

WELL-POSEDNESS FOR A DEEP-SEA MINING PIPELINE EQUATION WITH NONLOCAL STIFFNESS

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ABSTRACT. We study a fourth-order evolution equation describing the transverse vibrations of a deep-sea mining pipeline with a nonlocal stiffness coefficient. The bending rigidity depends on the total deformation of the pipeline through the integral

$$\int_0^L |v_s(\sigma, t)|^2 d\sigma.$$

The equation also contains axial loading, gyroscopic effects, submerged weight and linear velocity damping. We prove local-in-time existence of weak solutions by the Faedo-Galerkin method, using energy estimates, compactness arguments and the positivity of the nonlocal stiffness coefficient. We also derive a continuation criterion and establish a small-energy continuation result, which yields existence on arbitrary prescribed finite time intervals under a suitable smallness condition on the initial energy. Under additional regularity and compatibility assumptions, we prove the existence of local strong solutions. Finally, we establish uniqueness in the strong class and prove continuous dependence on the initial data in a regular energy class.

1. INTRODUCTION

Deep-sea mining systems rely on long vertical pipelines to transport a solid-liquid slurry from seabed collectors to surface vessels. These pipelines are slender, flexible and exposed to several interacting mechanical effects, including bending stiffness, internal slurry flow, axial loading, submerged weight and hydrodynamic resistance. As a consequence, transverse vibrations may arise during operation, and their control is essential for the safety, reliability and efficiency of the lifting system. From a mathematical point of view, such pipelines are commonly modeled by Euler-Bernoulli-type beam equations supplemented with terms describing the effects of internal flow, axial loading and external forces. The general theory of vibrations of slender structures interacting with axial flow is extensively developed in the fluid-structure interaction literature; see, for instance, [10]. A simplified linear model for the transverse displacement $v = v(s, t)$ of a deep-sea transport pipeline was given, in [15], by

$$mv_{tt} + EJv_{ssss} + a(s)v_{ss} + bv_{st} + G_0v_s + \delta v_t = 0, \quad (s, t) \in (0, L) \times (0, T), \quad (1.1)$$

where $m > 0$ is the effective mass per unit length, $EJ > 0$ is the bending rigidity, $a(s)$ denotes the axial loading coefficient, b is the gyroscopic coefficient associated with the internal slurry flow, G_0 is the submerged weight per unit length, and $\delta > 0$ is a linear damping coefficient. The term EJv_{ssss} represents the classical Euler–Bernoulli bending force, while $a(s)v_{ss}$, bv_{st} , and G_0v_s describe the effects of axial loading, internal flow and gravity, respectively. Mathematical methods for beam-type and higher-order evolution equations are closely related to the general theory of nonlinear evolution problems and compactness methods [3, 12, 13]; see, for example, Lions [6], Evans [4], Aubin [2], Simon [14], and Temam [16].

2020 *Mathematics Subject Classification.* 35A01, 35A02, 35D30, 35G31, 35B30, 35B35, 35B40, 74F10, 74K10.
Key words and phrases. Deep-sea mining pipeline; Euler-Bernoulli beam; nonlocal stiffness; Kirchhoff-type coefficient; weak solution; strong solutions; small-energy continuation; continuous dependence.
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 Submitted July 24, 2026. Published October 26, 2026.

Although (1.1) is useful for studying small oscillations and the onset of instability, it has an important limitation: the bending stiffness is assumed to be constant. This assumption is appropriate when the transverse deformation remains small and the structural response is essentially linear. In realistic deep-sea mining operations, however, large pipeline deflections may occur due to slurry transport, hydrodynamic forces and operational disturbances. In such regimes, the effective stiffness of the pipeline is no longer expected to be independent of the deformation. Instead, it may depend on the total stretching or bending state of the structure. Therefore, the purely linear model does not fully capture the global feedback between deformation and stiffness. To take this effect into account, we introduce a nonlocal stiffness correction of Kirchhoff type. More precisely, we replace the constant bending rigidity EJ by the deformation-dependent coefficient

$$M(v(t)) := EJ + \alpha \int_0^L |v_s(\sigma, t)|^2 d\sigma, \quad \alpha \geq 0.$$

This coefficient depends on the total deformation of the pipeline through the integral of $|v_s|^2$ over the whole interval $(0, L)$. Mechanically, the added term measures the global stretching effect induced by transverse displacement. Thus, when the pipeline deformation increases, the effective stiffness is modified accordingly. This gives a nonlinear feedback mechanism between the global deformation of the structure and its local bending response. Such nonlocal stiffness laws are in the spirit of Kirchhoff-type models, where the principal coefficient depends on an integral quantity measuring the global deformation of the system; see Kirchhoff [5], Lions [7], Arosio and Panizzi [1], and the survey-type works of Matsuyama and Ruzhansky [8, 9]. Accordingly, we study the nonlinear nonlocal model

$$mv_{tt} + \left(EJ + \alpha \int_0^L |v_s(\sigma, t)|^2 d\sigma \right) v_{ssss} + a(s)v_{ss} + bv_{st} + G_0 v_s + \delta v_t = 0, \quad (1.2)$$

for $(s, t) \in (0, L) \times (0, T)$, where

$$m > 0, \quad EJ > 0, \quad \alpha \geq 0, \quad \delta > 0,$$

and $a \in L^\infty(0, L)$. The equation is supplemented with the clamped boundary conditions

$$v(0, t) = v(L, t) = 0, \quad v_s(0, t) = v_s(L, t) = 0, \quad (1.3)$$

and the initial conditions

$$v(s, 0) = v_0(s), \quad v_t(s, 0) = v_1(s). \quad (1.4)$$

The additional nonlocal term $\alpha \left(\int_0^L |v_s(\sigma, t)|^2 d\sigma \right) v_{ssss}$ is the main novelty of the model. It closes the gap between the linear theory and the mechanically relevant situation in which the stiffness of the pipeline is affected by the total deformation. Unlike a purely pointwise nonlinearity, this term introduces a global coupling: the bending response at each point s depends on the deformation of the entire pipeline. Such nonlocal effects are natural in Kirchhoff-type beam models and are particularly appropriate for long slender structures where global stretching influences local bending dynamics. From the mathematical point of view, the coefficient

$$M(v(t)) = EJ + \alpha \int_0^L |v_s(\sigma, t)|^2 d\sigma$$

has two important features. First, it is positive, since $M(v(t)) \geq EJ > 0$, which preserves the elliptic character of the fourth-order bending operator. Second, it is nonlinear and nonlocal, which creates additional analytical difficulties in the energy estimates, in the compactness argument and in the proof of continuous dependence. In particular, when comparing two solutions v and w , one has to estimate the difference

$$M(v(t)) - M(w(t)) = \alpha \left(\|v_s(t)\|_{L^2(0, L)}^2 - \|w_s(t)\|_{L^2(0, L)}^2 \right).$$

This term couples the global deformations of two different solutions and requires estimates that are strong enough to control the fourth-order contribution appearing in the difference equation.

The aim of this paper is to establish a first well-posedness theory for (1.2)–(1.4). We first prove local-in-time existence of weak solutions by the Faedo–Galerkin approximation method. The proof relies on energy estimates, compactness arguments and the positivity of the nonlocal

stiffness coefficient. The compactness argument follows the classical approach of Aubin [2] and Simon [14], while the nonlinear energy estimates are inspired by standard methods for nonlinear evolution equations [6, 16]. We then derive a continuation criterion showing that a weak solution can be extended as long as its natural energy norm remains bounded. As a consequence, we obtain a small-energy continuation result: for every prescribed finite time interval $[0, T]$, weak solutions exist on the whole interval provided the initial energy is sufficiently small in terms of T and the structural parameters of the equation. This result is consistent with the role of smallness assumptions in several Kirchhoff-type evolution problems, where global or long-time solvability often depends on suitable a priori bounds or small initial data; see, for example, Arosio and Panizzi [1] and Matsuyama and Ruzhansky [8, 9]. The result should not be confused with unconditional global existence; rather, it gives long-time existence on arbitrary fixed finite intervals under a suitable smallness condition. Next, under additional regularity and a natural compatibility condition on the initial data, we prove the existence of local strong solutions. This step is necessary because the nonlocal stiffness coefficient depends on the solution itself, and differentiating the equation in time produces additional terms involving the derivative of $M(v(t))$. The strong-solution result provides the regularity needed to handle these terms rigorously. Finally, we establish uniqueness in the strong class and prove continuous dependence on the initial data in a regular energy class. The main difficulty is the nonlocal difference term $(M(v(t)) - M(w(t)))w_{ssss}$, which requires control of the fourth-order derivative of one solution. This explains why the uniqueness and continuous-dependence results are stated in a regular class. Together, these results provide a rigorous mathematical foundation for the study of deep-sea mining pipeline equations with deformation-dependent stiffness and prepare the ground for future investigations of stabilization, long-time dynamics [11] and more refined coupled fluid-structure models.

2. FUNCTIONAL SETTING AND WEAK FORMULATION

Let $I = (0, L)$, $H = L^2(I)$, and

$$V = H_0^2(I) = \{v \in H^2(I) : v(0) = v(L) = 0, v_s(0) = v_s(L) = 0\}.$$

We endow V with the equivalent norm

$$\|v\|_V = \|v_{ss}\|_{L^2(I)}.$$

We use the Poincaré-type inequality

$$\|v\|_{L^2(I)} + \|v_s\|_{L^2(I)} \leq C_P \|v_{ss}\|_{L^2(I)}, \quad v \in V. \quad (2.1)$$

For $v \in V$, set

$$M(v) := EJ + \alpha \|v_s\|_{L^2(I)}^2. \quad (2.2)$$

Clearly $M(v) \geq EJ > 0$.

Definition 2.1. A function v is called a weak solution of (1.2)–(1.4) on $[0, T]$ if

$$v \in L^\infty(0, T; V), \quad v_t \in L^\infty(0, T; H),$$

and, for every $\phi \in V$, the identity

$$m \langle v_{tt}, \phi \rangle + M(v(t))(v_{ss}, \phi_{ss}) + (a(s)v_{ss}, \phi) - b(v_t, \phi_s) + G_0(v_s, \phi) + \delta(v_t, \phi) = 0 \quad (2.3)$$

holds in the sense of distributions in time. Moreover,

$$v(0) = v_0, \quad v_t(0) = v_1.$$

3. EXISTENCE OF WEAK SOLUTIONS

The next result gives a local-in-time existence theorem for the nonlocal pipeline equation. The local character of the result is due to the fact that the coefficient multiplying the fourth-order operator depends on the deformation through the quantity $\|v_s(t)\|_{L^2(0, L)}^2$. This produces an additional nonlinear contribution in the energy estimate. Nevertheless, the positivity of the nonlocal stiffness coefficient and the compactness of the Galerkin sequence allow us to construct weak solutions.

Theorem 3.1 (Local existence). *Let $m > 0$, $EJ > 0$, $\alpha \geq 0$, $\delta > 0$, $a \in L^\infty(0, L)$, $v_0 \in V$, and $v_1 \in H$. Then there exists $T_* > 0$ such that problem (1.2)–(1.4) admits at least one weak solution on $[0, T_*]$.*

Proof. The proof is divided into several steps.

Step 1: Galerkin approximation. Let $\{w_k\}_{k \geq 1}$ be a basis of $V = H_0^2(0, L)$, chosen for instance from the eigenfunctions of the clamped biharmonic operator. For $n \geq 1$, set

$$V_n = \text{span}\{w_1, \dots, w_n\}.$$

We look for an approximate solution of the form

$$v_n(t) = \sum_{k=1}^n d_k^n(t) w_k.$$

The Galerkin approximation is defined by requiring, for $j = 1, \dots, n$,

$$\begin{aligned} m(v_{n,tt}, w_j) + M(v_n(t))(v_{n,ss}, w_{j,ss}) + (a(s)v_{n,ss}, w_j) \\ - b(v_{n,t}, w_{j,s}) + G_0(v_{n,s}, w_j) + \delta(v_{n,t}, w_j) = 0, \end{aligned} \quad (3.1)$$

where

$$M(v_n(t)) = EJ + \alpha \|v_{n,s}(t)\|_{L^2(0,L)}^2.$$

The initial data are chosen such that

$$v_n(0) = v_{n,0} \rightarrow v_0 \quad \text{in } V, \quad v_{n,t}(0) = v_{n,1} \rightarrow v_1 \quad \text{in } H.$$

Since (3.1) is a finite-dimensional system of ordinary differential equations with locally Lipschitz nonlinearities in the coefficients $d_k^n(t)$, it admits a local solution on some interval $[0, T_n)$.

At this level, the nonlocal term causes no difficulty because, in finite dimension,

$$v_n \mapsto M(v_n) = EJ + \alpha \|v_{n,s}\|_{L^2}^2$$

is a smooth function of the Galerkin coefficients.

Step 2: Energy estimate. Testing (3.1) by $v_{n,t}$, that is, multiplying by $\dot{d}_j^n(t)$ and summing over j , gives

$$\begin{aligned} \frac{m}{2} \frac{d}{dt} \|v_{n,t}\|_{L^2}^2 + M(v_n)(v_{n,ss}, v_{n,ts}) + \delta \|v_{n,t}\|_{L^2}^2 \\ = -(a(s)v_{n,ss}, v_{n,t}) - G_0(v_{n,s}, v_{n,t}). \end{aligned} \quad (3.2)$$

Note that the gyroscopic term vanishes;

$$(v_{n,t}, v_{n,ts}) = \frac{1}{2} \int_0^L \partial_s (v_{n,t}^2) ds = 0,$$

as

$$v_{n,t}(0, t) = v_{n,t}(L, t) = 0.$$

Let us define the modified energy

$$\mathcal{E}_n(t) := \frac{m}{2} \|v_{n,t}(t)\|_{L^2}^2 + \frac{1}{2} M(v_n(t)) \|v_{n,ss}(t)\|_{L^2}^2.$$

Since

$$M(v_n(t)) = EJ + \alpha \|v_{n,s}(t)\|_{L^2}^2,$$

we have

$$\frac{d}{dt} M(v_n(t)) = 2\alpha (v_{n,s}, v_{n,st}).$$

Using integration by parts and $v_{n,s} = 0$ on the boundary, we obtain

$$(v_{n,s}, v_{n,st}) = -(v_{n,ss}, v_{n,t}).$$

Hence

$$\frac{d}{dt} M(v_n(t)) = -2\alpha (v_{n,ss}, v_{n,t}).$$

Therefore,

$$\frac{d}{dt} \left(\frac{1}{2} M(v_n(t)) \|v_{n,ss}(t)\|_{L^2}^2 \right) = M(v_n)(v_{n,ss}, v_{n,tss}) - \alpha(v_{n,ss}, v_{n,t}) \|v_{n,ss}\|_{L^2}^2.$$

Substituting this identity into (3.2), we obtain

$$\frac{d}{dt} \mathcal{E}_n(t) + \delta \|v_{n,t}\|_{L^2}^2 = -(a(s)v_{n,ss}, v_{n,t}) - G_0(v_{n,s}, v_{n,t}) - \alpha(v_{n,ss}, v_{n,t}) \|v_{n,ss}\|_{L^2}^2. \quad (3.3)$$

The last term in (3.3) is the price paid for the nonlocal stiffness. It does not appear in the purely linear model. This term is responsible for the local nature of the result.

Step 3: Uniform local bounds. Since

$$M(v_n(t)) \geq EJ > 0,$$

the energy controls the natural norms:

$$\mathcal{E}_n(t) \geq \frac{m}{2} \|v_{n,t}(t)\|_{L^2}^2 + \frac{EJ}{2} \|v_{n,ss}(t)\|_{L^2}^2.$$

Using the Poincaré-type inequality

$$\|v_{n,s}\|_{L^2} \leq C_P \|v_{n,ss}\|_{L^2},$$

we estimate the lower-order terms:

$$\begin{aligned} |(a(s)v_{n,ss}, v_{n,t})| &\leq \|a\|_{L^\infty} \|v_{n,ss}\|_{L^2} \|v_{n,t}\|_{L^2} \leq C \mathcal{E}_n(t), \\ |G_0(v_{n,s}, v_{n,t})| &\leq |G_0| C_P \|v_{n,ss}\|_{L^2} \|v_{n,t}\|_{L^2} \leq C \mathcal{E}_n(t). \end{aligned}$$

For the nonlocal contribution, we use

$$\alpha|(v_{n,ss}, v_{n,t})| \|v_{n,ss}\|_{L^2}^2 \leq \alpha \|v_{n,ss}\|_{L^2}^3 \|v_{n,t}\|_{L^2}.$$

Since both $\|v_{n,ss}\|_{L^2}$ and $\|v_{n,t}\|_{L^2}$ are controlled by $\mathcal{E}_n(t)^{1/2}$, we obtain

$$\alpha|(v_{n,ss}, v_{n,t})| \|v_{n,ss}\|_{L^2}^2 \leq C \mathcal{E}_n(t)^2.$$

Consequently, (3.3) implies

$$\frac{d}{dt} \mathcal{E}_n(t) \leq C(\mathcal{E}_n(t) + \mathcal{E}_n(t)^2), \quad (3.4)$$

where $C > 0$ is independent of n .

By the comparison theorem for ordinary differential inequalities, there exists $T_* > 0$, depending only on the initial energy and on the structural parameters, such that

$$\sup_{0 \leq t \leq T_*} \mathcal{E}_n(t) \leq C_{T_*},$$

where C_{T_*} is independent of n . Hence

$$(v_n) \text{ is bounded in } L^\infty(0, T_*; V), \quad (3.5)$$

$$(v_{n,t}) \text{ is bounded in } L^\infty(0, T_*; H). \quad (3.6)$$

Step 4: Estimate of the second time derivative. For every $\phi \in V_n$,

$$\begin{aligned} m|\langle v_{n,tt}, \phi \rangle| &\leq M(v_n)|\langle v_{n,ss}, \phi_{ss} \rangle| + |\langle a(s)v_{n,ss}, \phi \rangle| + |\langle b(v_{n,t}, \phi_s) \\ &\quad + |G_0(v_{n,s}, \phi)| + \delta|\langle v_{n,t}, \phi \rangle|. \end{aligned}$$

Using (2.1), (3.5), (3.6), and the boundedness of $M(v_n)$ on $[0, T_*]$, we obtain

$$|\langle v_{n,tt}, \phi \rangle| \leq C_{T_*} \|\phi\|_V.$$

Thus

$$(v_{n,tt}) \text{ is bounded in } L^\infty(0, T_*; V'). \quad (3.7)$$

Step 5: Compactness. From (3.5) and (3.6), Banach-Alaoglu gives, up to a subsequence,

$$v_n \rightharpoonup^* v \quad \text{in } L^\infty(0, T_*; V),$$

$$v_{n,t} \rightharpoonup^* v_t \quad \text{in } L^\infty(0, T_*; H).$$

Moreover, by (3.6), the sequence $(v_{n,t})$ is bounded in $L^2(0, T_*; H)$. Since

$$V = H_0^2(0, L) \hookrightarrow H^1(0, L) \hookrightarrow H = L^2(0, L),$$

the Aubin-Simon compactness lemma yields

$$v_n \rightarrow v \quad \text{strongly in } L^2(0, T_*; H^1(0, L)). \quad (3.8)$$

In particular,

$$v_{n,s} \rightarrow v_s \quad \text{strongly in } L^2(0, T_*; L^2(0, L)).$$

Note that the strong convergence in H^1 is essential. Strong convergence only in $L^2(0, L)$ would not be enough, because the nonlocal coefficient depends on v_s .

Step 6: Convergence of the nonlocal coefficient. Using (3.8), we prove that

$$M(v_n) \rightarrow M(v) \quad \text{strongly in } L^1(0, T_*).$$

Indeed,

$$|M(v_n) - M(v)| = \alpha \left| \|v_{n,s}\|_{L^2}^2 - \|v_s\|_{L^2}^2 \right|.$$

Therefore,

$$|M(v_n) - M(v)| \leq \alpha (\|v_{n,s}\|_{L^2} + \|v_s\|_{L^2}) \|v_{n,s} - v_s\|_{L^2}.$$

Since (v_n) is bounded in $L^\infty(0, T_*; V)$, the sequence $(v_{n,s})$ is bounded in $L^\infty(0, T_*; L^2(0, L))$. Together with (3.8), this gives

$$M(v_n) \rightarrow M(v) \quad \text{strongly in } L^2(0, T_*), \quad (3.9)$$

and in particular strongly in $L^1(0, T_*)$.

Step 7: Passage to the limit. Let $\phi \in V$ and let $\eta \in C_c^\infty(0, T_*)$. Passing to the limit in the Galerkin identity tested by $\eta(t)\phi$, the linear terms follow directly from the weak convergences. We only focus on the nonlocal term $\int_0^{T_*} \eta(t) M(v_n(t))(v_{n,ss}(t), \phi_{ss}) dt$. We write

$$\begin{aligned} & \int_0^{T_*} \eta(t) [M(v_n(t))(v_{n,ss}(t), \phi_{ss}) - M(v(t))(v_{ss}(t), \phi_{ss})] dt \\ &= \int_0^{T_*} \eta(t) (M(v_n(t)) - M(v(t)))(v_{n,ss}(t), \phi_{ss}) dt \\ &+ \int_0^{T_*} \eta(t) M(v(t))(v_{n,ss}(t) - v_{ss}(t), \phi_{ss}) dt. \end{aligned}$$

The first term tends to zero by (3.9) and the uniform boundedness of $(v_{n,ss})$ in $L^\infty(0, T_*; L^2(0, L))$. The second term tends to zero by the weak-star convergence of v_n in $L^\infty(0, T_*; V)$. Therefore,

$$M(v_n)(v_{n,ss}, \phi_{ss}) \rightarrow M(v)(v_{ss}, \phi_{ss})$$

in the sense of distributions in time. Hence we can pass to the limit in (3.1) and obtain

$$m\langle v_{tt}, \phi \rangle + M(v(t))(v_{ss}, \phi_{ss}) + (a(s)v_{ss}, \phi) - b(v_t, \phi_s) + G_0(v_s, \phi) + \delta(v_t, \phi) = 0.$$

Thus v satisfies the weak formulation (2.3).

Step 8: Initial conditions. The conditions

$$v \in L^\infty(0, T_*; V), \quad v_t \in L^\infty(0, T_*; H), \quad v_{tt} \in L^\infty(0, T_*; V')$$

imply the standard weak continuity properties

$$v \in C_w([0, T_*]; V), \quad v_t \in C_w([0, T_*]; H).$$

Since $v_n(0) \rightarrow v_0$ in V and $v_{n,t}(0) \rightarrow v_1$ in H , we obtain

$$v(0) = v_0, \quad v_t(0) = v_1.$$

Therefore v is a weak solution of (1.2)–(1.4) on $[0, T_*]$. □

Remark 3.2. The proof shows that the main novelty is the passage to the limit in the nonlocal stiffness term. The compactness must be strong enough to guarantee convergence of $\|v_{n,s}(t)\|_{L^2}^2$. This is why the strong convergence in $L^2(0, T_*; H^1(0, L))$ is emphasized.

Remark 3.3. The differential inequality (3.4) explains why the result is stated locally in time. A global existence result may be obtained under additional a priori bounds, smallness assumptions, or by considering a modified nonlocal stiffness law that derives directly from a coercive potential.

4. SMALL-ENERGY CONTINUATION AND LONG-TIME EXISTENCE

The local existence result obtained in Theorem 3.1 follows from the differential inequality

$$\frac{d}{dt}\mathcal{E}_n(t) \leq C(\mathcal{E}_n(t) + \mathcal{E}_n(t)^2),$$

where $C > 0$ is independent of the Galerkin dimension n . This estimate also gives useful information on the continuation of solutions. In particular, small initial energy allows one to construct solutions on any prescribed finite time interval. The result below makes this observation precise.

Theorem 4.1 (Continuation criterion). *Let v be a weak solution of (1.2)–(1.4) defined on a maximal interval $[0, T_{\max})$, with $0 < T_{\max} \leq \infty$. Assume that*

$$\sup_{0 \leq t < T_{\max}} (\|v_t(t)\|_{L^2}^2 + \|v_{ss}(t)\|_{L^2}^2) < \infty.$$

Then $T_{\max} = \infty$.

Proof. The local existence proof shows that the time of existence depends only on the size of the initial energy and on the structural parameters of the equation. More precisely, if at some time $t_0 < T_{\max}$ one has

$$\|v_t(t_0)\|_{L^2}^2 + \|v_{ss}(t_0)\|_{L^2}^2 \leq R,$$

then the Galerkin construction can be restarted at $t = t_0$, with initial data $v(t_0)$, $v_t(t_0)$, and gives a solution on an interval $[t_0, t_0 + \tau_R]$, where $\tau_R > 0$ depends only on R and on the structural parameters, but not on t_0 . Suppose, by contradiction, that $T_{\max} < \infty$. By the assumed uniform bound, there exists $R > 0$ such that

$$\|v_t(t)\|_{L^2}^2 + \|v_{ss}(t)\|_{L^2}^2 \leq R, \quad 0 \leq t < T_{\max}.$$

Choose $t_0 < T_{\max}$ so close to $T_{\max}$ that $t_0 + \tau_R > T_{\max}$. Restarting the local existence argument at t_0 , we extend the solution beyond $T_{\max}$, contradicting the maximality of $T_{\max}$. Hence, $T_{\max} = \infty$. $\square$

Theorem 4.2 (Small-energy existence on arbitrary finite intervals). *Let $T > 0$ be fixed. There exists $\varepsilon_T > 0$, depending on T and on the structural parameters of the equation, such that if*

$$E(0) := \frac{m}{2}\|v_1\|_{L^2}^2 + \frac{1}{2}M(v_0)\|v_{0,ss}\|_{L^2}^2 \leq \varepsilon_T.$$

then problem (1.2)–(1.4) admits a weak solution on the whole interval $[0, T]$.

Proof. From the Galerkin energy estimate obtained in the proof of Theorem 3.1, we have

$$\frac{d}{dt}\mathcal{E}_n(t) \leq C(\mathcal{E}_n(t) + \mathcal{E}_n(t)^2), \tag{4.1}$$

where $C > 0$ is independent of n . Let $y(t)$ be the solution of the scalar comparison problem

$$y'(t) = C(y(t) + y(t)^2), \quad y(0) = y_0 > 0.$$

Solving this equation explicitly gives

$$y(t) = \frac{y_0 e^{Ct}}{1 + y_0 - y_0 e^{Ct}},$$

as long as the denominator is positive. Hence $y(t)$ remains finite on $[0, T]$ provided

$$1 + y_0 - y_0 e^{CT} > 0,$$

that is,

$$y_0 < \frac{1}{e^{CT} - 1}.$$

Choose

$$\varepsilon_T < \frac{1}{2(e^{CT} - 1)}.$$

If $\mathcal{E}_n(0) \leq \varepsilon_T$, then the comparison principle applied to (4.1) yields

$$\mathcal{E}_n(t) \leq \frac{\mathcal{E}_n(0)e^{Ct}}{1 + \mathcal{E}_n(0) - \mathcal{E}_n(0)e^{Ct}}, \quad 0 \leq t \leq T.$$

In particular,

$$\sup_{0 \leq t \leq T} \mathcal{E}_n(t) \leq C_T,$$

where $C_T > 0$ is independent of n . Therefore,

$$\begin{aligned} (v_n) &\text{ is bounded in } L^\infty(0, T; V), \\ (v_{n,t}) &\text{ is bounded in } L^\infty(0, T; H), \end{aligned}$$

uniformly with respect to n . The compactness and passage-to-the-limit arguments used in the proof of Theorem 3.1 can then be repeated on the whole interval $[0, T]$. Consequently, problem (1.2)–(1.4) admits a weak solution on $[0, T]$. $\square$

Remark 4.3. Theorem 4.2 gives long-time existence on any prescribed finite interval, provided the initial energy is sufficiently small relative to that interval. The smallness threshold ε_T decreases as T increases. Thus the result should not be confused with an unconditional global existence theorem.

5. HIGHER REGULARITY AND STRONG SOLUTIONS

In this section, we show that the local weak solution obtained above is in fact a strong solution whenever the initial data are more regular and satisfy the natural compatibility condition. Because the coefficient $M(v(t))$ depends on the solution itself, differentiating the equation in time produces additional terms involving $M'(v(t))$. This is the main difference with the linear stiffness case.

Theorem 5.1 (Local strong solutions). *Assume that $v_0 \in H^4(0, L) \cap V$, $v_1 \in V$, and that the compatibility condition $-\frac{1}{m}[M(v_0)v_{0,ssss} + a(s)v_{0,ss} + bv_{1,s} + G_0v_{0,s} + \delta v_1] \in H$ holds. Then there exists $T_s > 0$ such that the weak solution of (1.2)–(1.4) satisfies*

$$v \in L^\infty(0, T_s; H^4(0, L) \cap V), \quad v_t \in L^\infty(0, T_s; V), \quad v_{tt} \in L^\infty(0, T_s; H).$$

Proof. We argue at the Galerkin level and derive estimates independent of n . Let v_n be the Galerkin approximation constructed in the proof of Theorem 3.1, and set $w_n = v_{n,t}$. Differentiating the Galerkin system (3.1) with respect to t , we obtain, for every $\phi \in V_n$,

$$\begin{aligned} m(w_{n,tt}, \phi) + M(v_n)(w_{n,ss}, \phi_{ss}) + M'_n(t)(v_{n,ss}, \phi_{ss}) \\ + (a(s)w_{n,ss}, \phi) - b(w_{n,t}, \phi_s) + G_0(w_{n,s}, \phi) + \delta(w_{n,t}, \phi) = 0, \end{aligned} \quad (5.1)$$

where

$$M_n(t) := M(v_n(t)) = EJ + \alpha \|v_{n,s}(t)\|_{L^2}^2, \quad M'_n(t) = 2\alpha(v_{n,s}, v_{n,st}) = -2\alpha(v_{n,ss}, v_{n,t}).$$

Hence, on the local interval of existence obtained in Theorem 3.1, the previous energy estimates imply

$$|M'_n(t)| \leq C_{T_*}.$$

We now choose $\phi = w_{n,t} = v_{n,tt}$ in (5.1). Since the gyroscopic term vanishes,

$$(w_{n,t}, w_{n,ts}) = \frac{1}{2} \int_0^L \partial_s(w_{n,t}^2) ds = \frac{1}{2} w_{n,t}^2|_{s=0}^{s=L} = 0,$$

we obtain

$$\begin{aligned} & \frac{m}{2} \frac{d}{dt} \|w_{n,t}\|_{L^2}^2 + M(v_n)(w_{n,ss}, w_{n,tss}) + \delta \|w_{n,t}\|_{L^2}^2 \\ & = -(a(s)w_{n,ss}, w_{n,t}) - G_0(w_{n,s}, w_{n,t}) - M'_n(t)(v_{n,ss}, w_{n,tss}). \end{aligned} \quad (5.2)$$

We define the higher-order energy

$$\mathcal{F}_n(t) := \frac{m}{2} \|w_{n,t}(t)\|_{L^2}^2 + \frac{1}{2} M(v_n(t)) \|w_{n,ss}(t)\|_{L^2}^2.$$

Since $M(v_n(t)) \geq EJ > 0$, this energy controls

$$\|w_{n,t}(t)\|_{L^2}^2 + \|w_{n,ss}(t)\|_{L^2}^2.$$

Moreover,

$$\frac{d}{dt} \left(\frac{1}{2} M(v_n(t)) \|w_{n,ss}(t)\|_{L^2}^2 \right) = M(v_n)(w_{n,ss}, w_{n,tss}) + \frac{1}{2} M'_n(t) \|w_{n,ss}\|_{L^2}^2.$$

Therefore, from (5.2),

$$\begin{aligned} \frac{d}{dt} \mathcal{F}_n(t) + \delta \|w_{n,t}\|_{L^2}^2 & = -(a(s)w_{n,ss}, w_{n,t}) - G_0(w_{n,s}, w_{n,t}) \\ & \quad - M'_n(t)(v_{n,ss}, w_{n,tss}) + \frac{1}{2} M'_n(t) \|w_{n,ss}\|_{L^2}^2. \end{aligned} \quad (5.3)$$

The first two terms are estimated by the Poincaré inequality and Young's inequality:

$$|(a(s)w_{n,ss}, w_{n,t})| + |G_0(w_{n,s}, w_{n,t})| \leq C \mathcal{F}_n(t).$$

The last term in (5.3) satisfies

$$\left| \frac{1}{2} M'_n(t) \|w_{n,ss}\|_{L^2}^2 \right| \leq C_{T_*} \mathcal{F}_n(t).$$

It remains to estimate the term containing $w_{n,tss}$. Since

$$(v_{n,ss}, w_{n,tss}) = (v_{n,ssss}, w_{n,t}),$$

we have

$$|M'_n(t)(v_{n,ss}, w_{n,tss})| \leq C_{T_*} \|v_{n,ssss}\|_{L^2} \|w_{n,t}\|_{L^2}.$$

From the original Galerkin equation, we may write

$$M(v_n)v_{n,ssss} = -mv_{n,tt} - a(s)v_{n,ss} - bv_{n,st} - G_0v_{n,s} - \delta v_{n,t}.$$

Since $M(v_n) \geq EJ > 0$, it follows that

$$\|v_{n,ssss}\|_{L^2} \leq C (\|w_{n,t}\|_{L^2} + \|v_{n,ss}\|_{L^2} + \|w_{n,s}\|_{L^2} + \|v_{n,s}\|_{L^2} + \|w_n\|_{L^2}).$$

Using the local bounds from Theorem 3.1 and the Poincaré inequality applied to $w_n \in V_n$, we obtain

$$\|v_{n,ssss}\|_{L^2} \leq C_{T_*} (1 + \|w_{n,t}\|_{L^2} + \|w_{n,ss}\|_{L^2}).$$

Consequently,

$$|M'_n(t)(v_{n,ss}, w_{n,tss})| \leq C_{T_*} (1 + \mathcal{F}_n(t)).$$

Combining the preceding estimates in (5.3), we obtain

$$\frac{d}{dt} \mathcal{F}_n(t) \leq C_{T_*} (1 + \mathcal{F}_n(t)).$$

By Grönwall's inequality,

$$\sup_{0 \leq t \leq T_s} \mathcal{F}_n(t) \leq C_{T_s},$$

for some $0 < T_s \leq T_*$, where C_{T_s} is independent of n . The compatibility condition ensures that

$$\sup_n \|v_{n,tt}(0)\|_{L^2} < \infty,$$

and since $v_{n,t}(0) \rightarrow v_1$ in V , we also have

$$\sup_n \|v_{n,tss}(0)\|_{L^2} < \infty.$$

Thus the initial higher-order energy $\mathcal{F}_n(0)$ is uniformly bounded. We conclude that

$$\begin{aligned}(v_{n,t}) &\text{ is bounded in } L^\infty(0, T_s; V), \\ (v_{n,tt}) &\text{ is bounded in } L^\infty(0, T_s; H).\end{aligned}$$

Passing to the limit by weak-star compactness gives

$$v_t \in L^\infty(0, T_s; V), \quad v_{tt} \in L^\infty(0, T_s; H).$$

Finally, rewriting the equation as

$$M(v(t))v_{ssss} = -mv_{tt} - a(s)v_{ss} - bv_{st} - G_0v_s - \delta v_t,$$

and using $M(v(t)) \geq EJ > 0$, we see that the right-hand side belongs to $L^\infty(0, T_s; L^2(0, L))$. Hence

$$v_{ssss} \in L^\infty(0, T_s; L^2(0, L)).$$

Together with $v \in L^\infty(0, T_s; V)$, standard elliptic regularity for the clamped fourth-order operator yields $v \in L^\infty(0, T_s; H^4(0, L) \cap V)$. This completes the proof. $\square$

Remark 5.2. The compatibility condition in Theorem 5.1 is the natural requirement ensuring that the initial acceleration belongs to $L^2(0, L)$. The result is stated locally in time because the nonlocal stiffness coefficient generates terms depending on $M'(v(t))$, and the available a priori bounds are local in time.

6. UNIQUENESS IN THE STRONG CLASS

As a consequence of the higher regularity obtained in the previous section and of the continuous-dependence estimate below, we obtain uniqueness in the strong class. The point is that the additional H^4 -regularity allows us to control the nonlocal coefficient appearing in the difference of two solutions.

Theorem 6.1 (Uniqueness of strong solutions). *Let v and w be two strong solutions of (1.2)–(1.4) in the interval $0 \leq t \leq T$, such that*

$$v, w \in L^\infty(0, T; H^4(0, L) \cap V), \quad v_t, w_t \in L^\infty(0, T; V), \quad v_{tt}, w_{tt} \in L^\infty(0, T; H).$$

Assume that v and w have the same initial data, namely

$$v(0) = w(0), \quad v_t(0) = w_t(0).$$

Then $v = w$ in $0 \leq t \leq T$.

Proof. We set $z = v - w$. Then

$$z(0) = 0, \quad z_t(0) = 0.$$

Subtracting the equations satisfied by v and w , we obtain

$$mz_{tt} + M(v)z_{ssss} + (M(v) - M(w))w_{ssss} + a(s)z_{ss} + bz_{st} + G_0z_s + \delta z_t = 0.$$

Testing this equation by z_t , we obtain

$$\begin{aligned}\frac{m}{2} \frac{d}{dt} \|z_t\|_{L^2}^2 + M(v)(z_{ssss}, z_t) + \delta \|z_t\|_{L^2}^2 \\ = -(M(v) - M(w))(w_{ssss}, z_t) - (a(s)z_{ss}, z_t) - G_0(z_s, z_t).\end{aligned}\tag{6.1}$$

The gyroscopic term vanishes since

$$(z_t, z_{ts}) = \frac{1}{2} \int_0^L \partial_s(z_t^2) ds = \frac{1}{2} z_t^2 \Big|_{s=0}^{s=L} = 0.$$

We define the difference energy

$$\mathcal{G}(t) := \frac{m}{2} \|z_t(t)\|_{L^2}^2 + \frac{1}{2} M(v(t)) \|z_{ss}(t)\|_{L^2}^2.$$

Since $M(v(t)) \geq EJ > 0$, there exists $c_0 > 0$ such that

$$\mathcal{G}(t) \geq c_0 (\|z_t(t)\|_{L^2}^2 + \|z_{ss}(t)\|_{L^2}^2).$$

Moreover,

$$\frac{d}{dt} \left(\frac{1}{2} M(v(t)) \|z_{ss}(t)\|_{L^2}^2 \right) = M(v)(z_{ss}, z_{tss}) + \frac{1}{2} M'(v(t)) \|z_{ss}\|_{L^2}^2.$$

Since $(z_{ssss}, z_t) = (z_{ss}, z_{tss})$, we infer from (6.1) that

$$\begin{aligned} \frac{d}{dt} \mathcal{G}(t) + \delta \|z_t\|_{L^2}^2 &= -(M(v) - M(w))(w_{ssss}, z_t) - (a(s)z_{ss}, z_t) - G_0(z_s, z_t) \\ &\quad + \frac{1}{2} M'(v(t)) \|z_{ss}\|_{L^2}^2. \end{aligned} \quad (6.2)$$

Next we estimate the nonlocal coefficient difference. Since

$$M(v) - M(w) = \alpha (\|v_s\|_{L^2}^2 - \|w_s\|_{L^2}^2),$$

we have

$$M(v) - M(w) = \alpha(v_s + w_s, z_s).$$

Therefore, by the Poincaré inequality,

$$|M(v) - M(w)| \leq \alpha (\|v_s\|_{L^2} + \|w_s\|_{L^2}) \|z_s\|_{L^2} \leq C \|z_{ss}\|_{L^2}.$$

Using $w \in L^\infty(0, T; H^4(0, L))$, we obtain

$$|(M(v) - M(w))(w_{ssss}, z_t)| \leq C \|z_{ss}\|_{L^2} \|z_t\|_{L^2} \leq C \mathcal{G}(t).$$

The lower-order terms satisfy

$$|(a(s)z_{ss}, z_t)| + |G_0(z_s, z_t)| \leq C \mathcal{G}(t).$$

Furthermore, since v is a strong solution,

$$M'(v(t)) = -2\alpha(v_{ss}, v_t),$$

and hence $|M'(v(t))| \leq C$. Consequently,

$$\left| \frac{1}{2} M'(v(t)) \|z_{ss}\|_{L^2}^2 \right| \leq C \mathcal{G}(t).$$

Combining these estimates in (6.2), we obtain

$$\frac{d}{dt} \mathcal{G}(t) \leq C \mathcal{G}(t), \quad 0 \leq t \leq T.$$

Since v and w have the same initial data, $\mathcal{G}(0) = 0$. By Grönwall's inequality,

$$\mathcal{G}(t) = 0, \quad 0 \leq t \leq T.$$

Thus

$$z_t(t) = 0, \quad z_{ss}(t) = 0, \quad 0 \leq t \leq T.$$

Since $z(t) \in H_0^2(0, L)$, the identity $z_{ss}(t) = 0$ implies $z(t) = 0$. Therefore,

$$v(t) = w(t), \quad 0 \leq t \leq T.$$

This completes the proof. $\square$

Remark 6.2. This result is stated in the strong class because the estimate of the nonlocal term requires the control of w_{ssss} in $L^2(0, L)$. This is precisely why uniqueness in the general weak class is more delicate for the present nonlocal stiffness model.

7. CONTINUOUS DEPENDENCE IN THE REGULAR CLASS

We next prove continuous dependence with respect to the initial data. Since the stiffness coefficient is nonlocal and depends on the solution through the quantity

$$M(v(t)) = EJ + \alpha \|v_s(t)\|_{L^2(0,L)}^2,$$

the difference of two solutions contains the additional term $(M(v(t)) - M(w(t)))w_{ssss}$. This term is the main new difficulty compared with the corresponding linear problem. For this reason, the continuous-dependence result is stated in a regular energy class. This regularity allows us to control the fourth-order contribution w_{ssss} and to estimate the difference of the two nonlocal coefficients.

Theorem 7.1 (Continuous dependence). *Let v and w be two solutions of (1.2)–(1.4) on $[0, T]$ corresponding to initial data (v_0, v_1) and (w_0, w_1) , respectively. Assume that*

$$v, w \in L^\infty(0, T; H^4(0, L) \cap V), \quad v_t, w_t \in L^\infty(0, T; H),$$

and that, for some $R > 0$,

$$\|v\|_{L^\infty(0,T;H^4)} + \|w\|_{L^\infty(0,T;H^4)} + \|v_t\|_{L^\infty(0,T;H)} + \|w_t\|_{L^\infty(0,T;H)} \leq R.$$

Then there exists a constant $C_{T,R} > 0$ such that, for all $0 \leq t \leq T$,

$$\|v_t(t) - w_t(t)\|_{L^2}^2 + \|v_{ss}(t) - w_{ss}(t)\|_{L^2}^2 \leq C_{T,R} (\|v_1 - w_1\|_{L^2}^2 + \|v_{0,ss} - w_{0,ss}\|_{L^2}^2). \quad (7.1)$$

Proof. The proof is divided into several steps.

Step 1: Equation for the difference. We set $z = v - w$. Subtracting the equations satisfied by v and w , we obtain

$$mz_{tt} + M(v)z_{ssss} + (M(v) - M(w))w_{ssss} + a(s)z_{ss} + bz_{st} + G_0z_s + \delta z_t = 0. \quad (7.2)$$

The term $(M(v) - M(w))w_{ssss}$ is the only term which does not appear in the linear model. It comes from the fact that the bending coefficient depends on the solution itself. If the stiffness coefficient were constant, then the difference equation would contain only EJz_{ssss} . Here, because M depends on v , we must estimate the difference $M(v) - M(w)$.

Step 2: Difference energy identity. Testing (7.2) by z_t , we obtain

$$\begin{aligned} \frac{m}{2} \frac{d}{dt} \|z_t\|_{L^2}^2 + M(v)(z_{ssss}, z_t) + \delta \|z_t\|_{L^2}^2 \\ = - (M(v) - M(w))(w_{ssss}, z_t) - (a(s)z_{ss}, z_t) - G_0(z_s, z_t). \end{aligned} \quad (7.3)$$

The gyroscopic term vanishes because

$$(z_{st}, z_t) = \frac{1}{2} \int_0^L \partial_s(z_t^2) ds = 0,$$

using the clamped boundary condition, which implies $z_t(0, t) = z_t(L, t) = 0$. Again, we consider the difference energy

$$\mathcal{G}(t) := \frac{m}{2} \|z_t(t)\|_{L^2}^2 + \frac{1}{2} M(v(t)) \|z_{ss}(t)\|_{L^2}^2.$$

Since $M(v(t)) \geq EJ > 0$, this energy controls the natural norm:

$$\mathcal{G}(t) \geq \frac{m}{2} \|z_t(t)\|_{L^2}^2 + \frac{EJ}{2} \|z_{ss}(t)\|_{L^2}^2. \quad (7.4)$$

Step 3: Derivative of the nonlocal coefficient. We compute

$$M(v(t)) = EJ + \alpha \|v_s(t)\|_{L^2}^2.$$

Hence

$$\frac{d}{dt} M(v(t)) = 2\alpha (v_s, v_{st}).$$

Using integration by parts and the clamped boundary condition, we obtain

$$(v_s, v_{st}) = -(v_{ss}, v_t).$$

Therefore,

$$\frac{d}{dt}M(v(t)) = -2\alpha(v_{ss}, v_t). \quad (7.5)$$

By the regularity assumption,

$$\|v_{ss}\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^\infty(0,T;L^2)} \leq C_R.$$

Thus

$$\left| \frac{d}{dt}M(v(t)) \right| \leq C_R. \quad (7.6)$$

Step 4: Rewriting the fourth-order term. We now rewrite the term $M(v)(z_{ssss}, z_t)$. Since

$$(z_{ssss}, z_t) = (z_{ss}, z_{tss}),$$

we have

$$M(v)(z_{ssss}, z_t) = M(v)(z_{ss}, z_{tss}).$$

Moreover,

$$\frac{d}{dt} \left(\frac{1}{2} M(v) \|z_{ss}\|_{L^2}^2 \right) = M(v)(z_{ss}, z_{tss}) + \frac{1}{2} M'(v) \|z_{ss}\|_{L^2}^2.$$

Therefore,

$$M(v)(z_{ssss}, z_t) = \frac{d}{dt} \left(\frac{1}{2} M(v) \|z_{ss}\|_{L^2}^2 \right) - \frac{1}{2} M'(v) \|z_{ss}\|_{L^2}^2.$$

Substituting this identity into (7.3), we obtain

$$\begin{aligned} & \frac{d}{dt} \mathcal{G}(t) + \delta \|z_t\|_{L^2}^2 \\ &= - (M(v) - M(w))(w_{ssss}, z_t) - (a(s)z_{ss}, z_t) - G_0(z_s, z_t) + \frac{1}{2} M'(v) \|z_{ss}\|_{L^2}^2. \end{aligned} \quad (7.7)$$

Note that the additional term $\frac{1}{2} M'(v) \|z_{ss}\|_{L^2}^2$ appears because the coefficient $M(v(t))$ is time-dependent. This term is harmless in the regular class because $M'(v)$ is bounded.

Step 5: Estimate of the coefficient difference. We estimate the nonlocal difference:

$$M(v) - M(w) = \alpha (\|v_s\|_{L^2}^2 - \|w_s\|_{L^2}^2).$$

Thus

$$M(v) - M(w) = \alpha(v_s + w_s, z_s).$$

Consequently,

$$|M(v) - M(w)| \leq \alpha (\|v_s\|_{L^2} + \|w_s\|_{L^2}) \|z_s\|_{L^2}.$$

Using the regularity bound and the Poincaré inequality,

$$\|z_s\|_{L^2} \leq C_P \|z_{ss}\|_{L^2},$$

we obtain

$$|M(v) - M(w)| \leq C_R \|z_{ss}\|_{L^2}. \quad (7.8)$$

This estimate shows how the nonlocality is controlled. The difference of the global coefficients is reduced to the energy norm of z .

Step 6: Estimate of the nonlocal difference term. Using (7.8) and the assumption

$$\|w\|_{L^\infty(0,T;H^4)} \leq R,$$

we obtain

$$\begin{aligned} |(M(v) - M(w))(w_{ssss}, z_t)| &\leq |M(v) - M(w)| \|w_{ssss}\|_{L^2} \|z_t\|_{L^2} \\ &\leq C_R \|z_{ss}\|_{L^2} \|z_t\|_{L^2}. \end{aligned}$$

By Young's inequality,

$$|(M(v) - M(w))(w_{ssss}, z_t)| \leq C_R (\|z_{ss}\|_{L^2}^2 + \|z_t\|_{L^2}^2). \quad (7.9)$$

Using the coercivity of $\mathcal{G}(t)$, this becomes

$$|(M(v) - M(w))(w_{ssss}, z_t)| \leq C_R \mathcal{G}(t). \quad (7.10)$$

Step 7: Estimate of the lower-order terms. The axial-loading term satisfies

$$|(a(s)z_{ss}, z_t)| \leq \|a\|_{L^\infty} \|z_{ss}\|_{L^2} \|z_t\|_{L^2} \leq C (\|z_{ss}\|_{L^2}^2 + \|z_t\|_{L^2}^2).$$

Similarly, using the Poincaré inequality,

$$|G_0(z_s, z_t)| \leq |G_0| C_P \|z_{ss}\|_{L^2} \|z_t\|_{L^2} \leq C (\|z_{ss}\|_{L^2}^2 + \|z_t\|_{L^2}^2).$$

Thus

$$|(a(s)z_{ss}, z_t)| + |G_0(z_s, z_t)| \leq C\mathcal{G}(t). \quad (7.11)$$

Step 8: Estimate of the time-dependent coefficient term. From (7.6),

$$\left| \frac{1}{2} M'(v) \|z_{ss}\|_{L^2}^2 \right| \leq C_R \|z_{ss}\|_{L^2}^2 \leq C_R \mathcal{G}(t).$$

Step 9: Grönwall inequality. Combining (7.7), (7.10), (7.11), and the estimate of $M'(v)$, we obtain

$$\frac{d}{dt} \mathcal{G}(t) + \delta \|z_t\|_{L^2}^2 \leq C_R \mathcal{G}(t).$$

In particular,

$$\frac{d}{dt} \mathcal{G}(t) \leq C_R \mathcal{G}(t). \quad (7.12)$$

By Grönwall's lemma,

$$\mathcal{G}(t) \leq \mathcal{G}(0) e^{C_R t}, \quad 0 \leq t \leq T.$$

Since $M(v(t)) \geq EJ > 0$, we have

$$\|z_t(t)\|_{L^2}^2 + \|z_{ss}(t)\|_{L^2}^2 \leq C\mathcal{G}(t).$$

Moreover,

$$\mathcal{G}(0) \leq C_R (\|v_1 - w_1\|_{L^2}^2 + \|v_{0,ss} - w_{0,ss}\|_{L^2}^2).$$

Therefore,

$$\|v_t(t) - w_t(t)\|_{L^2}^2 + \|v_{ss}(t) - w_{ss}(t)\|_{L^2}^2 \leq C_{T,R} (\|v_1 - w_1\|_{L^2}^2 + \|v_{0,ss} - w_{0,ss}\|_{L^2}^2).$$

This is (7.1). □

Remark 7.2. The essential point in the proof is the estimate of the nonlocal coefficient difference:

$$|M(v) - M(w)| \leq C_R \|v_{ss} - w_{ss}\|_{L^2}.$$

This estimate converts the nonlocal term into a quantity controlled by the difference energy. The additional regularity assumption $w \in L^\infty(0, T; H^4(0, L))$ is used to control the factor w_{ssss} in $L^2(0, L)$.

Acknowledgements. The author extends his appreciation to the Deanship of Scientific Research at Northern Border University, Arar, KSA for funding this research work through the project number NBU-FFMRA-2026-2714-01.

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