that $\{(x, y) : d((x, y - y_0)) \leq 3r\} \subset \Omega$. Take $\eta \in C_c^\infty(\Omega)$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $\{d((x, y - y_0)) \leq 2r\}$, and $\eta = 0$ on $\{d((x, y - y_0)) \geq 3r\}$. Extend η by zero outside Ω . Define

$$U_{\varepsilon, z_0}(x, y) = \varepsilon^{-\frac{N_\gamma - 2}{2}} U\left(\frac{x}{\varepsilon}, \frac{y - y_0}{\varepsilon^{1+\gamma}}\right), \quad v_\varepsilon = \eta U_{\varepsilon, z_0}.$$

The full-space quantities are invariant under this rescaling:

$$\|U_{\varepsilon, z_0}\|_{\gamma, \mathbb{R}^N}^2 = S_{\gamma, \mu}^\infty, \quad \mathcal{B}_\infty(U_{\varepsilon, z_0}) = 1. \quad (3.6)$$

We first estimate the energy. The decay assumptions in (3.2), the volume growth $|\{d < R\}| = CR^{N_\gamma}$ and a dyadic decomposition give

$$\int_{d(z) > R} |\nabla_\gamma U|^2 dz \leq CR^{2-N_\gamma}, \quad \int_{d(z) > R} U^{2^*_\gamma} dz \leq CR^{-N_\gamma} \quad (R \geq 1).$$

On the fixed region where $\nabla_\gamma \eta$ is supported, the same assumptions and the scaling give

$$|U_{\varepsilon, z_0}| + |\nabla_\gamma U_{\varepsilon, z_0}| \leq C\varepsilon^{\frac{N_\gamma - 2}{2}}.$$

Expanding $|\nabla_\gamma(\eta U_{\varepsilon, z_0})|^2$, using the last two estimates and changing variables in the discarded tail, we obtain

$$\|v_\varepsilon\|_\gamma^2 = S_{\gamma, \mu}^\infty + O(\varepsilon^{N_\gamma - 2}). \quad (3.7)$$

The nonlocal truncation has a different rate. Set $t = \frac{2N_\gamma}{2N_\gamma - \mu}$; then $pt = 2^*$. Set $F_\varepsilon = U_{\varepsilon, z_0}^p$ and $G_\varepsilon = (1 - \eta^p)F_\varepsilon$ on $\mathbb{R}^N$. We have $\|F_\varepsilon\|_{L^t(\mathbb{R}^N)} = \|U^p\|_{L^t(\mathbb{R}^N)}$. The Hardy-Littlewood-Sobolev inequality yields

$$0 \leq \mathcal{B}_\infty(U_{\varepsilon, z_0}) - \mathcal{B}(v_\varepsilon) \leq C\|F_\varepsilon\|_{L^t(\mathbb{R}^N)}\|G_\varepsilon\|_{L^t(\mathbb{R}^N)}.$$

Moreover, (3.2) gives

$$\|G_\varepsilon\|_{L^t(\mathbb{R}^N)} \leq C\left(\int_{d(z) > r/\varepsilon} U(z)^{2^*_\gamma} dz\right)^{1/t} \leq C\varepsilon^{N_\gamma - \mu/2}.$$

Together with (3.6), this shows that

$$\mathcal{B}(v_\varepsilon) = 1 + O(\varepsilon^{N_\gamma - \mu/2}). \quad (3.8)$$

In particular, $\mathcal{B}(v_\varepsilon)$ stays bounded away from zero. Normalise by setting

$$\omega_\varepsilon = \frac{v_\varepsilon}{\mathcal{B}(v_\varepsilon)^{1/(2p)}}.$$

Then $\mathcal{B}(\omega_\varepsilon) = 1$, and Proposition 3.1, (3.7) and (3.8) give

$$\|\omega_\varepsilon\|_\gamma^2 = \frac{\|v_\varepsilon\|_\gamma^2}{\mathcal{B}(v_\varepsilon)^{1/p}} \leq S_{\gamma, \mu}(\Omega) + C\varepsilon^a.$$

Finally, the rescaled image of a fixed ball is contained in the region where $\eta = 1$ for small ε . The lower bound in (3.2) therefore gives

$$|v_\varepsilon|_q^q \geq \varepsilon^{N_\gamma - \frac{q(N_\gamma - 2)}{2}} \int_{d(z) < 1} U(z)^q dz \geq c\varepsilon^b.$$

The normalization factor tends to one by (3.8), so the same lower bound holds for ω_ε . This proves (3.4). Since

$$aq > 2b \iff q(N_\gamma - 2 + a) > 2N_\gamma,$$

condition (3.5) is exactly the exponent condition in $(\mathcal{C}_q)$. $\square$

Proof of Corollary 1.3. Proposition 3.2 verifies $(\mathcal{C}_q)$ in the stated range. The conclusion follows from the second part of Theorem 1.1. $\square$

Remark 3.3 (Scope of the whole-space input). For $\gamma = 0$, the sharp Hardy-Littlewood-Sobolev extremals and the Aubin-Talenti profile give (3.2). For $\gamma > 0$, the inequality in [19] bounds the Grushin Riesz form but does not establish attainment of (3.1). The decay result in [16] concerns the local equation $-\Delta_\gamma U = U^{2_\gamma^*-1}$. The equation in [7] contains a positive mass term. Neither result provides (3.2) for the massless critical equation (3.3). Corollary 1.3 therefore retains this profile assumption.

4. PURE POWER PERTURBATIONS

This section proves Theorems 1.1 and 1.4.

4.1. Concave range. We first estimate a concentrating path below the concave compactness threshold.

Lemma 4.1 (Concentrating level estimate). *Let $1 < q < 2$, assume $(\mathcal{C}_q)$, and let $\rho > 0$ be fixed. Then there exist $\lambda_1 = \lambda_1(\rho) > 0$ and, for every $0 < \lambda < \lambda_1$, a nonnegative function $e_\lambda \in H_\gamma(\Omega)$ such that*

$$\|e_\lambda\|_\gamma > \rho, \quad J_{\lambda,q}(e_\lambda) < 0,$$

and the mountain-pass level

$$d_{\lambda,q} = \inf_{\eta \in \Gamma_\lambda} \max_{t \in [0,1]} J_{\lambda,q}(\eta(t)),$$

where

$$\Gamma_\lambda = \{\eta \in C([0,1], H_\gamma(\Omega)) : \eta(0) = 0, \eta(1) = e_\lambda\},$$

satisfies

$$d_{\lambda,q} < c^* - D_q \lambda^{\frac{2}{2-q}},$$

where D_q is the constant in Lemma 2.4.

Proof. Let ω_ε be given by $(\mathcal{C}_q)$ and set $A_\varepsilon = \|\omega_\varepsilon\|_\gamma^2$. Since $\mathcal{B}(\omega_\varepsilon) = 1$, the maximum of

$$t \mapsto \frac{t^2}{2} A_\varepsilon - \frac{t^{2p}}{2p}$$

is

$$\frac{p-1}{2p} A_\varepsilon^{\frac{p}{p-1}} \leq c^* + C\varepsilon^a$$

for a constant $C > 0$ independent of small ε . We next justify the uniform use of the concave term. Choose $\lambda_1 > 0$ small enough that $D_q \lambda^{2/(2-q)} < c^*/2$ for $0 < \lambda < \lambda_1$. If $\max_{t \geq 0} J_{\lambda,q}(t\omega_\varepsilon) < c^*/2$, then

$$\max_{t \geq 0} J_{\lambda,q}(t\omega_\varepsilon) < \frac{c^*}{2} < c^* - D_q \lambda^{\frac{2}{2-q}}.$$

Otherwise, at a maximizing point $t_{\lambda,\varepsilon}$,

$$\frac{1}{2} A_\varepsilon t_{\lambda,\varepsilon}^2 \geq J_{\lambda,q}(t_{\lambda,\varepsilon} \omega_\varepsilon) \geq \frac{c^*}{2}.$$

The numbers A_ε are uniformly bounded. Thus $t_{\lambda,\varepsilon} \geq \tau > 0$, where τ is independent of small λ and ε . Condition $(\mathcal{C}_q)$ now gives

$$\max_{t \geq 0} J_{\lambda,q}(t\omega_\varepsilon) \leq c^* + C\varepsilon^a - c_1 \lambda \varepsilon^b \tag{4.1}$$

with $c_1 = c_0 \tau^q / q > 0$.

Choose $K > 0$ so large that $c_1 K^b > 2D_q$, and put $\varepsilon(\lambda) = K \lambda^{\frac{a}{b(2-q)}}$. Since $aq > 2b$,

$$C\varepsilon(\lambda)^a = CK^a \lambda^{\frac{aa}{b(2-q)}} = o\left(\lambda^{\frac{2}{2-q}}\right).$$

Decrease λ_1 so that $\varepsilon(\lambda) < \varepsilon_0$. In the second case, (4.1) yields

$$\max_{t \geq 0} J_{\lambda,q}(t\omega_{\varepsilon(\lambda)}) < c^* - D_q \lambda^{\frac{2}{2-q}}.$$

The same bound holds in the first case. Choose $T_\lambda > 0$ large so that

$$\|T_\lambda \omega_{\varepsilon(\lambda)}\|_\gamma > \rho, \quad J_{\lambda,q}(T_\lambda \omega_{\varepsilon(\lambda)}) < 0,$$

and set $e_\lambda = T_\lambda \omega_{\varepsilon(\lambda)}$. The path $s \mapsto s e_\lambda$ gives the stated upper bound. $\square$

Proof of Theorem 1.1. Fix $1 < q < 2$. By (2.1) and (2.4), for every $u \in H_\gamma(\Omega)$,

$$J_{\lambda,q}(u) \geq \frac{1}{2}\|u\|_\gamma^2 - \frac{\lambda C_q^q}{q}\|u\|_\gamma^q - \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p}\|u\|_\gamma^{2p}. \quad (4.2)$$

Choose $\rho > 0$ small so that

$$\begin{aligned} \frac{1}{2}\rho^2 - \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p}\rho^{2p} &\geq \frac{1}{3}\rho^2, \\ S_{\gamma,\mu}(\Omega)^{-p}\rho^{2p-2} &\leq \frac{1}{2}. \end{aligned}$$

Then choose $\lambda_0 > 0$ such that

$$\frac{\lambda C_q^q}{q}\rho^q \leq \frac{1}{6}\rho^2 \quad \text{for } 0 < \lambda < \lambda_0.$$

For such λ , (4.2) gives

$$J_{\lambda,q}(u) \geq \frac{1}{6}\rho^2 > 0 \quad \text{whenever } \|u\|_\gamma = \rho. \quad (4.3)$$

The preceding lower bound is uniform for $0 < \lambda < \lambda_0$. More precisely, the function

$$s \mapsto \frac{1}{2}s^2 - \frac{\lambda_0 C_q^q}{q}s^q - \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p}s^{2p}$$

is positive at $s = \rho$. Hence, after choosing $\eta \in (0, \rho)$ sufficiently small, the same estimate yields

$$J_{\lambda,q}(u) \geq \frac{1}{12}\rho^2 \quad \text{whenever } \rho - \eta \leq \|u\|_\gamma \leq \rho, \quad 0 < \lambda < \lambda_0. \quad (4.4)$$

Let $e \in C_c^\infty(\Omega)$, $e \geq 0$, $e \not\equiv 0$. Since $q < 2 < 2p$,

$$J_{\lambda,q}(te) = \frac{t^2}{2}\|e\|_\gamma^2 - \frac{\lambda t^q}{q}|e|_q^q - \frac{t^{2p}}{2p}\mathcal{B}(e) < 0$$

for all sufficiently small $t > 0$. Taking such t with $\|te\|_\gamma < \rho - \eta$, we find that

$$m_\lambda := \inf_{\overline{B}_\rho} J_{\lambda,q} < 0, \quad \overline{B}_\rho = \{u \in H_\gamma(\Omega) : \|u\|_\gamma \leq \rho\}.$$

By Ekeland's variational principle applied to $J_{\lambda,q}$ on the complete metric space $\overline{B}_\rho$, there exists a minimizing sequence $\{u_j\} \subset \overline{B}_\rho$ such that

$$\begin{aligned} J_{\lambda,q}(u_j) &\leq m_\lambda + \frac{1}{j}, \\ J_{\lambda,q}(v) &\geq J_{\lambda,q}(u_j) - \frac{1}{j}\|v - u_j\|_\gamma \quad \text{for every } v \in \overline{B}_\rho. \end{aligned}$$

By (4.4), $\|u_j\|_\gamma < \rho - \eta$ for all large j . For any $\varphi \in H_\gamma(\Omega)$ with $\|\varphi\|_\gamma = 1$, insert $v = u_j \pm t\varphi$ into the variational inequality and let $t \rightarrow 0^+$. This gives $|\langle J'_{\lambda,q}(u_j), \varphi \rangle| \leq 1/j$. Hence

$$J'_{\lambda,q}(u_j) \rightarrow 0 \quad \text{in } H_\gamma(\Omega)^*.$$

Thus $\{u_j\}$ is a Palais-Smale sequence at level m_λ .

Let D_q be the constant in Lemma 2.4. Decrease λ_0 , if necessary, so that

$$D_q \lambda_0^{\frac{2}{2-q}} < c^*.$$

For $0 < \lambda < \lambda_0$, since $m_\lambda < 0 < c^* - D_q \lambda^{2/(2-q)}$, Lemma 2.4 implies that $\{u_j\}$ converges strongly in $H_\gamma(\Omega)$, up to a subsequence, to some u_λ . Then

$$J_{\lambda,q}(u_\lambda) = m_\lambda < 0, \quad J'_{\lambda,q}(u_\lambda) = 0.$$

Therefore $u_\lambda \neq 0$. Testing $J'_{\lambda,q}(u_\lambda) = 0$ by u_λ^- gives $u_\lambda^- = 0$, hence $u_\lambda \geq 0$, and u_λ is a weak solution of (1.2). Since u_λ minimizes $J_{\lambda,q}$ on $\overline{B}_\rho$ and $\|u_\lambda\|_\gamma < \rho$, it is a local minimizer in $H_\gamma(\Omega)$.

It remains to prove the norm estimate for the first solution. Since u_λ is critical and nonnegative,

$$\|u_\lambda\|_\gamma^2 = \lambda|u_\lambda|_q^q + \mathcal{B}(u_\lambda). \quad (4.5)$$

As $|u_\lambda|_\gamma < \rho$ and $S_{\gamma,\mu}(\Omega)^{-p}\rho^{2p-2} \leq 1/2$, by (4.5), (2.1), and (2.4) we have

$$\|u_\lambda\|_\gamma^2 \leq \lambda C_q^q \|u_\lambda\|_\gamma^q + \frac{1}{2} \|u_\lambda\|_\gamma^2.$$

Thus

$$\|u_\lambda\|_\gamma^{2-q} \leq 2C_q^q \lambda,$$

and the asserted estimate follows.

Assume now that (C_q) holds. Apply Lemma 4.1 with the above value of ρ , and decrease λ_0 so that $\lambda_0 \leq \lambda_1(\rho)$. Let e_λ and $d_{\lambda,q}$ be given by that lemma. Since every path from 0 to e_λ crosses the sphere $\|u\|_\gamma = \rho$, (4.3) yields

$$d_{\lambda,q} \geq \inf_{\|u\|_\gamma=\rho} J_{\lambda,q}(u) > 0.$$

Moreover, Lemma 4.1 gives

$$0 < d_{\lambda,q} < c^* - D_q \lambda^{\frac{2}{2-q}}.$$

The mountain-pass theorem gives a Palais-Smale sequence at level $d_{\lambda,q}$; by Lemma 2.4 this sequence has a strongly convergent subsequence. Its limit will be denoted by v_λ . Then

$$J'_{\lambda,q}(v_\lambda) = 0, \quad J_{\lambda,q}(v_\lambda) = d_{\lambda,q} > 0.$$

Thus $v_\lambda \neq 0$. Testing the equation by v_λ^- gives $v_\lambda \geq 0$, and therefore v_λ is a nonnegative weak solution of (1.2). Since

$$J_{\lambda,q}(u_\lambda) < 0 < J_{\lambda,q}(v_\lambda),$$

the two solutions are distinct. □

4.2. Superlinear range.

Proof of Theorem 1.4. Let $2 < q < 2_\gamma^*$. From (2.1) and (2.4),

$$J_{\lambda,q}(u) \geq \frac{1}{2} \|u\|_\gamma^2 - \frac{\lambda C_q^q}{q} \|u\|_\gamma^q - \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p} \|u\|_\gamma^{2p}.$$

Since $2 < q < 2p$, for each fixed $\lambda > 0$ one can choose $\rho_\lambda > 0$ such that

$$\frac{\lambda C_q^q}{q} s^{q-2} + \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p} s^{2p-2} \leq \frac{1}{4} \quad \text{for } 0 \leq s \leq \rho_\lambda.$$

It follows that

$$J_{\lambda,q}(u) \geq \frac{1}{4} \|u\|_\gamma^2 \quad \text{for } \|u\|_\gamma \leq \rho_\lambda.$$

In particular, $J_{\lambda,q}(u) \geq \alpha_\lambda := \rho_\lambda^2/4$ on the sphere $\|u\|_\gamma = \rho_\lambda$. If $e \in C_c^\infty(\Omega)$, $e \geq 0$, $e \neq 0$, then

$$J_{\lambda,q}(te) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

Choose $T_\lambda > 0$ so that $\|T_\lambda e\|_\gamma > \rho_\lambda$ and $J_{\lambda,q}(T_\lambda e) < 0$. Thus $J_{\lambda,q}$ has the mountain-pass geometry. Define

$$e_\lambda = T_\lambda e, \quad \Gamma_\lambda = \{\eta \in C([0,1], H_\gamma(\Omega)) : \eta(0) = 0, \eta(1) = e_\lambda\},$$

$$c_{\lambda,q} = \inf_{\eta \in \Gamma_\lambda} \max_{s \in [0,1]} J_{\lambda,q}(\eta(s)).$$

Every path in Γ_λ crosses the sphere of radius ρ_λ . Hence $c_{\lambda,q} \geq \alpha_\lambda > 0$. The mountain-pass theorem yields a Palais-Smale sequence at level $c_{\lambda,q}$.

We now show that $c_{\lambda,q} < c^*$ for all large λ . Keep the above function e . The path $s \mapsto sT_\lambda e$ belongs to Γ_λ . Since the Choquard term is nonnegative,

$$c_{\lambda,q} \leq \max_{t \geq 0} J_{\lambda,q}(te) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \|e\|_\gamma^2 - \frac{\lambda t^q}{q} |e|_q^q \right\}.$$

The last maximum can be computed explicitly. If

$$M_\lambda(t) = \frac{a}{2}t^2 - \frac{\lambda b}{q}t^q, \quad a = \|e\|_\gamma^2, \quad b = |e|_q^q,$$

then the maximizing point is

$$t_\lambda = \left(\frac{a}{\lambda b}\right)^{\frac{1}{q-2}}, \quad \max_{t \geq 0} M_\lambda(t) = \left(\frac{1}{2} - \frac{1}{q}\right) a^{\frac{q}{q-2}} b^{-\frac{2}{q-2}} \lambda^{-\frac{2}{q-2}}.$$

Hence

$$c_{\lambda,q} \leq C\lambda^{-\frac{2}{q-2}} \rightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Choose $\Lambda_* > 0$ so large that $c_{\lambda,q} < c^*$ for $\lambda \geq \Lambda_*$. Lemma 2.4 then gives a strongly convergent Palais-Smale sequence. Its limit u_λ is a critical point of $J_{\lambda,q}$ at level $c_{\lambda,q} > 0$. Thus $u_\lambda \neq 0$. Testing by u_λ^- gives $u_\lambda \geq 0$, and hence u_λ is a weak solution of (1.2). The estimate $c_{\lambda,q} \rightarrow 0$ has been proved.

Finally, $J'_{\lambda,q}(u_\lambda) = 0$ and $q < 2p$ give

$$\begin{aligned} c_{\lambda,q} &= J_{\lambda,q}(u_\lambda) - \frac{1}{q} \langle J'_{\lambda,q}(u_\lambda), u_\lambda \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \|u_\lambda\|_\gamma^2 + \left(\frac{1}{q} - \frac{1}{2p}\right) \mathcal{B}(u_\lambda) \\ &\geq \frac{q-2}{2q} \|u_\lambda\|_\gamma^2. \end{aligned}$$

Thus $\|u_\lambda\|_\gamma \leq C\lambda^{-1/(q-2)} \rightarrow 0$ as $\lambda \rightarrow +\infty$. □

5. GENERAL SUPERLINEAR PERTURBATIONS

We now consider the parameterized general problem (1.3). For $\lambda > 0$ and $u \in H_\gamma(\Omega)$ define

$$I_\lambda(u) = \frac{1}{2} \|u\|_\gamma^2 - \lambda \int_\Omega F(z, u^+) dz - \frac{1}{2p} \mathcal{B}(u^+).$$

Under assumptions (A1)–(A2), $I_\lambda \in C^1(H_\gamma(\Omega), \mathbb{R})$ and

$$\langle I'_\lambda(u), \varphi \rangle = \int_\Omega \nabla_\gamma u \cdot \nabla_\gamma \varphi dz - \lambda \int_\Omega f(z, u^+) \varphi dz - \int_\Omega \left(\int_\Omega \frac{(u^+(w))^p}{d(z-w)^\mu} dw \right) (u^+)^{p-1} \varphi dz.$$

As before, every nonzero critical point of I_λ is nonnegative and solves (1.3).

Lemma 5.1. *Assume (A1)–(A3). For every fixed $\lambda > 0$, the functional I_λ has the mountain-pass geometry: there exist $\rho_\lambda, \alpha_\lambda > 0$ such that $I_\lambda(u) \geq \alpha_\lambda$ for $\|u\|_\gamma = \rho_\lambda$, and there exists $e \in H_\gamma(\Omega)$ with $\|e\|_\gamma > \rho_\lambda$ and $I_\lambda(e) < 0$.*

Proof. By (A1)–(A2), for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$F(z, t) \leq \varepsilon t^2 + C_\varepsilon t^r \quad \text{for a.e. } z \in \Omega, \quad t \geq 0.$$

Using the Sobolev embedding and (2.4),

$$I_\lambda(u) \geq \left(\frac{1}{2} - \lambda \varepsilon C_2^2\right) \|u\|_\gamma^2 - \lambda C_\varepsilon C_r^r \|u\|_\gamma^r - \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p} \|u\|_\gamma^{2p}.$$

Choose $\varepsilon > 0$ so that $\lambda \varepsilon C_2^2 \leq 1/8$. Since $r > 2$ and $2p > 2$, there is $\rho_\lambda > 0$ such that

$$\lambda C_\varepsilon C_r^r s^{r-2} + \frac{S_{\gamma,\mu}(\Omega)^{-p}}{2p} s^{2p-2} \leq \frac{1}{8} \quad \text{for } 0 \leq s \leq \rho_\lambda.$$

Thus

$$I_\lambda(u) \geq \frac{1}{4} \|u\|_\gamma^2 \quad \text{for } \|u\|_\gamma \leq \rho_\lambda.$$

In particular, $I_\lambda(u) \geq \alpha_\lambda := \rho_\lambda^2/4$ when $\|u\|_\gamma = \rho_\lambda$.

Let $0 \leq e \in C_c^\infty(\Omega)$, $e \not\equiv 0$. Since the kernel is positive and e is nontrivial, $\mathcal{B}(e) > 0$; moreover $2p > 2$, and therefore

$$I_\lambda(te) \leq \frac{t^2}{2} \|e\|_\gamma^2 - \frac{t^{2p}}{2p} \mathcal{B}(e) \rightarrow -\infty$$

as $t \rightarrow +\infty$. For large t , both $\|te\|_\gamma > \rho_\lambda$ and $I_\lambda(te) < 0$ hold. $\square$

Lemma 5.2. *Assume (A1)–(A3). For every fixed $\lambda > 0$, every Palais-Smale sequence of I_λ at a level $c < c^*$ has a strongly convergent subsequence in $H_\gamma(\Omega)$.*

Proof. Let $\{u_j\}$ be a Palais-Smale sequence for I_λ at level $c < c^*$. By (A3),

$$\begin{aligned} c + o(1) + o(1)\|u_j\|_\gamma &= I_\lambda(u_j) - \frac{1}{\theta} \langle I'_\lambda(u_j), u_j \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_j\|_\gamma^2 + \lambda \int_\Omega \left(\frac{1}{\theta} f(z, u_j^+) u_j^+ - F(z, u_j^+)\right) dz \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{2p}\right) \mathcal{B}(u_j^+). \end{aligned}$$

Because $2 < \theta \leq r < 2p$, the first and third coefficients are positive, and the integral term is nonnegative. Hence $\{u_j\}$ is bounded in $H_\gamma(\Omega)$.

As in Lemma 2.4, testing with u_j^- yields $\|u_j^-\|_\gamma \rightarrow 0$. Set $w_j = u_j^+$. Up to a subsequence,

$$w_j \rightharpoonup u \quad \text{in } H_\gamma(\Omega), \quad w_j \rightarrow u \quad \text{in } L^s(\Omega), \quad 1 \leq s < 2_\gamma^*, \quad w_j \rightarrow u \quad \text{a.e. in } \Omega,$$

with $u \geq 0$. The growth assumption (A2) gives

$$\int_\Omega F(z, w_j) dz = \int_\Omega F(z, u) dz + o(1), \quad (5.1)$$

because $w_j \rightarrow u$ in $L^r(\Omega)$ and $|F(z, t)| \leq C(t^2 + t^r)$. The Carathéodory property and (A2) give $|f(z, t)|^{r'} \leq C(t^{r'} + t^r)$ for $t \geq 0$. Since $w_j \rightarrow u$ in $L^r(\Omega)$ and $1 < r' < r$, the right-hand side is uniformly integrable along w_j . Vitali's theorem yields

$$f(\cdot, w_j) \rightarrow f(\cdot, u) \quad \text{in } L^{r'}(\Omega), \quad r' = \frac{r}{r-1}.$$

Consequently,

$$\int_\Omega f(z, w_j)(w_j - u) dz = o(1), \quad \int_\Omega f(z, w_j) w_j dz \rightarrow \int_\Omega f(z, u) u dz.$$

For every $\varphi \in H_\gamma(\Omega)$,

$$\int_\Omega f(z, w_j) \varphi dz \rightarrow \int_\Omega f(z, u) \varphi dz. \quad (5.2)$$

Since $H_\gamma(\Omega) \hookrightarrow L^r(\Omega)$, (5.2) follows from Hölder's inequality. Using Lemma 2.3, (5.2) and $I'_\lambda(u_j) \rightarrow 0$, we infer that $I'_\lambda(u) = 0$.

Put $v_j = w_j - u$. Then

$$\|u_j\|_\gamma^2 = \|v_j\|_\gamma^2 + \|u\|_\gamma^2 + o(1), \quad \mathcal{B}(w_j) = \mathcal{B}(v_j) + \mathcal{B}(u) + o(1),$$

and (5.1) holds. Hence

$$I_\lambda(u_j) = I_\lambda(u) + \frac{1}{2} \|v_j\|_\gamma^2 - \frac{1}{2p} \mathcal{B}(v_j) + o(1). \quad (5.3)$$

Moreover, subtracting $\langle I'_\lambda(u), u \rangle = 0$ from $\langle I'_\lambda(u_j), u_j \rangle = o(1)$ and using the strong convergence of the subcritical term gives

$$\|v_j\|_\gamma^2 - \mathcal{B}(v_j) = o(1). \quad (5.4)$$

As in the proof of Lemma 2.4, there exists $\ell \geq 0$ such that, up to a subsequence,

$$\|v_j\|_\gamma^2 \rightarrow \ell, \quad \mathcal{B}(v_j) \rightarrow \ell,$$

and either $\ell = 0$ or $\ell \geq S_{\gamma, \mu}(\Omega)^{p/(p-1)}$.

It remains to rule out $\ell > 0$. Since $I'_\lambda(u) = 0$, using (A3) we obtain

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{\theta} \langle I'_\lambda(u), u \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|_\gamma^2 + \lambda \int_\Omega \left(\frac{1}{\theta} f(z, u)u - F(z, u) \right) dz + \left(\frac{1}{\theta} - \frac{1}{2p} \right) \mathcal{B}(u) \geq 0. \end{aligned}$$

If $\ell > 0$, then by (5.3) and (5.4),

$$c = I_\lambda(u) + \frac{p-1}{2p} \ell \geq c^*,$$

contrary to $c < c^*$. Hence $\ell = 0$, and $u_j \rightarrow u$ strongly in $H_\gamma(\Omega)$. $\square$

Proof of Theorem 1.6. Fix $\lambda > 0$, and let ρ_λ and α_λ be given by Lemma 5.1. By (A4), choose $e \in C_c^\infty(U)$ with $e \geq 0$ and $e \not\equiv 0$. For every $t \geq 0$,

$$I_\lambda(te) \leq \frac{t^2}{2} \|e\|_\gamma^2 - \lambda b t^\sigma |e|_\sigma^\sigma.$$

The right-hand side tends to $-\infty$. Choose $T_\lambda > 0$ after its maximum so that

$$\|T_\lambda e\|_\gamma > \rho_\lambda, \quad I_\lambda(T_\lambda e) < 0.$$

Set $e_{\lambda,f} = T_\lambda e$ and

$$\Gamma_{\lambda,f} = \{\eta \in C([0, 1], H_\gamma(\Omega)) : \eta(0) = 0, \eta(1) = e_{\lambda,f}\}.$$

The mountain-pass level

$$c_{\lambda,f} = \inf_{\eta \in \Gamma_{\lambda,f}} \max_{s \in [0,1]} I_\lambda(\eta(s))$$

satisfies $c_{\lambda,f} \geq \alpha_\lambda > 0$, because every path in $\Gamma_{\lambda,f}$ crosses the sphere of radius ρ_λ . The path $s \mapsto sT_\lambda e$ belongs to $\Gamma_{\lambda,f}$. The maximum of the right-hand side over $t \geq 0$ is equal to

$$\frac{\sigma-2}{2\sigma} \|e\|_\gamma^2 \left(\frac{\|e\|_\gamma^2}{\lambda b \sigma |e|_\sigma^\sigma} \right)^{\frac{2}{\sigma-2}} = C_e \lambda^{-\frac{2}{\sigma-2}},$$

where $C_e > 0$ is independent of λ . Hence

$$0 < c_{\lambda,f} \leq \max_{t \geq 0} I_\lambda(te) \leq C_e \lambda^{-\frac{2}{\sigma-2}} < c^*$$

for all sufficiently large λ . The mountain-pass theorem gives a Palais-Smale sequence at level $c_{\lambda,f}$. Lemma 5.2 implies that this sequence converges strongly to a critical point $u \in H_\gamma(\Omega)$. Since $I_\lambda(u) = c_{\lambda,f} > 0$, $u \not\equiv 0$. Testing $I'_\lambda(u) = 0$ by u^- yields $u^- = 0$. Thus $u \geq 0$ and u solves (1.3). $\square$

Remark 5.3. For $f(z, t) = (t^+)^{q-1}$ with $2 < q < 2_\gamma^*$, conditions (A1)–(A4) hold with $r = \theta = \sigma = q$. Thus Theorem 1.6 includes the pure-power existence result. The proof in Section 4 also gives the bounds $c_{\lambda,q} \leq C\lambda^{-2/(q-2)}$ and $\|u_\lambda\|_\gamma \leq C\lambda^{-1/(q-2)}$.

Acknowledgments. This work was supported by the National Natural Science Foundation of China 12371121, and by the Fundamental Research Funds for the Central Universities SWU-KF26002.

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