Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 87, pp. 1–19.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.87

GROUND STATE AND MULTIPLE SOLUTIONS FOR MODIFIED AUTONOMOUS FOURTH-ORDER ELLIPTIC EQUATIONS UNDER BERESTYCKI-LIONS TYPE CONDITIONS

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Abstract. This article studies the modified fourth-order elliptic equation

$$\Delta^2 u - \Delta u + u - \frac{1}{2}u\Delta(u^2) = f(u), \quad \text{in } \mathbb{R}^N,$$
$$u \in H^2(\mathbb{R}^N),$$

where $N = 5$ or 6, and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies Berestycki-Lions type conditions. By using Jeanjean's augmented functional technique and constructing a Pohožaev-Palais-Smale sequence, we prove that the problem admits a ground state solution. Moreover, if f is odd, we prove, via a symmetric mountain-pass lemma, that the problem admits infinitely many radially symmetric solutions

1. Introduction

In this article, we consider the fourth-order elliptic equation

$$\begin{aligned} \Delta^2 u - \Delta u + u - \frac{1}{2}u\Delta(u^2) = f(u), \quad \text{in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{aligned} \tag{1.1}$$

where $N = 5$ or 6 and $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions specified below.

Equation (1.1) is a special case of the modified quasilinear fourth-order elliptic problem

$$\Delta^2 u - \lambda\Delta u + V(x)u - \kappa u\Delta(u^2) = g(x, u), \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

with $\kappa \geq 0$ and $\lambda \in \mathbb{R}$. When $\mathbb{R}^N$ in (1.2) is replaced by a bounded domain Ω and $\kappa = 0$, the equation models traveling waves in suspension bridges, as studied by Lazer and McKenna [9] and McKenna and Walter [15].

Another related topic is the normalized biharmonic Schrödinger equation

$$\begin{aligned} \Delta^2 u + \lambda u = h(x, u), \quad x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 \, dx = c, \end{aligned} \tag{1.3}$$

where λ is the Lagrange multiplier associated with the prescribed mass. Liu et al. [12] considered a spatially dependent nonlinearity

$$h(x, u) = a(\varepsilon x)|u|^{q-2}u + |u|^{p-2}u,$$

where $a : \mathbb{R}^N \to [0, \infty)$ is continuous and satisfies suitable conditions. Assuming $2 < q < 2 + \frac{8}{N} < p < 2_* := \frac{2N}{N-4}$ and $N \geq 5$, they established the existence of multiple normalized solutions of (1.3) via variational methods. Zhang et al. [21] extended these results to nonlinearities $h(x, u) = h(u)$ satisfying:

2020 *Mathematics Subject Classification.* 35A15, 35D30, 31B30, 58E05.
Key words and phrases. Biharmonic operator; Pohožaev identity; multidimensional odd paths; ground state; multiple solutions.
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Submitted August 4, 2026. Published November 3, 2026.

(H1) $h : \mathbb{R} \to \mathbb{R}$ is continuous and odd;

(H2) There exist constants $2 + \frac{8}{N} < \alpha \le \beta < 2_*$ such that

$$0 < \alpha H(s) \le h(s)s \le \beta H(s), \quad \forall s \ne 0,$$

where $H(s) = \int_0^s h(t)dt$;

(H3) The function defined by $\tilde{H}(s) := \frac{1}{2}h(s)s - H(s)$ is of class C^1 and satisfies

$$\tilde{H}'(s)s \ge \alpha \tilde{H}(s), \quad \forall s \in \mathbb{R},$$

where α is given in (H2).

Their constrained variational methods yielded normalized ground state solutions of (1.3) under general mass-supercritical assumptions.

Mederski and Siemianowski [16] considered the autonomous case

$$\Delta^2 u = h(u), \quad x \in \mathbb{R}^N,$$

where $N \ge 5$. Using the Pohožaev manifold method, they proved the existence of ground states for this equation when h has general subcritical growth at infinity, without assuming the Ambrosetti-Rabinowitz type condition. Further contributions can be found in [1, 5, 14, 19] and the references therein.

In the non-autonomous case ($V \not\equiv$ constant), Chen, Liu, and Wu [3] proved the existence of nontrivial and high-energy solutions for (1.2) under coercive potentials and for 3-superlinear nonlinearities g satisfying

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{t^4} = +\infty, \quad G(x,t) := \int_0^t g(x,s)\,ds,$$

together with other mild assumptions. These results were extended by Cheng and Tang [4]. Using sign-changing Nehari manifolds, the authors of [11] obtained least-energy nodal solutions of (1.2) in the critical-growth case. Liu and Zhao [13] used Morse theory to obtain nontrivial solutions of (1.2), while Yin and Jiang [20] studied indefinite potentials with nonlinearities $g(x,u) = g(u)$ satisfying

- There exists $\mu > 3$ such that $g(t)t \ge \mu \int_0^t g(s)ds > 0$ for all $t \ne 0$.

Motivated by these works, we establish ground state and multiplicity results for the autonomous problem (1.1). To the best of our knowledge, no prior results exist for (1.1) under Berestycki-Lions type conditions [2].

To state our main theorems, we assume that f satisfies the following conditions:

(H4) $f \in C(\mathbb{R})$, $\lim_{t\to 0} \frac{f(t)}{t} = 0$;

(H5) $\lim_{|t|\to\infty} \frac{f(t)}{|t|^{2_*-1}} = 0$, where $2_* := \frac{2N}{N-4}$;

(H6) there exists $\xi > 0$ such that $\widetilde{F}(\xi) := F(\xi) - \frac{1}{2}\xi^2 > 0$, where $F(t) := \int_0^t f(s)\,ds$ for $t \in \mathbb{R}$.

Our main results are

Theorem 1.1. *Assume that* (H4)–(H6) *hold. Then* (1.1) *admits at least one ground-state solution.*

Remark 1.2. Hypotheses (H4)–(H6) are of Berestycki-Lions type and, in particular, do not impose the Ambrosetti-Rabinowitz condition.

Theorem 1.3. *Assume that* (H4)–(H6) *hold and that f is odd. Then* (1.1) *admits infinitely many radially symmetric solutions.*

The proofs of Theorems 1.1 and 1.3 employ variational methods. For Theorem 1.1, the main difficulty in studying (1.1) when f satisfies only the Berestycki-Lions conditions is that the boundedness of Palais-Smale sequences, or even Cerami sequences, is not known a priori. Typically, such boundedness requires the Ambrosetti-Rabinowitz condition

- There exists $\mu > 4$ such that $f(t)t \ge \mu F(t) \ge 0$ for $t \in \mathbb{R}$.

To overcome this difficulty, we adapt Jeanjean's technique [8] by

- defining the scaling map $\varphi : E \to H^2(\mathbb{R}^N)$, $\varphi(s,v)(x) = v(x/e^s)$;

- constructing the augmented functional $\widetilde{I} = I \circ \varphi$;
- generating a Pohožaev-Palais-Smale sequence $\{u_n\} \subset H^2(\mathbb{R}^N)$ at level c (see (3.3)).

This framework yields a bounded Palais-Smale sequence $\{u_n\}$ for I.

For Theorem 1.3, we utilize the comparison functional J from (4.1). A key ingredient is the construction of $\tau : \pi_{k-1} \to H^2_{0,r}(B_R)$ satisfying the following properties:

$$0 \notin \tau(\pi_{k-1}), \quad \int_{\mathbb{R}^N} \widetilde{F}(\tau(\ell))dx \geq 1 \quad \forall \ell \in \pi_{k-1}.$$

This construction yields two essential results:

(i) Both I and J satisfy the symmetric mountain-pass geometry;
(ii) The minimax classes in (4.5)–(4.6) are nonempty, and the associated levels b_n and c_n are well defined. The criticality of c_n is addressed in Proposition 4.11, while that of b_n is proved later by means of the augmented functional.

The subsequent boundedness arguments follow the same strategy as in the proof of Theorem 1.1.

This paper is organized as follows: Section 2 presents preliminaries, Section 3 proves Theorem 1.1, and Section 4 establishes Theorem 1.3 via minimax methods.

2. Preliminaries

Throughout this article, we use the following notation:

- $L^p(\mathbb{R}^N)$: Lebesgue space with norm $\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p}$.
- $H^2(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \Delta u, \nabla u \in L^2(\mathbb{R}^N)\}$ endowed with the norm $\|u\| = \left(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2)dx\right)^{1/2}$.
- $H^2_r(\mathbb{R}^N)$: subspace of radially symmetric functions in $H^2(\mathbb{R}^N)$.
- $H^2_{0,r}(B_R) := H^2_0(B_R) \cap \{u : u \text{ is radial}\}$.
- $D^{2,2}(\mathbb{R}^N)$: completion of $C^\infty_0(\mathbb{R}^N)$ under $\|u\|_{D^{2,2}} = \|\Delta u\|_2$.
- $E = \mathbb{R} \times H^2(\mathbb{R}^N)$ with norm $\|(s,u)\|_E = (|s|^2 + \|u\|^2)^{1/2}$.
- E': dual space of E.
- $u_t(x) = u(x/t)$ for any $t > 0$.
- $C > 0$: generic constant that may vary between estimates.

Since $N = 5$ or 6, the Sobolev embeddings $H^2(\mathbb{R}^N) \hookrightarrow L^6(\mathbb{R}^N)$ and $H^2(\mathbb{R}^N) \hookrightarrow W^{1,3}(\mathbb{R}^N)$ hold. Hence, following the proof of [3, Lemma 2.2], we have the following estimate for all $u \in H^2\left(\mathbb{R}^N\right)$,

$$\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx \leq \|u\|_6^2 \|\nabla u\|_3^2 \leq \|u\|_6^2 \|u\|_{W^{1,3}}^2 \leq C\|u\|^4. \tag{2.1}$$

The energy functional $I : H^2(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + u^2)dx + \frac{1}{2}\int_{\mathbb{R}^N} u^2|\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u)dx$$

is well-defined and of class C^1 under assumptions (H4)–(H6) and (2.1). Its Fréchet derivative is given by

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^N} \left(\Delta u \Delta v + \nabla u \cdot \nabla v + uv + uv|\nabla u|^2 + u^2 \nabla u \cdot \nabla v\right) dx - \int_{\mathbb{R}^N} f(u)v dx$$

for $v \in H^2(\mathbb{R}^N)$.

Lemma 2.1. *Under assumptions* (A4)–(A6) *(A4), I satisfies the mountain-pass geometry:*

(i) *There exist $\rho, b > 0$, with ρ sufficiently small, such that* $\inf_{\|u\|=\rho} I(u) \geq b$,
(ii) *There exists $u_0 \in H^2(\mathbb{R}^N)$ with $\|u_0\| > \rho$ and $I(u_0) < 0$.*

Proof. (i) From (H4)–(H6), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(u)| \leq \varepsilon|u| + C_\varepsilon|u|^{2_*-1}. \tag{2.2}$$

Integrating (2.2) gives $|F(u)| \le \frac{\varepsilon}{2}u^2 + C_\varepsilon |u|^{2_*}$. Using this inequality with $\varepsilon = \frac{1}{2}$ and the Sobolev inequality, there exist $b, \rho > 0$ such that

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{4}\|u\|_2^2 - C\|u\|_{2_*}^{2_*} \\ &\ge \frac{1}{4}\|u\|^2 - C_1\|u\|^{2_*} \\ &= \|u\|^2\Big(\frac{1}{4} - C_1\rho^{2_*-2}\Big) = b > 0 \end{aligned}$$

for $\|u\| = \rho$ sufficiently small.

(ii) Following the cut-off construction in [2], let $R > 1$ be sufficiently large. Define η as

$$\eta(s) = \begin{cases} 1, & s \le 0, \\ 1 - 3s^2 + 2s^3, & 0 < s < 1, \\ 0, & s \ge 1. \end{cases}$$

We set

$$v(x) := \xi\eta(|x| - R). \tag{2.3}$$

Then $v \in H^2(\mathbb{R}^N)$, $v = \xi$ in B_R, $v = 0$ in $\mathbb{R}^N \setminus B_{R+1}$, and $0 \le v \le \xi$. By (f_3) and (2.3), one has

$$\begin{aligned} \int_{\mathbb{R}^N} \widetilde{F}(v)dx &\ge \widetilde{F}(\xi)|B_R| - |B_{R+1} \setminus B_R| \max_{t\in[0,\xi]} |\widetilde{F}(t)| \\ &\ge CR^N - C'R^{N-1} > 0 \end{aligned} \tag{2.4}$$

for R sufficiently large. By (2.4), we have

$$\begin{aligned} \lim_{t\to\infty} \frac{I(v_t)}{t^N} &= \lim_{t\to\infty} \frac{1}{t^N}\Big[\frac{1}{2}\left(t^{N-4}\|\Delta v\|_2^2 + t^{N-2}\|\nabla v\|_2^2 + t^N\|v\|_2^2\right)\Big] \\ &\quad + \lim_{t\to\infty} \frac{1}{t^N}\left[\frac{t^{N-2}}{2}\int v^2|\nabla v|^2dx - t^N\int F(v)dx\right] \\ &= -\int_{\mathbb{R}^N} \widetilde{F}(v)dx < 0. \end{aligned}$$

Thus $I(v_t) \to -\infty$ as $t \to \infty$. Choose t_0 sufficiently large so that

$$I(v_{t_0}) < 0, \quad \|v_{t_0}\| > \rho,$$

and set $u_0 := v_{t_0}$. □

We define

$$Q(u) = \int_{\mathbb{R}^N} u^2|\nabla u|^2\,dx.$$

If $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$, then [13, Lemma 4.1] gives

$$Q(u) \le \liminf_{n\to+\infty} Q(u_n), \quad \langle Q'(u_n), \phi\rangle \to \langle Q'(u), \phi\rangle, \tag{2.5}$$

for all $\phi \in H^2(\mathbb{R}^N)$.

Lemma 2.2. *Let $\{u_n\} \subset H^2(\mathbb{R}^N)$ be a bounded Palais-Smale sequence for I at a positive level $c > 0$. Then the set of critical points $\mathcal{N} = \{u \in H^2(\mathbb{R}^N) \setminus \{0\} : I'(u) = 0\}$ is nonempty.*

Proof. Since $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$, after passing to a subsequence, $u_n \rightharpoonup u$ in $H^2(\mathbb{R}^N)$ and $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for every $p \in [2, 2_*)$. Fix $r > 0$ and suppose that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^2dx = 0. \tag{2.6}$$

Then Lions' vanishing lemma yields

$$u_n \to 0 \quad \text{in } L^p(\mathbb{R}^N) \quad \text{for all } p \in (2, 2_*). \tag{2.7}$$

From (H4)–(H6), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(s)| \le \varepsilon\big(|s| + |s|^{2_*-1}\big) + C_\varepsilon |s|^{q-1}, \quad \text{for } q \in (2, 2_*). \tag{2.8}$$

Using (2.7), the boundedness of $\{u_n\}$ in $L^{2_*}(\mathbb{R}^N)$ and the arbitrariness of ε, we obtain

$$\int_{\mathbb{R}^N} f(u_n)u_n\,dx = o(1), \quad \int_{\mathbb{R}^N} F(u_n)\,dx = o(1).$$

Therefore,

$$I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle = \int_{\mathbb{R}^N} \Big(\frac{1}{2}f(u_n)u_n - F(u_n)\Big)dx - \frac{1}{2}\int_{\mathbb{R}^N} u_n^2|\nabla u_n|^2 dx \le o(1).$$

On the other hand, since $I(u_n) \to c > 0$ and $I'(u_n) \to 0$ while $\{u_n\}$ is bounded,

$$I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \to c > 0,$$

which is a contradiction. Hence (2.6) is false. Thus there exist $\eta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_r(y_n)} |u_n|^2 dx \ge \eta > 0.$$

Set $\widetilde{u}_n(x) = u_n(x + y_n)$. Then $\{\widetilde{u}_n\}$ is still a bounded Palais-Smale sequence at level c, and

$$\int_{B_r(0)} |\widetilde{u}_n|^2 dx \ge \eta > 0.$$

Passing to a subsequence, $\widetilde{u}_n \rightharpoonup u_0$ in $H^2(\mathbb{R}^N)$ and, by the compact embedding on bounded balls, $\widetilde{u}_n \to u_0$ in $L^2(B_r(0))$. Hence $u_0 \ne 0$. Moreover, using (2.5), $\widetilde{u}_n \to u_0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for every $p \in [2, 2_*)$, and (2.8), we may pass to the limit in $\langle I'(\widetilde{u}_n), \phi\rangle$ for every $\phi \in C_c^\infty(\mathbb{R}^N)$; density then gives $I'(u_0) = 0$. Consequently $\mathcal{N} \ne \emptyset$. □

3. Proof of Theorem 1.1

We need the following Pohožaev identity.

Proposition 3.1 (Pohožaev identity)**.** *Let $u \in H^2(\mathbb{R}^N)$ be a weak solution of* (1.1) *under assumptions* (A4) *and* (H5)*. Then $u \in H^3_{\mathrm{loc}}(\mathbb{R}^N)$ and*

$$P(u) := \int_{\mathbb{R}^N} \Big[\frac{N-4}{2}|\Delta u|^2 + \frac{N-2}{2}\left(|\nabla u|^2 + u^2|\nabla u|^2\right) + \frac{N}{2}u^2 - NF(u)\Big]dx = 0.$$

Proof. From [6, 10], we know $u \in H^3_{\mathrm{loc}}$. Here, for the reader's convenience, we simply give the proof of $u \in H^3_{\mathrm{loc}}$. We set

$$h(t) = \begin{cases} f(t)/t, & t \ne 0, \\ 0, & t = 0\,. \end{cases}$$

By (H4)–(H6), we have

$$|h(t)| \le C\big(1 + |t|^{8/(N-4)}\big).$$

For the fixed weak solution u, define the measurable functions

$$a_u = u\Delta u + |\nabla u|^2 - 1 + h(u), \quad b_u = \Delta u.$$

In the distributional sense, one has $\Delta^2 u = a_u u + b_u$. Using that $H^2(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ with $p \in [2, 2_*]$, we obtain

$$u \in L^{2N/(N-4)}_{\mathrm{loc}}, \quad \nabla u \in L^{2N/(N-2)}_{\mathrm{loc}}, \quad \Delta u \in L^2_{\mathrm{loc}}\,.$$

In particular, $u\Delta u$ and $|\nabla u|^2$ belong to $L^{N/(N-2)}_{\mathrm{loc}}$, while $h(u) \in L^{N/4}_{\mathrm{loc}}$. Since $N/(N-2) \ge N/4$ and $2 \ge N/4$ for $N = 5, 6$, we have $a_u, b_u \in L^{N/4}_{\mathrm{loc}}$. Thus the Carathéodory function $\mathfrak{g}(x,t) = a_u(x)t + b_u(x)$ satisfies

$$|\mathfrak{g}(x,t)| \le (|a_u(x)| + |b_u(x)|)(1 + |t|), \quad |a_u| + |b_u| \in L^{N/4}_{\mathrm{loc}}.$$

By local Brezis-Kato theorem [16, Theorem 1.1], $u \in L^q_{\mathrm{loc}}$ for every finite $q \ge 1$.

To obtain the required derivative regularity, we claim $\nabla u \in L^{48/13}_{\mathrm{loc}}$. Indeed, by $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^{24}_{\mathrm{loc}}(\mathbb{R}^N)$ and the Gagliardo-Nirenberg inequality, we obtain

$$\|\nabla u\|_{L^{48/13}_{\mathrm{loc}}} \le C\|D^2 u\|^{1/2}_{L^2_{\mathrm{loc}}} \|u\|^{1/2}_{L^{24}_{\mathrm{loc}}}.$$

Consequently,

$$u^2\Delta u,\ u|\nabla u|^2 \in L^{12/7}_{\mathrm{loc}}, \quad f(u) \in L^{12/7}_{\mathrm{loc}},$$

where the last assertion follows from (2.2) and $u \in L^q_{\mathrm{loc}}$ with $q \ge 1$. The equation and the interior L^p estimates for the biharmonic operator now imply

$$u \in W^{4,12/7}_{\mathrm{loc}}(\mathbb{R}^N) \hookrightarrow H^3_{\mathrm{loc}}(\mathbb{R}^N), \quad N = 5, 6.$$

The identity then follows directly from the Pohožaev variational principle applied to the functional I, as shown in [20, Lemma 3.2]. □

We define the mountain-pass level

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], H^2\left(\mathbb{R}^N\right)\right) : \gamma(0) = 0, \quad I(\gamma(1)) < 0\right\}.$$

By Lemma 2.1, $\Gamma \neq \emptyset$ and $c > 0$.

Proof of Theorem 1.1. Following Jeanjean [8], we introduce the mapping $\varphi : E \to H^2(\mathbb{R}^N)$ by

$$\varphi(s, v)(x) = v(x/e^s),$$

where $(s, v) \in E$. The composite functional $I \circ \varphi$ is given by

$$\begin{aligned}(I\circ\varphi)(s,v) = \frac{1}{2}\Big[e^{(N-4)s}\int_{\mathbb{R}^N} |\Delta v|^2 dx + e^{(N-2)s}\int_{\mathbb{R}^N} |\nabla v|^2 dx + e^{Ns}\int_{\mathbb{R}^N} v^2 dx\Big] \\ + \frac{e^{(N-2)s}}{2}\int_{\mathbb{R}^N} v^2|\nabla v|^2 dx - e^{Ns}\int_{\mathbb{R}^N} F(v)dx := \widetilde{I}(s,v).\end{aligned} \tag{3.1}$$

By (3.1), (A4), and (A5), the functional $I \circ \varphi$ is continuously Fréchet-differentiable on E. From Lemma 2.1, choose $v_0 \in H^2(\mathbb{R}^N)$ and $t_0 > 0$ such that $I((v_0)_{t_0}) < 0$; equivalently, with $s^* = \log t_0$, $(I \circ \varphi)(s^*, v_0) < 0$. Consequently,

$$\widetilde{\Gamma} := \{\widetilde{\gamma} \in C([0,1]; E) : \widetilde{\gamma}(0) = (0,0),\ (I\circ\varphi)(\widetilde{\gamma}(1)) < 0\} \neq \emptyset.$$

Thus, we can define the minimax level of $I \circ \varphi$,

$$\widetilde{c} = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}} \sup_{t\in[0,1]} (I\circ\varphi)(\widetilde{\gamma}(t)).$$

As $\Gamma = \{\varphi \circ \widetilde{\gamma} : \widetilde{\gamma} \in \widetilde{\Gamma}\}$, we have $c \le \widetilde{c}$. For every $\gamma \in \Gamma$, the path $t \mapsto (0, \gamma(t))$ belongs to $\widetilde{\Gamma}$, and then $\widetilde{c} \le c$. Hence $c = \widetilde{c}$. By the definition of c, for any $n \in \mathbb{N}$ there exists $\gamma_n \in \Gamma$, and we set $\widetilde{\gamma}_n(t) = (0, \gamma_n(t))$ such that

$$\max_{t\in[0,1]} (I\circ\varphi)(\widetilde{\gamma}_n(t)) = \max_{t\in[0,1]} I(\gamma_n(t)) \le c + \frac{1}{n}.$$

By the minimax principle [18, Theorem 2.8], there exists a sequence $\{(s_n, v_n)\}$ in E such that

(a) $(I \circ \varphi)(s_n, v_n) \to c$,

(b) $\|(I \circ \varphi)'(s_n, v_n)\|_{E'} \to 0$,

(c) $\min_{t\in[0,1]} \|(s_n, v_n) - \widetilde{\gamma}_n(t)\|_E \to 0$.

Condition (c) implies that $s_n \to 0$. Setting $u_n = \varphi(s_n, v_n)$, condition (a) yields $I(u_n) \to c$. Differentiating the explicit formula (3.1), we obtain, for any $(\mu, \omega) \in E$,

$$(I \circ \varphi)'(s_n, v_n)[\mu, \omega] = I'(\varphi(s_n, v_n))[\varphi(s_n, \omega)] + P(\varphi(s_n, v_n))\mu \to 0. \tag{3.2}$$

Taking $(\mu, \omega) = (1, 0)$ gives $P(u_n) \to 0$. To prove $I'(u_n) \to 0$, let $\psi \in H^2(\mathbb{R}^N)$ and choose $\omega_n(x) = \psi(e^{s_n}x)$; then $\varphi(s_n, \omega_n) = \psi$. Since $s_n \to 0$, the norms of the scaling operators $\psi \mapsto \omega_n$ are uniformly bounded. Hence (3.2) with $(\mu, \omega) = (0, \omega_n)$ implies $I'(u_n) \to 0$ in $(H^2)'$. Therefore

$$I(u_n) \to c, \quad I'(u_n) \to 0, \quad P(u_n) \to 0. \tag{3.3}$$

Note that

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{N}P(u_n) \\ &= \frac{2}{N}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + u_n^2|\nabla u_n|^2\right) dx \\ &\geq C\int_{\mathbb{R}^N} \left(|\Delta u_n|^2 + |\nabla u_n|^2\right) dx. \end{aligned}$$

Then

$$\int_{\mathbb{R}^N} |\Delta u_n|^2 dx, \quad \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \tag{3.4}$$

are bounded. Recall the Sobolev constant

$$S := \inf_{u \in D^{2,2}(\mathbb{R}^N)\, \|u\|_{2_*}=1} \int_{\mathbb{R}^N} |\Delta u|^2 dx.$$

This implies that $\|u_n\|_{2_*}$ is bounded by (3.4). By (2.2), $P(u_n) \to 0$ in (3.3) and the boundedness of $\|u_n\|_{2_*}$, we obtain

$$\int_{\mathbb{R}^N} \Big(\frac{N}{4}u_n^2 + CN|u_n|^{2_*}\Big) dx + o(1) \geq N\int_{\mathbb{R}^N} F(u_n) dx + o(1) \geq \frac{N}{2}\int_{\mathbb{R}^N} u_n^2 dx. \tag{3.5}$$

Hence $\|u_n\|_2$ is bounded. Therefore, $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. By Lemma 2.2, we obtain $\mathcal{N} \neq \emptyset$.

Finally, we establish the existence of a ground state solution of (1.1). Let

$$m_{gs} := \inf_{u \in \mathcal{N}} I(u).$$

For every $u \in \mathcal{N}$, Proposition 3.1 gives

$$I(u) = I(u) - \frac{1}{N}P(u) = \frac{2}{N}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2|\nabla u|^2\right) dx > 0,$$

and hence $0 \leq m_{gs} < \infty$. To complete the proof, it remains to show that m_{gs} is attained on $\mathcal{N}$. Let $\{u_n\} \subset \mathcal{N}$ be such that $I(u_n) \to m_{gs}$. Since $I'(u_n) = 0$, Proposition 3.1 gives $P(u_n) = 0$. The same estimates used above show that $\{u_n\}$ is bounded in $H^2(\mathbb{R}^N)$. Using $\langle I'(u_n), u_n\rangle = 0$, (2.2), and the Sobolev inequality, we obtain

$$\|u_n\|^2 \leq \int_{\mathbb{R}^N} f(u_n)u_n dx \leq \frac{1}{2}\|u_n\|^2 + C\|u_n\|^{2_*}. \tag{3.6}$$

From (3.6), there exists a constant $\alpha > 0$ such that

$$\liminf_{n\to\infty} \|u_n\| \geq \alpha > 0. \tag{3.7}$$

If $\{u_n\}$ vanishes, i.e., if for some $r > 0$,

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^2 dx = 0,$$

then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for every $t \in (2, 2_*)$ by Lions' lemma. Using (2.8), the boundedness of $\{u_n\}$ in $L^{2_*}(\mathbb{R}^N)$ and the arbitrariness of ε, we obtain

$$\int_{\mathbb{R}^N} f(u_n)u_n \, dx \to 0.$$

However, since $I'(u_n) = 0$,

$$\|u_n\|^2 + 2\int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 dx = \int_{\mathbb{R}^N} f(u_n)u_n \, dx \to 0,$$

which contradicts (3.7). As in Lemma 2.2, after suitable translations there exists $u \neq 0$ such that $I'(u) = 0$. Thus $u \in \mathcal{N}$, and the definition of m_{gs} gives $I(u) \geq m_{gs}$. It follows from $I'(u) = 0$, $I'(u_n) = 0$, weak lower semicontinuity of the norm and (2.5) that

$$\begin{aligned} m_{gs} \leq I(u) &= I(u) - \frac{1}{N}P(u) \\ &= \frac{2}{N}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2|\nabla u|^2\right) dx \\ &\leq \liminf_{n\to\infty} \Big\{ \frac{2}{N}\int_{\mathbb{R}^N} |\Delta u_n|^2 dx + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + u_n^2|\nabla u_n|^2\right) dx \Big\} \\ &\leq \liminf_{n\to\infty} \Big(I(u_n) - \frac{1}{N}P(u_n) \Big) = m_{gs}. \end{aligned}$$

Thus, $I(u) = m_{gs}$, which proves that u is a ground state solution. □

4. Proof of Theorem 1.3

To construct the required minimax levels, we use the augmented-dimension trick introduced in [7]. Choose an exponent $p_0 \in (3, 2_* - 1)$. For $s > 0$, define

$$m(s) = \sup_{0<\tau\leq s} \frac{\hat{f}(\tau)}{\tau^{p_0}},$$

where

$$\hat{f}(s) = \begin{cases} \max\{f(s) - \frac{s}{2}, 0\}, & \text{for } s \geq 0, \\ -\hat{f}(-s), & \text{for } s < 0. \end{cases}$$

We define

$$g(s) = \begin{cases} s^{p_0} m(s), & \text{for } s > 0, \\ 0, & \text{for } s = 0, \\ -g(-s), & \text{for } s < 0, \end{cases}$$

and set $G(s) = \int_0^s g(t)\, dt$.

We introduce the comparison functional $J : H_r^2(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + u^2|\nabla u|^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} G(u)\, dx, \tag{4.1}$$

for $u \in H_r^2(\mathbb{R}^N)$. Since $g(s) \geq (f(s) - s/2)^+$ for $s \geq 0$ and f is odd, we have

$$G(s) \geq F(s) - \frac{s^2}{4} \quad (s \in \mathbb{R}).$$

Consequently,

$$I(u) - J(u) = \int_{\mathbb{R}^N} \Big(G(u) - F(u) + \frac{1}{4}u^2 \Big) dx \geq 0,$$

that is, $I(u) \geq J(u)$ for all $u \in H_r^2(\mathbb{R}^N)$. Following the reasoning in [7, Lemma 2.1, Corollary 2.2, Lemma 2.4], we obtain the following lemma.

Lemma 4.1 ([7]). (i) $0 \leq (p_0 + 1)G(s) \leq sg(s)$,

(ii) $\lim_{|s|\to+\infty} \frac{g(s)}{|s|^{2_*-1}} = 0$,

(iii) *There exists* $\delta_0 > 0$ *such that* $g(s) = 0$ *for* $|s| \leq \delta_0$,

(iv) $J \in C^1(H_r^2(\mathbb{R}^N))$, *and, for all* $u, \varphi \in H_r^2(\mathbb{R}^N)$,

$$\langle J'(u), \varphi\rangle = \int_{\mathbb{R}^N} \Big(\Delta u \Delta\varphi + \nabla u \cdot \nabla\varphi + \frac{1}{2}u\varphi + u\varphi|\nabla u|^2 + u^2 \nabla u \cdot \nabla\varphi \Big) dx - \int_{\mathbb{R}^N} g(u)\varphi dx.$$

Definition 4.2. For each integer $k \geq 1$, we define the $(k-1)$-dimensional polyhedron

$$\pi_{k-1} = \left\{ \ell = (\ell_1, \dots, \ell_k) \in \mathbb{R}^k : \sum_{j=1}^{k} |\ell_j| = 1 \right\}$$

which is the ℓ^1-unit sphere in $\mathbb{R}^k$.

Remark 4.3. The polyhedron π_{k-1} possesses exactly $2k$ vertices located at $\pm \mathbf{e}_i$ (the standard basis vectors) and 2^k facets. Since π_{k-1} is homeomorphic to S^{k-1} by an odd homeomorphism (take for instance the radial projection $\pi_{k-1} \to S^{k-1}$), it is clear that $\operatorname{genus}(\pi_{k-1}) = k$, where $\operatorname{genus}(Y)$ is the Krasnosel'skii genus of Y, and

$$\operatorname{genus}(Y) := \min \left\{ n \in \mathbb{N}^+ \mid \exists \varphi : Y \to \mathbb{R}^n \setminus \{0\}, \varphi \text{ is odd and continuous} \right\}.$$

We set $\operatorname{genus}(Y) = \infty$ if such a map φ does not exist, and set $\operatorname{genus}(Y) = 0$ if $Y = \emptyset$.

Lemma 4.4. *There exists a function $\phi \in C^2(\mathbb{R})$ such that*

$$\phi(t) = \begin{cases} 0, & t \leq 0, \\ 10t^3 - 15t^4 + 6t^5, & 0 < t < 1, \\ 1, & t \geq 1. \end{cases}$$

Moreover,

$$0 \leq \phi \leq 1, \quad \phi(0) = 0, \quad \phi(1) = 1,$$
$$\phi'(0) = \phi'(1) = 0, \quad \phi''(0) = \phi''(1) = 0,$$

and there exists an constant $C_\phi > 0$ such that

$$\|\phi'\|_{L^\infty(\mathbb{R})} + \|\phi''\|_{L^\infty(\mathbb{R})} \leq C_\phi.$$

Proof. For $0 < t < 1$, we set

$$q(t) = 10t^3 - 15t^4 + 6t^5.$$

Then

$$q'(t) = 30t^2(1-t)^2, \quad q''(t) = 60t(1-t)(1-2t).$$

Thus $q(0) = 0$, $q(1) = 1$, and

$$q'(0) = q'(1) = q''(0) = q''(1) = 0.$$

Consequently, the extension defined above by constants on $(-\infty, 0]$ and $[1, \infty)$ belongs to $C^2(\mathbb{R})$. The stated bounds follow from the explicit formulas for q' and q''. □

Proposition 4.5. *Let $k \geq 1$. Assume that f is odd and (f_3) holds. Then there exists $R = R(k) > 0$ and an odd continuous mapping*

$$\tau : \pi_{k-1} \to H^2_{0,r}(B_R),$$

such that, for every $\ell \in \pi_{k-1}$, the function $u = \tau(\ell)$ satisfies

(i) $0 \notin \tau(\pi_{k-1})$;

(ii) *there exist constants $\varrho_k, C_k > 0$, depending on k but independent of ℓ, such that*

$$\varrho_k \leq \|\Delta u\|_{L^2(B_R)} \leq C_k;$$

(iii) $\int_{B_R} \widetilde{F}(u)dx \geq 1$.

Proof. Since f is odd, F and $\widetilde{F}$ are even. Hence

$$a := \widetilde{F}(\xi) = \widetilde{F}(-\xi) > 0.$$

We set $M := \max_{|s| \leq \xi} |\widetilde{F}(s)|$.

Step 1. A fixed-width smoothing procedure. Fix $h = 1$. Let $\varrho \in C_c^\infty((-1,1))$ be even, nonnegative, and satisfy

$$\int_{\mathbb{R}} \varrho(t)dt = 1,$$

and set $\varrho_h(t) = h^{-1}\varrho(t/h)$. We shall choose $R > 4h$ sufficiently large later. Define the boundary cut-off function

$$\chi_R(r) := \phi\Big(\frac{r-h}{h}\Big)\phi\Big(\frac{R-h-r}{h}\Big), \quad 0 \le r \le R.$$

Then $\chi_R \in C^2([0,R])$, $0 \le \chi_R \le 1$,

$$\chi_R(r) = 0 \quad \text{for } 0 \le r \le h \text{ and } R-h \le r \le R,$$

while $\chi_R(r) = 1$ for $2h \le r \le R-2h$. Moreover, $\chi_R'(0) = \chi_R'(R) = 0$, and

$$\|\chi_R\|_{C^2([0,R])} \le C_h,$$

where C_h is independent of R.

For $\ell = (\ell_1, \dots, \ell_k) \in \pi_{k-1}$, we define

$$\alpha_i(\ell) := R\sum_{j=1}^{i} |\ell_j|, \quad i = 0, 1, \dots, k,$$

with $\alpha_0 = 0$ and $\alpha_k = R$. Define a step function s_ℓ on $[0,R]$ by

$$s_\ell(r) = \operatorname{sgn}(\ell_i) \quad \text{for } r \in (\alpha_{i-1}(\ell), \alpha_i(\ell)) \text{ and } \ell_i \ne 0.$$

Intervals corresponding to $\ell_i = 0$ have length zero and are irrelevant. We extend s_ℓ by zero outside $[0,R]$ and set

$$v_\ell := \varrho_h * s_\ell \quad \text{on } \mathbb{R}.$$

Finally, define the radial function

$$\tau(\ell)(x) := \xi\, \chi_R(|x|) v_\ell(|x|), \quad x \in B_R.$$

Since χ_R vanishes in neighborhoods of both endpoints 0 and R, the function $x \mapsto \tau(\ell)(x)$ is C^2 in $\overline{B_R}$, vanishes near ∂B_R, and is radial. Therefore $\tau(\ell) \in H^2_{0,r}(B_R)$. The construction is odd because $s_{-\ell} = -s_\ell$, hence

$$\tau(-\ell) = -\tau(\ell).$$

Step 2. Continuity of τ. We claim that $\ell \mapsto s_\ell$ is continuous from π_{k-1} into $L^1(\mathbb{R})$. Indeed, if $\ell^m \to \ell$ in π_{k-1}, then the endpoints $\alpha_i(\ell^m)$ converge to $\alpha_i(\ell)$ for every i. For every index with $\ell_i \ne 0$, the signs of ℓ_i^m and ℓ_i agree for all sufficiently large m; the corresponding intervals differ only through their endpoints. Indices with $\ell_i = 0$ contribute intervals of length $R|\ell_i^m|$. Consequently,

$$\|s_{\ell^m} - s_\ell\|_{L^1(\mathbb{R})} \le 2\sum_{i=1}^{k-1} |\alpha_i(\ell^m) - \alpha_i(\ell)| + 2R \sum_{\{i:\ell_i=0\}} |\ell_i^m| \to 0.$$

Since convolution with the fixed smooth kernel ϱ_h maps $L^1(\mathbb{R})$ continuously into $C^2_{\mathrm{loc}}(\mathbb{R})$, we have

$$\|v_{\ell^m} - v_\ell\|_{C^2([0,R])} \le C_h \|s_{\ell^m} - s_\ell\|_{L^1(\mathbb{R})} \to 0.$$

Multiplication by the fixed cut-off χ_R then yields

$$\tau(\ell^m) \to \tau(\ell) \quad \text{in } H^2(B_R).$$

Thus $\tau : \pi_{k-1} \to H^2_{0,r}(B_R)$ is continuous.

Step 3. The nonlinear lower bound. Let

$$\mathcal{I}_\ell := \{\alpha_i(\ell) : i = 1, \cdots, k-1\}$$

be the set of interior interfaces. Define the exceptional radial set

$$D_\ell := \{r \in [0,R] : \operatorname{dist}(r, \{0,R\} \cup \mathcal{I}_\ell) \le 2h\}.$$

Outside D_ℓ, the support of the mollifier does not intersect any interface or endpoint and $\chi_R = 1$. Therefore

$$\tau(\ell)(x) = \pm\xi \quad \text{whenever } |x| \notin D_\ell.$$

The set D_ℓ is the union of at most $k+1$ intervals of length at most $4h$. Consequently, for $R \geq 1$,

$$|\{x \in B_R : |x| \in D_\ell\}| \leq C_N(k+1)hR^{N-1},$$

where $C_N > 0$ depends only on N. Since $|\tau(\ell)| \leq \xi$, we obtain

$$\int_{B_R} \widetilde{F}(\tau(\ell))dx \geq a\,|B_R| - (a+M)C_N(k+1)hR^{N-1}.$$

Choosing $R = R(k) > 4h$ sufficiently large gives

$$\int_{B_R} \widetilde{F}(\tau(\ell))dx \geq 1 \quad \text{for all } \ell \in \pi_{k-1}.$$

This proves item (iii) and also implies $0 \notin \tau(\pi_{k-1})$.

Step 4. Uniform upper and lower bounds for $\|\Delta\tau(\ell)\|_2$. The functions s_ℓ satisfy $|s_\ell| \leq 1$. Hence v_ℓ and its first two derivatives are uniformly bounded in terms of h and ϱ only. Together with the uniform C^2 bound for χ_R, this yields

$$\|\tau(\ell)\|_{C^2([0,R])} \leq C(\xi, h, \varrho) \quad \text{for all } \ell \in \pi_{k-1}.$$

Moreover, $\tau(\ell)$ vanishes on $[0,h]$. Therefore, for $r \geq h$,

$$|\Delta\tau(\ell)(r)| \leq |\tau(\ell)''(r)| + \frac{N-1}{r}|\tau(\ell)'(r)| \leq C(\xi, h, \varrho, N).$$

It follows that

$$\|\Delta\tau(\ell)\|_{L^2(B_R)} \leq C_k$$

for a constant $C_k > 0$ independent of ℓ; here C_k may depend on $R(k)$.

Finally, since π_{k-1} is compact and τ is continuous, the set $\tau(\pi_{k-1})$ is compact in $H^2_{0,r}(B_R)$. By item (iii), it does not contain the zero function. Since $\|\Delta\cdot\|_{L^2(B_R)}$ is an equivalent norm on $H^2_{0,r}(B_R)$, the continuous function

$$\ell \mapsto \|\Delta\tau(\ell)\|_{L^2(B_R)}$$

has a strictly positive minimum on π_{k-1}. Thus there exists $\varrho_k > 0$ such that

$$\varrho_k \leq \|\Delta\tau(\ell)\|_{L^2(B_R)} \leq C_k \quad \text{for all } \ell \in \pi_{k-1}.$$

The proof is complete. □

The functionals I and J satisfy the following geometric properties.

Proposition 4.6. *We have the following properties:*

(1) *There exist $\rho, \mu > 0$ such that*

$$I(u) \geq J(u) \geq 0 \quad \textit{for all } \|u\| \leq \rho,$$
$$I(u) \geq J(u) \geq \mu \quad \textit{for all } \|u\| = \rho;$$

(2) *For any $n \in \mathbb{N}^+$, there is an odd continuous mapping*

$$\gamma_{0n} : S^{n-1} \to H^2_r(\mathbb{R}^N)$$

where $S^{n-1} = \{\sigma \in \mathbb{R}^n \mid |\sigma| = 1\}$, such that

$$J(\gamma_{0n}(\sigma)) \leq I(\gamma_{0n}(\sigma)) < 0, \quad \textit{for all } \sigma \in S^{n-1}. \tag{4.2}$$

Proof. As in the proof of Lemma 2.1, by *(ii)* and *(iii)* of Lemma 4.1 there exist $\rho, \mu > 0$ such that

$$J(u) \geq \mu > 0 \quad \text{if } \|u\| = \rho, \quad J(u) \geq 0 \quad \text{if } \|u\| \leq \rho.$$

Together with $I \geq J$, this proves item (1).

We now prove item (2). By Proposition 4.5, applied with $k = n$, and by a fixed odd homeomorphism $h_n : S^{n-1} \to \pi_{n-1}$, there exists an odd continuous map

$$\zeta_n : S^{n-1} \to H^2_{0,r}(B_R) \subset H^2_r(\mathbb{R}^N)$$

such that $0 \notin \zeta_n(S^{n-1})$ and

$$\int_{\mathbb{R}^N} \widetilde{F}(\zeta_n(\sigma))dx \geq 1 \quad \text{for all } \sigma \in S^{n-1}.$$

For $\theta \geq 1$, define

$$\gamma_{0n}(\sigma)(x) = \zeta_n(\sigma)(x/\theta), \quad \sigma \in S^{n-1}.$$

Then γ_{0n} is odd and continuous. Since

$$\begin{aligned} I(\gamma_{0n}(\sigma)) &= \frac{1}{2}\theta^{N-4}\int_{\mathbb{R}^N} |\Delta\zeta_n(\sigma)|^2 dx + \frac{1}{2}\theta^{N-2}\int_{\mathbb{R}^N} \left(|\nabla\zeta_n(\sigma)|^2 + \zeta_n(\sigma)^2|\nabla\zeta_n(\sigma)|^2\right) dx \\ &\quad - \theta^N \int_{\mathbb{R}^N} \widetilde{F}(\zeta_n(\sigma))dx, \end{aligned}$$

and the positive terms above are uniformly bounded in σ, choosing $\theta = \theta_n$ sufficiently large gives

$$I(\gamma_{0n}(\sigma)) < 0 \quad \text{for all } \sigma \in S^{n-1}.$$

Because $J \leq I$, we also have $J(\gamma_{0n}(\sigma)) \leq I(\gamma_{0n}(\sigma)) < 0$. This proves item (2). □

Lemma 4.7. *The functional $J|_{H_r^2}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_m\} \subset H_r^2(\mathbb{R}^N)$ be a Palais-Smale sequence for J at level c, that is

$$J(u_m) \to c \quad \text{and} \quad \|J|'_{H_r^2}(u_m)\|_{(H_r^2)'} \to 0.$$

Put

$$A_m = \int_{\mathbb{R}^N} (|\Delta u_m|^2 + |\nabla u_m|^2)dx, \quad U_m = \int_{\mathbb{R}^N} u_m^2 dx.$$

Using Lemma 4.1 (i) and $p_0 \in (3, 2_* - 1)$, we obtain

$$\begin{aligned} c + o(1)(1 + \|u_m\|) &\geq J(u_m) - \frac{1}{p_0+1}\langle J'(u_m), u_m\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{p_0+1}\Big)A_m + \left(\frac{1}{4} - \frac{1}{2(p_0+1)}\right)U_m + \Big(\frac{1}{2} - \frac{2}{p_0+1}\Big)Q(u_m) \\ &\quad + \int_{\mathbb{R}^N} \Big(\frac{1}{p_0+1}g(u_m)u_m - G(u_m)\Big)dx \\ &\geq C\big(A_m + U_m + Q(u_m)\big). \end{aligned}$$

Hence $\{u_m\}$ is bounded in $H_r^2(\mathbb{R}^N)$. Up to a subsequence,

$$u_m \rightharpoonup u \quad \text{in } H_r^2(\mathbb{R}^N), \quad u_m \to u \quad \text{in } L^q(\mathbb{R}^N),\ 2 < q < 2_*,$$

and $u_m(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Since g satisfies (ii) and (iii) of Lemma 4.1, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|g(s)| \leq \varepsilon\big(|s|^{2_*-1} + |s|\big) + C_\varepsilon|s|^p \quad \text{for } p \in (1, 2_* - 1). \tag{4.3}$$

By (4.3), Hölder's inequality, and $u_m \to u$ in L^q with $q \in (2, 2_*)$, we obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} g(u_m)(u_m - u)dx\Big| &\leq \varepsilon\|u_m\|_2\|u_m - u\|_2 + \varepsilon\|u_m\|_{2_*}^{2_*-1}\|u_m - u\|_{2_*} + C_\varepsilon\|u_m\|_{p+1}^p\|u_m - u\|_{p+1} \\ &\leq C\varepsilon + o(1). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^N} g(u_m)(u_m - u)\,dx = o(1).$$

Since $u_m \rightharpoonup u$ in $H_r^2(\mathbb{R}^N)$, one can show in a standard way that

$$\int_{\mathbb{R}^N} (g(u_m) - g(u))u\,dx \to 0.$$

Consequently,

$$\int_{\mathbb{R}^N} g(u_m)u_m\,dx \to \int_{\mathbb{R}^N} g(u)u\,dx. \tag{4.4}$$

Passing to the limit in $\langle J|'_{H^2_r}(u_m), \phi\rangle$ for every radial $\phi \in C_c^\infty(\mathbb{R}^N)$ and then using density gives $J|'_{H^2_r}(u) = 0$. Hence, by (4.4) and $J|'_{H^2_r}(u) = 0$ we have

$$\begin{aligned} A + \frac{1}{2}U + 2Q(u) &= \int_{\mathbb{R}^N} g(u)u\,dx \\ &= \lim_{m\to\infty} \Big(A_m + \frac{1}{2}U_m + 2Q(u_m)\Big), \end{aligned}$$

where A and U are the corresponding quantities for u. Since the quadratic part and Q are weakly lower semicontinuous by (2.5), equality of the sums forces

$$A_m + \frac{1}{2}U_m \to A + \frac{1}{2}U, \quad Q(u_m) \to Q(u).$$

The weak convergence and convergence of the quadratic norm imply $u_m \to u$ strongly in $H^2_r(\mathbb{R}^N)$. Thus $J|_{H^2_r}$ satisfies the Palais-Smale condition. □

By Proposition 4.6, both functionals I and J exhibit the symmetric mountain-pass geometry, allowing us to define corresponding minimax values. Following the framework of [17, Chapter 9], we define for each $n \in \mathbb{N}^+$ the minimax values

$$b_n = \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} I(\gamma(\sigma)), \quad c_n = \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} J(\gamma(\sigma)), \tag{4.5}$$

where

- $D_n = \{\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n : |\sigma| \le 1\}$.
- Γ_n denotes the class of admissible mappings defined by

$$\begin{aligned} \Gamma_n = \Big\{ &\gamma \in C(D_n, H^2_r(\mathbb{R}^N)) : \gamma(-\sigma) = -\gamma(\sigma) \text{ for all } \sigma \in D_n, \\ &\gamma(\sigma) = \gamma_{0n}(\sigma) \text{ for all } \sigma \in \partial D_n \Big\}. \end{aligned} \tag{4.6}$$

where $\gamma_{0n} : \partial D_n = S^{n-1} \to H^2_r(\mathbb{R}^N)$ is the mapping constructed in Proposition 4.6.

We note that Γ_n is nonempty for all n, as it contains the explicit mapping:

$$\gamma(\sigma) = \begin{cases} |\sigma|\gamma_{0n}\big(\frac{\sigma}{|\sigma|}\big) & \text{for } \sigma \in D_n \setminus \{0\}, \\ 0 & \text{for } \sigma = 0, \end{cases}$$

which satisfies all required conditions.

Next, for each $j \in \mathbb{N}^+$, define the minimax families Λ_j by

$$\Lambda_j = \{h(\overline{D_m \setminus Y}) : h \in \Gamma_m,\, m \ge j,\, Y \in \mathcal{E}_m \text{ and } \operatorname{genus}(Y) \le m - j\},$$

where

$$\mathcal{E}_m := \{Y \subset \mathbb{R}^m \setminus \{0\} : Y \text{ is closed and symmetric}\}.$$

We also write $\mathcal{E}(H^2_r)$ for the family of closed symmetric subsets of $H^2_r(\mathbb{R}^N)\setminus\{0\}$. This construction is adapted from [17, Chapter 9, p. 56]. The next proposition records the basic properties of Λ_j.

Proposition 4.8. *The sets Λ_j possess the following properties:*

(i) *$\Lambda_j \neq \emptyset$ for all $j \in \mathbb{N}^+$;*
(ii) *(Monotonicity) $\Lambda_{j+1} \subset \Lambda_j$;*
(iii) *(Invariance) If $\varphi \in C(H^2_r, H^2_r)$ is odd and $\varphi = \mathrm{id}$ on $\gamma_{0m}(\partial D_m)$ for every $m \ge j$, then $\varphi(B) \in \Lambda_j$ for every $B \in \Lambda_j$;*
(iv) *(Excision) If $B \in \Lambda_j$ and $Z \in \mathcal{E}(H^2_r)$ satisfies $\operatorname{genus}(Z) \le i < j$, then $\overline{B\setminus Z} \in \Lambda_{j-i}$.*

Proof. (i) For every m, the cone extension

$$h_m(\sigma) = \begin{cases} |\sigma|\gamma_{0m}(\sigma/|\sigma|), & \sigma \neq 0, \\ 0, & \sigma = 0, \end{cases}$$

belongs to Γ_m. Taking $m = j$ and $Y = \emptyset$ shows that $\Lambda_j \neq \emptyset$.

(ii) If $B = h(\overline{D_m \setminus Y}) \in \Lambda_{j+1}$, by the definition of Λ_j we have

$$m \geq j+1 \geq j, \quad h \in \Gamma_m, \quad Y \in \mathcal{E}_m$$

and $\operatorname{genus}(Y) \leq m-(j+1) \leq m-j$. Therefore $B \in \Lambda_j$, that is, $\Lambda_{j+1} \subset \Lambda_j$.

(iii) If $B = h(\overline{D_m \setminus Y}) \in \Lambda_j$, then $\varphi \circ h$ is odd and continuous. On ∂D_m,

$$(\varphi \circ h)(\sigma) = \varphi(\gamma_{0m}(\sigma)) = \gamma_{0m}(\sigma),$$

so $\varphi \circ h \in \Gamma_m$ and $\varphi(B) = (\varphi \circ h)(\overline{D_m \setminus Y}) \in \Lambda_j$.

(iv) Let $B = h(\overline{D_m \setminus Y}) \in \Lambda_j$ and $Z \in \mathcal{E}(H_r^2)$ with $\operatorname{genus}(Z) \leq i < j$. We claim

$$\overline{B \setminus Z} = h\big(\overline{D_m \setminus (Y \cup h^{-1}(Z))}\big). \tag{4.7}$$

Since $h(0) = 0$ and $0 \notin Z$, the set $h^{-1}(Z)$ is a closed symmetric subset of $D_m \setminus \{0\}$. Moreover, by monotonicity of the genus under odd continuous maps,

$$\operatorname{genus}(h^{-1}(Z)) \leq \operatorname{genus}(Z) \leq i < j.$$

Thus

$$\begin{aligned} \operatorname{genus}\big(Y \cup h^{-1}(Z)\big) &\leq \operatorname{genus}(Y) + \operatorname{genus}(h^{-1}(Z)) \\ &\leq m - j + i = m - (j-i), \end{aligned}$$

which proves $\overline{B \setminus Z} \in \Lambda_{j-i}$ thanks to (4.7).

To prove (4.7), let $y \in h(\overline{D_m \setminus (Y \cup h^{-1}(Z))})$. Then there exists $x \in \overline{D_m \setminus (Y \cup h^{-1}(Z))} \subset D_m$ such that $y = h(x)$. By the definition of closure, there is a sequence $\{x_n\} \subset D_m \setminus (Y \cup h^{-1}(Z))$ such that $x_n \to x$. By continuity of h, $h(x_n) \to h(x) = y$. Since $x_n \in D_m \setminus (Y \cup h^{-1}(Z))$, we have

$$h(x_n) \in h(D_m \setminus Y) \setminus Z \subset \overline{B \setminus Z}.$$

This implies that $y \in \overline{B \setminus Z}$. Therefore

$$h\big(\overline{D_m \setminus (Y \cup h^{-1}(Z))}\big) \subset \overline{B \setminus Z}. \tag{4.8}$$

On the other hand, for any $y \in B \setminus Z$ there exists $y = h(w)$ where

$$w \in \overline{D_m \setminus Y} \setminus h^{-1}(Z) \subset \overline{D_m \setminus (Y \cup h^{-1}(Z))}.$$

Together with the continuity of h, we obtain

$$\overline{B \setminus Z} \subset h(\overline{D_m \setminus (Y \cup h^{-1}(Z))}). \tag{4.9}$$

The claim (4.7) follows from (4.8) and (4.9). □

We define another sequence of minimax values by

$$d_j = \inf_{A \in \Lambda_j} \max_{u \in A} J(u). \tag{4.10}$$

For the radius ρ fixed in Proposition 4.6, set

$$\mathbb{B}_\rho := \{u \in H_r^2(\mathbb{R}^N) : \|u\| < \rho\}.$$

Proposition 4.9. *If $B \in \Lambda_j$, then $B \cap \big(\partial \mathbb{B}_\rho \cap H_r^2\big) \neq \emptyset$.*

Proof. Set $B = h(\overline{D_m \setminus Y})$ where $h \in \Gamma_m$, $m \geq j$, $Y \in \mathcal{E}_m$ and $\operatorname{genus}(Y) \leq m-j$. For any $\sigma \in \partial D_m$, by (4.6) and (4.2) we obtain

$$J(h(\sigma)) = J(\gamma_{0m}(\sigma)) < 0.$$

Together with Proposition 4.6 (1), we have $\|h(\sigma)\| > \rho$. Let $\hat{O} = \{\sigma \in D_m \mid h(\sigma) \in \mathbb{B}_\rho\}$. Since h is odd, $0 \in \hat{O}$. Let O denote the component of $\hat{O}$ containing 0. Since $\hat{O}$ is symmetric and $0 \in O$, the uniqueness of the connected component containing 0 gives $-O = O$. Moreover, the boundary condition $\|h(\sigma)\| > \rho$ on ∂D_m implies $\overline{O} \subset \operatorname{int} D_m$. Thus O is a bounded symmetric neighborhood of 0 in $\mathbb{R}^m$. Therefore, $\operatorname{genus}(\partial O) = m$ and

$$\operatorname{genus}(\overline{\partial O \setminus Y}) \geq \operatorname{genus}(\partial O) - \operatorname{genus}(Y) \geq m - (m-j) = j \geq 1.$$

In particular, $\overline{\partial O \setminus Y} \neq \emptyset$. By the definition of O and continuity of h, $h(\partial O) \subset \partial \mathbb{B}_\rho$. Therefore, $h(\overline{\partial O \setminus Y}) \subset (B \cap \partial \mathbb{B}_\rho)$, that is, $B \cap \left(\partial \mathbb{B}_\rho \cap H_r^2\right) \neq \emptyset$. □

Propositions 4.8 and 4.9 imply that $0 < \mu \leq d_j < +\infty$, where μ is given in Proposition 4.6. For $d \in \mathbb{R}$ we set

$$K^d = \left\{u \in H_r^2(\mathbb{R}^N) : J(u) = d \text{ and } J|'_{H_r^2}(u) = 0\right\}.$$

Then K^d is compact by Lemma 4.7. For any $d \in \mathbb{R}$, we define the sublevel sets of J as follows:

$$J^d := \{u \in H_r^2(\mathbb{R}^N) : J(u) \leq d\}.$$

Lemma 4.10. *Let K^d be the critical set defined above. If*

$$d_j = d_{j+1} = \cdots = d_{j+i} := d \geq \mu > 0$$

for some $j \geq 1$ and $i \geq 0$, then

$$\operatorname{genus}(K^d) \geq i + 1.$$

In particular, if $i \geq 1$, then the critical set at level d is infinite.

Proof. Suppose, for a contradiction, that $\operatorname{genus}(K^d) \leq i$. By Lemma 4.7, K^d is compact. By item 5° of [17, Proposition 7.5], there exists a symmetric open neighborhood $\mathcal{N}_d$ of K^d, with $0 \notin \overline{\mathcal{N}_d}$, such that

$$\operatorname{genus}(\overline{\mathcal{N}_d}) = \operatorname{genus}(K^d) \leq i.$$

Apply the deformation theorem [17, Theorem A.4] with $\mathcal{O} = \overline{\mathcal{N}_d}$ and $\bar{\varepsilon} = \mu/2$. There exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in C([0,1] \times H_r^2, H_r^2)$ such that $\eta(1, \cdot)$ is odd,

$$\eta(1, J^{d+\varepsilon} \setminus \mathcal{O}) \subset J^{d-\varepsilon}, \tag{4.11}$$

and $\eta(1, u) = u$ for $u \in J^{d-\bar{\varepsilon}}$.

By the definition of $d_{j+i} = d$, choose $A \in \Lambda_{j+i}$ such that

$$\max_A J \leq d + \varepsilon,$$

that is $A \subset J^{d+\epsilon}$. Proposition 4.8 (iv), applied with $Z = \mathcal{O}$, yields a set $\overline{A \setminus \mathcal{O}} \in \Lambda_j$ satisfying

$$\overline{A \setminus \mathcal{O}} \subset \overline{J^{d+\varepsilon} \setminus \mathcal{O}}.$$

Therefore (4.11) gives

$$\eta(1, \overline{A \setminus \mathcal{O}}) \subset \eta(1, \overline{J^{d+\varepsilon} \setminus \mathcal{O}}) \subset \overline{\eta(1, J^{d+\varepsilon} \setminus \mathcal{O})} \subset J^{d-\varepsilon}. \tag{4.12}$$

and

$$\max_{\eta(1, \overline{A \setminus \mathcal{O}})} J \leq d - \epsilon < d.$$

For any $u \in \gamma_{0m}(\partial D_m)$ satisfying $J(u) < 0$, we have

$$J(u) \leq 0 < d - \bar{\varepsilon}.$$

Hence 2° of [17, Theorem A.4] and our choice of $\bar{\varepsilon}$ imply $\eta(1, u) = u$ on $J(u) < d - \bar{\varepsilon}$. Consequently, by (iii) of Proposition 4.8 we have $\eta(1, \overline{A \setminus \mathcal{O}}) \in \Lambda_j$. The definition of d_j and (4.11)-(4.12) then imply

$$d = d_j \leq \max_{u \in \eta(1, \overline{A \setminus \mathcal{O}})} J(u) \leq d - \varepsilon < d,$$

a contradiction. Thus $\operatorname{genus}(K^d) \geq i + 1$. If $i \geq 1$, then $\operatorname{genus}(K^d) \geq 2$; a finite symmetric set not containing zero has genus one, so K^d must be infinite. □

Proposition 4.11. *We have the following assertions:*

(a) *For every $n \in \mathbb{N}^+$, c_n is a critical value of J,*
(b) *$b_n \geq c_n \geq \mu$,*
(c) *$c_n \to \infty$ as $n \to \infty$.*

Proof. (a) Since J is even, satisfies the Palais-Smale condition by Lemma 4.7, and

$$\max_{\partial D_n} J(\gamma_{0n}) < 0 < c_n,$$

the standard symmetric minimax theorem [17, Chapter 9], applied to the homotopy-stable class Γ_n, shows that c_n is a critical value of J.

(b) For every $\gamma \in \Gamma_n$, oddness gives $\gamma(0) = 0$, whereas $\|\gamma(\sigma)\| > \rho$ on ∂D_n by Proposition 4.6. Hence, by continuity,

$$\gamma(D_n) \cap \{u \in H_r^2(\mathbb{R}^N) : \|u\| = \rho\} \neq \emptyset$$

for all $\gamma \in \Gamma_n$ and ρ given in Proposition 4.6. It follows from Proposition 4.6 and $I \geq J$ that

$$b_n \geq c_n \geq \mu > 0.$$

(c) Taking $m = n$, $h = \gamma$, and $Y = \emptyset$ in the definition of Λ_n gives $\gamma(D_n) \in \Lambda_n$ for every $\gamma \in \Gamma_n$. Therefore

$$d_n \leq \max_{u \in \gamma(D_n)} J(u) = \max_{\sigma \in D_n} J(\gamma(\sigma)).$$

Taking the infimum over all $\gamma \in \Gamma_n$, we obtain

$$d_n \leq c_n,$$

for all $n \in \mathbb{N}^+$. Proposition 4.8 (ii) and (4.10) imply that $\{d_n\}$ is nondecreasing,

$$d_1 \leq d_2 \leq \cdots \leq d_n \leq d_{n+1} \leq \cdots.$$

We claim that $d_n \to \infty$. Otherwise, $d_n \uparrow \overline{d} < \infty$. If $d_n = \overline{d}$ for all sufficiently large n, Lemma 4.10 implies $\operatorname{genus}(K^{\overline{d}}) = \infty$. But by Lemma 4.7, $K^{\overline{d}}$ is compact so $\operatorname{genus}(K^{\overline{d}}) < \infty$. Thus $\overline{d} > d_n$ for all $n \in \mathbb{N}^+$. Fix an arbitrary integer $k \geq 1$ (for example, $k = 1$). The set

$$\mathcal{K} := \{u \in H_r^2(\mathbb{R}^N) : J|'_{H_r^2}(u) = 0,\ d_{k+1} \leq J(u) \leq \overline{d}\}$$

is compact by the Palais-Smale condition in Lemma 4.7, symmetric, and does not contain zero. Hence $q := \operatorname{genus}(\mathcal{K}) < \infty$, and there is a symmetric open neighborhood $\mathcal{U}$ of $\mathcal{K}$, with $0 \notin \overline{\mathcal{U}}$, such that $\operatorname{genus}(\overline{\mathcal{U}}) = q$. Set $i = \max\{q, k+1\}$ and $\overline{\varepsilon} = \overline{d} - d_i > 0$. Since $d_i \geq d_{k+1} > 0$, every critical point with energy in $[d_i, \overline{d}]$ belongs to $\mathcal{K}$. The deformation theorem [17, Theorem A.4], applied with $c = \overline{d}$ and $\mathcal{O} = \overline{\mathcal{U}}$, yields $\varepsilon \in (0, \overline{\varepsilon})$ and an odd deformation η such that

$$\eta(1, J^{\overline{d}+\varepsilon} \setminus \mathcal{O}) \subset J^{\overline{d}-\varepsilon}.$$

Choose $n \in \mathbb{N}^+$ such that $d_n > \overline{d} - \varepsilon$ and $A \in \Lambda_{n+i}$ such that

$$\max_A J \leq \overline{d} + \varepsilon.$$

Here $\eta(1, u) = u$ whenever $J(u) \leq d_i$. In particular, the deformation fixes every boundary set $\gamma_{0m}(\partial D_m)$, whose energy is negative. Since $\operatorname{genus}(\mathcal{O}) = q \leq i$, the same excision and closure argument as in the proof of Lemma 4.10 gives $\overline{A \setminus \mathcal{O}} \in \Lambda_n$ and $\eta(1, \overline{A \setminus \mathcal{O}}) \in \Lambda_n$, with $J \leq \overline{d} - \varepsilon$ on the latter set. Therefore, by the choice of d_n,

$$d_n \leq \max_{u \in \eta(1, \overline{A \setminus \mathcal{O}})} J(u) \leq \overline{d} - \varepsilon,$$

a contradiction. Thus $d_n \to \infty$, and since $d_n \leq c_n$, we conclude that $c_n \to \infty$. □

The lack of a direct Palais-Smale condition for I is the main obstacle to proving that the levels b_n are critical values. To overcome this difficulty, introduce the augmented functionall $\widetilde{I} \in C^1(\mathbb{R} \times H_r^2)$ by

$$\begin{aligned}
\widetilde{I}(s, v) &:= I\big(v(e^{-s}\cdot)\big) \\
&= \frac{1}{2}\Big[e^{(N-4)s}\int_{\mathbb{R}^N} |\Delta v|^2 dx + e^{(N-2)s}\int_{\mathbb{R}^N} |\nabla v|^2 dx + e^{Ns}\int_{\mathbb{R}^N} v^2 dx\Big] \\
&\quad + \frac{e^{(N-2)s}}{2}\int_{\mathbb{R}^N} v^2 |\nabla v|^2 dx - e^{Ns}\int_{\mathbb{R}^N} F(v) dx.
\end{aligned}$$

Now, we can define the minimax values $\widetilde{b}_n$ for $\widetilde{I}$ by

$$\widetilde{b}_n = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}_n} \max_{\sigma\in D_n} \widetilde{I}(\widetilde{\gamma}(\sigma)),$$

where

$$\widetilde{\Gamma}_n = \Big\{\widetilde{\gamma} \in C(D_n, \mathbb{R}\times H_r^2(\mathbb{R}^N)) : \widetilde{\gamma} = (s,\eta), s : D_n \to \mathbb{R} \text{ is even},$$
$$\eta : D_n \to H_r^2(\mathbb{R}^N) \text{ is odd}, \widetilde{\gamma}|_{\partial D_n} = (0,\gamma_{0n}).$$

Lemma 4.12. *$\widetilde{b}_n = b_n$ for all $n \in \mathbb{N}^+$.*

Proof. If $\gamma \in \Gamma_n$, then the map $\sigma \mapsto (0,\gamma(\sigma))$ belongs to $\widetilde{\Gamma}_n$. Thus, by the definitions of b_n and $\widetilde{b}_n$, and by $\widetilde{I}(0,u) = I(u)$, we obtain

$$\max_{\sigma\in D_n} I(\gamma(\sigma)) = \max_{\sigma\in D_n} \widetilde{I}(0,\gamma(\sigma)). \tag{4.13}$$

This implies that $\widetilde{b}_n \le b_n$. On the other hand, if $\widetilde{\gamma}(\sigma) = (s(\sigma),\eta(\sigma)) \in \widetilde{\Gamma}_n$, we set

$$\gamma(\sigma) = \eta(\sigma)(e^{-s(\sigma)}x).$$

Since s is even and η is odd, $\gamma(-\sigma) = -\gamma(\sigma)$; on ∂D_n, $s = 0$ and $\eta = \gamma_{0n}$. Hence $\gamma \in \Gamma_n$, and $\widetilde{I}(\widetilde{\gamma}(\sigma)) = I(\gamma(\sigma))$ for all $\sigma \in D_n$. Therefore,

$$\max_{\sigma\in D_n} \widetilde{I}(\widetilde{\gamma}(\sigma)) = \max_{\sigma\in D_n} I(\gamma(\sigma))$$

and then

$$\widetilde{b}_n = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}_n} \max_{\sigma\in D_n} \widetilde{I}(\widetilde{\gamma}(\sigma)) \ge \inf_{\gamma\in\Gamma_n} \max_{\sigma\in D_n} I(\gamma(\sigma)) = b_n.$$

□

We now prove that, for every $n \in \mathbb{N}^+$, b_n is a critical value of I.

Lemma 4.13. *For each fixed $n \ge 1$, there exists a sequence $\{v_j\}_{j\ge1} \subset H_r^2(\mathbb{R}^N)$ such that*

(1) $I(v_j) \to b_n$;
(2) $\|I|'_{H_r^2}(v_j)\|_{(H_r^2)'} \to 0$;
(3) $P(v_j) \to 0$.

Proof. By the definition of b_n, for any $j \ge 1$ there exists $\gamma_j \in \Gamma_n$ such that

$$\max_{\sigma\in D_n} I(\gamma_j(\sigma)) \le b_n + \frac{1}{j^2}.$$

Setting $\widetilde{\gamma}_j = (0,\gamma_j) \in \widetilde{\Gamma}_n$, we deduce from (4.13) and Lemma 4.12 that

$$\max_{\sigma\in D_n} \widetilde{I}(\widetilde{\gamma}_j(\sigma)) \le b_n + \frac{1}{j^2}.$$

Applying the corresponding equivariant minimax principle (the equivariant form of [18, Theorem 2.8] and [7, Lemma 4.3, p. 266]), we obtain a sequence $\{(s_j,u_j)\} \subset \mathbb{R}\times H_r^2(\mathbb{R}^N)$ satisfying the following properties as $j \to \infty$,

(1) $\widetilde{I}(s_j,u_j) \to b_n$;
(2) $\operatorname{dist}((s_j,u_j),\widetilde{\gamma}_j(D_n)) \to 0$;
(3) $\|\widetilde{I}'(s_j,u_j)\|_{E_r'} \to 0$,

where $E_r := \mathbb{R}\times H_r^2(\mathbb{R}^N)$ and E_r' denotes the dual space of E_r. Because $\widetilde{\gamma}_j(D_n) \subset \{0\}\times H_r^2$, (b_2) implies $s_j \to 0$. Differentiating the explicit formula for $\widetilde{I}$, we obtain, for any $(\mu,\omega) \in E_r$,

$$(I\circ\varphi)'(s_j,u_j)[\mu,\omega] = I'(\varphi(s_j,u_j))[\varphi(s_j,\omega)] + P(\varphi(s_j,u_j))\mu \to 0.$$

Taking $\mu = 1$ and $\omega = 0$ in (b_3), we obtain

$$P(\varphi(s_j,u_j)) \to 0, \quad \text{as} \quad j\to\infty.$$

Set $w_j := \varphi(s_j, u_j)$. Then, as $j \to \infty$,

$$I(w_j) \to b_n, \quad \text{and} \quad P(w_j) \to 0.$$

Renaming w_j as v_j, we obtain (1) and (3). To prove (2), let $\omega \in H_r^2(\mathbb{R}^N)$ and define

$$\widetilde{\omega}_j(x) = \omega(e^{s_j} x).$$

Then $\varphi(s_j, \widetilde{\omega}_j) = \omega$. Since $s_j \to 0$, the scaling operators $\omega \mapsto \widetilde{\omega}_j$ are uniformly bounded on $H_r^2(\mathbb{R}^N)$. Therefore,

$$|\langle I|'_{H_r^2}(v_j), \omega\rangle| = |\widetilde{I}'(s_j, u_j)[(0, \widetilde{\omega}_j)]| \le o_j(1)\|\widetilde{\omega}_j\| \le o_j(1)\|\omega\|.$$

Hence $\|I|'_{H_r^2}(v_j)\|_{(H_r^2)'} \to 0$, and Lemma 4.13 is proved. □

Proof of Theorem 1.3. Fix $n \in \mathbb{N}^+$ and let $\{v_j\}$ be the sequence obtained in Lemma 4.13:

$$I(v_j) \to b_n > 0, \quad \|I|'_{H_r^2}(v_j)\|_{(H_r^2)'} \to 0, \quad P(v_j) \to 0.$$

Repeating the estimates (3.4)–(3.5), with v_j in place of u_n, shows that $\{v_j\}$ is bounded in $H_r^2(\mathbb{R}^N)$. Therefore, after passing to a subsequence,

$$v_j \rightharpoonup v \quad \text{in } H_r^2(\mathbb{R}^N), \quad v_j \to v \quad \text{in } L^q(\mathbb{R}^N) \quad \text{for every } q \in (2, 2_*),$$

and $v_j(x) \to v(x)$ a.e. in $\mathbb{R}^N$. By (2.8), Hölder's inequality, the boundedness of $\{v_j\}$ in $L^2 \cap L^{2_*}$, and $v_j \to v$ in L^q, $q \in (2, 2_*)$, we obtain, after first taking $j \to \infty$ and then $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} f(v_j)(v_j - v)\, dx \to 0.$$

Arguing as in the proof of (4.4), we obtain

$$\int_{\mathbb{R}^N} f(v_j) v_j\, dx \to \int_{\mathbb{R}^N} f(v) v\, dx.$$

Passing to the limit in $\langle I|'_{H_r^2}(v_j), \phi\rangle$ for radial $\phi \in C_c^\infty(\mathbb{R}^N)$ and then using density, we obtain $I|'_{H_r^2}(v) = 0$. Consequently,

$$\begin{aligned}
\|v_j\|^2 + 2\int_{\mathbb{R}^N} v_j^2 |\nabla v_j|^2 dx &= \int_{\mathbb{R}^N} f(v_j) v_j dx + o(1) \\
&= \int_{\mathbb{R}^N} f(v) v dx + o(1) \\
&= \|v\|^2 + 2\int_{\mathbb{R}^N} v^2 |\nabla v|^2 dx + o(1).
\end{aligned}$$

As in the proof of Lemma 4.7, weak lower semicontinuity of the quadratic part and (2.5), together with convergence of the sums above, yield convergence of the quadratic norms and hence $v_j \to v$ strongly in $H_r^2(\mathbb{R}^N)$. Hence

$$I(v) = b_n, \quad I|'_{H_r^2}(v) = 0.$$

Thus, by Palais' principle of symmetric criticality, we obtain fixed-point space, the principle of symmetric criticality gives $I'(v) = 0$. Since $b_n \ge c_n \ge \mu > 0$, this critical point v is nontrivial. Finally, by (b) and (c) of Proposition 4.11, we obtain $b_n \to \infty$. Therefore, equation (1.1) has infinitely many nontrivial radially symmetric solutions. □

Acknowledgments. L. F. Yin was supported by the National Natural Science Foundation of China (No. 12401140) and by the Natural Science Foundation of Sichuan Province (No. 2024NSFSC1343). F. Wang was supported by the Fundamental Research Funds for the Central Universities (2682026ZTPY010), China.

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