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IMPULSIVE ψ-CAPUTO FRACTIONAL EQUATIONS INVOLVING OPERATOR PAIRS IN BANACH SPACES

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Abstract. This article studies two classes of ψ-Caputo fractional equations with impulsive conditions and operator pairs in Banach spaces: one is the impulsive Sobolev-type integro-differential equations with operator pairs on bounded intervals, the other is the impulsive differential equations with operator pairs and finite delays on unbounded intervals. By combining the theory of propagating theory and the ψ-Laplace transform as well as fractional calculus, we firstly derive a proper definition of mild solutions for these impulsive equations. Depending on the definition, we establish existence and uniqueness theorems and a continuous dependence theorem for the impulsive integro-differential equations on bounded intervals. Next, we investigate the piecewise pseudo S-asymptotically (ω, c)-periodic solutions for the impulsive delay equations on unbounded intervals. Finally, some examples are given to illustrate the main results. Here we do not assume the operator B has bounded or compact inverse B^{-1}. Actually, our results are even new in the case when $B = I$ and $\psi(t) = t$.

1. Introduction

In recent years, fractional order equations have attracted the attention of many researchers mainly due to their in-depth and extensive development in both theory and application. For the theoretical and practical applications of fractional order equations readers may refer to [1, 12, 15, 17, 20, 25, 31, 33, 35, 37, 39] and the references therein.

Fractional order integro-differential equations of Sobolev type in Banach space are more accurate than integer order ones in dealing with certain problems and can be found in the fields of thermodynamics, bioengineering, fluid dynamics, automatic control systems, etc. For the study and applications of Sobolev type integro-differential equations, readers may refer to [2, 6, 21, 24, 26, 27, 28] and the references therein. Impulsive differential equations can be used for the mathematical simulation of processes with impulses in their development. Such processes can be observed in many fields of science and technology, such as population dynamics, control theory, neural networks, industrial robots, etc. For impulsive differential equations readers may refer to [7, 8, 9, 10, 11, 14, 26, 38] and the references therein. In [3], the authors introduced the ψ-Caputo fractional derivative by including a general increasing function, which extends the classic Caputo fractional derivative. In [28], the authors considered a class of ψ-Caputo fractional order integro-differential equations of Sobolev type with nonlocal conditions in Banach spaces. In [4], the authors examined the (ω, c)-periodic functions through analyzing solutions to Mathieu's equation $x'' + ax = 2q\cos(2t)x$, which represents a special case of Hill's equation containing just one harmonic mode. These solutions satisfy the functional relation $x(\cdot + \omega) = cx(\cdot)$ where $c \in \mathbb{C}$. Notably, when the parameter c takes values of 1 or -1, the (ω, c)-periodic functions reduce to conventional ω-periodic and ω-antiperiodic functions, respectively. In [13], the authors introduced the notion of the pseudo S-asymptotically (ω, c)-periodic function, established the completeness, convolution and superposition theorems for it in Banach spaces and applied the obtained results to the study of some evolution equations in abstract spaces.

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Motivated by the above works, in this paper, firstly, we will consider the following impulsive ψ-Caputo fractional integro-differential equation of Sobolev type in a Banach space X:

$$\begin{aligned}({}^cD^{q;\psi}(Bu))(t) &= Au(t) + Bf\Big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big), \quad t \in J',\\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k = 1, 2, \dots, m,\\ u(0) &= u_0,\end{aligned} \tag{1.1}$$

where ${}^cD^{q;\psi}, q \in (0,1)$, is the ψ-Caputo fractional derivative of order q with the lower limit 0. m is a positive integer, $T > 0$ is a constant, $J = [0,T]$. $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $J' = J \setminus \{0\}$. $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. The pair (A, B) generates a propagating family $\{W(t)\}_{t\geq 0}$. $f : J \times X \times X \to D(B)$, $\rho : \Xi \to \mathbb{R}$, $h : \Xi \times X \to X$ ($\Xi = \{(t,s) \in [0,T] \times [0,T] : t \geq s\}$) and $I_k : X \to D(B) (1 \leq k \leq m)$ are given functions to be specified later. $u_0 \in D(B)$.

Secondly, we consider the piecewise pseudo S-asymptotically (ω, c)-periodic solutions for the following impulsive differential equations on $\mathbb{R}_+$ in X with the ψ-Caputo fractional derivative and finite delay:

$$\begin{aligned}({}^cD^{q;\psi}(Bu))(t) &= Au(t) + Bf\big(t, u(t), u_t\big), \quad t \in \mathbb{R}_+ = [0,\infty),\\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k \in \mathbb{N}_+,\\ u(t) &= \varsigma(t), \quad t \in [-\sigma, 0],\end{aligned} \tag{1.2}$$

where ${}^cD^{q;\psi}, q \in (0,1)$, is the ψ-Caputo fractional derivative of order q with the lower limit 0, $\omega > 0$, $c \neq 0$, $\sigma > 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_n < \dots$, $\lim_{n\to\infty} t_n = \infty$, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$ where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. The pair (A, B) generates a propagating family $\{W(t)\}_{t\geq 0}$. $u_t \in PC([-\sigma, 0], X)$ is the history function defined by $u_t(s) = u(t+s)$ for $s \in [-\sigma, 0]$. The functions $f : \mathbb{R}_+ \times X \times PC([-\sigma, 0], X) \to D(B)$ and the impulsive function $I_k : X \to D(B) (k \in \mathbb{N}_+)$ are continuous functions to be specified later. $\varsigma(t) \in PC([-\sigma, 0], X)$.

This research builds upon the propagation family $\{W(t)\}_{t\geq 0}$ - an operator family generated by the pair (A, B) (Definition 2.12) - originally developed by Jin Liang and Ti-Jun Xiao ([29]). Compared to the requirements in the existing literature ([24, 26]) that the kernel function in the integral term be bounded, we weaken this condition and give results more general when the kernel function is an unbounded singular kernel. We also extend the notion of continuous pseudo S-asymptotically (ω, c)-periodic functions to that of piecewise continuous ones and investigate the existence and uniqueness of piecewise pseudo S-asymptotically (ω, c)-periodic solutions for impulsive fractional delay differential equations. The rest of this paper is organized as follows. In section 2, we introduce some notations and recall some basic known results. In section 3, we derive definitions of mild solutions for the impulsive problems (1.1) and (1.2). In section 4, we discuss the existence of mild solutions for the system (1.1) in the general case. In section 5, we give the existence and uniqueness theorem of mild solutions for the system (1.1) in the Lipschitz case. In section 6, we continue to prove the continuous dependence of solutions to the initial values when the solution is unique in the Lipschitz case. In section 7, we consider the existence and uniqueness of piecewise pseudo S-asymptotically (ω, c)-periodic solutions for the impulsive delay equations (1.2) on unbounded intervals. In section 8, we present some examples to illustrate the abstract results.

2. Preliminaries

We denote by X a Banach space with norm $\|\cdot\|$. We denote by $C(J, X)$ the space of all X-valued continuous functions on J with the natural norm $\|x\|_{C(J,X)} = \sup_{t\in J}\|x(t)\|$. Set $J_1 = [0, t_1], J_k = (t_{k-1}, t_k], k = 2, 3, \dots, m, J_{m+1} = (t_m, T]$. Let $PC(J, X) = \{u : J \to X|\ u(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \dots, m\}$. Obviously, $PC(J, X)$ is a Banach space with norm $\|u\|_{PC} = \sup_{t\in J}\|u(t)\|$. Let $\zeta(t)$ be the inverse function of $\psi(t)$. $0 < t_1 < t_2 < \cdots < t_n < \dots$, $\lim_{n\to\infty} t_n = \infty$. Let $PC(\mathbb{R}_+, X) = \{u : \mathbb{R}_+ \to X|\ u(t)$

is continuous at $t \neq t_n$, and left continuous at $t = t_n$, and $u(t_n^+)$ exists, $n \in \mathbb{N}_+\}$. The set $PBC(\mathbb{R}_+, X)$ denotes the Banach space formed by all piecewise bounded continuous functions $f : \mathbb{R}_+ \to X$ with sup-norm $\|f\|_\infty = \sup_{t\in\mathbb{R}_+} \|f\|$. Clearly, $PBC(\mathbb{R}_+, X) \subset PC(\mathbb{R}_+, X)$. Using the principal branch of the complex Logarithm, $c^{\frac{t}{\omega}} := \exp\left(\frac{t}{\omega}\mathrm{Log}(c)\right) = c^\wedge(t)$ and we will use notation $|c|^\wedge(t) = |c^\wedge(t)| = |c|^{\frac{t}{\omega}}$([4]). For given $c \in \mathbb{C}, \omega \in \mathbb{R}\backslash\{0\}$, we define the following set

$$PBC_{\omega,c}(\mathbb{R}_+, X) = \big\{f \in PC(\mathbb{R}_+, X) : \sup_{t\in\mathbb{R}_+} \|c^\wedge(-t)f(t)\| < \infty\big\}.$$

Lemma 2.1. *Let $f \in PC(\mathbb{R}_+, X)$. Then $f \in PBC_{\omega,c}(\mathbb{R}_+, X)$ if and only if $f(t) = c^\wedge(t)u(t)$, $u(t) \in PBC(\mathbb{R}_+, X)$.*

Theorem 2.2. *The set $PBC_{\omega,c}(\mathbb{R}_+, X)$ is a Banach space with the norm*

$$\|f\|_{pb\omega c} = \sup_{t\in\mathbb{R}_+} \|c^\wedge(-t)f(t)\|.$$

Definition 2.3 ([20])**.** The fractional integral of order q involving a general function ψ with the lower limit zero for a function f is defined as

$$(I^{q;\psi} f)(t) = \frac{1}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} f(s)\psi'(s)\, ds, \quad t > 0,\ 0 < q < 1,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.4 ([3])**.** The ψ-Caputo derivative of order q with the lower limit zero for a function f can be written as

$$({}^cD^{q;\psi} f)(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(\psi(t) - \psi(s))^q}\, ds, \quad t > 0,\ 0 < q < 1.$$

Remark 2.5. (1) Obviously, the ψ-Caputo fractional derivative of a constant is equal to zero. (2) When $\psi(t) = t$, the ψ-Caputo fractional derivative becomes the classic Caputo fractional derivative [20, 22, 37].

Definition 2.6 ([23])**.** Let X be a Banach space and $\mathcal{M}_X$ the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \mathcal{M}_X \to [0, \infty]$ defined by

$$\mu(\mathcal{I}) = \inf\{\epsilon > 0 : \mathcal{I} \subseteq \cup_{i=1}^n \mathcal{I}_i \quad \mathrm{diam}(\mathcal{I}_i) \leq \epsilon\},$$

where $\mathrm{diam}(\mathcal{I}_i) = \sup\{\|u - v\| : u, v \in \mathcal{I}_i\}$.

The Kuratowski measure of noncompactness satisfies the following properties:

Lemma 2.7 ([5])**.** Let $\Lambda_1, \Lambda_2 \subset X$ be bounded sets. Then we have:

(1) $\mu(\Lambda_1) = 0$ if and only if Λ_1 is relatively compact in X;
(2) $\mu(\Lambda_1) = \mu(\overline{\Lambda_1}) = \mu(conv\Lambda_1)$, where $conv\Lambda_1$ is the convex hull of Λ_1;
(3) $\mu(\Lambda_1) \leq \mu(\Lambda_2)$ when $\Lambda_1 \subseteq \Lambda_2$;
(4) $\mu(\Lambda_1 + \Lambda_2) \leq \mu(\Lambda_1) + \mu(\Lambda_2)$;
(5) $\mu(\Lambda_1 \cup \Lambda_2) = \max\{\mu(\Lambda_1), \mu(\Lambda_2)\}$;
(6) If $dimX = +\infty$, $\mu(B(0, R)) = 2R$, where $B(0, R)$ represents the closed ball centered at 0 with radius R in X.

The following Darbo's fixed point theorem will be used later.

Theorem 2.8 ([16])**.** *Let D be a bounded, convex and closed subset of a Banach space X, and let $A : D \to D$ be a continuous mapping. μ is a measure of noncompactness in X. Assume that there exists a constant $k \in [0, 1)$ such that*

$$\mu(AS) \leq k\mu(S) \tag{2.1}$$

for any nonempty set $S \subseteq D$. Then A has at least one fixed point in D.

Mappings satisfying the Darbo-condition (2.1) have been called k-set contractions. We will use the following ψ-Laplace transform.

Definition 2.9 ([28]). For $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+, X)$ and $\lambda \in \mathbb{C}$, the ψ-Laplace transform of f is defined by

$$\mathcal{L}_\psi[f](\lambda) = \widetilde{f}(\lambda) := \int_0^\infty e^{-\lambda\psi(t)} f(t)\psi'(t)dt,$$

as long as the integral on the right-hand side exists as a Bochner integral.

We introduce the following definition of piecewise pseudo S-asymptotically (ω, c)-periodic functions.

Definition 2.10. A function $f \in PBC_{\omega,c}(\mathbb{R}_+, X)$ is called piecewise (continuous) pseudo S-asymptotically (ω, c)-periodic if it satisfies

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r \|c^\wedge(-t)[f(t+\omega) - cf(t)]\|\, dt = 0.$$

$\mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, X)$ denotes the space of all piecewise pseudo S-asymptotically (ω, c)-periodic functions.

Lemma 2.11. *Let $f_1, f_2, f \in \mathrm{PPSAP}_{\omega,\mathrm{c}}(\mathbb{R}_+, \mathrm{X})$. Then the following results hold:*

(1) *$f_1 + f_2 \in \mathrm{PPSAP}_{\omega,\mathrm{c}}(\mathbb{R}_+, \mathrm{X})$, and $kf \in \mathrm{PPSAP}_{\omega,\mathrm{c}}(\mathbb{R}_+, \mathrm{X})$ for any $k \in \mathbb{C}$.*
(2) *The function $f_a(t) := f(t+a) \in \mathrm{PPSAP}_{\omega,\mathrm{c}}(\mathbb{R}_+, \mathrm{X})$ for each $a \in \mathbb{R}$.*

We set

$$\mathrm{PPSAP}_{\omega,c,0}(\mathbb{R}_+, X) = \{f \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, X) : f(0) = 0\}.$$

Clearly, $\mathrm{PPSAP}_{\omega,c,0}(\mathbb{R}_+, X)$ is a closed subspace of $\mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, X)$.

We recall the following abstract degenerate Cauchy problem as follows [29]:

$$\begin{aligned} \frac{d}{dt}Bu(t) &= Au(t), \quad t \in J, \\ Bu(0) &= Bu_0, \end{aligned} \tag{2.2}$$

Definition 2.12 ([29]). A strongly continuous operator family $\{W(t)\}_{t\geq 0}$ of $D(B)$ to a Banach space X, satisfying that $\{W(t)\}_{t\geq 0}$ is exponentially bounded, which means that for any $x \in D(B)$ there exist $\delta > 0, \mathscr{M} > 0$ such that

$$\|W(t)x\| \leq \mathscr{M} e^{\delta t}\|x\|, t \geq 0,$$

is called an exponentially bounded propagation family for (2.2) if for $\lambda > \delta$,

$$(\lambda B - A)^{-1}Bx = \int_0^\infty e^{-\lambda t} W(t)x\, dt, x \in D(B). \tag{2.3}$$

In this case, we also say that (2.2) has an exponentially bounded propagation family $\{W(t)\}_{t\geq 0}$. Moreover, if (2.3) holds, we also say that the pair (A, B) generates an exponentially bounded propagation family $\{W(t)\}_{t\geq 0}$.

In this article, we assume that $\{W(t)\}_{t\geq 0}$ is a norm continuous family for $t > 0$ and $\|W(t)\| \leq \mathscr{M}$.

3. Definitions of mild solutions to (1.1) and (1.2)

Lemma 3.1. *A function $u \in C(J, X)$ that is a mild solution to*

$$\begin{aligned} ({}^cD^{q;\psi}(Bu))(t) &= Au(t) + Bh(t), \quad t \geq 0, \\ u(0) &= u_0, \end{aligned} \tag{3.1}$$

satisfies the integral equation

$$u(t) = \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t) - \psi(s))^{q-1}\mathcal{R}(\psi(t) - \psi(s))h(s)\psi'(s)\, ds, \tag{3.2}$$

where

$$\mathcal{Q}(t) = \int_0^\infty \xi_q(\sigma)W(\sigma t^q)\, d\sigma, \quad \mathcal{R}(t) = q\int_0^\infty \sigma\xi_q(\sigma)W(\sigma t^q)\, d\sigma, \quad \xi_q(\sigma) = \frac{1}{q}\varpi_q(\sigma^{-1/q}), \sigma \in (0, \infty),$$

$$\varpi_q(\tau) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\tau^{-qn-1}\frac{\Gamma(qn+1)}{n!}\sin(n\pi q), \tau \in (0,\infty).$$

Proof. Let

$$H(\lambda) = \int_0^{\infty} e^{-\lambda\psi(t)}h(t)\psi'(t)\,dt.$$

Using Definitions 2.3 and 2.4, we can transform the first equation of (3.1) into the equivalent integral form

$$(Bu)(t) = Bu_0 + \frac{1}{\Gamma(q)}\int_0^t (\psi(t)-\psi(s))^{q-1}[Au(s)+Bh(s)]\psi'(s)\,ds. \tag{3.3}$$

An application of the ψ-Laplace transform to both sides of (3.3) yields

$$B\tilde{u}(\lambda) = \frac{Bu_0}{\lambda} + \frac{1}{\lambda^q}A\tilde{u}(\lambda) + \frac{1}{\lambda^q}BH(\lambda),$$

which implies

$$\tilde{u}(\lambda) = \lambda^{q-1}(\lambda^q B - A)^{-1}Bu_0 + (\lambda^q B - A)^{-1}BH(\lambda).$$

Then we have

$$\tilde{u}(\lambda) = \lambda^{q-1}\int_0^{\infty} e^{-\lambda^q s}W(s)u_0\,ds + \int_0^{\infty} e^{-\lambda^q s}W(s)H(\lambda)\,ds. \tag{3.4}$$

We recall that the Laplace transform of the one-sided stable probability density $\varpi_q(\tau)$ is

$$\int_0^{\infty} e^{-\lambda\tau}\varpi_q(\tau)\,d\tau = e^{-\lambda^q}, q \in (0,1). \tag{3.5}$$

Thus, by using (3.5) we obtain

$$\begin{aligned}
\lambda^{q-1}\int_0^{\infty} e^{-\lambda^q s}W(s)u_0\,ds &= q\int_0^{\infty}(\lambda t)^{q-1}e^{-(\lambda t)^q}W(t^q)u_0\,dt \quad (\text{set } s=t^q)\\
&= -\frac{1}{\lambda}\int_0^{\infty}\Big(\frac{d}{dt}e^{-(\lambda t)^q}\Big)W(t^q)u_0\,dt\\
&= \int_0^{\infty}\int_0^{\infty} e^{-\lambda t\tau}\tau\varpi_q(\tau)W(t^q)u_0\,d\tau\,dt\\
&= \int_0^{\infty}\int_0^{\infty} e^{-\lambda\sigma}\varpi_q(\tau)W\Big(\frac{\sigma^q}{\tau^q}\Big)u_0\,d\sigma\,d\tau \quad (\text{set } \sigma = t\tau)\\
&= \int_0^{\infty} e^{-\lambda s}\Big[\int_0^{\infty}\varpi_q(\tau)W\Big(\frac{s^q}{\tau^q}\Big)u_0\,d\tau\Big]\,ds\\
&= \int_0^{\infty} e^{-\lambda\psi(t)}\Big[\int_0^{\infty}\varpi_q(\tau)W\Big(\frac{\psi^q(t)}{\tau^q}\Big)u_0\,d\tau\Big]\psi'(t)\,dt,
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\int_0^{\infty} e^{-\lambda^q s}W(s)H(\lambda)\,ds\\
&= \int_0^{\infty} e^{-(\lambda\tau)^q}q\tau^{q-1}W(\tau^q)\Big(\int_0^{\infty} e^{-\lambda\psi(t)}h(t)\psi'(t)\,dt\Big)\,d\tau \quad (\text{set } s=\tau^q)\\
&= \int_0^{\infty}\int_0^{\infty} e^{-\lambda\tau\sigma}q\tau^{q-1}\varpi_q(\sigma)W(\tau^q)\Big(\int_0^{\infty} e^{-\lambda t}h(\zeta(t))\,dt\Big)\,d\sigma\,d\tau\\
&= q\int_0^{\infty}\int_0^{\infty} e^{-\lambda\vartheta}\frac{\vartheta^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{\vartheta^q}{\sigma^q}\Big)\Big(\int_0^{\infty} e^{-\lambda t}h(\zeta(t))\,dt\Big)d\vartheta\,d\sigma \quad (\text{set } \vartheta=\sigma\tau)\\
&= q\int_0^{\infty}\Big(\int_0^{\infty}\int_t^{\infty} e^{-\lambda\tau}\frac{(\tau-t)^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(\tau-t)^q}{\sigma^q}\Big)h(\zeta(t))\,d\tau\,dt\Big)\,d\sigma \ (\text{set} \quad \tau=\vartheta+t)\\
&= q\int_0^{\infty}\Big(\int_0^{\infty}\int_0^{\tau} e^{-\lambda\tau}\frac{(\tau-t)^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(\tau-t)^q}{\sigma^q}\Big)h(\zeta(t))\,dt\,d\tau\Big)\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda t}\Big[q\int_0^t\int_0^\infty \frac{(t-s)^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(t-s)^q}{\sigma^q}\Big)h(\zeta(s))\,d\sigma\,ds\Big]\,dt\\
&= \int_0^\infty e^{-\lambda\psi(t)}\Big[q\int_0^{\psi(t)}\int_0^\infty \frac{(\psi(t)-s)^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(\psi(t)-s)^q}{\sigma^q}\Big)h(\zeta(s))\,d\sigma\,ds\Big]\psi'(t)\,dt\\
&= \int_0^\infty e^{-\lambda\psi(t)}\Big[q\int_0^t\int_0^\infty \frac{(\psi(t)-\psi(s))^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(\psi(t)-\psi(s))^q}{\sigma^q}\Big)h(s)\psi'(s)\,d\sigma\,ds\Big]\psi'(t)\,dt.
\end{aligned}
\tag{3.7}
$$

Hence, from (3.4), (3.6) and (3.7) we can obtain

$$
\begin{aligned}
\tilde{u}(\lambda) = \int_0^\infty e^{-\lambda\psi(t)}\Big[\int_0^\infty \varpi_q(\tau)W\Big(\frac{\psi^q(t)}{\tau^q}\Big)u_0\,d\tau\\
+ q\int_0^t\int_0^\infty \frac{(\psi(t)-\psi(s))^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(\psi(t)-\psi(s))^q}{\sigma^q}\Big)h(s)\psi'(s)\,d\sigma\,ds\Big]\psi'(t)\,dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
u(t) &= \int_0^\infty \varpi_q(\tau)W\Big(\frac{\psi^q(t)}{\tau^q}\Big)u_0\,d\tau + q\int_0^t\int_0^\infty \frac{(\psi(t)-\psi(s))^{q-1}}{\sigma^q}\varpi_q(\sigma)W\Big(\frac{(\psi(t)-\psi(s))^q}{\sigma^q}\Big)\\
h(s)\psi'(s)\,d\sigma\,ds &= \int_0^\infty \xi_q(\sigma)W(\sigma\psi^q(t))u_0\,d\sigma\\
&\quad + q\int_0^t\int_0^\infty (\psi(t)-\psi(s))^{q-1}\sigma\xi_q(\sigma)W(\sigma(\psi(t)-\psi(s))^q)h(s)\psi'(s)\,d\sigma\,ds,
\end{aligned}
$$

where ξ_q is a probability density function defined on $(0,\infty)$ such that

$$
\xi_q(\sigma) = \frac{1}{q}\varpi_q(\sigma^{-1/q}) \geq 0.
$$

The above analysis yields the integral equation (3.2) satisfied by mild solutions to the equations (3.1), which completes the proof. □

Next, we derive a definition of mild solutions for (1.1) and (1.2). We first consider the fractional equation with impulses,

$$
\begin{aligned}
({}^cD^{q;\psi}(Bu))(t) &= Au(t) + B\mathscr{G}(t), t \in J',\\
\Delta u|_{t=t_k} &= \mathscr{Y}_k, \quad k = 1,2,\dots,m,\\
u(0) &= u_0,
\end{aligned}
\tag{3.8}
$$

where $\mathscr{G}(t) \in PC(J, D(B)), \mathscr{Y}_k \in D(B), u_0 \in D(B)$.

We can decompose $u(\cdot)$ which is a solution of the equation (3.8) to $\mathscr{V}(\cdot) + \mathscr{W}(\cdot)$ where $\mathscr{V}$ is the mild solution of

$$
\begin{aligned}
({}^cD^{q;\psi}(B\mathscr{V}))(t) &= A\mathscr{V}(t) + B\mathscr{G}(t), \quad t \in J',\\
\mathscr{V}(0) &= u_0,
\end{aligned}
\tag{3.9}
$$

on J, and $\mathscr{W}$ is the mild solution of

$$
\begin{aligned}
({}^cD^{q;\psi}(B\mathscr{W}))(t) &= A\mathscr{W}(t), \quad t \in J',\\
\Delta\mathscr{W}|_{t=t_k} &= \mathscr{Y}_k, \quad k = 1,2,\dots,m,\\
\mathscr{W}(0) &= 0,
\end{aligned}
\tag{3.10}
$$

By Lemma 3.1, (3.9) has the mild solution

$$
\mathscr{V}(t) = \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))\mathscr{G}(s)\psi'(s)\,ds, t \in J.
$$

Now we rewrite (3.10) in the integral form

$$
B\mathscr{W}(t) = \sum_{i=1}^m B\mathscr{X}_i(t)\mathscr{Y}_i + \frac{1}{\Gamma(q)}\int_0^t (\psi(t)-\psi(s))^{q-1}A\mathscr{W}(s)\psi'(s)\,ds, t \in J,
\tag{3.11}
$$

where

$$\mathscr{X}_i(t) = \begin{cases} 0, & t \in [0, t_i), \\ 1, & t \in [t_i, T]. \end{cases}$$

We apply the ψ-Laplace transform for (3.11) to obtain

$$B\widetilde{\mathscr{W}}(\lambda) = \sum_{i=1}^{m} \frac{e^{-\lambda\psi(t_i)}}{\lambda} B\mathscr{Y}_i + \frac{1}{\lambda^q} A\widetilde{\mathscr{W}}(\lambda),$$

which implies

$$\widetilde{\mathscr{W}}(\lambda) = \sum_{i=1}^{m} e^{-\lambda\psi(t_i)} \lambda^{q-1} (\lambda^q B - A)^{-1} B\mathscr{Y}_i.$$

By the process in proof of Lemma 3.1, we know that the ψ-Laplace transform of $\mathcal{Q}(\psi(t))\mathscr{Y}_i$ is $\lambda^{q-1}(\lambda^q B - A)^{-1}B\mathscr{Y}_i$, then we can derive the mild solution of (3.10) as

$$\mathscr{W}(t) = \sum_{i=1}^{m} \mathscr{X}_i(t)\mathcal{Q}(\psi(t) - \psi(t_i))\mathscr{Y}_i = \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k))\mathscr{Y}_k.$$

Summarizing, the mild solution of (3.8) is given by

$$u(t) = \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t) - \psi(s))^{q-1} \mathcal{R}(\psi(t) - \psi(s))\mathscr{G}(s)\psi'(s)\, ds + \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k))\mathscr{Y}_k.$$

Through the above analysis, we introduced the following definitions of the mild solution for the systems (1.1) and (1.2).

Definition 3.2. By a mild solution of the system (1.1), we mean a function $u \in PC(J, X)$ satisfying the integral equation

$$\begin{aligned} u(t) = \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t) - \psi(s))^{q-1} \mathcal{R}(\psi(t) - \psi(s)) f\Big(s, u(s), \int_0^s \rho(s,\tau) h(s,\tau,u(\tau))\, d\tau\Big) \\ \times \psi'(s)\, ds + \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k)) I_k(u(t_k)), \end{aligned}$$

where

$$\mathcal{Q}(t) = \int_0^\infty \xi_q(\sigma) W(\sigma t^q)\, d\sigma, \quad \mathcal{R}(t) = q\int_0^\infty \sigma\xi_q(\sigma) W(\sigma t^q)\, d\sigma.$$

Here ξ_q is a probability density function defined on $(0,\infty)$ [34] such that

$$\xi_q(\sigma) = \frac{1}{q}\varpi_q(\sigma^{-1/q}), \quad \sigma \in (0,\infty),$$

$$\varpi_q(\tau) = \frac{1}{\pi}\sum_{n=1}^{\infty} (-1)^{n-1} \tau^{-qn-1} \frac{\Gamma(qn+1)}{n!} \sin(n\pi q), \quad \tau \in (0,\infty).$$

Remark 3.3. If we assume that the propagating family $\{W(t)\}_{t\geq 0}$ is uniformly bounded, that is, there exists $\mathscr{M} > 0$ such that $\|W(t)\| \leq \mathscr{M}, t \geq 0$, then by [28], we have

$$\|\mathcal{Q}(t)\| \leq \mathscr{M}, \|\mathcal{R}(t)\| \leq \frac{\mathscr{M}}{\Gamma(q)}, t \geq 0. \tag{3.12}$$

Definition 3.4. By a mild solution of the system (1.2), we mean a function $u \in PC([-\sigma,\infty), X)$ satisfying the integral equation

$$u(t) = \begin{cases} \varsigma(t), & t \in [-\sigma, 0], \\ \mathcal{Q}(\psi(t))\varsigma(0) + \int_0^t (\psi(t) - \psi(s))^{q-1} \mathcal{R}(\psi(t) - \psi(s)) f\left(s, u(s), u_s\right) \psi'(s)\, ds \\ + \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k)) I_k(u(t_k)), & t > 0. \end{cases} \tag{3.13}$$

Remark 3.5. If we assume that the propagating family $\{W(t)\}_{t\geq 0}$ is uniformly exponentially stable, that is, there exist $\mathscr{M} > 0, \mu > 0$ such that $\|W(t)\| \leq \mathscr{M}e^{-\mu t}, t \geq 0$, then we have $\lim_{n\to\infty} \|\mathcal{Q}(t)\| = 0$.

4. Existence of solutions in the general case for(1.1)

In this section, we will consider the existence of mild solutions of the problem (1.1). By Definition 3.2, we define an operator P by

$$\begin{aligned}&(Pu)(t)\\&= \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))f\Big(s, u(s), \int_0^s \rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\\ &\psi'(s)\,ds\\&\quad + \sum_{0<t_k<t} \mathcal{Q}(\psi(t)-\psi(t_k))I_k(u(t_k)).\end{aligned} \tag{4.1}$$

Obviously, P maps $PC(J,X)$ into $PC(J,X)$.

We se $\mathscr{B}_r = \{u \in PC(J,X) |\ \|u\|_{PC} \le r\}$, and state the following hypotheses.

(H1) The operator family $\{W(t)\}_{t\ge 0}$ is norm-continuous for $t>0$.

(H2) $f(\cdot,\mathcal{V},\mathcal{W}) : J \to D(B) \subset X$ is measurable for all $(\mathcal{V},\mathcal{W}) \in X\times X$ and $f(t,\cdot,\cdot) : X\times X \to D(B)$ is continuous for a.e. $t\in J$, and there exist a function $\boldsymbol{\mu}(\cdot) \in L^{p_1}(J,\mathbb{R}_+)$ $\Big(p_1 > \frac{1}{q} > 1\Big)$ and a continuous function $\boldsymbol{\nu}(\cdot)$ such that

$$\|f(t,\mathcal{V},\mathcal{W})\| \le \boldsymbol{\mu}(t)\|\mathcal{V}\| + \boldsymbol{\nu}(t)\|\mathcal{W}\|$$

for almost all $t\in J$. $\boldsymbol{\nu}^* = \sup_{t\in J}\boldsymbol{\nu}(t)$.

(H3) There exists a function $\phi(\cdot) \in L^{p_2}(J,\mathbb{R}_+)$ such that for any bounded sets $E_i \subset X$ $(i=1,2)$,

$$\mu(f(t,E_1,E_2)) \le \phi(t)(\mu(E_1)+\mu(E_2)), \quad \text{a.e. } t\in J.$$

(H4) The function $h(t,s,\cdot) : X\to X$ is continuous for a.e. $(t,s)\in\Xi$, and for each $\mathcal{V}\in X$, the function $h(\cdot,\cdot,\mathcal{V}) : \Xi \to X$ is measurable. Moreover, there exists a function $\mathbf{h} : \Xi \to \mathbb{R}_+$ with $\sup_{t\in J}\int_0^t \mathbf{h}(t,s)\,ds := \mathbf{h}^* < \infty$ such that

$$\|h(t,s,\mathcal{V})\| \le \mathbf{h}(t,s)\|\mathcal{V}\|, \mathcal{V}\in X.$$

(H5) For any bounded set $E\subset X$ and $(t,s)\in\Xi$, there exists a function $\boldsymbol{\zeta} : \Xi \to \mathbb{R}_+$ such that

$$\mu(h(t,s,E)) \le \boldsymbol{\zeta}(t,s)\mu(E),$$

with $\sup_{t\in J}\int_0^t \boldsymbol{\zeta}(t,s)\,ds := \boldsymbol{\zeta}^* < \infty$.

(H6) $I_k :\ X \to D(B)$ $(k=1,\dots,m)$ is continuous, and there exist positive constants $\mathbb{L}_k$ and $\mathbb{N}_k$ $(1\le k\le m)$ such that

$$\|I_k(x)\| \le \mathbb{L}_k\|x\| + \mathbb{N}_k, \quad x\in X,\ k=1,\dots,m.$$

(H7) There exist positive constants $\mathbb{M}_k$ $(1\le k\le m)$ such that for any bounded set $E\subset X$,

$$\mu(I_k(E)) \le \mathbb{M}_k\mu(E).$$

(H8) $\rho(t,\cdot)$ is measurable on $[0,t]$ for any $t\in J$, and $\wp(t) := \operatorname{ess\,sup}\{|\rho(t,s)|, 0\le s\le t\}$ is bounded on J. The map $t\to\wp_t$: $J\to L^\infty(J,\mathbb{R})$ is continuous, where $\wp_t(s)=\rho(t,s)$. Denote $\sup_{t\in J}\wp(t)$ by $\wp$.

(H9)
$$\frac{\mathscr{M}}{\Gamma(q)}\left(\ell_{p_1,q}T^{q-\frac{1}{p_1}}\|\boldsymbol{\mu}\|_{L^{p_1}} + \frac{\psi^q(T)\wp\mathbf{h}^*\boldsymbol{\nu}^*}{q}\right) + \mathscr{M}\sum_{k=1}^m \mathbb{L}_k < 1. \tag{4.2}$$

(H10)
$$\frac{2\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_2,q}T^{q-\frac{1}{p_2}}\|\phi\|_{L^{p_2}} + \frac{2\wp\boldsymbol{\zeta}^*\psi^q(T)}{q}\Big) + \mathscr{M}\sum_{k=1}^m \mathbb{M}_k < \frac{1}{2}. \tag{4.3}$$

Here $\ell_{p_i,q} := \big(\frac{p_i-1}{p_i q-1}\big)^{\frac{p_i-1}{p_i}}$ $(i=1,2)$.

Lemma 4.1. *If* (H2), (H4), (H6), (H9) *hold, then there exists some* $r>0$ *such that* $P(\mathscr{B}_r)\subset\mathscr{B}_r$.

Proof. Suppose this is not true. Then for each $r > 0$, there exist $u_r(\cdot) \in \mathscr{B}_r$ and some $t \in J$ such that

$$\|(Pu_r)(t)\| > r.$$

It follows from (H2), (H4) and (H6) that

$$\begin{aligned}
r &< \|\mathcal{Q}(\psi(t))u_0\| + \int_0^t (\psi(t)-\psi(s))^{q-1}\Big\|\mathcal{R}(\psi(t)-\psi(s))f\Big(s, u_r(s), \int_0^s \rho(s,\tau)h(s,\tau,u_r(\tau))\,d\tau\Big)\Big\| \\
&\quad \times \psi'(s)\,ds + \sum_{0<t_k<t} \|\mathcal{Q}(\psi(t)-\psi(t_k))I_k(u_r(t_k))\| \\
&\le \mathscr{M}\|u_0\| + \frac{\mathscr{M}}{\Gamma(q)}\int_0^t (\psi(t)-\psi(s))^{q-1}\Big[\boldsymbol{\mu}(s)r + \boldsymbol{\nu}(s)\wp r\int_0^s \mathbf{h}(s,\tau)\,d\tau\Big]\psi'(s)\,ds \\
&\quad + \mathscr{M}\sum_{k=1}^m \Big(r\mathbb{L}_k + \mathbb{N}_k\Big) \\
&\le \mathscr{M}\|u_0\| + \frac{\mathscr{M}r}{\Gamma(q)}\Big(\int_0^t (\psi(t)-\psi(s))^{q-1}\boldsymbol{\mu}(s)\psi'(s)\,ds + \frac{\psi^q(T)\wp\mathbf{h}^*\boldsymbol{\nu}^*}{q}\Big) + \mathscr{M}r\sum_{k=1}^m \mathbb{L}_k + \mathscr{M}\sum_{k=1}^m \mathbb{N}_k.
\end{aligned}$$

Moreover, by Hölder inequality, we have

$$\int_0^t (\psi(t)-\psi(s))^{q-1}\boldsymbol{\mu}(s)\psi'(s)\,ds \le (\psi(t))^{\frac{qp_1-1}{p_1}}\ell_{p_1,q}\|\boldsymbol{\mu}\|_{L^{p_1}} \le \ell_{p_1,q}(\psi(T))^{q-\frac{1}{p_1}}\|\boldsymbol{\mu}\|_{L^{p_1}}.$$

Thus

$$r < \mathscr{M}\|u_0\| + \frac{\mathscr{M}r}{\Gamma(q)}\Big(\ell_{p_1,q}T^{q-\frac{1}{p_1}}\|\boldsymbol{\mu}\|_{L^{p_1}} + \frac{\psi^q(T)\wp\mathbf{h}^*\boldsymbol{\nu}^*}{q}\Big) + \mathscr{M}r\sum_{k=1}^m \mathbb{L}_k + \mathscr{M}\sum_{k=1}^m \mathbb{N}_k. \tag{4.4}$$

Dividing both sides of (4.4) by r, and taking $r \to \infty$, we obtain

$$\frac{\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_1,q}T^{q-\frac{1}{p_1}}\|\boldsymbol{\mu}\|_{L^{p_1}} + \frac{\psi^q(T)\wp\mathbf{h}^*\boldsymbol{\nu}^*}{q}\Big) + \mathscr{M}\sum_{k=1}^m \mathbb{L}_k \ge 1.$$

This contradicts (H9). Hence for some positive number r, $P\mathscr{B}_r \subset \mathscr{B}_r$. □

Lemma 4.2. *If* (H2), (H4), (H6) *hold, then* $P : \mathscr{B}_r \to \mathscr{B}_r$ *is continuous.*

Proof. Let $u_n \in \mathscr{B}_r$, $n = 1, 2, \ldots$, be a sequence such that $u_n \to u \in \mathscr{B}_r$ in $PC(J, X)$ as $n \to \infty$. By (H2) and (H4), we see that for almost every $t \in J$,

$$f\Big(t, u_n(t), \int_0^t \rho(t,s)h(t,s,u_n(s))\,ds\Big) \to f\Big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big),$$

as $n \to \infty$. Noting that $u_n \to u$ in $PC(J, X)$, we infer that there exists $\varepsilon > 0$ such that $\|u_n - u\|_{PC} \le \varepsilon$ for k sufficiently large. Therefore,

$$\begin{aligned}
&\Big\|f\Big(t, u_n(t), \int_0^t \rho(t,s)h(t,s,u_n(s))\,ds\Big) - f\Big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big)\Big\| \\
&\le \boldsymbol{\mu}(t)(\|u_n(t)\| + \|u(t)\|) + \boldsymbol{\nu}(t)\Big(\int_0^t \|\rho(t,s)h(t,s,u_n(s))\|\,ds + \int_0^t \|\rho(t,s)h(t,s,u(s))\|\,ds\Big) \\
&\le \boldsymbol{\mu}(t)(\|u_n(t)-u(t)\| + 2\|u(t)\|) + \boldsymbol{\nu}(t)\wp\int_0^t \mathbf{h}(t,s)(\|u_n(s)-u(s)\| + 2\|u(s)\|)\,ds \\
&\le (\boldsymbol{\mu}(t) + \boldsymbol{\nu}^*\wp\mathbf{h}^*)\Big(\varepsilon + 2\sup_{t\in J}\|u(t)\|\Big).
\end{aligned}$$

It follows from the continuity of I_k $(k = 1, \ldots, m)$ by (H6) and the Lebesgue's dominated convergence theorem that

$$\|(Pu_n)(t) - (Pu)(t)\|$$

$$
\begin{aligned}
&\le \int_0^t (\psi(t)-\psi(s))^{q-1}\Big\|\mathcal{R}(\psi(t)-\psi(s))\Big[f\Big(t,u_n(t),\int_0^t \rho(t,s)h(t,s,u_n(s))\,ds\Big)\\
&\quad - f\Big(t,u(t),\int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big)\Big]\Big\|\psi'(s)\,ds + \sum_{0<t_k<t}\|\mathcal{Q}(\psi(t)-\psi(t_k))\\
&\quad \times [I_k(u_n(t_k))-I_k(u(t_k))]\|\\
&\le \frac{\mathscr{M}}{\Gamma(q)}\int_0^t (\psi(t)-\psi(s))^{q-1}\Big\|f\Big(t,u_n(t),\int_0^t \rho(t,s)h(t,s,u_n(s))\,ds\Big)\\
&\quad - f\Big(t,u(t),\int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big)\Big\|\psi'(s)\,ds + \mathscr{M}\sum_{0<t_k<t}\|I_k(u_n(t_k))-I_k(u(t_k))\|\\
&\to 0, \quad \text{as} n\to\infty.
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}\|Pu_n - Pu\|_{PC}=0$, which shows that P is continuous. □

The proof of the following lemma is obvious, so we omit it.

Lemma 4.3. *If* (H1) *holds, then* $\{\mathcal{Q}(t)\}_{t\ge0}$ *and* $\{\mathcal{R}(t)\}_{t\ge0}$ *are norm continuous for* $t>0$.

Lemma 4.4. *If* (H2), (H4), (H6) *hold, then* $\{Pu : u\in\mathscr{B}_r\}$ *is a family of equi-continuous functions.*

Proof. Without loss of generality, we suppose $t_j\le\iota_2<\iota_1<t_{j+1}$, $u\in\mathscr{B}_r$, then we have

$$
\begin{aligned}
&\Big\|\int_0^{\iota_1}(\psi(\iota_1)-\psi(s))^{q-1}\mathcal{R}(\psi(\iota_1)-\psi(s))f\Big(s,u(s),\int_0^s\rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\psi'(s)\,ds\\
&-\int_0^{\iota_2}(\psi(\iota_2)-\psi(s))^{q-1}\mathcal{R}(\psi(\iota_2)-\psi(s))f\Big(s,u(s),\int_0^s\rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\psi'(s)\,ds\Big\|\\
&\le\int_0^{\iota_2}\Big\|\Big[(\psi(\iota_1)-\psi(s))^{q-1}\mathcal{R}(\psi(\iota_1)-\psi(s))-(\psi(\iota_2)-\psi(s))^{q-1}\mathcal{R}(\psi(\iota_2)-\psi(s))\Big]\\
&\quad\times f\Big(s,u(s),\int_0^s\rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\psi'(s)\Big\|\,ds\\
&\quad+\int_{\iota_2}^{\iota_1}(\psi(\iota_1)-\psi(s))^{q-1}\|\mathcal{R}(\psi(\iota_1)-\psi(s))\|\Big\|f\Big(s,u(s),\int_0^s\rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\Big\|\psi'(s)\,ds\\
&=:\mathbb{I}_1+\mathbb{I}_2.
\end{aligned}
$$

For $\mathbb{I}_1$, we have

$$
\begin{aligned}
\mathbb{I}_1&\le\int_0^{\iota_2}|(\psi(\iota_1)-\psi(s))^{q-1}-(\psi(\iota_2)-\psi(s))^{q-1}|\|\mathcal{R}(\psi(\iota_1)-\psi(s))\|\\
&\quad\times\Big\|f\Big(s,u(s),\int_0^s\rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\Big\|\psi'(s)\,ds\\
&\quad+\int_0^{\iota_2}(\psi(\iota_2)-\psi(s))^{q-1}\cdot\|\mathcal{R}(\psi(\iota_1)-\psi(s))-\mathcal{R}(\psi(\iota_2)-\psi(s))\|\\
&\quad\times\Big\|f\Big(s,u(s),\int_0^s\rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\Big\|\psi'(s)\,ds\\
&\le\frac{\mathscr{M}}{\Gamma(q)}\int_0^{\iota_2}|(\psi(\iota_1)-\psi(s))^{q-1}-(\psi(\iota_2)-\psi(s))^{q-1}|\\
&\quad\times\Big(\boldsymbol{\mu}(s)\|u(s)\|+\boldsymbol{\nu}^*\wp\mathbf{h}^*\sup_{\tau\in[0,s]}\|u(\tau)\|\Big)\psi'(s)\,ds\\
&\quad+\int_0^{\iota_2}(\psi(\iota_2)-\psi(s))^{q-1}\|\mathcal{R}(\psi(\iota_1)-\psi(s))-\mathcal{R}(\psi(\iota_2)-\psi(s))\|\\
&\quad\times\Big(\boldsymbol{\mu}(s)\|u(s)\|+\boldsymbol{\nu}^*\wp\mathbf{h}^*\sup_{\tau\in[0,s]}\|u(\tau)\|\Big)\psi'(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\mathscr{M} r}{\Gamma(q)} \int_0^{\iota_2} |(\psi(\iota_1)-\psi(s))^{q-1} - (\psi(\iota_2)-\psi(s))^{q-1}| \boldsymbol{\mu}(s)\psi'(s)\,ds \\
&\quad + r \int_0^{\iota_2} \boldsymbol{\mu}(s)(\psi(\iota_2)-\psi(s))^{q-1} \|\mathcal{R}(\psi(\iota_1)-\psi(s)) - \mathcal{R}(\psi(\iota_2)-\psi(s))\| \psi'(s)\,ds \\
&\quad + \frac{\mathscr{M}\wp \mathbf{h}^* \boldsymbol{\nu}^* r}{\Gamma(q)} \int_0^{\iota_2} |(\psi(\iota_1)-\psi(s))^{q-1} - (\psi(\iota_2)-\psi(s))^{q-1}| \psi'(s)\,ds \\
&\quad + \wp \mathbf{h}^* \boldsymbol{\nu}^* r \int_0^{\iota_2} (\psi(\iota_2)-\psi(s))^{q-1} \|\mathcal{R}(\psi(\iota_1)-\psi(s)) - \mathcal{R}(\psi(\iota_2)-\psi(s))\| \psi'(s)\,ds \\
&=: \mathbb{I}_3 + \mathbb{I}_4 + \mathbb{I}_5 + \mathbb{I}_6.
\end{aligned}
$$

Clearly, $\mathbb{I}_3$ tends to 0 as $\iota_2 \to \iota_1$. $\mathbb{I}_4$ tends to 0 as $\iota_2 \to \iota_1$ by Lemma 4.1. Similarly, $\mathbb{I}_5$ and $\mathbb{I}_6$ tend to 0 as $\iota_2 \to \iota_1$, respectively.

For $\mathbb{I}_2$, we have

$$
\mathbb{I}_2 \le \frac{\mathscr{M} r}{\Gamma(q)} \Big[\int_{\iota_2}^{\iota_1} (\psi(\iota_1)-\psi(s))^{q-1} \boldsymbol{\mu}(s)\psi'(s)\,ds + \wp \mathbf{h}^* \boldsymbol{\nu}^* \int_{\iota_2}^{\iota_1} (\psi(\iota_1)-\psi(s))^{q-1} \psi'(s)\,ds \Big] \to 0,
$$

as $\iota_2 \to \iota_1$. In conclusion,

$$
\|(Pu)(\iota_2) - (Pu)(\iota_1)\| \to 0, \quad \text{as } \iota_2 \to \iota_1,
$$

which means that the operator $P : \mathscr{B}_r \to \mathscr{B}_r$ is equicontinuous.

Let $\mathscr{H} = \overline{\operatorname{conv} P(\mathscr{B}_r)}$. Then it is easy to see that P maps $\mathscr{H}$ into itself and $\mathscr{H} \subset \mathscr{B}_r$ is equicontinuous. □

Lemma 4.5 ([30]). *Let X be a Banach space, and let $\Lambda \subset X$ be bounded. Then there exists a countable set $\Lambda_1 \subset \Lambda$, such that $\mu(\Lambda) \le 2\mu(\Lambda_1)$.*

Lemma 4.6 ([18]). *Let X be a Banach space, and let $\Upsilon \subset PC(J,X)$ be equicontinuous and bounded. Then $\mu(\Upsilon(t)) \in PC(J,\mathbb{R}_+)$, and $\mu(\Upsilon) = \max_{t\in J} \mu(\Upsilon(t))$.*

Lemma 4.7 ([19]). *Let $\{u_n(t)\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X. Suppose that there exists $\mathcal{U}(t) \in L^1(J,\mathbb{R}_+)$ such that $\|u_n(t)\| \le \mathcal{U}(t)$ for a.e. $t \in J$, and each $n \in \mathbb{N}_+$. Then the function $\mathscr{U}(t) := \mu(\{u_n\}_{n=1}^{\infty}) \in L^1(J,\mathbb{R}_+)$ and satisfies*

$$
\mu\Big(\Big\{ \int_0^t u_n(s)\,ds : \ n \ge 1 \Big\}\Big) \le 2 \int_0^t \mathscr{U}(s)\,ds.
$$

Lemma 4.8. *If* (H3), (H5), (H7), (H10) *hold, then $P : \mathscr{H} \to \mathscr{H}$ is a k-set contraction.*

Proof. For any $E \subset \mathscr{H}$, by Lemma 4.5, there exists a countable set $E_0 = \{u_n\} \subset E$, such that

$$
\mu(P(E)) \le 2\mu(P(E_0)).
$$

By the equicontinuity of $\mathscr{H}$, we know that $E_0 \subset E$ is also equicontinuous.

For $t \in J$, by Lemmas 2.7 and 4.7, we have

$$
\begin{aligned}
&\mu(P(E_0)(t)) \\
&= \mu\Big(\Big\{ \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t)-\psi(s))^{q-1} \mathcal{R}(\psi(t)-\psi(s)) \\
&\quad \times f\Big(s, u_n(s), \int_0^s \rho(s,\tau) h(s,\tau,u_n(\tau))\,d\tau\Big)\psi'(s)\,ds + \sum_{0<t_k<t} \mathcal{Q}(\psi(t)-\psi(t_k)) I_k(u_n(t_k)) \Big\}\Big) \\
&\le \frac{2\mathscr{M}}{\Gamma(q)} \int_0^t (\psi(t)-\psi(s))^{q-1} \mu\Big(\Big\{ f\Big(s, u_n(s), \int_0^s \rho(s,\tau) h(s,\tau,u_n(\tau))\,d\tau\Big)\Big\}\Big)\psi'(s)\,ds \\
&\quad + \mathscr{M} \sum_{0<t_k<t} \mu\Big(\Big\{ I_k(u_n(t_k)) \Big\}\Big) \\
&\le \frac{2\mathscr{M}}{\Gamma(q)} \int_0^t (\psi(t)-\psi(s))^{q-1} \Big[\phi(s) \Big(\mu(E_0) + 2\wp \int_0^s \boldsymbol{\zeta}(s,\tau)\,d\tau \mu(E_0) \Big) \Big] \psi'(s)\,ds + \mathscr{M} \sum_{k=1}^{m} \mathbb{M}_k \mu(E_0)
\end{aligned}
$$

$$\le \Big\{\frac{2\mathscr{M}}{\Gamma(q)}\int_0^t (\psi(t)-\psi(s))^{q-1}\Big(\phi(s)+2\wp\boldsymbol{\zeta}^*\Big)\psi'(s)\,ds+\mathscr{M}\sum_{k=1}^m \mathbb{M}_k\Big\}\mu(E_0)$$
$$\le 2\Big[\frac{2\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_2,q}T^{q-\frac{1}{p_2}}\|\phi\|_{L^{p_2}}+\frac{2\wp\boldsymbol{\zeta}^*\psi^q(T)}{q}\Big)+\mathscr{M}\sum_{k=1}^m \mathbb{M}_k\Big]\mu(E).$$

Since $P(E_0)\subset\mathscr{H}$ is bounded and equicontinuous, we know from Lemma 4.6 that

$$\mu(P(E_0))=\max_{t\in J}\mu(P(E_0)(t)).$$

Therefore, by (H10) we have

$$\mu(P(E))\le 2\Big[\frac{2\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_2,q}T^{q-\frac{1}{p_2}}\|\phi\|_{L^{p_2}}+\frac{2\wp\boldsymbol{\zeta}^*\psi^q(T)}{q}\Big)+\mathscr{M}\sum_{k=1}^m \mathbb{M}_k\Big]\mu(E):=k\mu(E), k\in[0,1).$$

Thus $P:\mathscr{H}\to\mathscr{H}$ a k-set contraction. □

Theorem 4.9. *Assume that* (H1)–(H10) *are satisfied. Then system* (1.1) *has at least one mild solution on* J.

Proof. It follows from Lemma 4.1, lemma 4.2, 4.4, 4.8 and Theorem 2.8 that P has at least one fixed point $u\in\mathscr{H}$, which is just a mild solution of the system (1.1). □

When $B=I$, then $D(B)=X$. We assume that A generates a norm continuous semigroup $\{W(t)\}_{t\ge0}$ of uniformly bounded linear operators on X, then we have the following theorem.

Theorem 4.10. *Assume that* (H1)–(H10) *are satisfied. Then the system*

$$\begin{gathered}({}^cD^{q;\psi}u)(t)=Au(t)+f\Big(t,u(t),\int_0^t\rho(t,s)h(t,s,u(s))\,ds\Big),t\in J',\\ \Delta u|_{t=t_k}=I_k(u(t_k)),k=1,2,\dots,m,\\ u(0)=u_0,\end{gathered}\tag{4.5}$$

has at least one mild solution on J.

5. Existence and uniqueness of a solution in the Lipschitz case for (1.1)

In this section we consider the Lipschitz case for the problem (1.1) when the integral term contains an unbounded singular kernel. First we propose a more refined sequence than that in [26], and give its asymptotic estimation.

Lemma 5.1. *For the sequence*

$$\mathcal{T}_n=\sum_{\substack{0\le i,j_1,j_2\le n\\ i+j_1+j_2=n}}\frac{\binom{n}{i,j_1,j_2}b^i b_1^{j_1}b_2^{j_2}}{\Gamma(j_1\alpha_1+j_2\alpha_2+1)},n\in\mathbb{N}_+,$$

where $0<b<1, b_i,\alpha_i\ (i=1,2)>0$ *are constants,* $\binom{n}{i,j_1,j_2}=\frac{n!}{i!j_1!j_2!}$, *it has the asymptotic estimation:*

$$\mathcal{T}_n=o\Big(\frac{1}{n^{\gamma+1}}\Big),n\to\infty,\tag{5.1}$$

where γ *can be an arbitrary positive constant.*

Proof. By the fact

$$\lim_{t\to+\infty}\Big(b^{t-1}t\Big(\frac{t}{t-1}\Big)^{t-1}\Big)^{1/t}=b<1,$$

we see that there exists $\mathbb{T}>2$ such that

$$\tau:=\Big(b^{\mathbb{T}-1}\mathbb{T}\Big(\frac{\mathbb{T}}{\mathbb{T}-1}\Big)^{\mathbb{T}-1}\Big)^{1/\mathbb{T}}<1.\tag{5.2}$$

For each positive integer $n > 2\mathbb{T}$, set $n = j\mathbb{T} + p$ $(0 \le p < \mathbb{T})$, where $\mathbb{T}$ is given above. Clearly, $j = [n/\mathbb{T}] < [n/2]$, then for enough large positive integer $n > 2\mathbb{T}$, it follows from the Stirling formula

$$n! = \Big(\frac{n}{e}\Big)^n \sqrt{2\pi n}\Big(1 + O\Big(\frac{1}{n}\Big)\Big)$$

and the equality (5.2) that

$$\begin{aligned}
\mathcal{T}_n^1 &\equiv b^n + \sum_{\substack{0\le j_1,j_2\le 1\\ j_1+j_2=1}} \frac{\binom{n}{n-1,j_1,j_2} b^{n-1} b_1^{j_1} b_2^{j_2}}{\Gamma(j_1\alpha_1 + j_2\alpha_2 + 1)} + \sum_{\substack{0\le j_1,j_2\le 2\\ j_1+j_2=2}} \frac{\binom{n}{n-2,j_1,j_2} b^{n-2} b_1^{j_1} b_2^{j_2}}{\Gamma(j_1\alpha_1 + j_2\alpha_2 + 1)} + \cdots \\
&\quad + \sum_{\substack{0\le j_1,j_2\le j\\ j_1+j_2=j}} \frac{\binom{n}{n-j,j_1,j_2} b^{n-j} b_1^{j_1} b_2^{j_2}}{\Gamma(j_1\alpha_1 + j_2\alpha_2 + 1)} \\
&\le C_n^0 b^n + \frac{C_n^1 b^{n-1}\delta^1}{\Gamma(\alpha+1)} + \frac{C_n^2 b^{n-2}\delta^2}{\Gamma(2\alpha+1)} + \cdots + \frac{C_n^j b^{n-j}\delta^j}{\Gamma(j\alpha+1)} \\
&\le C_n^j b^{n-j}\Big[1 + \frac{\delta^1}{\Gamma(\alpha+1)} + \frac{\delta^2}{\Gamma(2\alpha+1)} + \cdots + \frac{\delta^j}{\Gamma(j\alpha+1)}\Big] \\
&= b^{n-j}\frac{n!}{j!(n-j)!} O(1) \\
&= \frac{O(1) b^{n-j} n^n \sqrt{2\pi n}}{j^j \sqrt{2\pi j}\sqrt{2\pi(n-j)}(n-j)^{n-j}} \\
&= O\Big(\frac{\mathbb{T}^j}{\sqrt{j}}\Big)\Big(\frac{b\mathbb{T}}{\mathbb{T}-1}\Big)^{(\mathbb{T}-1)j} \\
&= O\Big(\frac{\big(b^{\mathbb{T}-1}\mathbb{T}\big(\frac{\mathbb{T}}{\mathbb{T}-1}\big)^{\mathbb{T}-1}\big)^j}{\sqrt{j}}\Big) \\
&= O\Big(\frac{\tau^{j\mathbb{T}}}{\sqrt{j}}\Big) \\
&= O\Big(\frac{\tau^n}{\sqrt{n}}\Big),
\end{aligned}$$

where $\delta = b_1 + b_2$, $\alpha = \max\{\alpha_1, \alpha_2\}$ and $C_n^m = \frac{n!}{m!(n-m)!}$ is the binomial coefficient.

On the other side, without loss of generality, we can suppose that $b > 1$. By the Stirling formula

$$\Gamma(z+1) = \sqrt{2\pi z}\Big(\frac{z}{e}\Big)^z \Big(1 + O\Big(\frac{1}{z}\Big)\Big), z \to +\infty$$

and

$$C_n^{[\frac{n}{2}]} = O(2^n/\sqrt{n})$$

(see [32]), we have

$$\begin{aligned}
\mathcal{T}_n^2 &\equiv \sum_{\substack{0\le j_1,j_2\le j+1\\ j_1+j_2=j+1}} \frac{\binom{n}{n-j-1,j_1,j_2} b^{n-j-1} b_1^{j_1} b_2^{j_2}}{\Gamma(j_1\alpha_1 + j_2\alpha_2 + 1)} + \cdots + \sum_{\substack{0\le j_1,j_2\le n\\ j_1+j_2=n}} \frac{\binom{n}{0,j_1,j_2} b^{n-n} b_1^{j_1} b_2^{j_2}}{\Gamma(j_1\alpha_1 + j_2\alpha_2 + 1)} \\
&\le \frac{C_n^{j+1} b^{n-j-1}\delta^{j+1}}{\Gamma((j+1)\alpha+1)} + \cdots + \frac{C_n^n b^{n-n}\delta^n}{\Gamma(n\alpha+1)} \\
&\le \frac{1}{\Gamma((j+1)\alpha+1)} C_n^{[\frac{n}{2}]}(b^{n-j-1}\delta^{j+1} + \cdots + b^{n-n}\delta^n) \\
&\le \frac{1}{\Gamma((j+1)\alpha+1)} C_n^{[\frac{n}{2}]}(b^{n-j-1}\delta^{j+1} + \cdots + b^{n-n}\delta^n)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{O\Big(\frac{2^n}{\sqrt{n}}\Big)e^{(j+1)\alpha}(b^{n-j-1}\delta^{j+1}+\cdots+b^{n-n}\delta^n)}{\sqrt{2\pi(j+1)\alpha}((j+1)\alpha)^{(j+1)\alpha}\Big(1+O\Big(\frac{1}{(j+1)\alpha}\Big)\Big)} \\
&\le \frac{O\Big(\frac{2^n}{\sqrt{n}}\Big)e^{(j+1)\alpha}(1+\delta+\delta^2+\cdots+\delta^{j+1}+\cdots+\delta^n)}{\sqrt{2\pi(j+1)\alpha}((j+1)\alpha)^{(j+1)\alpha}} \\
&\le \frac{O(1)2^n e^{(j+1)\alpha}\delta^{n+1}}{\sqrt{n}\sqrt{(j+1)\alpha}((j+1)\alpha)^{(j+1)\alpha}} \\
&\le \frac{O(1)2^n\Big(\frac{e}{\alpha}\Big)^{(j+1)\alpha}\delta^{n+1}}{j^{(j+1)\alpha+1}} \\
&= o\Big(\frac{1}{j^{\gamma+1}}\Big) \\
&= o\Big(\frac{1}{n^{\gamma+1}}\Big)(n\to\infty),
\end{aligned}
$$

where γ can be an arbitrary positive constant. Therefore,

$$
\begin{aligned}
\mathcal{T}_n &= \mathcal{T}_n^1 + \mathcal{T}_n^2 \\
&= O\Big(\frac{\tau^n}{\sqrt{n}}\Big) + o\Big(\frac{1}{n^{\gamma+1}}\Big) \\
&= o\Big(\frac{1}{n^{\gamma+1}}\Big)(n\to\infty),
\end{aligned}
$$

where γ can be an arbitrary positive constant. We have thus proved this lemma. □

TABLE 1. $b = 0.2$, $b_1 = 1.5$, $b_2 = 2.5$, $\alpha_1 = 0.75$, $\alpha_2 = 0.8$

n	1	2	10	11	12	13
$\mathcal{T}_n$	4.52	13.27	323.01	341.66	344.95	333.88
n	l5	30	32	33	35	40
$\mathcal{T}_n$	279.60	2.97	1.20	0.75	0.28	0.02

TABLE 2. $b = 0.3$, $b_1 = 2$, $b_2 = 3$, $\alpha_1 = 0.5$, $\alpha_2 = 5/6$

n	1	2	10	50	100	130
$\mathcal{T}_n$	5.56	29.78	43877.09	136868944.00	53206.70	8.63
n	l35	136	137	138	l39	140
$\mathcal{T}_n$	1.65	1.18	0.84	0.60	0.42	0.30

TABLE 3. $b = 0.4$, $b_1 = 3$, $b_2 = 4$, $\alpha_1 = 0.8$, $\alpha_2 = 0.9$

n	1	2	10	25	50	75
$\mathcal{T}_n$	7.78	37.44	35890.58	2686178.50	129564.25	8030.80
n	90	100	103	104	105	110
$\mathcal{T}_n$	112.45	4.49	1.63	1.15	0.82	0.14

To illustrate the above result, we calculate some of the numerical values of $\mathcal{T}_n$ in some cases; see Tables 1–4. The above numerical calculations verify Lemma 5.1 about the asymptotic estimation of the sequence $\mathcal{T}_n$ in (5.1), also we can see from the values of $\mathcal{T}_n$ in the tables that it reaches its maximum value at a positive integer as n increases and then gradually decreases towards 0.

TABLE 4. $b = 0.8$, $b_1 = 1$, $b_2 = 1.2$, $\alpha_1 = 0.9$, $\alpha_2 = 1.8$

n	1	2	10	25	50	75
$\mathcal{T}_n$	2.56	4.73	44.75	147.49	109.24	25.57
n	100	110	111	112	113	120
$\mathcal{T}_n$	3.26	1.27	1.15	1.04	0.95	0.47

Theorem 5.2. *Assume the kernel function is*

$$\rho(t,s) = (\psi(t) - \psi(s))^{r-1}\psi'(s) \ (0 < r \le 1)$$

and the following assumptions hold:

(H11) $f : J \times X \times X \to D(B)$ *is continuous, and there exist nonnegative constants* $\ell_i (i = 1, 2)$ *such that*

$$\|f(t,x_1,x_2) - f(t,y_1,y_2)\| \le \ell_1\|x_1 - y_1\| + \ell_2\|x_2 - y_2\|, \quad t \in J,\ x_i, y_i \in X,\ i = 1, 2;$$

(H12) $h : \Xi \times X \to X$ *is continuous, and there exists a nonnegative constant* ℓ_3 *such that*

$$\|h(t,s,x) - h(t,s,y)\| \le \ell_3\|x - y\|, \quad (t,s) \in \Xi,\ x, y \in X;$$

(H13) $I_k : X \to D(B) (1 \le k \le m)$ *is continuous, and there exists a nonnegative constant* $\mathscr{I}_k$ $(1 \le k \le m)$ *such that*

$$\|I_k(x) - I_k(y)\| \le \mathscr{I}_k\|x - y\|, \quad 1 \le k \le m,\ x, y \in X;$$

(H14) $\mathscr{M}\sum_{k=1}^{m} \mathscr{I}_k < 1$.

Then problem (1.1) *has a unique mild solution on* J.

Proof. We consider the operator $P : PC(J,X) \to PC(J,X)$ defined in (4.1). We want to prove (5.3) below by induction: For each $t \in J$ and $u, v \in PC(J,X)$, we have

$$\begin{aligned}
&\|(P^n u)(t) - (P^n v)(t)\| \\
&\le \mathscr{M}^n \sum_{\substack{0 \le i,j_1,j_2 \le n \\ i+j_1+j_2 = n}} \frac{\binom{n}{i,j_1,j_2}\Big(\sum_{k=1}^{m} \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r) + 1)} \|u - v\|_{PC}, n \in \mathbb{N}_+.
\end{aligned} \tag{5.3}$$

For $n = 1$, by (3.12), we have

$$\begin{aligned}
&\|(Pu)(t) - (Pv)(t)\| \\
&\le \sum_{0 < t_k < t} \mathscr{M}\mathscr{I}_k\|u(t_k) - v(t_k)\| + \frac{\mathscr{M}}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} \\
&\quad \times \Big[\ell_1\|u(s) - v(s)\| + \ell_2\ell_3 \int_0^s (\psi(s) - \psi(\tau))^{r-1}\psi'(\tau)\|u(\tau) - v(\tau)\|\, d\tau\Big]\psi'(s)\, ds \\
&\le \mathscr{M}\Big[\sum_{k=1}^{m} \mathscr{I}_k + \frac{\ell_1(\psi(t))^q}{\Gamma(q+1)} + \frac{\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r}}{\Gamma(q+r+1)}\Big]\|u - v\|_{PC},
\end{aligned}$$

so (5.3) holds for $n = 1$.

Suppose (5.3) holds for $n = \mathscr{N}$, that is,

$$\begin{aligned}
&\|(P^{\mathscr{N}} u)(t) - (P^{\mathscr{N}} v)(t)\| \\
&\le \mathscr{M}^{\mathscr{N}} \sum_{\substack{0 \le i,j_1,j_2 \le \mathscr{N} \\ i+j_1+j_2 = \mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2}\Big(\sum_{k=1}^{m} \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r) + 1)} \|u - v\|_{PC},
\end{aligned}$$

then we have

$$\|(P^{\mathscr{N}+1} u)(t) - (P^{\mathscr{N}+1} v)(t)\|$$

$$
\begin{aligned}
&\le \sum_{0<t_k<t} \mathscr{M}\mathscr{I}_k \|(P^{\mathscr{N}}u)(t_k) - (P^{\mathscr{N}}v)(t_k)\| + \frac{\mathscr{M}}{\Gamma(q)} \int_0^t (\psi(t)-\psi(s))^{q-1} \Big[\ell_1 \|(P^{\mathscr{N}}u)(s) \\
&\quad - (P^{\mathscr{N}}v)(s)\| + \ell_2\ell_3 \int_0^s (\psi(s)-\psi(\tau))^{r-1}\psi'(\tau) \|(P^{\mathscr{N}}u)(\tau) - (P^{\mathscr{N}}v)(\tau)\| \, d\tau \Big] \psi'(s)\, ds \\
&\le \mathscr{M} \sum_{k=1}^m \mathscr{I}_k \mathscr{M}^{\mathscr{N}} \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N} \\ i+j_1+j_2=\mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r)+1)} \|u-v\|_{PC} \\
&\quad + \frac{\mathscr{M}\ell_1}{\Gamma(q)} \int_0^t (\psi(t)-\psi(s))^{q-1} \cdot \mathscr{M}^{\mathscr{N}} \\
&\quad \times \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N} \\ i+j_1+j_2=\mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(s))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(s))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r)+1)} \|u-v\|_{PC} \cdot \psi'(s)\, ds \\
&\quad + \frac{\mathscr{M}\ell_2\ell_3}{\Gamma(q)} \int_0^t (\psi(t)-\psi(s))^{q-1}\psi'(s) \Big[\int_0^s (\psi(s)-\psi(\tau))^{r-1}\psi'(\tau) \mathscr{M}^{\mathscr{N}} \\
&\quad \times \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N} \\ i+j_1+j_2=\mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(\tau))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(\tau))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r)+1)} \|u-v\|_{PC} \, d\tau \Big] \, ds,
\end{aligned}
$$

where we use the properties of the Beta function $B(\cdot,\cdot)$

$$
\int_0^t (t-s)^{\alpha-1} s^{m\alpha}\, ds = t^{(m+1)\alpha} B(\alpha, m\alpha+1) \quad (m\in\mathbb{N}, \alpha>0),
$$

and

$$
B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\alpha,\beta>0),
$$

so we have

$$
\begin{aligned}
&\|(P^{\mathscr{N}+1}u)(t) - (P^{\mathscr{N}+1}v)(t)\| \\
&\le \mathscr{M}^{\mathscr{N}+1} \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N} \\ i+j_1+j_2=\mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^{i+1} (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r)+1)} \|u-v\|_{PC} \\
&\quad + \mathscr{M}^{\mathscr{N}+1} \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N} \\ i+j_1+j_2=\mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1+1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma((j_1+1) q + j_2(q+r)+1)} \|u-v\|_{PC} \\
&\quad + \mathscr{M}^{\mathscr{N}+1} \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N} \\ i+j_1+j_2=\mathscr{N}}} \frac{\binom{\mathscr{N}}{i,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2+1}}{\Gamma(j_1 q + (j_2+1)(q+r)+1)} \|u-v\|_{PC} \\
&= \mathscr{M}^{\mathscr{N}+1} \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+1 \\ i+j_1+j_2=\mathscr{N}+1}} \frac{\binom{\mathscr{N}}{i-1,j_1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r)+1)} \|u-v\|_{PC} \\
&\quad + \mathscr{M}^{\mathscr{N}+1} \sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+1 \\ i+j_1+j_2=\mathscr{N}+1}} \frac{\binom{\mathscr{N}}{i,j_1-1,j_2} \Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1} (\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1 q + j_2(q+r)+1)} \|u-v\|_{PC}
\end{aligned}
$$

$$+\mathscr{M}^{\mathscr{N}+1}\sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+1\\ i+j_1+j_2=\mathscr{N}+1}}\frac{\binom{\mathscr{N}}{i,j_1,j_2-1}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}$$

$$=\mathscr{M}^{\mathscr{N}+1}\sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+1\\ i+j_1+j_2=\mathscr{N}+1}}\frac{\binom{\mathscr{N}+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC},$$

where we use the property

$$\binom{\mathscr{N}}{i-1,j_1,j_2}+\binom{\mathscr{N}}{i,j_1-1,j_2}+\binom{\mathscr{N}}{i,j_1,j_2-1}=\binom{\mathscr{N}+1}{i,j_1,j_2},$$

and we take $\binom{\mathscr{N}}{l_1,l_2,l_3}=0$ when $l_i=-1$ for some i. Thus we have proved that

$$\|(P^nu)(t)-(P^nv)(t)\|$$

$$\le\mathscr{M}^n\sum_{\substack{0\le i,j_1,j_2\le n\\ i+j_1+j_2=n}}\frac{\binom{n}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC},n\in\mathbb{N}_+,$$

so we have that

$$\|P^nu-P^nv\|_{PC}$$

$$\le\mathscr{M}^n\sum_{\substack{0\le i,j_1,j_2\le n\\ i+j_1+j_2=n}}\frac{\binom{n}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(T))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC},n\in\mathbb{N}_+.$$

We set

$$b=\mathscr{M}\sum_{k=1}^m\mathscr{I}_k<1,\quad b_1=\mathscr{M}\ell_1(\psi(T))^q,\quad b_2=\mathscr{M}\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r},\quad \alpha_1=q,\quad \alpha_2=q+r,$$

using Lemma 5.1, we know that there exists a positive integer n_0 such that

$$\mathscr{M}^{n_0}\sum_{\substack{0\le i,j_1,j_2\le n_0\\ i+j_1+j_2=n_0}}\frac{\binom{n_0}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(T))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}<1,$$

so P^{n_0} is a contraction mapping on $PC(J,X)$, then by a well-known extension of the Banach contraction mapping theorem, P has a unique fixed point $u(t)$ on $PC(J,X)$, which is the unique mild solution of the system (1.1). □

When $B=I$, then $D(B)=X$. If we assume that A generates a norm continuous semigroup $\{W(t)\}_{t\ge0}$ of uniformly bounded linear operators on X, we then have the following theorem.

Theorem 5.3. *Assume that the kernel function is*

$$\rho(t,s)=(\psi(t)-\psi(s))^{r-1}\psi'(s)\ (0<r\le1)$$

and that (H11)–(H14) *are satisfied. Then system* (4.5) *has a unique mild solution on* J.

6. Continuous dependence of solutions to initial values for (1.1)

In this section, we consider the continuous dependence of a mild solution to problem (1.1) in the Lipschitz case.

Theorem 6.1. *Suppose that*

$$\rho(t,s) = (\psi(t)-\psi(s))^{r-1}\psi'(s) \quad (0<r\le 1)$$

and the conditions (H11)–(H14) *hold. Let* $u(t)$ *be the unique mild solution of* (1.1) *and* $v(t)$ *be the unique mild solution of*

$$\begin{aligned} ({}^cD^{q;\psi}(Bu))(t) &= Au(t) + Bf\Big(t,u(t),\int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big), \quad t\in J',\\ \Delta u|_{t=t_k} &= I_k(u(t_k)), \quad k=1,2,\dots,m,\\ u(0) &= v_0, \end{aligned} \tag{6.1}$$

where $v_0\in X$. *Then there exists a positive constant* M *such that*

$$\|u-v\|_{PC}\le M\|u_0-v_0\|.$$

Proof. $u(t)$ and $v(t)$ satisfy the following equations, respectively,

$$\begin{aligned} u(t) &= \mathcal{Q}(\psi(t))u_0 + \int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))\\ &\quad\times f\Big(s,u(s),\int_0^s \rho(s,\tau)h(s,\tau,u(\tau))\,d\tau\Big)\psi'(s)\,ds + \sum_{0<t_k<t}\mathcal{Q}(\psi(t)-\psi(t_k))I_k(u(t_k)), \end{aligned}$$

and

$$\begin{aligned} v(t) &= \mathcal{Q}(\psi(t))v_0 + \int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))\\ &\quad\times f\Big(s,v(s),\int_0^s \rho(s,\tau)h(s,\tau,v(\tau))\,d\tau\Big)\psi'(s)\,ds + \sum_{0<t_k<t}\mathcal{Q}(\psi(t)-\psi(t_k))I_k(v(t_k)). \end{aligned}$$

We want to prove (6.2) by induction:

$$\begin{aligned} &\|u(t)-v(t)\|\\ &\le \mathscr{M}\Big[1+\sum_{p=1}^{n}\mathscr{M}^p \sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}} \frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\|u_0-v_0\|\\ &\quad+\mathscr{M}^{n+1}\sum_{\substack{0\le i,j_1,j_2\le n+1\\ i+j_1+j_2=n+1}} \frac{\binom{n+1}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i (\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}, \end{aligned} \tag{6.2}$$

for $n\in\mathbb{N}_+$. First, we have

$$\begin{aligned} &\|u(t)-v(t)\|\\ &\le \mathscr{M}\|u_0-v_0\| + \frac{\mathscr{M}}{\Gamma(q)}\int_0^t(\psi(t)-\psi(s))^{q-1}\Big[\ell_1+\ell_2\ell_3\int_0^s(\psi(s)-\psi(\tau))^{r-1}\psi'(\tau)\,d\tau\Big]\\ &\quad\times\psi'(s)\,ds\|u-v\|_{PC} + \mathscr{M}\sum_{k=1}^m\mathscr{I}_k\|u-v\|_{PC}\\ &= \mathscr{M}\|u_0-v_0\| + \mathscr{M}\Big[\sum_{k=1}^m\mathscr{I}_k + \frac{\ell_1(\psi(t))^q}{\Gamma(q+1)} + \frac{\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r}}{\Gamma(q+r+1)}\Big]\|u-v\|_{PC}. \end{aligned}$$

For $n=1$,

$$\begin{aligned} &\|u(t)-v(t)\|\\ &\le \mathscr{M}\|u_0-v_0\| + \frac{\mathscr{M}\ell_1}{\Gamma(q)}\int_0^t(\psi(t)-\psi(s))^{q-1}\Big\{\mathscr{M}\|u_0-v_0\| + \mathscr{M}\Big[\sum_{k=1}^m\mathscr{I}_k + \frac{\ell_1(\psi(s))^q}{\Gamma(q+1)}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\ell_2\ell_3\Gamma(r)(\psi(s))^{q+r}}{\Gamma(q+r+1)}\Big]\|u-v\|_{PC}\Big\}\psi'(s)\,ds+\frac{\mathscr{M}\ell_2\ell_3}{\Gamma(q)}\int_0^t(\psi(t)-\psi(s))^{q-1}\int_0^s(\psi(s)-\psi(\tau))^{r-1}\\
&\times\psi'(\tau)\Big\{\mathscr{M}\|u_0-v_0\|+\mathscr{M}\Big[\sum_{k=1}^m\mathscr{I}_k+\frac{\ell_1(\psi(\tau))^q}{\Gamma(q+1)}+\frac{\ell_2\ell_3\Gamma(r)(\psi(\tau))^{q+r}}{\Gamma(q+r+1)}\Big]\|u-v\|_{PC}\Big\}\,d\tau\psi'(s)\,ds\\
&+\mathscr{M}\sum_{k=1}^m\mathscr{I}_k\Big\{\mathscr{M}\|u_0-v_0\|+\mathscr{M}\Big[\sum_{k=1}^m\mathscr{I}_k+\frac{\ell_1(\psi(t))^q}{\Gamma(q+1)}+\frac{\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r}}{\Gamma(q+r+1)}\Big]\|u-v\|_{PC}\Big\}\\
&=\mathscr{M}\Big[1+\mathscr{M}\sum_{k=1}^m\mathscr{I}_k+\frac{\mathscr{M}\ell_1(\psi(t))^q}{\Gamma(q+1)}+\frac{\mathscr{M}\ell_2\ell_3(\psi(t))^{q+r}}{\Gamma(q+r+1)}\Big]\|u_0-v_0\|+\mathscr{M}^2\Big[\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^2\\
&+\frac{(\ell_1(\psi(t))^q)^2}{\Gamma(2q+1)}+\frac{(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^2}{\Gamma(2(q+r)+1)}+\frac{2(\ell_1(\psi(t))^q)(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})}{\Gamma(q+(q+r)+1)}\\
&+\frac{2\left(\sum_{k=1}^m\mathscr{I}_k\right)(\ell_1(\psi(t))^q)}{\Gamma(q+1)}+\frac{2\big(\sum_{k=1}^m\mathscr{I}_k\big)(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})}{\Gamma(q+r+1)}\Big]\|u-v\|_{PC},
\end{aligned}
$$

so (6.2) holds for $n=1$.

Suppose (6.2) holds for $n=\mathscr{N}$, that is,

$$
\begin{aligned}
&\|u(t)-v(t)\|\\
&\le\mathscr{M}\Big[1+\sum_{p=1}^{\mathscr{N}}\mathscr{M}^p\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\|u_0-v_0\|\\
&\quad+\mathscr{M}^{\mathscr{N}+1}\sum_{\substack{0\le i,j_1,j_2\le\mathscr{N}+1\\ i+j_1+j_2=\mathscr{N}+1}}\frac{\binom{l+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC},
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\|u(t)-v(t)\|\\
&\le\mathscr{M}\|u_0-v_0\|+\frac{\mathscr{M}\ell_1}{\Gamma(q)}\int_0^t(\psi(t)-\psi(s))^{q-1}\Big\{\mathscr{M}\Big[1\\
&\quad+\sum_{p=1}^{\mathscr{N}}\mathscr{M}^p\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(s))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(s))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\|u_0-v_0\|\\
&\quad+\mathscr{M}^{\mathscr{N}+1}\sum_{\substack{0\le i,j_1,j_2\le\mathscr{N}+1\\ i+j_1+j_2=\mathscr{N}+1}}\frac{\binom{\mathscr{N}+1}{i,j_1,j_2}\big(\sum_{k=1}^m\mathscr{I}_k\big)^i(\ell_1(\psi(s))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(s))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}\Big\}\\
&\quad\times\psi'(s)\,ds+\frac{\mathscr{M}\ell_2\ell_3}{\Gamma(q)}\int_0^t(\psi(t)-\psi(s))^{q-1}\int_0^s(\psi(s)-\psi(\tau))^{r-1}\psi'(\tau)\Big\{\mathscr{M}\Big[1\\
&\quad+\sum_{p=1}^{\mathscr{N}}\mathscr{M}^p\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\big(\sum_{k=1}^m\mathscr{I}_k\big)^i(\ell_1(\psi(\tau))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(\tau))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\|u_0-v_0\|\\
&\quad+\mathscr{M}^{\mathscr{N}+1}\sum_{\substack{0\le i,j_1,j_2\le\mathscr{N}+1\\ i+j_1+j_2=\mathscr{N}+1}}\frac{\binom{\mathscr{N}+1}{i,j_1,j_2}\big(\sum_{k=1}^m\mathscr{I}_k\big)^i(\ell_1(\psi(\tau))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(\tau))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}\Big\}\\
&\quad\times\psi'(s)\,d\tau\,ds+\mathscr{M}\sum_{k=1}^m\mathscr{I}_k\Big\{\mathscr{M}\Big[1
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{p=1}^{\mathcal{N}}\mathscr{M}^p\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\|u_0-v_0\|\\
&+\mathscr{M}^{\mathcal{N}+1}\sum_{\substack{0\le i,j_1,j_2\le \mathcal{N}+1\\ i+j_1+j_2=\mathcal{N}+1}}\frac{\binom{\mathcal{N}+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}\Big\}\\
&=\mathscr{M}\|u_0-v_0\|+\frac{\mathscr{M}^2\ell_1(\psi(t))^q}{\Gamma(q+1)}\|u_0-v_0\|\\
&+\mathscr{M}\sum_{p=1}^{\mathcal{N}}\mathscr{M}^{p+1}\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1+1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma((j_1+1)q+j_2(q+r)+1)}\|u_0-v_0\|\\
&+\mathscr{M}^{\mathcal{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathcal{N}+1\\ i+j_1+j_2=\mathcal{N}+1}}\frac{\binom{\mathcal{N}+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1+1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma((j_1+1)q+j_2(q+r)+1)}\|u-v\|_{PC}\\
&+\frac{\mathscr{M}^2\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r}}{\Gamma(q+r+1)}\|u_0-v_0\|+\mathscr{M}\sum_{p=1}^{\mathcal{N}}\mathscr{M}^{p+1}\\
&\times\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2+1}}{\Gamma(j_1q+(j_2+1)(q+r)+1)}\|u_0-v_0\|\\
&+\mathscr{M}^{\mathcal{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathcal{N}+1\\ i+j_1+j_2=\mathcal{N}+1}}\frac{\binom{\mathcal{N}+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2+1}}{\Gamma(j_1q+(j_2+1)(q+r)+1)}\|u-v\|_{PC}\\
&+\mathscr{M}^2\sum_{k=1}^m\mathscr{I}_k\|u_0-v_0\|\\
&+\mathscr{M}\sum_{p=1}^{\mathcal{N}}\mathscr{M}^{p+1}\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}}\frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^{i+1}(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u_0-v_0\|\\
&+\mathscr{M}^{\mathcal{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathcal{N}+1\\ i+j_1+j_2=\mathcal{N}+1}}\frac{\binom{\mathcal{N}+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^{i+1}(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}\\
&=\mathscr{M}\|u_0-v_0\|+\frac{\mathscr{M}^2\ell_1(\psi(t))^q}{\Gamma(q+1)}\|u_0-v_0\|+\mathscr{M}\sum_{p=1}^{\mathcal{N}}\mathscr{M}^{p+1}\\
&\times\sum_{\substack{0\le i,j_1,j_2\le p+1\\ i+j_1+j_2=p+1}}\frac{\binom{p}{i,j_1-1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u_0-v_0\|\\
&+\mathscr{M}^{\mathcal{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathcal{N}+2\\ i+j_1+j_2=\mathcal{N}+2}}\frac{\binom{\mathcal{N}+1}{i,j_1-1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \|u-v\|_{PC} + \frac{\mathscr{M}^2\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r}}{\Gamma(q+r+1)}\|u_0-v_0\| + \mathscr{M}\sum_{p=1}^{\mathscr{N}}\mathscr{M}^{p+1}\\
&\times \sum_{\substack{0\le i,j_1,j_2\le p+1\\ i+j_1+j_2=p+1}} \frac{\binom{p}{i,j_1,j_2-1}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u_0-v_0\|\\
&+ \mathscr{M}^{\mathscr{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+2\\ i+j_1+j_2=\mathscr{N}+2}} \frac{\binom{\mathscr{N}+1}{i,j_1,j_2-1}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\\
&\times \|u-v\|_{PC} + \mathscr{M}^2\sum_{k=1}^m\mathscr{I}_k\|u_0-v_0\| + \mathscr{M}\sum_{p=1}^{\mathscr{N}}\mathscr{M}^{p+1}\\
&\times \sum_{\substack{0\le i,j_1,j_2\le p+1\\ i+j_1+j_2=p+1}} \frac{\binom{p}{i-1,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u_0-v_0\|\\
&+ \mathscr{M}^{\mathscr{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+2\\ i+j_1+j_2=\mathscr{N}+2}} \frac{\binom{\mathscr{N}+1}{i-1,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC}\\
&= \mathscr{M}\|u_0-v_0\| + \frac{\mathscr{M}^2\ell_1(\psi(t))^q}{\Gamma(q+1)}\|u_0-v_0\| + \frac{\mathscr{M}^2\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r}}{\Gamma(q+r+1)}\|u_0-v_0\| + \mathscr{M}^2\sum_{k=1}^m\mathscr{I}_k\\
&\times \|u_0-v_0\| + \mathscr{M}\sum_{p=1}^{\mathscr{N}}\mathscr{M}^{p+1}\sum_{\substack{0\le i,j_1,j_2\le p+1\\ i+j_1+j_2=p+1}} \frac{\binom{p+1}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\\
&\times \|u_0-v_0\| + \mathscr{M}^{\mathscr{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+2\\ i+j_1+j_2=\mathscr{N}+2}} \frac{\binom{\mathscr{N}+2}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\\
&\times \|u-v\|_{PC}\\
&= \mathscr{M}\Big[1+\sum_{p=1}^{\mathscr{N}+1}\mathscr{M}^p\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}} \frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\\
&\times \|u_0-v_0\| + \mathscr{M}^{\mathscr{N}+2}\sum_{\substack{0\le i,j_1,j_2\le \mathscr{N}+2\\ i+j_1+j_2=\mathscr{N}+2}} \frac{\binom{\mathscr{N}+2}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(t))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(t))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\\
&\times \|u-v\|_{PC},
\end{aligned}
$$

so (6.2) holds for $n=\mathscr{N}+1$.

We have thus proved (6.2) by induction. So we have

$$
\begin{aligned}
&\|u-v\|_{PC}\\
&\le \mathscr{M}\Big[1+\sum_{p=1}^{n}\mathscr{M}^p\sum_{\substack{0\le i,j_1,j_2\le p\\ i+j_1+j_2=p}} \frac{\binom{p}{i,j_1,j_2}\Big(\sum_{k=1}^m \mathscr{I}_k\Big)^i(\ell_1(\psi(T))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\Big]\|u_0-v_0\|
\end{aligned}
$$

$$+\mathscr{M}^{n+1}\sum_{\substack{0\le i,j_1,j_2\le n+1\\ i+j_1+j_2=n+1}}\frac{\binom{n+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(T))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}\|u-v\|_{PC},$$

for $n\in\mathbb{N}_+$. By Lemma 5.1, we can set

$$\tilde{M}=\sum_{n=1}^{\infty}\mathscr{M}^n\sum_{\substack{0\le i,j_1,j_2\le n\\ i+j_1+j_2=n}}\frac{\binom{n}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(T))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)},$$

and we have that

$$\lim_{n\to\infty}\mathscr{M}^{n+1}\sum_{\substack{0\le i,j_1,j_2\le n+1\\ i+j_1+j_2=n+1}}\frac{\binom{n+1}{i,j_1,j_2}\Big(\sum_{k=1}^m\mathscr{I}_k\Big)^i(\ell_1(\psi(T))^q)^{j_1}(\ell_2\ell_3\Gamma(r)(\psi(T))^{q+r})^{j_2}}{\Gamma(j_1q+j_2(q+r)+1)}=0.$$

Then we obtain

$$\|u-v\|_{PC}\le\mathscr{M}(1+\widetilde{M})\|u_0-v_0\|.$$

If we set

$$M=\mathscr{M}(1+\widetilde{M}),$$

then the proof of Theorem 6.1 is complete. □

I $B=I$, then $D(B)=X$. We assume that A generates a norm continuous semigroup $\{W(t)\}_{t\ge0}$ of uniformly bounded linear operators on X, then we have the following theorem.

Theorem 6.2. *Assume that the kernel function*

$$\rho(t,s)=(\psi(t)-\psi(s))^{r-1}\psi'(s)\ (0<r\le1)$$

and that $(I_1)-(I_4)$ are satisfied. Let $u(t)$ be the unique mild solution of (4.5) and and $v(t)$ be the unique mild solution of

$$\begin{gathered}({}^cD^{q;\psi}u)(t)=Au(t)+f\Big(t,u(t),\int_0^t\rho(t,s)h(t,s,u(s))\,ds\Big),\quad t\in J',\\ \Delta u|_{t=t_k}=I_k(u(t_k)),\quad k=1,2,\dots,m,\\ u(0)=v_0,\end{gathered}\tag{6.3}$$

where $v_0\in X$. Then there exists a positive constant M such that

$$\|u-v\|_{PC}\le M\|u_0-v_0\|.$$

7. Existence and uniqueness of a piecewise pseudo S-asymptotically (ω,c)-periodic solution for (1.2)

Before presenting the main existence and uniqueness theorems in this section, we prove some superposition results.

Lemma 7.1. *Assume that*

$$\int_0^t\|c^\wedge(-s)(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))\psi'(t-s)\|\,ds$$

is uniformly bounded on $\mathbb{R}_+$. If $f\in\mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,D(B))$,

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{r}\int_0^r\int_0^t c^\wedge(-s)\|&(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(t+\omega-s))\psi'(t+\omega-s)\\ &-(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))\psi'(t-s)\|\,ds\,dt=0\end{aligned}$$

and there exists a locally integrable $g(t)$ such that

$$(\psi(t)-\psi(t-s))^{q-1}\psi'(t-s)\le g(s),\quad\forall t\ge s.\tag{7.1}$$

Then

$$u(t) := \int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))f(s)\psi'(s)\,ds \in \mathrm{PSAP}_{\omega,c,0}(\mathbb{R}_+, X).$$

Proof. Since $f \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, D(B))$, we have

$$f(t) = c^{\wedge}(t)p(t), p(t) \in PBC(\mathbb{R}_+, D(B)).$$

Hence

$$\begin{aligned} u(t) &= \int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))f(s)\psi'(s)\,ds \\ &= c^{\wedge}(t)\int_0^t c^{\wedge}(-s)(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))p(t-s)\psi'(t-s)\,ds, \end{aligned}$$

and

$$\begin{aligned} &\Big\| \int_0^t c^{\wedge}(-s)(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))p(t-s)\psi'(t-s)\,ds\Big\| \\ &\le \int_0^t \|c^{\wedge}(-s)(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))p(t-s)\psi'(t-s)\|\,ds \\ &\le \|p\|_\infty \int_0^t \|c^{\wedge}(-s)(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))\psi'(t-s)\|\,ds < \infty, \end{aligned}$$

which hints that $\int_0^t c^{\wedge}(-s)(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))p(t-s)\psi'(t-s)\,ds \in BC(\mathbb{R}_+, X), u(t) \in BC_{\omega,c}(\mathbb{R}_+, X)$. It also follows from the Fubini's theorem that

$$\begin{aligned} &\frac{1}{r}\int_0^r \|c^{\wedge}(-t)[u(t+\omega)-cu(t)\|\,dt \\ &= \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\Big[\int_0^{t+\omega}(\psi(t+\omega)-\psi(s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(s))f(s)\psi'(s)\,ds \\ &\quad - c\int_0^t (\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))f(s)\psi'(s)\,ds\Big]\Big\|\,dt \\ &= \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\Big[\int_0^{t+\omega}(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(t+\omega-s))f(t+\omega-s) \\ &\quad \times \psi'(t+\omega-s)\,ds - c\int_0^t (\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))f(t-s)\psi'(t-s)\,ds\Big]\Big\|\,dt \\ &= \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\Big[\int_0^{t+\omega}(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(t+\omega-s))f(t+\omega-s) \\ &\quad \times \psi'(t+\omega-s)\,ds - \int_0^t (\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))f(t+\omega-s)\psi'(t-s)\,ds\Big]\Big\|\,dt \\ &\quad + \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\Big[\int_0^t (\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))f(t+\omega-s)\psi'(t-s)\,ds \\ &\quad - c\int_0^t (\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))f(t-s)\psi'(t-s)\,ds\Big]\Big\|\,dt \\ &\le \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\int_t^{t+\omega}(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(t+\omega-s))f(t+\omega-s) \\ &\quad \times \psi'(t+\omega-s)\,ds\Big\|\,dt + \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\int_0^t [(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(t+\omega-s)) \\ &\quad \times \psi'(t+\omega-s) - (\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))\psi'(t-s)]f(t+\omega-s)\,ds\Big\|\,dt \\ &\quad + \frac{1}{r}\int_0^r \Big\|c^{\wedge}(-t)\int_0^t (\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))[f(t+\omega-s)-cf(t-s)]\psi'(t-s)\,ds\Big\|\,dt \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\|f\|_{pb\omega c}}{r}\int_0^r\int_t^{t+\omega}\|c^\wedge(\omega-s)(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)\\
&\quad-\psi(t+\omega-s))\psi'(t+\omega-s)\,ds\|\,dt\\
&\quad+\frac{\|f\|_{pb\omega c}}{r}\int_0^r\int_0^t c^\wedge(\omega-s)\|(\psi(t+\omega)-\psi(t+\omega-s))^{q-1}\mathcal{R}(\psi(t+\omega)-\psi(t+\omega-s))\\
&\quad\times\psi'(t+\omega-s)-(\psi(t)-\psi(t-s))^{q-1}\mathcal{R}(\psi(t)-\psi(t-s))\psi'(t-s)\|\,ds\,dt\\
&\quad+\frac{\Gamma(q)}{M}\int_0^r\|c^\wedge(-s)g(s)\|\Big[\frac{1}{r}\int_s^r\|c^\wedge(s-t)[f(t+\omega-s)-cf(t-s)]\|\,dt\Big]\,ds.
\end{aligned}
$$

From $f\in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,D(B))$, Lemma 2.11 and the dominated convergence theorem, we have

$$\lim_{r\to\infty}\frac{1}{r}\int_0^r\|c^\wedge(-t)[u(t+\omega)-cu(t)\|\,dt=0.$$

Thus $u\in \mathrm{PSAP}_{\omega,c,0}(\mathbb{R}_+,X)$. □

Remark 7.2. It can be verified that $\psi(t)$ satisfies the condition (7.1) when $\psi(t)=t$ or arctan$t$.

Lemma 7.3. *Let $f\in C(\mathbb{R}_+\times X\times PC([-r,0],D(B)))$ satisfy the following conditions:*

(H15) (a) $\sup_{t\in\mathbb{R}_+}\|c^\wedge(-t)f(t,x,\phi)\|<+\infty$ *uniformly for $x\in X$, $\phi\in PC([-r,0],X)$.*
(b) $\lim_{r\to\infty}\frac{1}{r}\int_0^r\|c^\wedge(-t)[f(t+\omega,cx,c\phi)-cf(t,x,\phi)]\|\,dt=0$ *uniformly for $x\in X$, $\phi\in PC([-r,0],X)$.*

(H16) *There exist constants $l_i>0$ such that for all $t\in\mathbb{R}_+$, $x_i\in X$ and $\phi_i\in PC([-r,0],X)(i=1,2)$,*

$$\|f(t,x_1,\phi_1)-f(t,x_2,\phi_2)\|\le l_1\|x_1-x_2\|+l_2\|\phi_1-\phi_2\|_{PC}.$$

Then for each $u\in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,X)$, $f(\cdot,u(\cdot),u.)\in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,D(B))$.

Proof. If follows by (H15)(a) that $f(\cdot,u(\cdot),u.)\in PBC_{\omega,c}(\mathbb{R}_+,D(B))$. For each $u\in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,X)$, we have

$$\lim_{r\to\infty}\frac{1}{r}\int_0^r\|c^\wedge(-t)[u(t+\omega)-cu(t)]\|\,dt=0.$$

And we have

$$
\begin{aligned}
&\frac{1}{r}\int_0^r\|c^\wedge(-t)[f(t+\omega,u(t+\omega),u_{t+\omega})-cf(t,u(t),u_t)]\|\,dt\\
&=\frac{1}{r}\int_0^r\Big\|c^\wedge(-t)\Big[f(t+\omega,u(t+\omega),u_{t+\omega})-cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)\\
&\quad+cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-cf(t,u(t),u_t)\Big]\Big\|\,dt\\
&\le\frac{1}{r}\int_0^r\Big\|c^\wedge(-t)\Big[f(t+\omega,u(t+\omega),u_{t+\omega})-cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)\Big]\Big\|\,dt\\
&\quad+\frac{1}{r}\int_0^r\Big\|c^\wedge(-t)\Big[cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-cf(t,u(t),u_t)\Big]\Big\|\,dt\\
&:=I_1+I_2.
\end{aligned}
$$

By (H15)(b), we have $I_1\to0$ as $r\to0$. For I_2, we have

$$
\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r}\int_0^r\Big\|c^\wedge(-t)\Big[cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-cf(t,u(t),u_t)\Big]\Big\|\,dt\\
&=\lim_{r\to\infty}\frac{1}{r}\int_0^r|c^\wedge(-t+\omega)|\Big\|f\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-f(t,u(t),u_t)\Big\|\,dt\\
&\le\lim_{r\to\infty}\frac{1}{r}\int_0^r|c^\wedge(-t+\omega)|\Big[l_1\Big\|\frac{1}{c}u(t+\omega)-u(t)\Big\|+l_2\Big\|\frac{1}{c}u_{t+\omega}-u_t\Big\|_{PC}\Big]\,dt\\
&=l_1\lim_{r\to\infty}\frac{1}{r}\int_0^r\|c^\wedge(-t)[u(t+\omega)-cu(t)]\|\,dt+l_2\lim_{r\to\infty}\frac{1}{r}\int_0^r\|c^\wedge(-t)[u_{t+\omega}-cu_t]\|_{PC}\,dt
\end{aligned}
$$

$$\le l_1 \lim_{r\to\infty}\frac{1}{r}\int_0^r \|c^\wedge(-t)[u(t+\omega)-cu(t)]\|\,dt$$
$$+\sup_{s\in[-\sigma,0]} l_2 \lim_{r\to\infty}\frac{1}{r}\int_0^r \|c^\wedge(-t)[u(t+\omega+s)-cu(t+s)]\|\,dt = 0.$$

Thus, we have proved $f(\cdot,u(\cdot),u_\cdot)\in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,D(B))$. □

Lemma 7.4. *Let $f\in PBC_{\omega,c}(\mathbb{R}_+,D(B))$. Then the following 2 assertions are equivalent:*

(a) $\lim_{r\to\infty}\frac{1}{r}\int_0^r \|c^\wedge(-t)[f(t+\omega)-cf(t)]\|\,dt=0$.
(b) *For each $\varepsilon>0$, $\lim_{r\to\infty}\frac{1}{r}\int_{M_{r,\varepsilon}(f)}dt=0$, where*

$$M_{r,\varepsilon}(f)=\{t\in[0,r]:\|c^\wedge(-t)[f(t+\omega)-cf(t)]\|\ge\varepsilon\}.$$

Let $\mathcal{H}=\big\{u:\mathbb{R}_+\to\mathbb{R}$ be locally integrable and $\lim_{r\to\infty}\frac{1}{r}\int_0^r u(t)\,dt<\infty\big\}$. Using Hölder's inequality, we can prove that $L^p(\mathbb{R}_+,\mathbb{R})\subset\mathcal{H}$ for $p\ge 1$.

Theorem 7.5. *Let $f\in C(\mathbb{R}_+\times X\times PC([-r,0],D(B)))$ satisfy* (H15) *and the following condition:*

(H17) *There exists a function $\mathbb{H}(t)\in\mathcal{H}$ such that for any $\varepsilon>0$, there is a constant $\delta>0$ such that*

$$\|c^\wedge(-t)[f(t,c^\wedge(t)x_1,c^\wedge(t)\phi_1)-f(t,c^\wedge(t)x_2,c^\wedge(t)\phi_2)]\|\le\mathbb{H}(t)\varepsilon,$$

for $x_i\in X,\phi_i\in PC([-r,0],X)(i=1,2)$ with $\|x_1-x_2\|<\delta,\|\phi_1-\phi_2\|_{PC}<\delta$ and $t\in\mathbb{R}_+$.

Then for each $u\in\mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,X)$, $f(\cdot,u(\cdot),u_\cdot)\in\mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+,D(B))$.

Proof. From (H15)(a), we have $f(\cdot,u(\cdot),u_\cdot)\in PBC_{\omega,c}(\mathbb{R}_+,D(B))$ and there exists a constant $M>0$ such that $\sup_{t\in\mathbb{R}_+}\|c^\wedge(-t)f(t,x,\phi)\|\le M$ for all $x\in X,\phi\in PC([-r,0],X)$. And we have

$$\begin{aligned}
&\frac{1}{r}\int_0^r \|c^\wedge(-t)[f(t+\omega,u(t+\omega),u_{t+\omega})-cf(t,u(t),u_t)]\|\,dt\\
&=\frac{1}{r}\int_0^r \Big\|c^\wedge(-t)\Big[f(t+\omega,u(t+\omega),u_{t+\omega})-cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)\\
&\quad+cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-cf(t,u(t),u_t)\Big]\Big\|\,dt\\
&\le\frac{1}{r}\int_0^r \Big\|c^\wedge(-t)\Big[f(t+\omega,u(t+\omega),u_{t+\omega})-cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)\Big]\Big\|\,dt\\
&\quad+\frac{1}{r}\int_0^r \Big\|c^\wedge(-t)\Big[cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-cf(t,u(t),u_t)\Big]\Big\|\,dt\\
&:=I_1+I_2.
\end{aligned}$$

Obviously, we have $I_1\to 0$ as $r\to 0$. For I_2, if $\|c^\wedge(-t)[u(t+\omega)-cu(t)]\|<\max\{|c|\delta,|c|^{1-\sigma/\omega}\}:=\tilde{c}$, then

$$\Big\|c^\wedge(-t)\frac{1}{c}u(t+\omega)-c^\wedge(-t)u(t)\Big\|=\Big\|\frac{1}{c}c^\wedge(-t)[u(t+\omega)-cu(t)]\Big\|<\delta,$$

and

$$\Big\|c^\wedge(-t)\frac{1}{c}u_{t+\omega}-c^\wedge(-t)u_t\Big\|_{PC}=\sup_{\theta\in[-\sigma,0]}\Big\|\frac{1}{c}c^\wedge(-t)[u(t+\omega+\theta)-cu(t+\theta)]\Big\|<\delta.$$

From (H17), we have

$$\begin{aligned}
&\frac{1}{r}\int_0^r \Big\|c^\wedge(-t)\Big[cf\Big(t,\frac{1}{c}u(t+\omega),\frac{1}{c}u_{t+\omega}\Big)-cf(t,u(t),u_t)\Big]\Big\|\,dt\\
&=|c|\frac{1}{r}\int_0^r \Big\|c^\wedge(-t)\Big[f\Big(t,c^\wedge(t)c^\wedge(-t)\frac{1}{c}u(t+\omega),c^\wedge(t)c^\wedge(-t)\frac{1}{c}u_{t+\omega}\Big)\\
&\quad-f(t,c^\wedge(t)c^\wedge(-t)u(t),c^\wedge(t)c^\wedge(-t)u_t)\Big]\Big\|\,dt\\
&=|c|\frac{1}{r}\int_{[0,r]\setminus M_{r,\tilde{c}}(u)} \Big\|c^\wedge(-t)\Big[f\Big(t,c^\wedge(t)c^\wedge(-t)\frac{1}{c}u(t+\omega),c^\wedge(t)c^\wedge(-t)\frac{1}{c}u_{t+\omega}\Big)
\end{aligned}$$

$$
\begin{aligned}
&- f(t, c^\wedge(t)c^\wedge(-t)u(t), c^\wedge(t)c^\wedge(-t)u_t)\Big]\Big\| \, dt \\
&+ |c|\frac{1}{r}\int_{M_{r,\tilde{c}}(u)} \Big\|c^\wedge(-t)\Big[f\Big(t, c^\wedge(t)c^\wedge(-t)\frac{1}{c}u(t+\omega), c^\wedge(t)c^\wedge(-t)\frac{1}{c}u_{t+\omega}\Big) \\
&- f(t, c^\wedge(t)c^\wedge(-t)u(t), c^\wedge(t)c^\wedge(-t)u_t)\Big]\Big\| \, dt \\
&\le |c|\varepsilon\frac{1}{r}\int_0^r \mathbb{H}(t)\, dt + 2M|c|\frac{1}{r}\int_{M_{r,\tilde{c}}(u)} dt.
\end{aligned}
$$

From Lemma 7.4, $\mathbb{H}(t) \in \mathcal{H}$ and the arbitrariness of ε, we have $\lim_{r\to\infty} I_2 = 0$, and thus

$$
\lim_{r\to\infty}\frac{1}{r}\int_0^r \|c^\wedge(-t)[f(t+\omega, u(t+\omega), u_{t+\omega}) - cf(t, u(t), u_t)]\|\, dt = 0,
$$

which means $f(\cdot, u(\cdot), u_\cdot) \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, D(B))$. □

Corollary 7.6. *Let $f \in C(\mathbb{R}_+ \times X \times PC([-r,0], D(B)))$ satisfy* (H15) *and the following condition:*

(H18) *There exists a function $\mathbb{H}(t) \in \mathcal{H}$ such that*

$$
\|c^\wedge(-t)[f(t, c^\wedge(t)x_1, c^\wedge(t)\phi_1) - f(t, c^\wedge(t)x_2, c^\wedge(t)\phi_2)]\| \le \mathbb{H}(t)[\|x_1 - x_2\| + \|\phi_1 - \phi_2\|_{PC}],
$$

for $x_i \in X$, $\phi_i \in PC([-r,0], X)(i = 1, 2)$, and $t \in \mathbb{R}_+$.

Then for each $u \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, X)$ and $f(\cdot, u(\cdot), u_\cdot) \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, D(B))$.

Proof. We can check that (H17) in Lemma 7.5 holds if (H18) is satisfied. Hence, the conclusion can be directly derived from Theorem 7.5. □

Corollary 7.7. *Let $f \in C(\mathbb{R}_+ \times X \times PC([-r,0], D(B)))$ satisfy* (H15) *and the following condition:*

(H19) *There exists a function $\mathbb{H}(t) \in L^p(\mathbb{R}_+, \mathbb{R}_+)(1 \le p < \infty)$ such that*

$$
\|c^\wedge(-t)[f(t, c^\wedge(t)x_1, c^\wedge(t)\phi_1) - f(t, c^\wedge(t)x_2, c^\wedge(t)\phi_2)]\| \le \mathbb{H}(t)[\|x_1 - x_2\| + \|\phi_1 - \phi_2\|_{PC}],
$$

for $x_i \in X$, $\phi_i \in PC([-r,0], X)(i = 1, 2)$ and $t \in \mathbb{R}_+$.

Then for each $u \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, X)$, $f(\cdot, u(\cdot), u_\cdot) \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, D(B))$.

Proof. It can be directly proved that if $\mathbb{H}(t) \in L^p(\mathbb{R}_+, \mathbb{R}_+)(1 \le p < \infty)$ then $\mathbb{H}(t) \in \mathcal{H}$. So the result is a direct consequence of Corollary 7.6. □

Theorem 7.8. *Suppose that $\mathcal{Q}(t) \in \mathrm{PSAP}_{\omega,c}(\mathbb{R}_+, \mathcal{B}(X))$,*

$$
\sup_{t\in\mathbb{R}_+}\int_0^t c^\wedge(-t+s)(\psi(t) - \psi(s))^{q-1}\psi'(s)\, ds := \Psi < \infty,
$$

and that there exists $\mathfrak{i}_k$ such that $\|I_k(x) - I_k(y)\| \le \mathfrak{i}_k\|x - y\|$ for all $x, y \in X$ and $k \in \mathbb{N}_+$. If

$$
\sup_{t\in\mathbb{R}_+}\sum_{0<t_k<t} \mathfrak{i}_k c^\wedge(-t+t_k) := \mathcal{I} < \infty,
$$

and

$$
\frac{l_1\Psi}{\Gamma(q)} + \mathcal{I} < \frac{1}{\mathscr{M}},
$$

then problem (1.2) *has a unique mild solution $u \in \mathrm{PPSAP}_{\omega,\mathrm{c}}(\mathbb{R}_+, \mathrm{X})$.*

Proof. For $\varsigma \in PC([-\sigma, 0], X)$, we define the function $y(t) = \varsigma(t)$ for $t \in [-\sigma, 0]$, and $y(t) = \mathcal{Q}(\psi(t))\varsigma(0)$ for $t \ge 0$. Then $y \in PC([-\sigma, \infty), X)$. We set $u(t) = x(t) + y(t), t \in [-\sigma, +\infty)$. Also we set

$$
\widetilde{PBC}(X) = \{x \in PBC([-\sigma, +\infty), X) : x|_{t\ge 0} \in PBC(\mathbb{R}_+, X), x|_{[-\sigma,0]} = 0\}.
$$

It is obvious that u satisfies (3.13) if and only if x satisfies $x_0 = 0$ and for $t \ge 0$,

$$
\begin{aligned}
x(t) = &\int_0^t (\psi(t) - \psi(s))^{q-1}\mathcal{R}(\psi(t) - \psi(s))f\left(s, x(s) + y(s), x_s + y_s\right)\psi'(s)\, ds \\
&+ \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k))I_k(x(t_k) + y(t_k)).
\end{aligned}
$$

We define the operator $F : \widetilde{PBC}(X) \to \widetilde{PBC}(X)$ by

$$x(t) = \begin{cases} 0, & t \in [-\sigma, 0], \\ \int_0^t (\psi(t) - \psi(s))^{q-1} \mathcal{R}(\psi(t) - \psi(s)) f\left(s, x(s) + y(s), x_s + y_s\right) \psi'(s)\, ds \\ + \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k)) I_k(x(t_k) + y(t_k)), & t \geq 0. \end{cases} \tag{7.2}$$

By Lemmas 7.1, 7.3 and the condition that $\mathcal{Q}(t) \in \mathrm{PSAP}_{\omega,c}(\mathbb{R}_+, \mathcal{B}(X))$, we have

$$F(\mathrm{PPSAP}_{\omega,c,0}(\mathbb{R}_+, X)) \subseteq \mathrm{PPSAP}_{\omega,c,0}(\mathbb{R}_+, X).$$

For each $x, \tilde{x} \in \mathrm{PSAP}_{\omega,c,0}(\mathbb{R}_+, \mathcal{B}(X))$ and all $t \in \mathbb{R}_+$, we have

$$\begin{aligned} \|c^\wedge(-t)[F(x)(t) - F(\tilde{x})(t)]\| &\leq \frac{\mathscr{M} l_1}{\Gamma(q)} \int_0^t c^\wedge(-t+s)(\psi(t) - \psi(s))^{q-1} \|c^\wedge(-s)[x(s) - \tilde{x}(s)]\| \psi'(s)\, ds \\ &\quad + \mathscr{M} \sum_{0<t_k<t} \mathfrak{i}_k c^\wedge(-t+t_k) \|c^\wedge(-t_k)[x(t_k) - \tilde{x}(t_k)]\| \\ &\leq \mathscr{M}\Big(\frac{l_1 \Psi}{\Gamma(q)} + \mathcal{I}\Big) \|x - \tilde{x}\|_{pb\omega c}. \end{aligned}$$

Thus

$$\|Fx - F\tilde{x}\|_{pb\omega c} \leq \mathscr{M}\Big(\frac{l_1 \Psi}{\Gamma(q)} + \mathcal{I}\Big) \|x - \tilde{x}\|_{pb\omega c},$$

which shows that F is a contraction mapping by $\frac{l_1 \Psi}{\Gamma(q)} + \mathcal{I} < \frac{1}{\mathscr{M}}$. Then the proof can now be completed by using the contraction mapping principle. □

Remark 7.9. If we assume that the propagating family $\{W(t)\}_{t\geq 0}$ is uniformly exponentially stable and $|c| \geq 1$, it can be easily seen that $\mathcal{Q}(t) \in \mathrm{PSAP}_{\omega,c}(\mathbb{R}_+, \mathcal{B}(X))$ by Remark 3.5.

Theorem 7.10. *Suppose that $\mathcal{Q}(t) \in \mathrm{PSAP}_{\omega,c}(\mathbb{R}_+, \mathcal{B}(X))$. There exists positive constants N and λ such that*

$$\|c^\wedge(-t+s)(\psi(t) - \psi(s))^{q-1} \mathcal{R}(\psi(t) - \psi(s)) \psi'(s)\| \leq N e^{-\lambda(t-s)} \tag{7.3}$$

for each $t \geq s \geq 0$, and there exists $\mathfrak{i}_k$ such that $\|I_k(x) - I_k(y)\| \leq \mathfrak{i}_k \|x - y\|$ for all $x, y \in X$ and $k \in \mathbb{N}_+$. Also suppose that $f \in C(\mathbb{R}_+ \times X \times PC([-r, 0], D(B)))$ satisfies (H15) *and*

$$\|c^\wedge(-t)[f(t, c^\wedge(t)x_1, c^\wedge(t)\phi_1) - f(t, c^\wedge(t)x_2, c^\wedge(t)\phi_2)]\| \leq \mathcal{H}(t)[\|x_1 - x_2\| + \|\phi_1 - \phi_2\|_{PC}] \tag{7.4}$$

with $\mathcal{H}(t) \in L^p(\mathbb{R}_+, \mathbb{R}_+)(1 < p < \infty)$ for each $x_i \in X, \phi_i \in PC([-r, 0], X)(i = 1, 2)$ and all $t \in \mathbb{R}_+$. If

$$\sup_{t\in\mathbb{R}_+} \sum_{0<t_k<t} \mathfrak{i}_k c^\wedge(-t+t_k) := \mathcal{I} < \infty$$

and $\mathcal{I}\mathscr{M} < 1$, then problem (1.2) *has a unique mild solution $u \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, \mathrm{X})$.*

Proof. Let $F : \mathrm{PPSAP}_{\omega,c,0}(\mathbb{R}_+, X) \to \mathrm{PPSAP}_{\omega,c,0}(\mathbb{R}_+, X)$ as (7.2). By Lemma 7.1, Corollary 7.7, and the condition that $\mathcal{Q}(t) \in \mathrm{PSAP}_{\omega,c}(\mathbb{R}_+, \mathcal{B}(X))$, F is well-defined.

Let $\eta(t) = \int_{-\infty}^t \mathcal{H}^p(s)\, ds$. Define the following norm on $\mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, \mathrm{X})$ by

$$\|u\|_\eta = \sup_{t\in\mathbb{R}_+} \left\{e^{-\zeta\eta(t)} \|u\|_{pb\omega c}\right\}, \quad u \in \mathrm{PPSAP}_{\omega,c}(\mathbb{R}_+, \mathrm{X}),$$

where ζ is a large enough positive constant. We set $F(t) = F_1(t) + F_2(t)$, where

$$F_1(t) = \int_0^t (\psi(t) - \psi(s))^{q-1} \mathcal{R}(\psi(t) - \psi(s)) f\left(s, x(s) + y(s), x_s + y_s\right) \psi'(s)\, ds,$$

$$F_2(t) = \sum_{0<t_k<t} \mathcal{Q}(\psi(t) - \psi(t_k)) I_k(x(t_k) + y(t_k))$$

for $t \geq 0$. For each $x, \tilde{x} \in \mathrm{PSAP}_{\omega,c,0}(\mathbb{R}_+, \mathcal{B}(X))$ and all $t \in \mathbb{R}_+$, on one hand, we have

$$\|c^\wedge(-t)[F_2(x)(t) - F_2(\tilde{x})(t)]\| \leq \mathscr{M} \sum_{0<t_k<t} \mathfrak{i}_k c^\wedge(-t+t_k) \|c^\wedge(-t_k)[x(t_k) - \tilde{x}(t_k)]\| \leq \mathscr{M}\mathcal{I} \|x - \tilde{x}\|_{pb\omega c}.$$

Hence,

$$\|F_2x - F_2\tilde{x}\|_\eta \le \mathscr{M}\mathcal{I}\|x-\tilde{x}\|_\eta.$$

On the other hand, we have

$$\begin{aligned}
&\|c^\wedge(-t)[F_1(x)(t) - F_1(\tilde{x})(t)]\| \\
&\le \int_0^t \|c^\wedge(-t+s)(\psi(t)-\psi(s))^{q-1}\mathcal{R}(\psi(t)-\psi(s))c^\wedge(-s)[f\,(s,x(s)+y(s),x_s+y_s) \\
&\quad - f\,(s,\tilde{x}(s)+y(s),\tilde{x}_s+y_s)]\|\psi'(s)\,ds \\
&\le N\int_0^t e^{-\lambda(t-s)}\mathcal{H}(s)\|c^\wedge(-s)[x(s)-\tilde{x}(s)]\|\,ds \\
&\le N\int_0^t e^{-\lambda(t-s)}\mathcal{H}(s)e^{\zeta\eta(s)}\|x-\tilde{x}\|_\eta\,ds \\
&\le N\Big[\int_0^t e^{\zeta p\eta(s)}\mathcal{H}^p(s)\,ds\Big]^{\frac{1}{p}}\Big[\int_0^t e^{-\lambda\frac{p(t-s)}{p-1}}\,ds\Big]^{\frac{p-1}{p}}\|x-\tilde{x}\|_\eta \\
&\le N\Big(\lambda\frac{p}{p-1}\Big)^{\frac{1-p}{p}}\Big[\int_0^t e^{\zeta p\eta(s)}d\eta(s)\Big]^{\frac{1}{p}}\|x-\tilde{x}\|_\eta \\
&\le N\Big(\lambda\frac{p}{p-1}\Big)^{\frac{1-p}{p}}(p\zeta)^{-\frac{1}{p}}e^{\zeta\eta(t)}\|x-\tilde{x}\|_\eta.
\end{aligned}$$

Therefore,

$$\|Fx-F\tilde{x}\|_\eta \le \Big[\mathscr{M}\mathcal{I} + N\Big(\lambda\frac{p}{p-1}\Big)^{\frac{1-p}{p}}(p\zeta)^{-\frac{1}{p}}\Big]\|x-\tilde{x}\|_\eta.$$

It can be seen clearly that F is a contraction mapping by the condition $\mathcal{I}\mathscr{M} < 1$ when ζ is sufficiently large. So again the contraction mapping principle guarantees that the problem (1.2) has a unique mild solution $u \in \mathrm{PPSAP}_{\omega,\mathrm{c}}(\mathbb{R}_+,\mathrm{X})$. □

When $B = I$ and A generates a norm continuous semigroup $\{W(t)\}_{t\ge0}$ of uniformly bounded linear operators on X, there are corresponding conclusions of Theorems 7.8 and 7.10, which we will not elaborate further here.

8. Examples

In this section, we give some examples to illustrate the main results obtained in previous sections.

Example 8.1. Consider the following fractional diffusion equations of Sobolev type with impulsive conditions involving the ψ-Caputo fractional derivative:

$$\begin{aligned}
\frac{\partial^{q;\psi}}{\partial t^q}\Big(u(t,x)-\frac{\partial^2u(t,x)}{\partial x^2}\Big) &= \frac{\partial^2u(t,x)}{\partial x^2} + f\Big(t,u(t,x),\int_0^t\rho(t,s)h(t,s,u(s,x))\,ds\Big) \\
&\quad - \frac{\partial^2}{\partial x^2}f\Big(t,u(t,x),\int_0^t\rho(t,s)h(t,s,u(s,x))\,ds\Big), t\in J', \qquad (8.1)\\
\Delta u|_{t=t_k} &= I_k(u(t_k,x)),\quad k=1,2,\dots,m,\\
u(0,x) &= u_0(x).
\end{aligned}$$

Let $X = L^2(\mathbb{R})$, and define $D(A) = D(B) = H^2(\mathbb{R})$,

$$Au = \frac{\partial^2u}{\partial x^2},\quad Bu = u - \frac{\partial^2u}{\partial x^2}.$$

In view of [24, Theorem 4.], it can be seen easily that the pair (A,B) generates a uniformly bounded propagation family $W(t)$. Similarly to the proofs of [29, (2.15), (2.16) and (2.17)], we can see that $\{W(t)\}_{t\ge0}$ is norm continuous for $t>0$ and $\|W(t)\|\le1$.

To study system (8.1), we assume the conditions (H1)–(H8) hold. By Theorem 4.9, we can obtain the following theorem.

Theorem 8.2. *Assume that the conditions* (H1)–(H8) *are satisfied. If*

$$\max\Big\{\frac{\mathscr{M}}{\Gamma(q)}\left(\ell_{p_1,q}T^{q-\frac{1}{p_1}}\|\boldsymbol{\mu}\|_{L^{p_1}}+\frac{\psi^q(T)\wp\mathbf{h}^*\boldsymbol{\nu}^*}{q}\right)+\mathscr{M}\sum_{k=1}^{m}\mathbb{L}_k,$$
$$\frac{4\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_2,q}T^{q-\frac{1}{p_2}}\|\phi\|_{L^{p_2}}+\frac{2\wp\boldsymbol{\zeta}^*\psi^q(T)}{q}\Big)+2\mathscr{M}\sum_{k=1}^{m}\mathbb{M}_k\Big\}<1,$$

then there exists at least one mild solution for problem (8.1).

Let $X=\{u(t)|\ u(t)\in L^2[0,\pi], u(t)$ is a real function$\}$. Define the inner product and norm on X for $\mathbf{u}_1,\mathbf{u}_2\in X$ by

$$\langle\mathbf{u}_1,\mathbf{u}_2\rangle=\int_0^\pi \mathbf{u}_1(t)\mathbf{u}_2(t)\,dt, \|\mathbf{u}_1\|=\Big(\int_0^\pi \mathbf{u}_1^2(t)\,dt\Big)^{\frac{1}{2}}.$$

Define the operator $A:D(A)\subset X\to X$ by

$$D(A):=\{\mathbf{v}\in X:\mathbf{v}''\in X,\mathbf{v}(0)=\mathbf{v}(\pi)=0\}, Au=\frac{\partial^2 u}{\partial x^2}.$$

From [36] we know that $-A$ has eigenvalues of the form n^2 ($n\in\mathbb{N}_+$), and the corresponding normalized eigenfunctions are given by $e_n=\sqrt{\frac{2}{\pi}}\sin(nx)$ ($n\in\mathbb{N}_+$). Moreover, A generates a compact analytic semigroup $\{W(t)\}_{t\geq 0}$ on X, and

$$W(t)u=\sum_{n=1}^{\infty}\mathrm{e}^{-n^2t}\langle u,e_n\rangle e_n.$$

We can verify that $\|W(t)\|\leq \mathrm{e}^{-t}$ for all $t\geq 0$, and take $\mathscr{M}=1$. Furthermore, by [36] we know that $\{W(t)\}_{t\geq 0}$ is continuous in the uniform operator topology for $t>0$.

Example 8.3. *Consider the system*

$$\begin{gathered}({}^cD^{q;\psi}u)(t)=Au(t)+f\left(t,u(t),\int_0^t\rho(t,s)h(t,s,u(s))\,ds\right),t\in J',\\ \Delta u|_{t=t_k}=I_k(u(t_k)),\quad k=1,2,\dots,m,\\ u(0)=u_0,\end{gathered}\tag{8.2}$$

where $q=2/5$, $T=2$, $\psi(t)=t^2$, $\rho(t,s)\equiv 1$, $h(t,s,u(s,x))=\frac{1}{5}\sin(s^3)\cos(u(s,x)/t^2)$,

$$f(t,u,v)=\frac{1}{6k\cdot\sqrt[k]{t}}u+\frac{t^4}{16k^2}v,$$

$m=3$, $t_1=\frac{1}{2}$, $t_2=1$, $t_3=3/2$, $I_1(u)=\frac{1}{10^4}\arctan(u)$, $I_2(u)=10^{-4}\cos(u)$, $I_3(u)\equiv u_1$, *where*

$$\frac{1}{\sqrt[k]{t}}\in L^{p_i}([0,1],\mathbb{R}^+)\ (p_i>2,i=1,2),$$

and $u_0,u_1\in H^2(\mathbb{R}^3)$.

We have

$$\|f(t,\mathcal{V},\mathcal{W})\|\leq\frac{1}{6k\sqrt[k]{t}}\|\mathcal{V}\|+\frac{t^4}{16k^2}\|\mathcal{W}\|:=\boldsymbol{\mu}(t)\|\mathcal{V}\|+\boldsymbol{\nu}(t)\|\mathcal{W}\|,$$

and if we take $k=7$, then for any bounded sets $E_i\subset X(i=1,2)$,

$$\mu(f(t,E_1,E_2))\leq\frac{1}{6k\cdot\sqrt[k]{t}}(\mu(E_1)+\mu(E_2)):=\phi(t)(\mu(E_1)+\mu(E_2)),t\in(0,1].$$

Moreover,

$$\|h(t,s,\mathcal{V})\|\leq\frac{s^3}{5t^2}\|\mathcal{V}\|:=\mathbf{h}(t,s)\|\mathcal{V}\|,$$
$$\mathbf{h}^*:=\sup_{t\in[0,1]}\int_0^t\mathbf{h}(t,s)\,ds=\sup_{t\in[0,1]}\int_0^t\frac{s^3}{5t^2}\,ds=\frac{1}{20}.$$

For $\mathbf{w}_1, \mathbf{w}_2 \in X$, we have

$$\|h(t,s,\mathbf{w}_1) - h(t,s,\mathbf{w}_2)\| \le \frac{s^3}{5t^2}\|\mathbf{w}_1 - \mathbf{w}_2\|.$$

So, for any bounded set $E \subset X$,

$$\mu(h(t,s,E)) \le \frac{s^3}{5t^2}\mu(E) := \boldsymbol{\zeta}(t,s)\mu(E),$$
$$\boldsymbol{\zeta}^* := \sup_{t\in[0,1]}\int_0^t \boldsymbol{\zeta}(t,s)\,ds = \sup_{t\in[0,1]}\int_0^t \frac{s^3}{5t^2}\,ds = \frac{1}{20}.$$

If we take $p_1 = 3$ and $p_2 = 4$, then

$$\|\boldsymbol{\mu}\|_{L^{p_1}} \approx 0.03148, \quad \|\phi\|_{L^{p_2}} \approx 0.02905, \boldsymbol{\nu}^* = \frac{1}{49}.$$

Noting that

$$\Gamma\Big(\frac{2}{5}\Big) \approx 2.218, \quad \ell_{p_1,q} = \Big(\frac{p_1-1}{p_1 q - 1}\Big)^{\frac{p_1-1}{p_1}} = 10^{2/3} \approx 4.642, \quad \ell_{p_2,q} = \Big(\frac{p_2-1}{p_2 q - 1}\Big)^{\frac{p_2-1}{p_2}} = 5^{\frac{3}{4}} \approx 3.344,$$
$$\mathbb{L}_1 = \mathbb{M}_1 = 10^{-4}, \quad \mathbb{L}_2 = \mathbb{M}_2 = 10^{-4}, \quad \mathbb{L}_3 = \mathbb{M}_3 = 0,$$

we have

$$\begin{aligned}
&\frac{\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_1,q}T^{q-\frac{1}{p_1}}\|\boldsymbol{\mu}\|_{L^{p_1}} + \frac{\psi^q(T)\wp\mathbf{h}^*\boldsymbol{\nu}^*}{q}\Big) + \mathscr{M}\sum_{k=1}^m \mathbb{L}_k,\\
&\approx \frac{1}{2.218}\Big(4.642\times 2^{\frac{1}{15}}\times 0.03148 + 2.5\times 4^{\frac{2}{5}}\times\frac{1}{20}\times\frac{1}{49}\Big) + 2\times 10^{-4}\\
&\approx 0.0712 < 1
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2\mathscr{M}}{\Gamma(q)}\Big(\ell_{p_2,q}T^{q-\frac{1}{p_2}}\|\phi\|_{L^{p_2}} + \frac{2\wp\boldsymbol{\zeta}^*\psi^q(T)}{q}\Big) + \mathscr{M}\sum_{k=1}^m \mathbb{M}_k\\
&\approx \frac{2}{2.218}\Big(3.344\times 2^{\frac{3}{20}}\times 0.02905 + 5\times\frac{1}{20}\times 4^{\frac{2}{5}}\Big) + 2\times 10^{-4}\\
&\approx 0.4899 < 0.5\,.
\end{aligned}$$

Hence (H1)–(H10) are satisfied. This means that system (8.2) has mild solutions by Theorem 4.10.

Example 8.4. Consider the following equations on a bounded interval:

$$\begin{gathered}
({}^cD^{q;\psi}u)(t) = Au(t) + f\Big(t, u(t), \int_0^t \rho(t,s)h(t,s,u(s))\,ds\Big), \quad t \in J',\\
\Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1,2,\dots,m,\\
u(0) = u_0,
\end{gathered} \tag{8.3}$$

where $q = 1/2$, $\psi(t) = t^2$, $X = \mathbb{R}$, $J = [0,T]$, $T = 1$, $A = -1$, $u_0 = 25$, $f(t,x_1,x_2) = \sin t + x_1 - x_2$, $h(t,s,x) = t + \frac{1}{10}x$, $\rho(t,s) = (\psi(t)-\psi(s))^{r-1}\psi'(s)$, $r = 1/2$, $m = 2$, $I_1(x) = I_2(x) = \frac{1}{200}x$.

Then $\mathscr{M} = 1$, $\ell_1 = \ell_2 = 1, \ell_3 = \frac{1}{10}$, $\mathscr{I}_1 = \mathscr{I}_2 = \frac{1}{200}$. Consider the operator P defined in (4.1). From the fact that $\mathscr{M}(\mathscr{I}_1 + \mathscr{I}_2) < 1$, we can deduce that (8.3) has a unique solution $u(t)$ by Theorem 5.3. If we take the initial function $u_0(t) = 25 + \frac{1}{8}\sin(6\pi t)$, then Figure 1 shows the curves of $(P^i u_0)(t)(i = 1, 5, 10, 15)$. We can see that the curves of $(P^{10}u_0)(t)$ and $(P^{15}u_0)(t)$ almost coincide, and they can be regarded as the unique numerical solution $u(t)$ of the equations (8.3). If we take $u_0 = 25, 24.9, 25.1$ respectively, we can obtain the curves of the unique solutions $u(t), \bar{u}(t)$

and $\hat{u}(t)$ corresponding to these different initial values in Figure 2. The values of $\mathcal{T}_n (1 \le n \le 20)$ are demonstrated in table 5. We can compute

$$\sum_{n=1}^{\infty} \mathcal{T}_n \approx \sum_{n=1}^{13} \mathcal{T}_n \approx \sum_{n=1}^{20} \mathcal{T}_n \approx 5.67.$$

By Theorem 6.2, we see that

$$|u(t) - \bar{u}(t)| \le (1 + \sum_{n=1}^{\infty} \mathcal{T}_n) \times 0.1 \approx 0.67, t \in [0, 1]$$

and likewise

$$|u(t) - \hat{u}(t)| \le (1 + \sum_{n=1}^{\infty} \mathcal{T}_n) \times 0.1 \approx 0.67, t \in [0, 1],$$

which are consistent with the graphs in Figure 2.

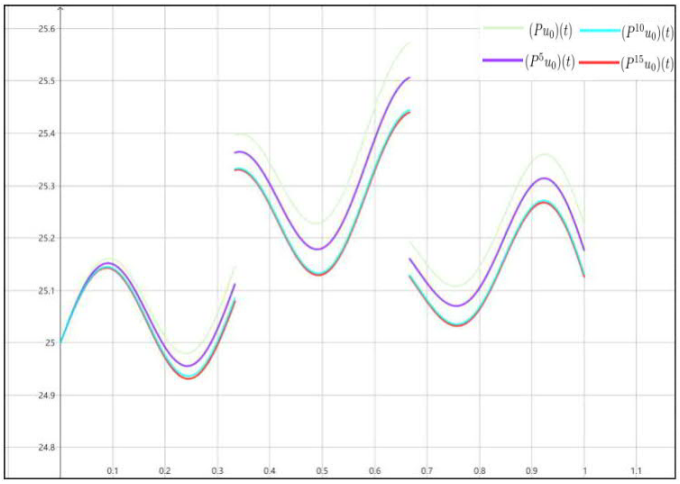


FIGURE 1. $(P^i u_0)(t)$ $(i = 1, 5, 10, 15)$

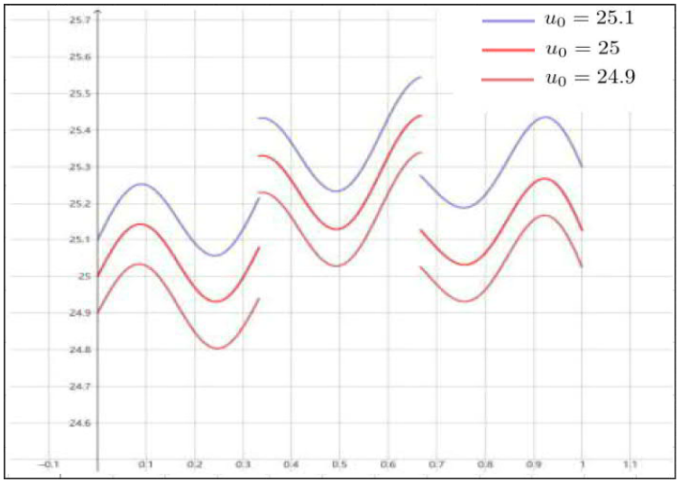


FIGURE 2. Solutions for (8.3) with initial values $u_0 = 25$, 24.9, and 25.1

TABLE 5. Values of $\mathcal{T}_n$ $(1 \le n \le 20)$

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{T}_n$	131:5625	1.308587	1.086272	0.789376	0.516426	0.309808	O.72651	0.090245	0.044573	0.020926
n	11	12	l3	14	15	16	17	18	19	20
$\mathcal{T}_n$	0.009384	0.004035	0.001669	0.000666	0.000257	0.000096	0.000035	0.000012	0.000004	0.000001

Example 8.5. Consider the equations (1.2). Suppose that $\mathcal{Q}(t) \in \mathrm{PSAP}_{\omega,c}(\mathbb{R}_+, \mathcal{B}(X))$, and that the function $f(t)$ satisfies the condition (A'). The impulse points $t_k = k (k \in \mathbb{N}_+)$ and the impulsive function I_k satisfies the Lipschitz condition with $\mathfrak{i}_k = 1$ for each $k \in \mathbb{N}_+$.

Suppose that $c = 2, \omega = 1$ and $\psi(t) = t$, and we can verify that

$$\begin{aligned}\Psi &= \sup_{t\in\mathbb{R}_+} \int_0^t c^{\wedge}(-t+s)(\psi(t)-\psi(s))^{q-1}\psi'(s)\,ds \\ &= \sup_{t\in\mathbb{R}_+} \int_0^t 2^{-t+s}(t-s)^{q-1}\,ds \\ &= \sup_{t\in\mathbb{R}_+} \int_0^t 2^{-s}s^{q-1}\,ds.\end{aligned}$$

Then we have

$$\begin{aligned}\sup_{t\in[0,1]} \int_0^t 2^{-s}s^{q-1}\,ds &\le \int_0^1 2^{-s}s^{q-1}\,ds \le \frac{1}{2q}, \\ \sup_{t\in(1,+\infty)} \int_0^t 2^{-s}s^{q-1}\,ds &\le \int_0^1 2^{-s}s^{q-1}\,ds + \int_1^t 2^{-s}s^{q-1}\,ds \\ &\le \frac{1}{2q} + \int_0^t 2^{-s}\,ds = \frac{1}{2q} + \frac{1}{\ln 2}.\end{aligned}$$

So we obtain $\Psi \le \frac{1}{2q} + \frac{1}{\ln 2}$. Moreover, we see that

$$\mathcal{I} = \sup_{t\in\mathbb{R}_+} \sum_{0<t_k<t} \mathfrak{i}_k c^{\wedge}(-t+t_k) = \sup_{t\in\mathbb{R}_+} \sum_{0<k<t} 2^{-t+k}.$$

When $t \in (k-1, k]$ $(k \in \mathbb{N}_+)$, we have

$$\sup_{t\in(k-1,k]} \sum_{0<k'<t} 2^{-t+k'} \le 2^{-k+1} \sum_{k'=1}^{k-1} 2^{k'} = 2 - 2^{-k+1}.$$

Hence, $\mathcal{I} \le 1$. So we conclude that if $\mathscr{M}\big[\frac{l_1}{\Gamma(q)}\big(\frac{1}{2q} + \frac{1}{\ln 2}\big) + 1\big] < 1$, then (1.2) has a unique mild solution $u \in \mathrm{PPSAP}_{1,2}(\mathbb{R}_+, \mathrm{X})$ by Theorem 7.8. Furthermore, we assume $\mathcal{R}(t)$ and $f(t)$ satisfy the conditions (7.3) and (7.4), respectively. If $\mathscr{M} < 1$, then Theorem 7.10 guarantees (1.2) has a unique mild solution $u \in \mathrm{PPSAP}_{1,2}(\mathbb{R}_+, \mathrm{X})$.

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