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DECAY RATES FOR NONLINEAR DIFFUSION WAVES OF p-SYSTEMS WITH NONLINEAR AND TIME-DEPENDENT DAMPING

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Abstract. In this article, we study the Cauchy problem for a p-system with both nonlinear and time-dependent damping. By combining the a priori estimates with energy methods, we prove that under sufficiently small initial data, the Cauchy problem admits a unique global solution. Furthermore, we show that the solution converges time- asymptotically to the corresponding nonlinear diffusion wave governed by the classical Darcy's law. In particular, L^∞ convergence rates are established via weighted energy estimates. This result fills a significant gap in the theoretical framework by explicitly accounting for the combined effects of nonlinear and time-dependent damping, an aspect that has not been adequately addressed in previous studies.

1. Introduction

In this article, we consider the Cauchy problem of the p-system with nonlinear and time-dependent damping,

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= -\frac{\alpha u + g(u)}{(1+t)^\lambda}, \quad x \in \mathbb{R},\ t > 0, \end{aligned} \tag{1.1}$$

with the initial data

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)) \to (v_\pm, u_\pm), \quad v_+ \neq v_-, \quad x \to \pm\infty. \tag{1.2}$$

This p-system models the 1D compressible flow through porous media in the Lagrangian coordinate, where $u = u(x,t)$ is the velocity of the flow at time t and position x, $v(>0)$ is the specific volume, $p(>0)$ is the pressure with $p'(v) < 0$, the damping coefficient $\alpha > 0$ is a constant, $g(u)$ is a nonlinear damping and $0 \le \lambda < 1$.

The p-system is a fundamental model in continuum mechanics and gas dynamics, governing the one-dimensional motion of compressible fluids or elastic materials. Without damping, the system is hyperbolic and can develop shock waves in finite time. When damping is present, it introduces dissipation that suppresses shock formation and drives the solution toward a nonlinear diffusion wave, a phenomenon analogous to the Darcy flow in porous media. An important physical scenario is the flow of gas through a porous medium, where the damping coefficient may depend nonlinearly on the velocity (e.g., a Forchheimer-type correction) and also vary with time due to changes in the medium permeability or external conditions. This combination of nonlinearity and time dependence in the damping term gives rise to new physical phenomena and significant mathematical challenges.

There exists a substantial body of literature on the global existence, large-time behavior, and convergence rates of smooth solutions to the p-system under linear, nonlinear, space-dependent, and spatiotemporal damping. cf. [1, 3, 5, 13, 15, 16, 18, 20, 21, 22, 23, 25, 27, 28, 30].

For the Cauchy problem with linear damping, the existence of smooth global solutions with small initial data has been studied by many authors, cf. [4, 14, 19, 29]. The large time behavior

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of the solutions was carried out by Hsiao and Liu in [6, 7]. They showed that the solutions of the Cauchy problem of (1.1) time-asymptotically behave as those of the system

$$\begin{aligned} \bar{v}_t - \bar{u}_x &= 0, \\ p(\bar{v})_x &= -\alpha \bar{u}, \end{aligned} \tag{1.3}$$

or

$$\begin{aligned} \bar{v}_t &= -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ \bar{u} &= -\frac{1}{\alpha} p(\bar{v})_x, \end{aligned} \tag{1.4}$$

with the same end states as $v_0(x)$:

$$\bar{v}(\pm\infty, t) = v_{\pm}. \tag{1.5}$$

Here the well-known porous media equation is obtained by Darcy's law.

For system (1.1) with time-dependent damping $\frac{\alpha u}{(1+t)^{\lambda}}$ ($\alpha > 0$ and λ are constants), where the damping effect $\frac{\alpha u}{(1+t)^{\lambda}}$ is called under damping for $\lambda > 0$ and over damping for $\lambda < 0$. Pan [24] and Hou-Yin [8, 9] first investigated the well-posedness and convergence for the 1-D and n-D under damping respectively. The optimal convergence rates were showed by Ji and Mei [10] by Fourier analysis method. For the over damping case with $\lambda < 0$, Li, Li, Mei and Zhang [12], Ji and Mei [11] first investigated the large-time behavior of the solutions in 1-D and n-D cases respectively. For the Cauchy problem of the system (1.1) with time-dependent damping when $0 \le \lambda < 1$, Cui, Yin, Zhang and Zhu in [2] proved that the solutions time-asymptotically converge to the diffusion wave whose profile is self-similar solution to the corresponding parabolic equation.

However, to the best of our knowledge, only a limited number of studies have addressed the case involving simultaneous nonlinear and time-dependent damping.

From a practical standpoint, this gap is far from trivial. Such a combined damping model is not merely a theoretical curiosity; it arises naturally in many real-world systems. For instance, in smart materials, adaptive vibration control, and mechanical structures subject to variable energy dissipation, the damping mechanisms genuinely depend on both the amplitude of the response (nonlinearity) and the evolving system conditions (time dependence). Resolving this open problem provides a more accurate and physically realistic mathematical framework for predicting, analyzing, and designing such systems.

Accordingly, the primary objective of this paper is to provide a positive resolution to this open problem. The presence of both nonlinear and time-dependent damping introduces several new phenomena and poses severe mathematical challenges, which arise from two main aspects.

On the one hand, it is necessary to identify a suitable asymptotic profile that properly captures the combined effects of nonlinear and time-dependent damping: a task that becomes ambiguous and highly nontrivial due to their mutual modulation. On the other hand, the inclusion of nonlinear damping significantly complicates the energy estimates, rendering them far more intricate and demanding than in conventional settings. Unlike standard treatments where damping is purely linear or purely time-invariant, the nonlinear term forces the construction of more delicate energy functionals and more refined estimates.

Throughout this paper, we assume the following:

(H1) $p(v) \in C^4(\mathbb{R}^+)$, $p'(v) \le -C_0 < 0 (C_0 : \text{constant})$, for all $v > 0$;
(H2) $v_{\pm} > 0$, $u_+ = u_- = 0$;
(H3) $g(u) \in C^3(\mathbb{R})$, $g(0) = g'(0) = g''(0) = 0$.

The rest of this paper is organized as follows. In Section 2, we reformulate the problem (1.1)-(1.2) and state the main theorem. In Section 3, we collect some fundamental properties of the nonlinear diffusion wave $(\bar{v}(x,t), \bar{u}(x,t))$ of (2.1). In Section 4, we obtain the a *priori* estimates on finite time intervals. In Section 5, we prove the decay rates.

Notation. Hereafter, we use C or c to denote generic positive constants that are independent of the data and time t and may change from line to line. The symbols ε and δ will always represent

sufficiently small positive constants. $L^p = L^p(\mathbb{R})$ $(1 \le p \le \infty)$ denotes the usual Lebesgue space with the norms

$$\|f\|_{L^p} = \Big(\int_{\mathbb{R}} |f(x)|^p dx \Big)^{1/p}, \quad 1 \le p < \infty,$$
$$\|f\|_\infty = \|f\|_{L^\infty} = \sup_{\mathbb{R}} |f(x)|.$$

The region of integration, $\mathbb{R}$, will be omitted without causing any confusion. $H^l (l \ge 0)$ denotes the usual l-th order Sobolev space with the norm

$$\|f\|_l = \Big(\sum_{j=0}^{l} \|\partial_x^j f\|^2 \Big)^{1/2},$$

where $\|\cdot\| = \|\cdot\|_0 = \|\cdot\|_{L^2}$. For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_l$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_l$, respectively. The set of k-times continuously differentiable functions with compact support on $\mathbb{R}$ is denoted by $C_0^k(\mathbb{R})$. The space $C^k([0,T];\, X)$ denotes the set of k-times differentiable functions from $[0,T]$ to the Hilbert space X.

2. Main results

From asymptotic analysis, it is well known that, as $t \to \infty$, the first term u_t and the nonlinear damping term $g(u)$ in $(1.1)_2$ decay to zero faster than the other terms. As in [6, 22], the solutions of (1.1) behave time-asymptotically like those of Darcy's law.

$$\begin{aligned} \bar{v}_t - \bar{u}_x &= 0, \\ p(\bar{v})_x &= -\frac{\alpha \bar{u}}{(1+t)^\lambda}, \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} \bar{v}_t &= -\frac{(1+t)^\lambda}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x &= -\frac{\alpha}{(1+t)^\lambda} \bar{u}. \end{aligned} \tag{2.2}$$

We denote by the so-called nonlinear diffusion wave the self-similar solution of (2.1) or (2.2), in the form of $\phi\big(\frac{x}{(1+t)^{\frac{\lambda+1}{2}}}\big)$ by $\bar{v}(x,t)$ with the same end states as $v(x,0)$

$$\bar{v}(\pm\infty, t) = v_\pm, \tag{2.3}$$

and set

$$\bar{u}(x,t) = -\frac{(1+t)^\lambda}{\alpha} p(\bar{v})_x, \tag{2.4}$$

which is dictated by Darcy's law.

We set

$$\begin{aligned} w(x,t) &= \int_{-\infty}^{x} (v(y,t) - \bar{v}(y + x_0, t))\, \mathrm{d}y, \\ z(x,t) &= u(x,t) - \bar{u}(x + x_0, t), \end{aligned} \tag{2.5}$$

where x_0 is a constant uniquely defined by

$$\int_{-\infty}^{\infty} \big(v_0(x) - \bar{v}(x + x_0, 0)\big)\, \mathrm{d}x = \frac{u_+ - u_-}{-\alpha},$$

i.e.

$$x_0 = \frac{1}{v_+ - v_-} \Big\{ \frac{u_+ - u_-}{\alpha} + \int_{-\infty}^{\infty} (v_0(x) - \bar{v}(x,0))\, \mathrm{d}x \Big\}.$$

We define

$$\begin{aligned} w_0(x) = w(x,0) &= \int_{-\infty}^{x} (v_0(y) - \bar{v}\,(y + x_0, 0))\, \mathrm{d}y, \\ z_0(x) = z(x,0) &= u_0(x) - \bar{u}\,(x + x_0, 0)\,. \end{aligned} \tag{2.6}$$

Then, from (1.1), (2.1), (2.5), (2.6) and (H2), it follows that $(w(x,t), z(x,t))$ satisfies the Cauchy problem

$$\begin{aligned} w_t - z &= 0, \\ z_t + \left(p\left(w_x + \bar{v}\right) - p(\bar{v})\right)_x + \frac{\alpha}{(1+t)^\lambda} z + \bar{u}_t &= \frac{1}{(1+t)^\lambda} g(z+\bar{u}), \end{aligned} \tag{2.7}$$

with the initial data

$$(w(x,0), z(x,0)) = (w_0(x), z_0(x)) \to (0,0) \quad \text{as } x \to \pm\infty. \tag{2.8}$$

We rewrite (2.7) and (2.8) as

$$w_{tt} + \left(p'(\bar{v}) w_x\right)_x + \frac{\alpha}{(1+t)^\lambda} w_t = F + \frac{1}{(1+t)^\lambda} g(w_t + \bar{u}), \tag{2.9}$$

with the initial data

$$(w(x,0), w_t(x,0)) = (w_0(x), z_0(x)), \tag{2.10}$$

where

$$F = \frac{1}{\alpha}(1+t)^\lambda p(\bar{v})_{xt} + \frac{\lambda}{\alpha}(1+t)^{\lambda-1} p(\bar{v})_x - \left(p\left(w_x + \bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x\right)_x. \tag{2.11}$$

Our main result is as follows.

Theorem 2.1. *Under the assumptions* (H1)–(H3)*, problem* (2.7)-(2.8) *admits a unique global-in-time solution, provided that* $\delta := |v_+ - v_-|$ *and that* $(w_0, z_0) \in H^3 \times H^2$ *with* $\|w_0\|_3^2 + \|z_0\|_2^2 + \delta$ *sufficiently small. This solution satisfies*

$$\begin{aligned} w &\in C^k\left((0,\infty), H^{3-k}(\mathbb{R})\right), \quad k = 0,1,2,3; \\ w_t &\in C^k\left((0,\infty), H^{2-k}(\mathbb{R})\right), \quad k = 0,1,2. \end{aligned}$$

Furthermore, when $0 \le \lambda < 1/7$*, we have*

$$\begin{aligned} &\sum_{k=0}^{3} (1+t)^{(\lambda+1)k} \|\partial_x^k w(t)\|^2 + \sum_{k=0}^{2} (1+t)^{(\lambda+1)k+2} \|\partial_x^k w_t(t)\|^2 \\ &+ \int_0^t \Big(\sum_{j=1}^{3} (1+s)^{(\lambda+1)j-1} \|\partial_x^j w(s)\|^2 + \sum_{j=0}^{2} (1+s)^{(\lambda+1)j+1} \|\partial_x^j w_t(s)\|^2 \Big) \, \mathrm{d}s \\ &\le C\left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right); \end{aligned} \tag{2.12}$$

when $1/7 < \lambda < 1/3$*, we have*

$$\begin{aligned} &\sum_{k=0}^{3} (1+t)^{(\lambda+1)k+\frac{1}{2}-\frac{7\lambda}{2}} \|\partial_x^k w(t)\|^2 + \sum_{k=0}^{2} (1+t)^{(\lambda+1)k+\frac{5}{2}-\frac{7\lambda}{2}} \|\partial_x^k w_t(t)\|^2 \\ &\le C\left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right); \end{aligned} \tag{2.13}$$

and when $\beta \in (\frac{1}{2} - \frac{5\lambda}{2}, \lambda)$*, we have*

$$\begin{aligned} &\int_0^t \Big(\sum_{j=0}^{3} (1+s)^{(\lambda+1)(j-1)+\beta} \|\partial_x^j w(s)\|^2 + \sum_{j=0}^{2} (1+s)^{(\lambda+1)j+\beta-\lambda+1} \|\partial_x^j w_t(s)\|^2 \Big) \, \mathrm{d}s \\ &\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}} \Big(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta \Big); \end{aligned} \tag{2.14}$$

when $\lambda = 1/7$*, we have that for sufficiently small* $\varepsilon > 0$*,*

$$\begin{aligned} &\sum_{k=0}^{3} (1+t)^{\frac{8k}{7}} \|\partial_x^k w(t)\|^2 + \sum_{k=0}^{2} (1+t)^{\frac{8k}{7}+2} \|\partial_x^k w_t(t)\|^2 \\ &+ \int_0^t \Big(\sum_{j=1}^{3} (1+s)^{\frac{8j}{7}-1} \|\partial_x^j w(s)\|^2 + \sum_{j=0}^{2} (1+s)^{\frac{8j}{7}+1} \|\partial_x^j w_t(s)\|^2 \Big) \, \mathrm{d}s \\ &\le C(1+t)^{\varepsilon} \left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right). \end{aligned} \tag{2.15}$$

Noting that $w_x = v - \bar{v}$ and $z = u - \bar{u}$, and applying Theorem 2.1 and Sobolev's inequality, we immediately obtain the following convergence rates.

Corollary 2.2. *Under the assumptions of Theorem 2.1, there exists a unique global-in-time solution $(v,u)(x,t)$ to the system (1.1)–(1.2), satisfying*

$$\|\int_{-\infty}^{x}(v(y,t)-\bar{v}(y+x_0,t))\,\mathrm{d}y\|_{L^\infty}\le\begin{cases}C(1+t)^{-\frac{\lambda+1}{4}}, & 0\le\lambda<1/7,\\ C(1+t)^{-\frac{2}{7}+\frac{\varepsilon}{2}}, & \lambda=1/7,\\ C(1+t)^{\frac{3\lambda-1}{2}}, & 1/7<\lambda<1/3,\end{cases}\tag{2.16}$$

$$\|(v-\bar{v})(\cdot,t)\|_{L^\infty}\le\begin{cases}C(1+t)^{-\frac{3}{4}(\lambda+1)}, & 0\le\lambda<1/7,\\ C(1+t)^{-\frac{6}{7}+\frac{\varepsilon}{2}}, & \lambda=1/7,\\ C(1+t)^{\lambda-1}, & 1/7<\lambda<1/3,\end{cases}\tag{2.17}$$

$$\|(u-\bar{u})(\cdot,t)\|_{L^\infty}\le\begin{cases}C(1+t)^{-\frac{\lambda+5}{4}}, & 0\le\lambda<1/7,\\ C(1+t)^{-\frac{9}{7}+\frac{\varepsilon}{2}}, & \lambda=1/7,\\ C(1+t)^{\frac{3(\lambda-1)}{2}}, & 1/7<\lambda<1/3.\end{cases}\tag{2.18}$$

Remark 2.3. Because of the nonlinear damping, we need to assume that $u_- = u_+$. When $u_- \neq u_+$, the values of $u(x,t)$ at $x = \pm\infty$ must be eliminated in order to apply the energy method. Following the ideas in [6, 7], it is natural to construct correction functions. However, how to construct such correction functions in this case remains an open problem.

3. Preliminaries

In this section, we collect some fundamental properties of the nonlinear diffusion wave $(\bar{v},\bar{u})(x,t)$ of (2.1). Our interest lies in the dissipative nature of the nonlinear diffusion wave.

Following the approach in [2, 6], we first derive the self-similar solutions to the nonlinear diffusion equation $(2.2)_1$. Let

$$\begin{aligned}\bar{v}(x,t)&=\phi(\xi),\\ \phi(\pm\infty)&=v_\pm,\end{aligned}\tag{3.1}$$

where

$$\xi=\frac{x}{(1+t)^{\frac{\lambda+1}{2}}}.\tag{3.2}$$

Then $\phi(\xi)$ satisfies

$$(p'(\phi)\phi')'=\frac{\lambda+1}{2}\alpha\xi\phi'.\tag{3.3}$$

Hence, for an arbitrary $\xi_0\in\mathbb{R}$, we have

$$\phi'(\xi)=\frac{\phi'(\xi_0)\,p'(\phi(\xi_0))}{p'(\phi(\xi))}e^{\int_{\xi_0}^{\xi}\frac{\lambda+1}{2p'(\phi(\eta))}\alpha\eta d\eta}.\tag{3.4}$$

A direct computation shows that

$$\sum_{k=1}^{4}\big|\frac{d^k}{d\xi^k}\phi(\xi)\big|+|\phi(\xi)-v_+|_{\{\xi>0\}}+|\phi(\xi)-v_-|_{\{\xi<0\}}\le C|v_+-v_-|e^{-c|\xi|^2}.\tag{3.5}$$

Furthermore, from the form of the function $\bar{v}(x,t)$, it follows that

$$\begin{aligned}&\bar{v}_x=\phi'(\xi)(1+t)^{-\frac{\lambda+1}{2}},\quad \bar{v}_t=-\frac{\lambda+1}{2}\xi\phi'(\xi)(1+t)^{-1},\quad \bar{v}_{xx}=\phi''(\xi)(1+t)^{-(\lambda+1)},\\ &\bar{v}_{xt}=-\frac{\lambda+1}{2}(1+t)^{-\frac{\lambda+3}{2}}\left(\phi'(\xi)+\xi\phi''(\xi)\right),\quad \bar{v}_{xxx}=\phi'''(\xi)(1+t)^{-\frac{3}{2}(\lambda+1)},\\ &\bar{v}_{tt}=\frac{(\lambda+1)^2\phi''(\xi)\xi^2+(\lambda+1)(\lambda+3)\phi'(\xi)\xi}{4(1+t)^2}.\end{aligned}\tag{3.6}$$

Consequently, a direct calculation yields the following lemma.

Lemma 3.1. *Assume that $p(v) \in C^4(\mathbb{R}^+)$ and $p'(v) < 0$ for $v \in \mathbb{R}^+$. Then, for each $p \in [1, \infty]$, the solution $(\bar{v}, \bar{u})(x,t)$ of (2.2)-(2.3) satisfies the following dissipative estimates*

$$\begin{aligned}
&\min\{v_+, v_-\} \le \bar{v}(x,t) \le \max\{v_+, v_-\},\\
&\|\partial_x^k \bar{v}(t)\|_{L^p(\mathbb{R})} \le C|v_+ - v_-|(1+t)^{-\frac{(\lambda+1)k}{2}+\frac{\lambda+1}{2p}}, \quad k = 1,2,3,\dots,\\
&\|\partial_x^k \bar{u}(t)\|_{L^p(\mathbb{R})} \le C|v_+ - v_-|(1+t)^{-1-\frac{(\lambda+1)(k-1)}{2}+\frac{\lambda+1}{2p}}, \quad k = 0,1,2,3,\dots,\\
&\|\partial_x^k \bar{v}_t(t)\|_{L^p(\mathbb{R})} \le C|v_+ - v_-|(1+t)^{-1-\frac{(\lambda+1)k}{2}+\frac{\lambda+1}{2p}}, \quad k = 0,1,2,3,\dots,\\
&\|\partial_x^k \bar{u}_t(t)\|_{L^p(\mathbb{R})} \le C|v_+ - v_-|(1+t)^{\frac{\lambda-3}{2}-\frac{(\lambda+1)k}{2}+\frac{\lambda+1}{2p}}, \quad k = 0,1,2,3,\dots.
\end{aligned}$$

4. A priori estimates on finite time intervals

We analyze the existence and asymptotic behavior of the unique global-in-time solution $w(x,t)$ to (2.9)-(2.10). Under assumptions (H1)–(H3), the existence of local-in-time solutions to (2.9)-(2.10) can be readily established by the standard iteration argument (see Matsumura [17]). To extend local solutions to be global in time, it suffices to derive uniform a priori estimates of solutions to (2.9)-(2.10). Because of the influence of the time-dependent damping, such estimates cannot be obtained directly in a single step. Following the ideas from [2], we consequently divide the interval $(0,t)$ into two subintervals, which overcomes this technical difficulty.

In this section, we present the a priori estimates of the solution to the initial problem (2.9)-(2.10) on finite time intervals.

Theorem 4.1. *Under the assumptions of Theorem 2.1, for any given $T > 0$, $0 \le \lambda < 1$, $\alpha > 0$, the solution $(w(x,t), z(x,t))$ to the initial problem (2.9)-(2.10) on $[0,T]$ satisfies*

$$\|w\|_3^2 + \|w_t\|_2^2 + \int_0^t \left(\|w_t\|_2^2 + \|w_x\|^2 + \|w_{xx}\|^2 + \|w_{xxx}\|^2\right) \mathrm{d}s \le C(T)\left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right). \tag{4.1}$$

To prove this theorem, make the a priori assumption

$$N(T) = \sup_{0\le t\le T}\Big\{\sum_{k=0}^{2} ||\partial_x^k w(t)||_{L^\infty} + \sum_{k=0}^{1} ||\partial_x^k z(t)||_{L^\infty}\Big\} \le \varepsilon, \tag{4.2}$$

where $0 < \varepsilon \ll 1$. Then we prove the theorem through a series of lemmas.

Lemma 4.2. *Under the assumptions of Theorem 2.1, for any given $T > 0$, $0 \le \lambda < 1$, $\alpha > 0$, the solution $(w(x,t), w_t(x,t))$ to the initial problem (2.9)-(2.10) on $[0,T]$ satisfies*

$$\|w\|^2 + \|w_t\|^2 + \|w_x\|^2 + \int_0^t \left(\|w_t\|^2 + \|w_x\|^2\right) \mathrm{d}s \le C(T)\left(\|w_0\|_1^2 + \|z_0\|^2 + \delta\right). \tag{4.3}$$

Proof. **Step 1.** Multiplying (2.9) by w and then integrating with respect to x over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left[ww_t + \frac{\alpha}{2}(1+t)^{-\lambda}w^2\right]\mathrm{d}x + \frac{\alpha\lambda}{2}\int (1+t)^{-\lambda-1}w^2\,\mathrm{d}x - \int p'(\bar{v})w_x^2\,\mathrm{d}x\\
&= \int w_t^2\,\mathrm{d}x + \int Fw\,\mathrm{d}x + \int \frac{g(w_t+\bar{u})}{(1+t)^\lambda}w\,\mathrm{d}x.
\end{aligned} \tag{4.4}$$

By (2.11), the second term on the right-hand side of (4.4) can be bounded,

$$\int Fw\,\mathrm{d}x = \int \left[\frac{1}{\alpha}(1+t)^\lambda p(\bar{v})_{xt} + \frac{\lambda}{\alpha}(1+t)^{\lambda-1}p(\bar{v})_x - \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v})w_x\right)_x\right]w\,\mathrm{d}x. \tag{4.5}$$

With $p'(v) \le -C_0 < 0$ (where C_0 is a positive constant), applying Lemma 3.1 and Cauchy-Schwarz's inequality yields

$$\begin{aligned}\int \frac{1}{\alpha}(1+t)^{\lambda}p(\bar{v})_{xt}w\,\mathrm{d}x &= -\int \frac{1}{\alpha}(1+t)^{\lambda}p(\bar{v})_t w_x\,\mathrm{d}x\\ &\le \frac{C_0}{4}\int w_x^2\,\mathrm{d}x + C\int (1+t)^{2\lambda}|\bar{v}_t|^2\,\mathrm{d}x\\ &\le \frac{C_0}{4}\int w_x^2\,\mathrm{d}x + C(T)\delta^2,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\int \frac{\lambda}{\alpha}(1+t)^{\lambda-1}p(\bar{v})_x w\,\mathrm{d}x &\le \frac{\alpha\lambda}{4}\int (1+t)^{-\lambda-1}w^2\,\mathrm{d}x + C\int (1+t)^{3\lambda-1}|\bar{v}_x|^2\,\mathrm{d}x\\ &\le \frac{\alpha\lambda}{4}\int (1+t)^{-\lambda-1}w^2\,\mathrm{d}x + C(T)\delta^2.\end{aligned} \tag{4.7}$$

Invoking Lemma 3.1 and the a priori assumption (4.2) yields

$$\begin{aligned}-\int \left(p\left(w_x+\bar{v}\right)-p(\bar{v})-p'(\bar{v})w_x\right)_x w\,\mathrm{d}x &= \int \left(p\left(w_x+\bar{v}\right)-p(\bar{v})-p'(\bar{v})w_x\right)w_x\,\mathrm{d}x\\ &= \frac{1}{2}\int p''(\bar{v}+\theta_1 w_x)w_x^3\,\mathrm{d}x\\ &\le C\varepsilon\int w_x^2\,\mathrm{d}x,\end{aligned} \tag{4.8}$$

where $0<\theta_1<1$.

The last term on the right-hand side of (4.4) remains to be estimated. With assumption (H3), Lemma 3.1, the a priori assumption (4.2), and Cauchy-Schwarz's inequality, we derive

$$\begin{aligned}\int \frac{g\left(w_t+\bar{u}\right)}{(1+t)^{\lambda}}w\,\mathrm{d}x &= \frac{1}{6}\int g'''\left(\theta_2(w_t+\bar{u})\right)(1+t)^{-\lambda}\left(w_t+\bar{u}\right)^3 w\,\mathrm{d}x\\ &\le C\int (1+t)^{-\lambda}\left(|w_t|^3+|\bar{u}|^3\right)|w|\,\mathrm{d}x\\ &\le C\varepsilon^2\int (1+t)^{-\lambda}w_t^2\,\mathrm{d}x + C(T)\delta^2\\ &\le C\varepsilon^2\int w_t^2\,\mathrm{d}x + C(T)\delta^2,\end{aligned} \tag{4.9}$$

where $0<\theta_2<1$.

From (4.6)-(4.9) and (4.4), and using the smallness of ε, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \left[ww_t+\frac{\alpha}{2}(1+t)^{-\lambda}w^2\right]\mathrm{d}x + \frac{\alpha\lambda}{4}\int (1+t)^{-\lambda-1}w^2\,\mathrm{d}x + \frac{C_0}{2}\int w_x^2\,\mathrm{d}x\\ &\le 2\int w_t^2\,\mathrm{d}x + C(T)\delta^2.\end{aligned} \tag{4.10}$$

Step 2. Multiplying (2.9) by $(1+t)^{\lambda}w_t$ and then integrating the resulting equation with respect to x over $\mathbb{R}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\lambda}w_t^2-(1+t)^{\lambda}p'(\bar{v})w_x^2\right]\mathrm{d}x + \int \alpha w_t^2\,\mathrm{d}x\\ &= -\frac{1}{2}\int (1+t)^{\lambda}p''(\bar{v})\bar{v}_t w_x^2\,\mathrm{d}x - \frac{\lambda}{2}\int (1+t)^{\lambda-1}p'(\bar{v})w_x^2\,\mathrm{d}x\\ &\quad + \frac{\lambda}{2}\int (1+t)^{\lambda-1}w_t^2\,\mathrm{d}x + \int (1+t)^{\lambda}Fw_t\,\mathrm{d}x + \int g(w_t+\bar{u})w_t\,\mathrm{d}x\\ &\le C\int (1+t)^{\lambda-1}\left(w_x^2+w_t^2\right)\mathrm{d}x + \int (1+t)^{\lambda}Fw_t\,\mathrm{d}x + \int g(w_t+\bar{u})w_t\,\mathrm{d}x.\end{aligned} \tag{4.11}$$

Using (2.11), we can estimate the second term on the right-hand side of (4.11) as

$$\begin{aligned}\int (1+t)^{\lambda} F w_t \,\mathrm{d}x = \int \Big[&\frac{1}{\alpha}(1+t)^{2\lambda} p(\bar{v})_{xt} + \frac{\lambda}{\alpha}(1+t)^{2\lambda-1} p(\bar{v})_x \\ &- (1+t)^{\lambda} \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x \right)_x \Big] w_t \,\mathrm{d}x.\end{aligned} \tag{4.12}$$

Using Lemma 3.1 and the a priori assumption (4.2), we obtain

$$\begin{aligned}\int \frac{1}{\alpha}(1+t)^{2\lambda} p(\bar{v})_{xt} w_t \,\mathrm{d}x &\le \frac{\alpha}{6}\int w_t^2 \,\mathrm{d}x + C\int (1+t)^{4\lambda} \left(|\bar{v}_{xt}|^2 + |\bar{v}_x|^2 |\bar{v}_t|^2 \right) \mathrm{d}x \\ &\le \frac{\alpha}{6}\int w_t^2 \,\mathrm{d}x + C(T)\delta^2,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\int \frac{\lambda}{\alpha}(1+t)^{2\lambda-1} p(\bar{v})_x w_t \,\mathrm{d}x &\le \frac{\alpha}{6}\int w_t^2 \,\mathrm{d}x + C\int (1+t)^{4\lambda-2} |\bar{v}_x|^2 \,\mathrm{d}x \\ &\le \frac{\alpha}{6}\int w_t^2 \,\mathrm{d}x + C(T)\delta^2,\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&- \int (1+t)^{\lambda} \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x \right)_x w_t \,\mathrm{d}x \\
&= \int (1+t)^{\lambda} \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x \right) w_{xt} \,\mathrm{d}x \\
&= (1+t)^{\lambda} \frac{\mathrm{d}}{\mathrm{d}t} \int \Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v}) w_x - \frac{p'(\bar{v})}{2} w_x^2 \Big) \,\mathrm{d}x \\
&- \int (1+t)^{\lambda} \bar{v}_t \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x - \frac{p''(\bar{v})}{2} w_x^2 \right) \mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t} \int (1+t)^{\lambda} \Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v}) w_x - \frac{p'(\bar{v})}{2} w_x^2 \Big) \mathrm{d}x \\
&- \lambda \int (1+t)^{\lambda-1} \Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v}) w_x - \frac{p'(\bar{v})}{2} w_x^2 \Big) \mathrm{d}x \\
&+ C\delta \int (1+t)^{\lambda-1} |w_x|^3 \,\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t} \int (1+t)^{\lambda} \Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v}) w_x - \frac{p'(\bar{v})}{2} w_x^2 \Big) \mathrm{d}x \\
&+ C(\varepsilon+\delta) \int (1+t)^{\lambda-1} w_x^2 \,\mathrm{d}x.
\end{aligned} \tag{4.15}$$

Using assumption (H3), Lemma 3.1, the a priori assumption (4.2) and Cauchy-Schwarz's inequality, the last term on the right-hand side of (4.11) can be estimated

$$\begin{aligned}\int g\left(w_t+\bar{u}\right) w_t \,\mathrm{d}x &\le C\int \left(|w_t|^3 + |\bar{u}|^3 \right) |w_t| \,\mathrm{d}x \\ &\le C\left(\varepsilon^2+\delta\right) \int w_t^2 \,\mathrm{d}x + \frac{C}{\delta}\int |\bar{u}|^6 \,\mathrm{d}x \\ &\le C\left(\varepsilon^2+\delta\right) \int w_t^2 \,\mathrm{d}x + C(T)\delta^5.\end{aligned} \tag{4.16}$$

By combining (4.13)-(4.16) and (4.11), and exploiting the smallness of ε and δ, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int \left[(1+t)^{\lambda} w_t^2 - (1+t)^{\lambda} p'(\bar{v}) w_x^2 \right] \mathrm{d}x + \frac{\alpha}{2}\int w_t^2 \,\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t} \int (1+t)^{\lambda} \Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v}) w_x - \frac{p'(\bar{v})}{2} w_x^2 \Big) \mathrm{d}x \\
&+ C\int (1+t)^{\lambda-1} \left(w_x^2 + w_t^2 \right) \mathrm{d}x + C(T)\delta.
\end{aligned} \tag{4.17}$$

Computing (4.17) $\times\, h$ + (4.10) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \Big[ww_t + \frac{\alpha}{2}(1+t)^{-\lambda}w^2 + \frac{h}{2}(1+t)^{\lambda}w_t^2 - \frac{h}{2}(1+t)^{\lambda}p'(\bar v)w_x^2\Big]\,\mathrm{d}x\\
&+ \frac{\alpha\lambda}{4}\int (1+t)^{-\lambda-1}w^2\,\mathrm{d}x + \frac{C_0}{2}\int w_x^2\,\mathrm{d}x + \Big(\frac{\alpha h}{2} - 2\Big)\int w_t^2\,\mathrm{d}x\\
&\le h\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\lambda}\Big(\int_{\bar v}^{w_x+\bar v} p(s)\,\mathrm{d}s - p(\bar v)w_x - \frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\
&\quad + Ch\int (1+t)^{\lambda-1}\left(w_x^2 + w_t^2\right)\,\mathrm{d}x + C(T)\delta.
\end{aligned} \tag{4.18}$$

By choosing $h = 6/\alpha$, applying Gronwall's inequality to ((4.18) on $[0,t]$, and using the fact that ε is small, we derive

$$\|w\|^2 + \|w_t\|^2 + \|w_x\|^2 + \int_0^t \left(\|w_t\|^2 + \|w_x\|^2\right)\,\mathrm{d}s \le C(T)\left(\|w_0\|_1^2 + \|z_0\|^2 + \delta\right). \tag{4.19}$$

This completes the proof. □

Lemma 4.3. *Under the assumptions of Theorem 2.1, for any given $T > 0$, $0 \le \lambda < 1$, $\alpha > 0$, the solution $(w(x,t), w_t(x,t))$ to the initial problem (2.9)-(2.10) on $[0,T]$ satisfies*

$$\|w_{xx}\|^2 + \|w_{xt}\|^2 + \int_0^t \left(\|w_{xx}\|^2 + \|w_{xt}\|^2\right)\,\mathrm{d}s \le C(T)\left(\|w_0\|_2^2 + \|z_0\|_1^2 + \delta\right), \tag{4.20}$$

$$\|w_{xxx}\|^2 + \|w_{xxt}\|^2 + \int_0^t \left(\|w_{xxx}\|^2 + \|w_{xxt}\|^2\right)\,\mathrm{d}s \le C(T)\left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right). \tag{4.21}$$

Proof. Differentiating (2.9) in x,

$$w_{xtt} + (p'(\bar v)w_x)_{xx} + \frac{\alpha}{(1+t)^{\lambda}}w_{xt} = F_x + \frac{g(w_t+\bar u)_x}{(1+t)^{\lambda}}. \tag{4.22}$$

Step 1. Multiplying (4.22) by $(1+t)^{\lambda}w_{xt}$ and integrating the result over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\lambda}w_{xt}^2 - (1+t)^{\lambda}p'(\bar v)w_{xx}^2\right]\,\mathrm{d}x + \int \alpha w_{xt}^2\,\mathrm{d}x\\
&= \frac{\lambda}{2}\int (1+t)^{\lambda-1}w_{xt}^2\,\mathrm{d}x - \frac{\lambda}{2}\int (1+t)^{\lambda-1}p'(\bar v)w_{xx}^2\,\mathrm{d}x\\
&\quad - \frac{1}{2}\int (1+t)^{\lambda}p''(\bar v)\bar v_t w_{xx}^2\,\mathrm{d}x - \int (1+t)^{\lambda}p'''(\bar v)|\bar v_x|^2 w_x w_{xt}\,\mathrm{d}x\\
&\quad - \int (1+t)^{\lambda}p''(\bar v)\bar v_{xx} w_x w_{xt}\,\mathrm{d}x - \int (1+t)^{\lambda}p''(\bar v)\bar v_x w_{xx} w_{xt}\,\mathrm{d}x\\
&\quad + \int (1+t)^{\lambda}F_x w_{xt}\,\mathrm{d}x + \int g'(w_t+\bar u)(w_{xt}+\bar u_x)w_{xt}\,\mathrm{d}x\\
&:= \sum_{k=1}^{8} I_k.
\end{aligned} \tag{4.23}$$

By using Cauchy Schwarz's inequality, Lemma 3.1, and the a priori estimate (4.2), we can estimate the right-hand side as follows

$$I_1 + I_2 \le C\int (1+t)^{\lambda-1}\left(w_{xt}^2 + w_{xx}^2\right)\,\mathrm{d}x, \tag{4.24}$$

$$I_3 \le C\delta\int (1+t)^{\lambda-1}w_{xx}^2\,\mathrm{d}x, \tag{4.25}$$

$$I_4 \le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{2\lambda}|\bar v_x|^4 w_x^2\,\mathrm{d}x \le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C(T)\delta^2, \tag{4.26}$$

$$I_5 \le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{2\lambda}|\bar v_{xx}|^2 w_x^2\,\mathrm{d}x \le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C(T)\delta^2, \tag{4.27}$$

$$I_6 \le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{2\lambda}|\bar v_x|^2 w_{xx}^2\,\mathrm{d}x \le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C(T)\delta^2. \tag{4.28}$$

Also we have the estimated

$$\begin{aligned} I_7 = \int \Big[&\frac{1}{\alpha}(1+t)^{2\lambda}p(\bar v)_{xxt} + \frac{\lambda}{\alpha}(1+t)^{2\lambda-1}p(\bar v)_{xx} \\ &-(1+t)^{\lambda}\left(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x\right)_{xx}\Big] w_{xt}\,\mathrm{d}x. \end{aligned} \tag{4.29}$$

By using Lemma 3.1 and the a priori assumption (4.2), we obtain

$$\begin{aligned} &\int \frac{1}{\alpha}(1+t)^{2\lambda}p(\bar v)_{xxt}w_{xt}\,\mathrm{d}x \\ &\le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{4\lambda}\left(|\bar v_{xxt}|^2 + |\bar v_x|^2|\bar v_{xt}|^2 + |\bar v_{xx}|^2|\bar v_t|^2 + |\bar v_x|^4|\bar v_t|^2\right)\mathrm{d}x \\ &\le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C(T)\delta^2, \end{aligned} \tag{4.30}$$

$$\begin{aligned} \int \frac{\lambda}{\alpha}(1+t)^{2\lambda-1}p(\bar v)_{xx}w_{xt}\,\mathrm{d}x &\le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{4\lambda-2}\left(|\bar v_{xx}|^2+|\bar v_x|^4\right)\mathrm{d}x \\ &\le \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x + C(T)\delta^2, \end{aligned} \tag{4.31}$$

$$\begin{aligned} &-\int (1+t)^{\lambda}\left(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x\right)_{xx} w_{xt}\,\mathrm{d}x \\ &= \int (1+t)^{\lambda}\left(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x\right)_{x} w_{xxt}\,\mathrm{d}x \\ &= (1+t)^{\lambda}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)w_{xx}^2\,\mathrm{d}x \\ &\quad -\frac{1}{2}\int (1+t)^{\lambda}\left(p''\left(w_x+\bar v\right)\left(w_{xt}+\bar v_t\right)-p''(\bar v)\bar v_t\right)w_{xx}^2\,\mathrm{d}x \\ &\quad +\int (1+t)^{\lambda}\bar v_x w_{xxt}\left(p'\left(w_x+\bar v\right)-p'(\bar v)-p''(\bar v)w_x\right)\mathrm{d}x \\ &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\lambda}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)w_{xx}^2\,\mathrm{d}x \\ &\quad -\frac{\lambda}{2}\int (1+t)^{\lambda-1}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)w_{xx}^2\,\mathrm{d}x + C(T)\left(\varepsilon+\delta\right)\int (1+t)^{\lambda-1}w_{xx}^2\,\mathrm{d}x \\ &\quad +\int (1+t)^{\lambda}\bar v_x w_{xxt}\left(p'\left(w_x+\bar v\right)-p'(\bar v)-p''(\bar v)w_x\right)\mathrm{d}x \\ &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\lambda}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)w_{xx}^2\,\mathrm{d}x + \frac{\alpha}{16}\int w_{xt}^2\,\mathrm{d}x \\ &\quad +C(T)\left(\varepsilon+\delta\right)\int (1+t)^{\lambda-1}w_{xx}^2\,\mathrm{d}x + C(T)\delta. \end{aligned} \tag{4.32}$$

By applying assumption (H3), Lemma 3.1, the a priori assumption (4.2) and Cauchy-Schwarz's inequality, we can estimate the last term on the right-hand side of (4.23) as

$$\begin{aligned} \int g'\left(w_t+\bar u\right)\left(w_{xt}+\bar u_x\right)w_{xt}\,\mathrm{d}x &\le C\int \left(w_t+\bar u\right)^2\left(w_{xt}^2+|\bar u_x w_{xt}|\right)\mathrm{d}x \\ &\le C(\varepsilon^2+\delta^2)\int w_{xt}^2\,\mathrm{d}x + C\int \left(w_t+\bar u\right)^2\bar u_x w_{xt}\,\mathrm{d}x \\ &\le C(\varepsilon^2+\delta^2)\int w_{xt}^2\,\mathrm{d}x + C(\varepsilon^2+\delta^2)\int \bar u_x^2\,\mathrm{d}x \\ &\le C(\varepsilon^2+\delta^2)\int w_{xt}^2\,\mathrm{d}x + C(T)\delta^2. \end{aligned} \tag{4.33}$$

Combining (4.24) -(4.33) with (4.23), and using the smallness of ε, δ, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\lambda} w_{xt}^2 - (1+t)^{\lambda} p'(\bar{v}) w_{xx}^2\right] \mathrm{d}x + \frac{\alpha}{2}\int w_{xt}^2 \,\mathrm{d}x \\
&\leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\lambda}\left(p'\left(w_x+\bar{v}\right) - p'(\bar{v})\right) w_{xx}^2 \,\mathrm{d}x \\
&\quad + C\int (1+t)^{\lambda-1}\left(w_{xx}^2 + w_{xt}^2\right) \mathrm{d}x + C(T)\delta.
\end{aligned} \tag{4.34}$$

Step 2. Multiplying (2.9) by $-w_{xx}$, and integrating the resulting equation over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left[w_x w_{xt} + \frac{\alpha}{2}(1+t)^{-\lambda} w_x^2\right] \mathrm{d}x + \frac{\alpha\lambda}{2}\int (1+t)^{-\lambda-1} w_x^2 \,\mathrm{d}x - \int p'(\bar{v}) w_{xx}^2 \,\mathrm{d}x \\
&= \int w_{xt}^2 \,\mathrm{d}x + \int p''(\bar{v})\bar{v}_x w_x w_{xx} \,\mathrm{d}x - \int F w_{xx} \,\mathrm{d}x - \int (1+t)^{-\lambda} g(w_t+\bar{u}) w_{xx} \,\mathrm{d}x.
\end{aligned} \tag{4.35}$$

By applying Cauchy-Schwarz's inequality together with the a priori assumption (4.2), we obtain

$$\int p''(\bar{v})\bar{v}_x w_x w_{xx} \,\mathrm{d}x \leq \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C\int |\bar{v}_x|^2 w_x^2 \,\mathrm{d}x \leq \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C(T)\delta^2. \tag{4.36}$$

We estimate the third term of the right-hand side on (4.35) by using (2.11) as follows

$$\begin{aligned}
-\int F w_{xx} \,\mathrm{d}x = -\int \Big[&\frac{1}{\alpha}(1+t)^{\lambda} p(\bar{v})_{xt} + \frac{\lambda}{\alpha}(1+t)^{\lambda-1} p(\bar{v})_x \\
&- \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x\right)_x\Big] w_{xx} \,\mathrm{d}x.
\end{aligned} \tag{4.37}$$

Applying Lemma 3.1, the a priori assumption (4.2), and Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
-\int \frac{1}{\alpha}(1+t)^{\lambda} p(\bar{v})_{xt} w_{xx} \,\mathrm{d}x &\leq \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C\int (1+t)^{2\lambda}\left(|\bar{v}_{xt}|^2 + |\bar{v}_x|^2|\bar{v}_t|^2\right) \mathrm{d}x \\
&\leq \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C(T)\delta^2,
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
-\int \frac{\lambda}{\alpha}(1+t)^{\lambda-1} p(\bar{v})_x w_{xx} \,\mathrm{d}x &\leq \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C\int (1+t)^{2\lambda-2}|\bar{v}_x|^2 \,\mathrm{d}x \\
&\leq \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C(T)\delta^2,
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
&\int \left(p\left(w_x+\bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x\right)_x w_{xx} \,\mathrm{d}x \\
&= \int \left(p'\left(w_x+\bar{v}\right) - p'(\bar{v})\right) w_{xx}^2 \,\mathrm{d}x + \int \left(p'\left(w_x+\bar{v}\right) - p'(\bar{v}) - p''(\bar{v}) w_x\right)\bar{v}_x w_{xx} \,\mathrm{d}x \\
&\leq C\int |w_x| w_{xx}^2 \,\mathrm{d}x + C\int w_x^2 |\bar{v}_x||w_{xx}| \,\mathrm{d}x \\
&\leq C\varepsilon\int w_{xx}^2 \,\mathrm{d}x + \frac{C_0}{10}\int w_{xx}^2 \,\mathrm{d}x + C(T)\delta^2.
\end{aligned} \tag{4.40}$$

By using the assumption (H3), Lemma 3.1, the a priori assumption (4.2) and Cauchy-Schwarz's inequality, the last term on the right-hand side of (4.35) can be estimated as

$$\begin{aligned}
-\int \frac{g\left(w_t+\bar{u}\right)}{(1+t)^{\lambda}} w_{xx} \,\mathrm{d}x &\leq C\int (1+t)^{-\lambda}\left(|w_t| + |\bar{u}|\right)^3 |w_{xx}| \,\mathrm{d}x \\
&\leq C(\varepsilon+\delta)\int w_{xx}^2 \,\mathrm{d}x + C(T)\delta^2.
\end{aligned} \tag{4.41}$$

Combining (4.36)-(4.41), (4.35), and using the smallness of δ and ε, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[w_xw_{xt}+\frac{\alpha}{2}(1+t)^{-\lambda}w_x^2\Big]\,\mathrm{d}x+\frac{\alpha\lambda}{2}\int(1+t)^{-\lambda-1}w_x^2\,\mathrm{d}x+\frac{C_0}{2}\int w_{xx}^2\,\mathrm{d}x\\&\leq\int w_{xt}^2\,\mathrm{d}x+C\delta^2.\end{aligned}\tag{4.42}$$

Computing (4.34) $\times\, h$ + (4.42) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[w_xw_{xt}+\frac{\alpha}{2}(1+t)^{-\lambda}w_x^2+\frac{h}{2}(1+t)^{\lambda}w_{xt}^2-\frac{h}{2}(1+t)^{\lambda}p'(\bar v)w_{xx}^2\Big]\,\mathrm{d}x\\&+\frac{\alpha\lambda}{2}\int(1+t)^{-\lambda-1}w_x^2\,\mathrm{d}x+\frac{C_0}{2}\int w_{xx}^2\,\mathrm{d}x+\big(\frac{\alpha h}{2}-1\big)\int w_{xt}^2\,\mathrm{d}x\\&\leq\frac{h}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\lambda}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)w_{xx}^2\,\mathrm{d}x\\&\quad+Ch\int(1+t)^{\lambda-1}\left(w_{xx}^2+w_{xt}^2\right)\,\mathrm{d}x+C\delta.\end{aligned}\tag{4.43}$$

Choosing $h=6/\alpha$, applying Gronwall's inequality to (4.43), and using (4.3), we obtain

$$\|w_{xx}\|^2+\|w_{xt}\|^2+\int_0^t\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right)\,\mathrm{d}s\leq C(T)\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\tag{4.44}$$

This completes the proof. □

Similar calculations to (4.3) and (4.20) yield (4.21) (whose proof is omitted). Consequently, (4.1) follows from (4.3), (4.20) and (4.21).

Finally, it remains to show that the a priori assumption (4.2) can be closed. Under this a priori assumption, we have proved that (4.1) holds provided ε is sufficiently small. Therefore, by a continuity argument, the assumption (4.2) is always satisfied whenever $\|w_0\|_3^2+\|z_0\|_2^2+\delta$ is sufficiently small. This completes the proof of Theorem 4.1.

5. Decay rates

The aim of this section is to derive the decay rates for the solution $(w,z)(x,t)$. To this end, we introduce a new a priori assumption

$$\begin{aligned}&\sup_{0\leq t\leq T}\big\{\|w\|_{L^\infty}+\|w_x(\cdot,t)\|_{L^\infty}+\|w_t(\cdot,t)\|_{L^\infty}\\&+(1+t)\|w_{xt}(\cdot,t)\|_{L^\infty}+(1+t)\|w_{xx}(\cdot,t)\|_{L^\infty}\big\}\leq\varepsilon_1,\end{aligned}\tag{5.1}$$

where $0<\varepsilon_1\ll 1$.

Theorem 5.1. *Under assumptions* (H1)–(H3)*, let* $\delta:=|v_+-v_-|$*, when* $(w_0,z_0)\in H^3\times H^2$ *and* $\|w_0\|_3^2+\|z_0\|_2^2+\delta$ *is sufficiently small. If* $w\in C^k\left((0,T),H^{3-k}(\mathbb{R})\right)(k=0,1,2,3)$ *is a smooth solution of* (2.9)-(2.10)*, then there exists a constant* $C>0$ *independent of* t *such that when* $0\leq\lambda<\frac{1}{7}$*, we have*

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^{(\lambda+1)k}\|\partial_x^kw(t)\|^2+\sum_{k=0}^{2}(1+t)^{(\lambda+1)k+2}\|\partial_x^kw_t(t)\|^2\\&+\int_0^t\Big(\sum_{j=1}^{3}(1+s)^{(\lambda+1)j-1}\|\partial_x^jw(s)\|^2+\sum_{j=0}^{2}(1+s)^{(\lambda+1)j+1}\|\partial_x^jw_t(s)\|^2\Big)\,\mathrm{d}s\\&\leq C\big(\|w_0\|_3^2+\|z_0\|_2^2+\delta\big),\end{aligned}\tag{5.2}$$

when $1/7<\lambda<1/3$*, we have*

$$\begin{aligned}&\sum_{k=0}^{3}(1+t)^{(\lambda+1)k+\frac{1}{2}-\frac{7\lambda}{2}}\|\partial_x^kw(t)\|^2+\sum_{k=0}^{2}(1+t)^{(\lambda+1)k+\frac{5}{2}-\frac{7\lambda}{2}}\|\partial_x^kw_t(t)\|^2\\&\leq C\left(\|w_0\|_3^2+\|z_0\|_2^2+\delta\right),\end{aligned}\tag{5.3}$$

when $\beta \in (\frac{1}{2} - \frac{5\lambda}{2}, \lambda)$, *we have*

$$\begin{aligned}&\int_0^t \Big(\sum_{j=0}^{3} (1+s)^{(\lambda+1)(j-1)+\beta} \|\partial_x^j w(s)\|^2 + \sum_{j=0}^{2} (1+s)^{(\lambda+1)j+\beta-\lambda+1} \|\partial_x^j w_t(s)\|^2 \Big) \,\mathrm{d}s \\ &\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}} \left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta \right),\end{aligned} \tag{5.4}$$

and when $\lambda = 1/7$, *we have for any sufficiently small* $\varepsilon > 0$,

$$\begin{aligned}&\sum_{k=0}^{3} (1+t)^{\frac{8k}{7}} \|\partial_x^k w(t)\|^2 + \sum_{k=0}^{2} (1+t)^{\frac{8k}{7}+2} \|\partial_x^k w_t(t)\|^2 \\ &+ \int_0^t \Big(\sum_{j=1}^{3} (1+s)^{\frac{8j}{7}-1} \|\partial_x^j w(s)\|^2 + \sum_{j=0}^{2} (1+s)^{\frac{8j}{7}+1} \|\partial_x^j w_t(s)\|^2 \Big) \,\mathrm{d}s \\ &\le C(1+t)^{\varepsilon} \left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta \right).\end{aligned} \tag{5.5}$$

To prove Theorem 5.1, we establish the following series of lemmas.

Lemma 5.2. *Under the assumptions of Theorem 5.1, it holds that*

$$\begin{aligned}&\|w\|^2 + (1+t)^{2\lambda} \left(\|w_t\|^2 + \|w_x\|^2 \right) + \int_0^t (1+s)^{\lambda} \left(\|w_t\|^2 + \|w_x\|^2 \right) \mathrm{d}s \\ &\le C \left(\|w_0\|_1^2 + \|z_0\|^2 + \delta \right),\end{aligned} \tag{5.6}$$

for $0 \le \lambda < 1/7$.

Proof. Multiplying (2.9) by $(1+t)^{\beta} w$ and integrating it over $\mathbb{R}$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \int \Big[(1+t)^{\beta} w w_t + \frac{\alpha}{2} (1+t)^{\beta-\lambda} w^2 \Big] \,\mathrm{d}x + \frac{a(\lambda-\beta)}{2} \int (1+t)^{\beta-\lambda-1} w^2 \,\mathrm{d}x \\ &- \int (1+t)^{\beta} p'(\bar{v}) w_x^2 \,\mathrm{d}x \\ &= \int (1+t)^{\beta} w_t^2 \,\mathrm{d}x + \beta \int (1+t)^{\beta-1} w w_t \,\mathrm{d}x + \int (1+t)^{\beta} F w \,\mathrm{d}x \\ &\quad + \int (1+t)^{\beta-\lambda} g \left(w_t + \bar{u} \right) w \,\mathrm{d}x.\end{aligned} \tag{5.7}$$

By applying Cauchy-Schwarz's inequality, the right-hand side of (5.7) can be estimated as

$$\beta \int (1+t)^{\beta-1} w w_t \,\mathrm{d}x \le C \int (1+t)^{\beta-2} w^2 \,\mathrm{d}x + \int (1+t)^{\beta} w_t^2 \,\mathrm{d}x. \tag{5.8}$$

From (2.11), we can derive the third term on the right-hand side of (5.7),

$$\begin{aligned}\int (1+t)^{\beta} F w \,\mathrm{d}x = \int [&\frac{1}{\alpha} (1+t)^{\beta+\lambda} p(\bar{v})_{xt} + \frac{\lambda}{\alpha} (1+t)^{\beta+\lambda-1} p(\bar{v})_x \\ &- (1+t)^{\beta} \left(p\left(w_x + \bar{v}\right) - p(\bar{v}) - p'(\bar{v}) w_x \right)_x] w \,\mathrm{d}x.\end{aligned} \tag{5.9}$$

Applying Lemma 3.1, Cauchy-Schwarz's inequality and the fact that $p'(v) \le -C_0 < 0$, we obtain

$$\begin{aligned}\int \frac{1}{\alpha} (1+t)^{\beta+\lambda} p(\bar{v})_{xt} w \,\mathrm{d}x &= -\int \frac{1}{\alpha} (1+t)^{\beta+\lambda} p(\bar{v})_t w_x \,\mathrm{d}x \\ &\le \frac{C_0}{4} \int (1+t)^{\beta} w_x^2 \,\mathrm{d}x + C \int (1+t)^{\beta+2\lambda} |\bar{v}_t|^2 \,\mathrm{d}x \\ &\le \frac{C_0}{4} \int (1+t)^{\beta} w_x^2 \,\mathrm{d}x + C\delta^2 (1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}},\end{aligned} \tag{5.10}$$

$$\begin{aligned}\int \frac{\lambda}{\alpha} (1+t)^{\beta+\lambda-1} p(\bar{v})_x w \,\mathrm{d}x &\le \frac{\nu}{2} \int (1+t)^{-\kappa} w^2 \,\mathrm{d}x + \frac{C}{\nu} \int (1+t)^{2\beta+2\lambda+\kappa-2} |\bar{v}_x|^2 \,\mathrm{d}x \\ &\le \frac{\nu}{2} \int (1+t)^{-\kappa} w^2 \,\mathrm{d}x + \frac{C\delta^2}{\nu} (1+t)^{2\beta+\frac{3\lambda}{2}+\kappa-\frac{5}{2}},\end{aligned} \tag{5.11}$$

where $\kappa > 1$, $\upsilon > 0$ are constants to be determined later.

Applying Lemma 3.1 and (5.1), we obtain

$$\begin{aligned}
&-\int (1+t)^{\beta}\left(p\left(w_x+\bar{v}\right)-p(\bar{v})-p'(\bar{v})w_x\right)_x w\,\mathrm{d}x\\
&=\int (1+t)^{\beta}\left(p\left(w_x+\bar{v}\right)-p(\bar{v})-p'(\bar{v})w_x\right)w_x\,\mathrm{d}x\\
&\le C\varepsilon_1\int (1+t)^{\beta}w_x^2\,\mathrm{d}x.
\end{aligned}\tag{5.12}$$

Applying assumption (H3), Lemma 3.1, (5.1) and Cauchy-Schwarz's inequality, we can estimate the last term on the right-hand side of (5.7) as

$$\begin{aligned}
&\int (1+t)^{\beta-\lambda}g\left(w_t+\bar{u}\right)w\,\mathrm{d}x\\
&\le C\int (1+t)^{\beta-\lambda}\left(|w_t|^3+|\bar{u}|^3\right)|w|\,\mathrm{d}x\\
&\le \frac{\nu}{2}\int (1+t)^{-\kappa}w^2\,\mathrm{d}x+C\varepsilon_1\int (1+t)^{\beta-\lambda}w_t^2\,\mathrm{d}x+C\delta^2(1+t)^{2\beta+\frac{3\lambda}{2}-\frac{5}{2}+\kappa}.
\end{aligned}\tag{5.13}$$

From (5.8)-(5.13) and (5.7), it follows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[(1+t)^{\beta}ww_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}w^2\right]\mathrm{d}x+\frac{a(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1}w^2\,\mathrm{d}x\\
&+\frac{C_0}{2}\int (1+t)^{\beta}w_x^2\,\mathrm{d}x\\
&\le 3\int (1+t)^{\beta}w_t^2\,\mathrm{d}x+\nu\int (1+t)^{-\kappa}w^2\,\mathrm{d}x+C\int (1+t)^{\beta-2}w^2\,\mathrm{d}x\\
&\quad+C\delta^2(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}+C\delta^2(1+t)^{2\beta+\frac{3\lambda}{2}+\kappa-\frac{5}{2}}.
\end{aligned}\tag{5.14}$$

Multiplying (2.9) by $(1+t)^{\beta+\lambda}w_t$ and integrating over $\mathbb{R}$, we apply Lemma 3.1 to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left[(1+t)^{\beta+\lambda}w_t^2-(1+t)^{\beta+\lambda}p'(\bar{v})w_x^2\right]\mathrm{d}x+\int\alpha(1+t)^{\beta}w_t^2\,\mathrm{d}x\\
&=-\frac{1}{2}\int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_tw_x^2\,\mathrm{d}x-\frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}p'(\bar{v})w_x^2\,\mathrm{d}x\\
&\quad+\frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}w_t^2\,\mathrm{d}x+\int (1+t)^{\beta+\lambda}Fw_t\,\mathrm{d}x\\
&\quad+\int (1+t)^{\beta}g\left(w_t+\bar{u}\right)w_t\,\mathrm{d}x\\
&\le C\int (1+t)^{\beta+\lambda-1}(w_x^2+w_t^2)\,\mathrm{d}x+\int (1+t)^{\beta+\lambda}Fw_t\,\mathrm{d}x\\
&\quad+\int (1+t)^{\beta}g\left(w_t+\bar{u}\right)w_t\,\mathrm{d}x.
\end{aligned}\tag{5.15}$$

The second term on the right-hand side of (5.15) admits the following estimate

$$\begin{aligned}
\int (1+t)^{\beta+\lambda}Fw_t\,\mathrm{d}x=\int\Big[&\frac{1}{\alpha}(1+t)^{\beta+2\lambda}p(\bar{v})_{xt}+\frac{\lambda}{\alpha}(1+t)^{\beta+2\lambda-1}p(\bar{v})_x\\
&-(1+t)^{\beta+\lambda}\left(p\left(w_x+\bar{v}\right)-p(\bar{v})-p'(\bar{v})w_x\right)_x\Big]w_t\,\mathrm{d}x.
\end{aligned}\tag{5.16}$$

By applying Lemma 3.1 together with Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
\int\frac{1}{\alpha}(1+t)^{\beta+2\lambda}p(\bar{v})_{xt}w_t\,\mathrm{d}x&\le\frac{\alpha}{8}\int (1+t)^{\beta}w_t^2\,\mathrm{d}x+C\int (1+t)^{\beta+4\lambda}\left(|\bar{v}_{xt}|^2+|\bar{v}_x|^2|\bar{v}_t|^2\right)\mathrm{d}x\\
&\le\frac{\alpha}{8}\int (1+t)^{\beta}w_t^2\,\mathrm{d}x+C\delta^2(1+t)^{\beta+\frac{7\lambda}{2}-\frac{5}{2}},
\end{aligned}\tag{5.17}$$

$$\begin{aligned}\int \frac{\lambda}{\alpha}(1+t)^{\beta+2\lambda-1}p(\bar v)_x w_t\,\mathrm{d}x &\le \frac{\alpha}{8}\int (1+t)^{\beta}w_t^2\,\mathrm{d}x + C\int (1+t)^{\beta+4\lambda-2}|\bar v_x|^2\,\mathrm{d}x\\ &\le \frac{\alpha}{8}\int (1+t)^{\beta}w_t^2\,\mathrm{d}x + C\delta^2(1+t)^{\beta+\frac{7\lambda}{2}-\frac{5}{2}}.\end{aligned} \tag{5.18}$$

By Lemma 3.1 and (5.1), we obtain

$$\begin{aligned}&-\int (1+t)^{\beta+\lambda}\left(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x\right)_x w_t\,\mathrm{d}x\\ &=\int (1+t)^{\beta+\lambda}\left(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x\right)w_{xt}\,\mathrm{d}x\\ &=(1+t)^{\beta+\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\int\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\ &\quad-\int (1+t)^{\beta+\lambda}\bar v_t\Big(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x-\frac{p''(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\ &\le\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\ &\quad-(\beta+\lambda)\int (1+t)^{\beta+\lambda-1}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\ &\quad+C\delta\int (1+t)^{\beta+\lambda-1}|w_x|^3\,\mathrm{d}x\\ &\le\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\ &\quad+C(\delta+\varepsilon_1)\int (1+t)^{\beta+\lambda-1}w_x^2\,\mathrm{d}x.\end{aligned} \tag{5.19}$$

Applying assumption (H3), Lemma 3.1, (5.1) and Cauchy-Schwarz's inequality, we can estimate the last term on the right-hand side of (5.15) as

$$\begin{aligned}&\int (1+t)^{\beta}g\left(w_t+\bar u\right)w_t\,\mathrm{d}x\\ &\le C\int (1+t)^{\beta}(|w_t|^3+|\bar u|^3)|w_t|\,\mathrm{d}x\\ &\le C\varepsilon_1\int (1+t)^{\beta}w_t^2\,\mathrm{d}x+\frac{\alpha}{8}\int (1+t)^{\beta}w_t^2\,\mathrm{d}x+C\delta^2(1+t)^{\beta+\frac{7\lambda}{2}-\frac{5}{2}}.\end{aligned} \tag{5.20}$$

From (5.16)-(5.20) with (5.15), we derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left[(1+t)^{\beta+\lambda}w_t^2-(1+t)^{\beta+\lambda}p'(\bar v)w_x^2\right]\,\mathrm{d}x+\frac{\alpha}{2}\int (1+t)^{\beta}w_t^2\,\mathrm{d}x\\ &\le\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\ &\quad+C\int (1+t)^{\beta+\lambda-1}\left(w_x^2+w_t^2\right)\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{7\lambda}{2}-\frac{5}{2}}.\end{aligned} \tag{5.21}$$

From (5.21) $\times h$ + (5.14), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[(1+t)^{\beta}ww_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}w^2+\frac{h}{2}(1+t)^{\beta+\lambda}w_t^2-\frac{h}{2}(1+t)^{\beta+\lambda}p'(\bar v)w_x^2\Big]\,\mathrm{d}x\\
&+\frac{a(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}w^2\,\mathrm{d}x+\frac{C_0}{2}\int(1+t)^{\beta}w_x^2\,\mathrm{d}x\\
&+\Big(\frac{\alpha h}{2}-3\Big)\int(1+t)^{\beta}w_t^2\,\mathrm{d}x\\
&\le h\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\
&\quad+\nu\int(1+t)^{-\kappa}w^2\,\mathrm{d}x+C\int(1+t)^{\beta-2}w^2\,\mathrm{d}x+Ch\int(1+t)^{\beta+\lambda-1}\left(w_x^2+w_t^2\right)\,\mathrm{d}x\\
&\quad+C\delta^2(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}+C\delta^2(1+t)^{2\beta+\frac{3\lambda}{2}-\frac{5}{2}+\kappa}.
\end{aligned}\tag{5.22}$$

Since $0\le\lambda<\frac{1}{7}$, we have $1<\frac{3}{2}-\frac{7\lambda}{2}$. We set $\beta=\lambda,\upsilon=1$, and choose a constant κ such that $1<\kappa<\frac{3}{2}-\frac{7\lambda}{2}$. Then we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[(1+t)^{\lambda}ww_t+\frac{\alpha}{2}w^2+\frac{h}{2}(1+t)^{2\lambda}w_t^2-\frac{h}{2}(1+t)^{2\lambda}p'(\bar v)w_x^2\Big]\,\mathrm{d}x\\
&+\frac{C_0}{2}\int(1+t)^{\lambda}w_x^2\,\mathrm{d}x+\Big(\frac{\alpha h}{2}-3\Big)\int(1+t)^{\lambda}w_t^2\,\mathrm{d}x\\
&\le h\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{2\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x\\
&\quad+C\int(1+t)^{-\kappa}w^2\,\mathrm{d}x+C\int(1+t)^{\lambda-2}w^2\,\mathrm{d}x\\
&\quad+Ch\int(1+t)^{2\lambda-1}\left(w_x^2+w_t^2\right)\,\mathrm{d}x+C\delta(1+t)^{\frac{7\lambda}{2}+\kappa-\frac{5}{2}}.
\end{aligned}\tag{5.23}$$

We choose $T_0\gg1$ so that for $t\ge T_0$,

$$\begin{aligned}
Ch(1+t)^{\lambda-1}&\le\frac{C_0}{4},\\
Ch(1+t)^{\lambda-1}&\le\frac{1}{2}(\frac{\alpha h}{2}-3).
\end{aligned}\tag{5.24}$$

We define

$$H(t)\sim\|w\|^2+(1+t)^{2\lambda}\|w_x\|^2+(1+t)^{2\lambda}\|w_t\|^2.$$

Taking $h=\frac{10}{\alpha}$, from $\kappa<\frac{3}{2}-\frac{7\lambda}{2}<2-\lambda$ we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H(t)+\frac{C_0}{4}\int(1+t)^{\lambda}w_x^2\,\mathrm{d}x+\int(1+t)^{\lambda}w_t^2\,\mathrm{d}x\\
&\le h\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{2\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\,\mathrm{d}x+C(1+t)^{-\kappa}H(t)\\
&\quad+C\delta(1+t)^{\frac{7\lambda}{2}-\frac{5}{2}+\kappa}.
\end{aligned}\tag{5.25}$$

Applying Gronwall's inequality to (5.25) on $[T_0,t]$ and noting that $1<\kappa<\frac{3}{2}-\frac{7\lambda}{2}$, with $0\le\lambda<\frac{1}{7}$, we obtain

$$H(t)+C\int_{T_0}^{t}(1+s)^{\lambda}\left(\|w_x\|^2+\|w_t\|^2\right)\,\mathrm{d}x\le C\varepsilon_1\int(1+t)^{2\lambda}w_x^2\,\mathrm{d}x+C\left(H\left(T_0\right)+\delta\right).\tag{5.26}$$

From (4.3), (5.26) and Theorem 4.1, we obtain (5.6), The proof is complete. □

Lemma 5.3. *Under the assumptions of Theorem 5.1, it holds that*

$$(1+t)^{\frac{1}{2}-\frac{7\lambda}{2}}\|w\|^2+(1+t)^{\frac{1}{2}-\frac{3\lambda}{2}}\left(\|w_x\|^2+\|w_t\|^2\right)\le C\left(\|w_0\|_1^2+\|z_0\|^2+\delta\right),\tag{5.27}$$

$$\begin{aligned}&\int_0^t \left[(1+s)^{\beta-\lambda-1}\|w\|^2+(1+s)^{\beta}\left(\|w_x\|^2+\|w_t\|^2\right)\right]\mathrm{d}s\\&\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_1^2+\|z_0\|^2+\delta\right),\quad \text{for } \frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda.\end{aligned}\tag{5.28}$$

Proof. As in the proof of Lemma 5.2, observing that $\lambda>\frac{1}{2}-\frac{5\lambda}{2}$ holds because of $\frac{1}{7}<\lambda<\frac{1}{3}$, we let $\frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda, \kappa=\lambda-\beta+1$, and $\nu=\frac{\alpha(\lambda-\beta)}{4}>0$ in (5.22) accordingly. Thereby we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[(1+t)^{\beta}ww_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}w^2+\frac{h}{2}(1+t)^{\beta+\lambda}w_t^2\\&-\frac{h}{2}(1+t)^{\beta+\lambda}p'(\bar v)w_x^2\Big]\mathrm{d}x+\frac{\alpha(\lambda-\beta)}{4}\int(1+t)^{\beta-\lambda-1}w^2\,\mathrm{d}x\\&+\frac{C_0}{2}\int(1+t)^{\beta}w_x^2\,\mathrm{d}x+\Big(\frac{\alpha h}{2}-3\Big)\int(1+t)^{\beta}w_t^2\,\mathrm{d}x\\&\le h\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\mathrm{d}x\\&\quad+C\int(1+t)^{\beta-2}w^2\,\mathrm{d}x+Ch\int(1+t)^{\beta+\lambda-1}\left(w_x^2+w_t^2\right)\mathrm{d}x+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}.\end{aligned}\tag{5.29}$$

We define

$$H_1(t)\sim(1+t)^{\beta-\lambda}\|w\|^2+(1+t)^{\beta+\lambda}\|w_x\|^2+(1+t)^{\beta+\lambda}\|w_t\|^2.$$

Taking $h=\frac{10}{\alpha}$, we choose $T_1\gg1$ so that if $t\ge T_1$, then

$$Ch(1+t)^{\lambda-1}\le\min\Big\{\frac{C_0}{4},\frac{1}{2}\Big(\frac{\alpha h}{2}-3\Big)\Big\}.$$

Then we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\frac{C_0}{4}\int(1+t)^{\beta}w_x^2\,\mathrm{d}x+\int(1+t)^{\beta}w_t^2\,\mathrm{d}x+c\int(1+t)^{\beta-\lambda-1}w^2\,\mathrm{d}x\\&\le h\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda}\Big(\int_{\bar v}^{w_x+\bar v}p(s)\,\mathrm{d}s-p(\bar v)w_x-\frac{p'(\bar v)}{2}w_x^2\Big)\mathrm{d}x\\&\quad+C(1+t)^{\lambda-2}H_1(t)+C\delta^2(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}.\end{aligned}\tag{5.30}$$

Using Gronwall's inequality on the interval $[T_1,t]$ for (5.30), with the fact that $\beta+\frac{5\lambda}{2}-\frac{3}{2}>-1$ and $1/7<\lambda<1/3$, we conclude that

$$\begin{aligned}&H_1(t)+\int_{T_1}^t\left[(1+s)^{\beta}\left(\|w_x\|^2+\|w_t\|^2\right)+(1+s)^{\beta-\lambda-1}\|w\|^2\right]\mathrm{d}s\\&\le C\varepsilon_1\int(1+t)^{\beta+\lambda}w_x^2\,\mathrm{d}x+H(T_1)+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}},\end{aligned}\tag{5.31}$$

which, in conjunction with (4.3), and given that $\beta+\frac{5\lambda}{2}-\frac{1}{2}>0$, gives (5.27) and (5.28). This completes the proof. □

In the subsequent lemmas, we establish the improved decay rates for w_x and w_t.

Lemma 5.4. *Under the assumptions of Theorem 5.1, it holds that for* $0\le\lambda<1/7$,

$$(1+t)^{\lambda+1}\left(\|w_x\|^2+\|w_t\|^2\right)+\int_0^t(1+s)\|w_t\|^2\,\mathrm{d}s\le C\left(\|w_0\|_1^2+\|z_0\|^2+\delta\right),\tag{5.32}$$

and for $1/7<\lambda<1/3$,

$$\begin{aligned}&(1+t)^{\frac{3}{2}-\frac{5\lambda}{2}}\left(\|w_x\|^2+\|w_t\|^2\right)\le C\left(\|w_0\|_1^2+\|z_0\|^2+\delta\right),\\&\int_0^t(1+s)^{\beta-\lambda+1}\|w_t\|^2\,\mathrm{d}s\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_1^2+\|z_0\|^2+\delta\right),\quad\frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda.\end{aligned}\tag{5.33}$$

Proof. When $0 \le \lambda < 1/7$, multiplying (5.21) by $(1+t)^{1-\lambda}$, and taking $\beta = \lambda$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\lambda+1}w_t^2 - (1+t)^{\lambda+1}p'(\bar{v})w_x^2\right]\,\mathrm{d}x + c\int (1+t)w_t^2\,\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\lambda+1}\Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v})w_x - \frac{p'(\bar{v})}{2}w_x^2\Big)\,\mathrm{d}x \\
&\quad - (1-\lambda)\int (1+t)^{\lambda}\Big(\int_{\bar{v}}^{w_x+\bar{v}} p(s)\,\mathrm{d}s - p(\bar{v})w_x - \frac{p'(\bar{v})}{2}w_x^2\Big)\,\mathrm{d}x \\
&\quad + C\int \left((1+t)^{\lambda}w_t^2 + (1+t)^{\lambda}w_x^2\right)\,\mathrm{d}x + C\delta(1+t)^{\frac{7\lambda}{2}-\frac{3}{2}}.
\end{aligned} \tag{5.34}$$

By integrating the above inequality over $[0,t]$ and using (5.6), we arrive at the improved decay rate

$$(1+t)^{\lambda+1}\left(\|w_x\|^2 + \|w_t\|^2\right) + \int_0^t (1+s)\|w_t\|^2\,\mathrm{d}s \le C\left(\|w_0\|_1^2 + \|z_0\|^2 + \delta\right). \tag{5.35}$$

The proof of (5.32) is complete. A similar calculation yields (5.33), and its proof is omitted. □

Lemma 5.5. *Under the assumptions of Theorem 5.1, for the case of $0 \le \lambda < 1/7$, it holds that*

$$\begin{aligned}
&(1+t)^{2\lambda+2}\left(\|w_{xx}\|^2 + \|w_{xt}\|^2\right) + \int_0^t \left((1+s)^{2\lambda+1}\|w_{xx}\|^2 + (1+s)^{\lambda+2}\|w_{xt}\|^2\right)\,\mathrm{d}s \\
&\le C\left(\|w_0\|_2^2 + \|z_0\|_1^2 + \delta\right).
\end{aligned} \tag{5.36}$$

Proof. Multiplying (4.22) by $(1+t)^{\beta+\lambda}w_{xt}$ and integrating over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\beta+\lambda}w_{xt}^2 - (1+t)^{\beta+\lambda}p'(\bar{v})w_{xx}^2\right]\,\mathrm{d}x + \int \alpha(1+t)^{\beta}w_{xt}^2\,\mathrm{d}x \\
&= \frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}w_{xt}^2\,\mathrm{d}x - \frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}p'(\bar{v})w_{xx}^2\,\mathrm{d}x \\
&\quad - \frac{1}{2}\int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_t w_{xx}^2\,\mathrm{d}x - \int (1+t)^{\beta+\lambda}p'''(\bar{v})|\bar{v}_x|^2 w_x w_{xt}\,\mathrm{d}x \\
&\quad - \int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_{xx}w_x w_{xt}\,\mathrm{d}x - \int (1+t)^{\beta+\lambda}p''(\bar{v})\bar{v}_x w_{xx}w_{xt}\,\mathrm{d}x \\
&\quad + \int (1+t)^{\beta+\lambda}F_x w_{xt}\,\mathrm{d}x + \int (1+t)^{\beta}g'(w_t+\bar{u})(w_{xt}+\bar{u}_x)w_{xt}\,\mathrm{d}x \\
&:= \sum_{k=1}^{8} J_k.
\end{aligned} \tag{5.37}$$

By applying Cauchy-Schwarz's inequality, Lemma 3.1, and (5.1), we obtain

$$J_1 \le C\int (1+t)^{\beta+\lambda-1}w_{xt}^2\,\mathrm{d}x, \tag{5.38}$$

$$J_2 \le C\int (1+t)^{\beta+\lambda-1}w_{xx}^2\,\mathrm{d}x, \tag{5.39}$$

$$J_3 \le C\delta\int (1+t)^{\beta+\lambda-1}w_{xx}^2\,\mathrm{d}x, \tag{5.40}$$

$$\begin{aligned}
J_4 &\le \frac{\alpha}{16}\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{\beta+2\lambda}|\bar{v}_x|^4 w_x^2\,\mathrm{d}x \\
&\le \frac{\alpha}{16}\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x + C\delta^4\int (1+t)^{\beta-2}w_x^2\,\mathrm{d}x,
\end{aligned} \tag{5.41}$$

$$\begin{aligned}
J_5 &\le \frac{\alpha}{16}\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{\beta+2\lambda}|\bar{v}_{xx}|^2 w_x^2\,\mathrm{d}x \\
&\le \frac{\alpha}{16}\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x + C\delta^2\int (1+t)^{\beta-2}w_x^2\,\mathrm{d}x,
\end{aligned} \tag{5.42}$$

$$\begin{aligned} J_6 &\le \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2 \,\mathrm{d}x + C\int (1+t)^{\beta+2\lambda}|\bar v_x|^2 w_{xx}^2 \,\mathrm{d}x \\ &\le \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2 \,\mathrm{d}x + C\delta^2\int (1+t)^{\beta+\lambda-1} w_{xx}^2 \,\mathrm{d}x. \end{aligned} \tag{5.43}$$

Also we have the estimate

$$\begin{aligned} J_7 = \int \Big[&\frac{1}{\alpha}(1+t)^{\beta+2\lambda} p(\bar v)_{xxt} + \frac{\lambda}{\alpha}(1+t)^{\beta+2\lambda-1} p(\bar v)_{xx} \\ &- (1+t)^{\beta+\lambda}\left(p\left(w_x+\bar v\right) - p(\bar v) - p'(\bar v)w_x\right)_{xx}\Big] w_{xt}\,\mathrm{d}x. \end{aligned} \tag{5.44}$$

Using Lemma 3.1 and (5.1), the right-hand side of (5.44) can be estimated term by term as follows

$$\begin{aligned} &\int \frac{1}{\alpha}(1+t)^{\beta+2\lambda} p(\bar v)_{xxt} w_{xt}\,\mathrm{d}x \\ &\le C\int (1+t)^{\beta+4\lambda}\left(|\bar v_{xxt}|^2 + |\bar v_x|^2|\bar v_{xt}|^2 + |\bar v_{xx}|^2|\bar v_t|^2 + |\bar v_x|^4|\bar v_t|^2\right)\,\mathrm{d}x \\ &\quad + \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x + C\delta^2(1+t)^{\beta+\frac{5\lambda}{2}-\frac{7}{2}}, \end{aligned} \tag{5.45}$$

$$\begin{aligned} &\int \frac{\lambda}{\alpha}(1+t)^{\beta+2\lambda-1} p(\bar v)_{xx} w_{xt}\,\mathrm{d}x \\ &\le \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{\beta+4\lambda-2}\left(|\bar v_{xx}|^2 + |\bar v_x|^4\right)\,\mathrm{d}x \\ &\le \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x + C\delta^2(1+t)^{\beta+\frac{5\lambda}{2}-\frac{7}{2}}, \end{aligned} \tag{5.46}$$

$$\begin{aligned} &-\int (1+t)^{\beta+\lambda}\left(p\left(w_x+\bar v\right) - p(\bar v) - p'(\bar v)w_x\right)_{xx} w_{xt}\,\mathrm{d}x \\ &= \int (1+t)^{\beta+\lambda}\left(p\left(w_x+\bar v\right) - p(\bar v) - p'(\bar v)w_x\right)_{x} w_{xxt}\,\mathrm{d}x \\ &= (1+t)^{\beta+\lambda}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(p'\left(w_x+\bar v\right) - p'(\bar v)\right) w_{xx}^2\,\mathrm{d}x \\ &\quad - \frac{1}{2}\int (1+t)^{\beta+\lambda}\left(p''\left(w_x+\bar v\right)\left(w_{xt}+\bar v_t\right) - p''(\bar v)\bar v_t\right) w_{xx}^2\,\mathrm{d}x \\ &\quad + \int (1+t)^{\beta+\lambda}\bar v_x w_{xxt}\left(p'\left(w_x+\bar v\right) - p'(\bar v) - p''(\bar v)w_x\right)\,\mathrm{d}x \\ &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\left(p'\left(w_x+\bar v\right) - p'(\bar v)\right) w_{xx}^2\,\mathrm{d}x \\ &\quad - \frac{\beta+\lambda}{2}\int (1+t)^{\beta+\lambda-1}\left(p'\left(w_x+\bar v\right) - p'(\bar v)\right) w_{xx}^2\,\mathrm{d}x \\ &\quad + C(\varepsilon_1+\delta)\int (1+t)^{\beta+\lambda-1} w_{xx}^2\,\mathrm{d}x \\ &\quad - \int (1+t)^{\beta+\lambda} w_{xt}\bar v_{xx}\left(p'\left(w_x+\bar v\right) - p'(\bar v) - p''(\bar v)w_x\right)\,\mathrm{d}x \\ &\quad - \int (1+t)^{\beta+\lambda} w_{xt}\bar v_x\left(p''\left(w_x+\bar v\right) - p''(\bar v)\right) w_{xx}\,\mathrm{d}x \\ &\quad - \int (1+t)^{\beta+\lambda} w_{xt}\bar v_x^2\left(p''\left(w_x+\bar v\right) - p''(\bar v) - p'''(\bar v)w_x\right)\,\mathrm{d}x \\ &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\left(p'\left(w_x+\bar v\right) - p'(\bar v)\right) w_{xx}^2\,\mathrm{d}x + \frac{\alpha}{16}\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x \\ &\quad + C(\varepsilon_1+\delta)\int (1+t)^{\beta+\lambda-1} w_{xx}^2\,\mathrm{d}x + C\delta\int (1+t)^{\beta-2} w_x^2\,\mathrm{d}x. \end{aligned} \tag{5.47}$$

Applying assumption (H3), Lemma 3.1, (5.1) and Cauchy-Schwarz's inequality, the last term on the right-hand side of (5.37) can be estimated as follows

$$\begin{aligned} J_8 &\le C\int (1+t)^{\beta}\left(w_t^2+\bar{u}^2\right)\left(w_{xt}^2+|\bar{u}_x||w_{xt}|\right)\,\mathrm{d}x \\ &\le C\left(\varepsilon_1+\delta\right)\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x + C\delta^2\int (1+t)^{\beta-2}w_t^2\,\mathrm{d}x \\ &\quad + \frac{\alpha}{16}\int (1+t)^{\beta}w_{xt}^2 + C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{7}{2}}. \end{aligned} \tag{5.48}$$

Substituting (5.38)-(5.48) into (5.37) and choosing $\varepsilon_1 \ll 1, \delta \ll 1$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\beta+\lambda}w_{xt}^2-(1+t)^{\beta+\lambda}p'(\bar{v})w_{xx}^2\right]\,\mathrm{d}x+\frac{\alpha}{2}\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x \\ &\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right)w_{xx}^2\,\mathrm{d}x \\ &\quad + C\int (1+t)^{\beta+\lambda-1}\left(w_{xx}^2+w_{xt}^2\right)\,\mathrm{d}x + C\delta\int (1+t)^{\beta-2}w_x^2\,\mathrm{d}x \\ &\quad + C\delta\int (1+t)^{\beta-2}w_t^2\,\mathrm{d}x + C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{7}{2}}. \end{aligned} \tag{5.49}$$

Multiplying (2.9) by $-(1+t)^{\beta}w_{xx}$ and then integrating the resulting equation over $\mathbb{R}$ yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int \Big[(1+t)^{\beta}w_xw_{xt}+\frac{\alpha}{2}(1+t)^{\beta-\lambda}w_x^2\Big]\,\mathrm{d}x \\ &+\frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1}w_x^2\,\mathrm{d}x-\int (1+t)^{\beta}p'(\bar{v})w_{xx}^2\,\mathrm{d}x \\ &=\int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x+\beta\int (1+t)^{\beta-1}w_xw_{xt}\,\mathrm{d}x+\int (1+t)^{\beta}p''(\bar{v})\bar{v}_xw_xw_{xx}\,\mathrm{d}x \\ &\quad -\int (1+t)^{\beta}Fw_{xx}\,\mathrm{d}x-\int (1+t)^{\beta-\lambda}g(w_t+\bar{u})w_{xx}\,\mathrm{d}x. \end{aligned} \tag{5.50}$$

Applying Cauchy-Schwarz's inequality and Lemma 3.1, the second and third term on the right-hand side of (5.50) can be estimated as follows

$$\beta\int (1+t)^{\beta-1}w_xw_{xt}\,\mathrm{d}x \le \int (1+t)^{\beta}w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{\beta-2}w_x^2\,\mathrm{d}x, \tag{5.51}$$

$$\begin{aligned} \int (1+t)^{\beta}p''(\bar{v})\bar{v}_xw_xw_{xx}\,\mathrm{d}x &\le \frac{C_0}{12}\int (1+t)^{\beta}w_{xx}^2\,\mathrm{d}x + C\int (1+t)^{\beta}|\bar{v}_x|^2w_x^2\,\mathrm{d}x \\ &\le \frac{C_0}{12}\int (1+t)^{\beta}w_{xx}^2\,\mathrm{d}x + C\delta^2\int (1+t)^{\beta-\lambda-1}w_x^2\,\mathrm{d}x. \end{aligned} \tag{5.52}$$

By (2.11), we obtain

$$\begin{aligned} -\int (1+t)^{\beta}Fw_{xx}\,\mathrm{d}x = -\int \Big[&\frac{1}{\alpha}(1+t)^{\beta+\lambda}p(\bar{v})_{xt}+\frac{\lambda}{\alpha}(1+t)^{\beta+\lambda-1}p(\bar{v})_x \\ &-(1+t)^{\beta}\left(p\left(w_x+\bar{v}\right)-p(\bar{v})-p'(\bar{v})w_x\right)_x\Big]w_{xx}\,\mathrm{d}x. \end{aligned} \tag{5.53}$$

By applying Lemma 3.1, (5.1) and Cauchy-Schwarz's inequality, the right-hand side of (5.53) can be estimated term by term as follows

$$\begin{aligned} -\int \frac{1}{\alpha}(1+t)^{\beta+\lambda}p(\bar{v})_{xt}w_{xx}\,\mathrm{d}x &\le \frac{C_0}{12}\int (1+t)^{\beta}w_{xx}^2\,\mathrm{d}x + C\int (1+t)^{\beta+2\lambda}\left(|\bar{v}_{xt}|^2+|\bar{v}_x|^2|\bar{v}_t|^2\right)\,\mathrm{d}x \\ &\le \frac{C_0}{12}\int (1+t)^{\beta}w_{xx}^2\,\mathrm{d}x + C\delta^2(1+t)^{\beta+\frac{3\lambda}{2}-\frac{5}{2}}, \end{aligned} \tag{5.54}$$

$$
\begin{aligned}
-\int \frac{\lambda}{\alpha}(1+t)^{\beta+\lambda-1}p(\bar v)_x w_{xx}\,\mathrm{d}x &\le \frac{C_0}{12}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x + C\int (1+t)^{\beta+2\lambda-2}|\bar v_x|^2\,\mathrm{d}x\\
&\le \frac{C_0}{12}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x + C\delta^2(1+t)^{\beta+\frac{3\lambda}{2}-\frac{5}{2}},
\end{aligned}
\tag{5.55}
$$

$$
\begin{aligned}
&\int (1+t)^{\beta}\left(p\left(w_x+\bar v\right)-p(\bar v)-p'(\bar v)w_x\right)_x w_{xx}\,\mathrm{d}x\\
&= \int (1+t)^{\beta}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right) w_{xx}^2\,\mathrm{d}x\\
&\quad + \int (1+t)^{\beta}\left(p'\left(w_x+\bar v\right)-p'(\bar v)-p''(\bar v)w_x\right)\bar v_x w_{xx}\,\mathrm{d}x\\
&\le C\int (1+t)^{\beta}|w_x| w_{xx}^2\,\mathrm{d}x + \frac{C_0}{12}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x\\
&\quad + \int (1+t)^{\beta}|\bar v_x|^2|w_x|^4\,\mathrm{d}x\\
&\le C\varepsilon_1\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x + \frac{C_0}{12}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x\\
&\quad + C\varepsilon_1^2\delta^2\int (1+t)^{\beta-\lambda-1} w_x^2\,\mathrm{d}x.
\end{aligned}
\tag{5.56}
$$

By applying assumption (H3), Lemma 3.1, (5.1) and Cauchy-Schwarz's inequality, the last term on the right-hand side of (5.50) can be estimated as

$$
\begin{aligned}
-\int (1+t)^{\beta-\lambda} g\left(w_t+\bar u\right) w_{xx}\,\mathrm{d}x &\le C\int (1+t)^{\beta-\lambda}\left(|w_t|^3+|\bar u|^3\right)|w_{xx}|\,\mathrm{d}x\\
&\le \frac{C_0}{12}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x + C\varepsilon_1^2\int (1+t)^{\beta-\lambda-1} w_t^2\,\mathrm{d}x\\
&\quad + C\delta(1+t)^{\beta+\frac{3\lambda}{2}-\frac{5}{2}}.
\end{aligned}
\tag{5.57}
$$

Applying (5.51)-(5.57) to (5.50) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[(1+t)^{\beta} w_x w_{xt}+\frac{\alpha}{2}(1+t)^{\beta-\lambda} w_x^2\right]\mathrm{d}x\\
&+\frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1} w_x^2\,\mathrm{d}x + \frac{C_0}{2}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x\\
&\le 2\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x + C\int (1+t)^{\beta-2} w_x^2\,\mathrm{d}x\\
&\quad + C\delta^2\int (1+t)^{\beta-\lambda-1} w_x^2\,\mathrm{d}x + C\varepsilon_1^2\int (1+t)^{\beta-\lambda-1} w_t^2\,\mathrm{d}x + C\delta^2(1+t)^{\beta+\frac{3\lambda}{2}-\frac{5}{2}}.
\end{aligned}
\tag{5.58}
$$

Combining (5.49) $\times h$ and (5.58) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \Big[(1+t)^{\beta} w_x w_{xt} + \frac{\alpha}{2}(1+t)^{\beta-\lambda} w_x^2 \\
&+ \frac{h}{2}(1+t)^{\beta+\lambda} w_{xt}^2 - \frac{h}{2}(1+t)^{\beta+\lambda} p'(\bar{v}) w_{xx}^2\Big]\,\mathrm{d}x \\
&+ \frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta-\lambda-1} w_x^2\,\mathrm{d}x \\
&+ \frac{C_0}{2}\int (1+t)^{\beta} w_{xx}^2\,\mathrm{d}x + \Big(\frac{\alpha h}{2}-2\Big)\int (1+t)^{\beta} w_{xt}^2\,\mathrm{d}x \\
&\le \frac{h}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+\lambda}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right) w_{xx}^2\,\mathrm{d}x \\
&\quad + C\int (1+t)^{\beta-\lambda-1} w_x^2\,\mathrm{d}x + C\int (1+t)^{\beta-\lambda-1} w_t^2\,\mathrm{d}x \\
&\quad + Ch\int (1+t)^{\beta+\lambda-1}\left(w_{xx}^2 + w_{xt}^2\right)\,\mathrm{d}x + C\delta(1+t)^{\beta+\frac{3\lambda}{2}-\frac{5}{2}}.
\end{aligned} \tag{5.59}$$

Multiplying (5.59) by $(1+t)^{\lambda+1}$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \Big[(1+t)^{\beta+\lambda+1} w_x w_{xt} + \frac{\alpha}{2}(1+t)^{\beta+1} w_x^2 + \frac{h}{2}(1+t)^{\beta+2\lambda+1} w_{xt}^2 \\
&- \frac{h}{2}(1+t)^{\beta+2\lambda+1} p'(\bar{v}) w_{xx}^2\Big]\,\mathrm{d}x + \frac{\alpha(\lambda-\beta)}{2}\int (1+t)^{\beta} w_x^2\,\mathrm{d}x \\
&+ \frac{C_0}{2}\int (1+t)^{\beta+\lambda+1} w_{xx}^2\,\mathrm{d}x + \left(\frac{\alpha h}{2}-2\right)\int (1+t)^{\beta+\lambda+1} w_{xt}^2\,\mathrm{d}x \\
&\le \frac{h}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{\beta+2\lambda+1}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right) w_{xx}^2\,\mathrm{d}x + C\int (1+t)^{\beta} w_t^2\,\mathrm{d}x \\
&\quad + C\int (1+t)^{\beta+2\lambda}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right) w_{xx}^2\,\mathrm{d}x + C\int (1+t)^{\beta} w_x^2\,\mathrm{d}x \\
&\quad + Ch\int (1+t)^{\beta+2\lambda}\left(w_{xx}^2 + w_{xt}^2\right)\,\mathrm{d}x + C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}.
\end{aligned} \tag{5.60}$$

We define

$$H_2(t) \sim (1+t)^{\lambda+1}\|w_x\|^2 + (1+t)^{3\lambda+1}\|w_{xx}\|^2 + (1+t)^{3\lambda+1}\|w_{xt}\|^2.$$

Setting $\beta = \lambda$, $h = 8/\alpha$, and choosing $T_2 \gg 1$ such that for any $t \ge T_2$, $Ch(1+t)^{\lambda-1} \le min\{\frac{C_0}{4}, \frac{1}{2}(\frac{\alpha h}{2}-1)\}$, we obtain from (5.60) that, for $t \ge T_2$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) + C\int (1+t)^{2\lambda+1}\left(w_{xx}^2 + w_{xt}^2\right)\,\mathrm{d}x \\
&\le C\frac{\mathrm{d}}{\mathrm{d}t}\int (1+t)^{3\lambda+1}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right) w_{xx}^2\,\mathrm{d}x + C\int (1+t)^{\lambda} w_x^2\,\mathrm{d}x \\
&\quad + C\int (1+t)^{\lambda} w_t^2\,\mathrm{d}x + C\delta(1+t)^{\frac{7\lambda}{2}-\frac{3}{2}}.
\end{aligned} \tag{5.61}$$

Integrating (5.61) from T_2 to t, and using (5.6) and (5.32) together with $0 \le \lambda < \frac{1}{7}$, we obtain

$$\begin{aligned}
&(1+t)^{3\lambda+1}\left(\|w_{xx}(t)\|^2 + \|w_{xt}(t)\|^2\right) + \int_{T_2}^{t}(1+s)^{2\lambda+1}\left(\|w_{xx}\|^2 + \|w_{xt}\|^2\right)\,\mathrm{d}s \\
&\le C\left(H_2\left(T_2\right) + \|w_0\|_1^2 + \|z_0\|^2 + \delta\right).
\end{aligned} \tag{5.62}$$

For each $t > 0$, combining (5.62), (4.3) and (4.20) yields

$$\begin{aligned}
&(1+t)^{3\lambda+1}\left(\|w_{xx}(t)\|^2 + \|w_{xt}(t)\|^2\right) + \int_{0}^{t}(1+s)^{2\lambda+1}\left(\|w_{xx}\|^2 + \|w_{xt}\|^2\right)\,\mathrm{d}s \\
&\le C\left(\|w_0\|_2^2 + \|z_0\|_1^2 + \delta\right).
\end{aligned} \tag{5.63}$$

Multiplying (5.49) by $(1+t)^2$, with $\beta=\lambda$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left[(1+t)^{2\lambda+2}w_{xt}^2-(1+t)^{2\lambda+2}p'(\bar{v})w_{xx}^2\right]\mathrm{d}x+\frac{\alpha}{2}\int(1+t)^{\lambda+2}w_{xt}^2\,\mathrm{d}x\\
&\le\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{2\lambda+2}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right)w_{xx}^2\,\mathrm{d}x\\
&\quad+C\int(1+t)^{2\lambda+1}\left(w_{xx}^2+w_{xt}^2\right)\mathrm{d}x+C\delta^2\int(1+t)^{\lambda}w_x^2\,\mathrm{d}x\\
&\quad+C\delta\int(1+t)^{\lambda}w_t^2\,\mathrm{d}x+C\delta(1+t)^{\frac{7\lambda}{2}-\frac{3}{2}}.
\end{aligned}\tag{5.64}$$

Integrating (5.64) over $[0,t]$ and using (5.6), we obtain

$$(1+t)^{2\lambda+2}\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right)+\int_0^t(1+t)^{\lambda+2}\|w_{xt}\|^2\,\mathrm{d}s\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\tag{5.65}$$

Consequently, in conjunction with (5.63), this implies (5.36), which completes the proof. □

Lemma 5.6. *Under the assumptions of Theorem 5.1, when $1/7<\lambda<1/3$, it holds that*

$$\begin{aligned}
&(1+t)^{\frac{5}{2}-\frac{3\lambda}{2}}\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right)\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right),\\
&\int_0^t\left((1+s)^{\beta+\lambda+1}\|w_{xx}\|^2+(1+s)^{\beta+2}\|w_{xt}\|^2\right)\mathrm{d}s\\
&\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right),\quad \frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda.
\end{aligned}\tag{5.66}$$

Proof. Define

$$H_3(t)\sim(1+t)^{\beta+1}\|w_x\|^2+(1+t)^{\beta+2\lambda+1}\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right).$$

Taking $\frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda$. Let $T_3\gg1$ be sufficiently large such that for all $t\ge T_3$, $Ch(1+t)^{\lambda-1}\le min\{\frac{C_0}{4},\frac{1}{2}(\frac{\alpha h}{2}-1)\}$, where we take $h=\frac{8}{\alpha}$. Consequently, if $t\ge T_3$, it follows from (5.60) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H_3(t)+c\int(1+t)^{\beta+\lambda+1}w_{xx}^2\,\mathrm{d}x+c\int(1+t)^{\beta+\lambda+1}w_{xt}^2\,\mathrm{d}x+c\int(1+t)^{\beta}w_x^2\,\mathrm{d}x\\
&\le C\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+2\lambda+1}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right)w_{xx}^2\,\mathrm{d}x+C\int(1+t)^{\beta}w_t^2\,\mathrm{d}x\\
&\quad+C\int(1+t)^{\beta}w_x^2\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}.
\end{aligned}\tag{5.67}$$

By integrating (5.67) over $[T_3,t]$ and invoking the a priori estimate (5.1), (5.27) and (5.28), we obtain

$$\begin{aligned}
&H_3(t)+\int_{T_3}^t(1+s)^{\beta+\lambda+1}\left(\|w_{xt}\|^2+\|w_{xx}\|^2\right)\mathrm{d}s\\
&\le H_3\left(T_3\right)+C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_1^2+\|z_0\|^2+\delta\right),
\end{aligned}\tag{5.68}$$

which, combined with (4.3), (4.20) and the inequality $\beta+\frac{5\lambda}{2}-\frac{1}{2}>0$, gives

$$\begin{aligned}
&(1+t)^{\beta+2\lambda+1}\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right)+\int_0^t(1+s)^{\beta+\lambda+1}\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right)\mathrm{d}s\\
&\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}\tag{5.69}$$

Finally, multiplying (5.49) by $(1+t)^2$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\beta+\lambda+2}w_{xt}^2-(1+t)^{\beta+\lambda+2}p'(\bar{v})w_{xx}^2\right]\mathrm{d}x+\frac{\alpha}{2}\int(1+t)^{\beta+2}w_{xt}^2\,\mathrm{d}x\\
&\le C\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda+2}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right)w_{xx}^2\,\mathrm{d}x\\
&\quad+C\int(1+t)^{\beta+\lambda+1}\left(w_{xx}^2+w_{xt}^2\right)\mathrm{d}x+C\int(1+t)^{\beta}w_x^2\,\mathrm{d}x\\
&\quad+C\int(1+t)^{\beta}w_t^2\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}.
\end{aligned}\tag{5.70}$$

Integrating (5.70) over $[0,t]$, then applying (5.27), (5.28) and (5.69), we can obtain

$$\begin{aligned}
&(1+t)^{\beta+\lambda+2}\left(\|w_{xx}\|^2+\|w_{xt}\|^2\right)+\int_0^t(1+s)^{\beta+2}\|w_{xt}\|^2\,\mathrm{d}s\\
&\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}\tag{5.71}$$

Combining this with (5.69) yields (5.66). This completes the proof. □

The proof of the next lemma is analogous to that of Lemma 5.6, so we omit it.

Lemma 5.7. *Under the assumptions of Theorem 5.1, when $0\le\lambda<1/7$, it holds that*

$$\begin{aligned}
&(1+t)^{3\lambda+3}\left(\|w_{xxx}\|^2+\|w_{xxt}\|^2\right)+\int_0^t\left((1+s)^{3\lambda+2}\|w_{xxx}\|^2+(1+s)^{2\lambda+3}\|w_{xxt}\|^2\right)\mathrm{d}s\\
&\le C\left(\|w_0\|_3^2+\|z_0\|_2^2+\delta\right),
\end{aligned}\tag{5.72}$$

and for $1/7<\lambda<1/3$,

$$\begin{aligned}
&(1+t)^{\frac{7}{2}-\frac{\lambda}{2}}\left(\|w_{xxx}\|^2+\|w_{xxt}\|^2\right)\le C\left(\|w_0\|_3^2+\|z_0\|_2^2+\delta\right),\\
&\int_0^t\left((1+s)^{\beta+2\lambda+2}\|w_{xxx}\|^2+(1+s)^{\beta+\lambda+3}\|w_{xxt}\|^2\right)\mathrm{d}s\\
&\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_3^2+\|z_0\|_2^2+\delta\right),\quad \frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda.
\end{aligned}\tag{5.73}$$

Next, we present the decay rates of $z=w_t$.

Lemma 5.8. *Under the assumptions of Theorem 5.1, when $0\le\lambda<1/7$, it holds that*

$$\begin{aligned}
&(1+t)^2\|z\|^2+(1+t)^{\lambda+3}\left(\|z_x\|^2+\|z_t\|^2\right)+\int_0^t\left((1+s)^{\lambda+2}\|z_x\|^2+(1+s)^3\|z_t\|^2\right)\mathrm{d}s\\
&\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}\tag{5.74}$$

Proof. Differentiating (2.9) with respected to t yields

$$\begin{aligned}
&z_{tt}+\left(p'(\bar{v})w_x\right)_{xt}+\frac{\alpha}{(1+t)^{\lambda}}z_t\\
&=\frac{\alpha\lambda}{(1+t)^{\lambda+1}}z+F_t+\frac{(1+t)^{\lambda}g'(z+\bar{u})(z_t+\bar{u}_t)-\lambda(1+t)^{\lambda-1}g(z+\bar{u})}{(1+t)^{2\lambda}}.
\end{aligned}\tag{5.75}$$

Multiplying (5.75) by $(1+t)^{\beta+\lambda}z_t$, integrating over $\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left[(1+t)^{\beta+\lambda}z_t^2-(1+t)^{\beta+\lambda}p'(\bar{v})z_x^2\right]\,\mathrm{d}x+\frac{\alpha}{2}\int(1+t)^{\beta}z_t^2\,\mathrm{d}x\\
&\le \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right)z_x^2\,\mathrm{d}x\\
&\quad+C\int(1+t)^{\beta-2}z^2\,\mathrm{d}x+C\int(1+t)^{\beta+\lambda-1}z_x^2\,\mathrm{d}x\\
&\quad+\frac{\beta+\lambda}{2}\int(1+t)^{\beta+\lambda-1}z_t^2\,\mathrm{d}x+C\delta^2\int(1+t)^{\beta+\lambda-3}w_x^2\,\mathrm{d}x\\
&\quad+C\delta^2\int(1+t)^{\beta+2\lambda-2}w_{xx}^2\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{7\lambda}{2}-\frac{9}{2}}.
\end{aligned}\tag{5.76}$$

Multiplying (5.75) by $(1+t)^{\beta}z$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[(1+t)^{\beta}zz_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}z^2\right]\,\mathrm{d}x\\
&+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}z^2\,\mathrm{d}x+\frac{C_0}{2}\int(1+t)^{\beta}z_x^2\,\mathrm{d}x\\
&\le 2\int(1+t)^{\beta}z_t^2\,\mathrm{d}x+C\int(1+t)^{\beta-2}z^2\,\mathrm{d}x\\
&\quad+C\delta^2\int(1+t)^{\beta-2}w_x^2\,\mathrm{d}x+C\delta^2(1+t)^{\beta+\frac{5\lambda}{2}-\frac{7}{2}}.
\end{aligned}\tag{5.77}$$

Combining (5.76) $\times\, h$ + (5.77) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[(1+t)^{\beta}zz_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda}z^2\\
&+\frac{h}{2}(1+t)^{\beta+\lambda}z_t^2-\frac{h}{2}(1+t)^{\beta+\lambda}p'(\bar{v})z_x^2\Big]\,\mathrm{d}x\\
&+\frac{\alpha(\lambda-\beta)}{2}\int(1+t)^{\beta-\lambda-1}z^2\,\mathrm{d}x+\frac{C_0}{2}\int(1+t)^{\beta}z_x^2\,\mathrm{d}x\\
&+\Big(\frac{\alpha h}{2}-2\Big)\int(1+t)^{\beta}z_t^2\,\mathrm{d}x\\
&\le\frac{h}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda}\left(p'\left(w_x+\bar{v}\right)-p'(\bar{v})\right)z_x^2\,\mathrm{d}x+C\int(1+t)^{\beta-2}z^2\,\mathrm{d}x\\
&\quad+C\int(1+t)^{\beta+\lambda-1}\left(z_x^2+z_t^2\right)\,\mathrm{d}x+C\delta^2\int(1+t)^{\beta-2}w_x^2\,\mathrm{d}x\\
&\quad+C\delta^2\int(1+t)^{\beta+2\lambda-2}w_{xx}^2\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{7}{2}}.
\end{aligned}\tag{5.78}$$

Integrating (5.78) on $[0,t]$ with $\beta=\lambda$, $h=\frac{8}{\alpha}$ gives

$$\|z\|+(1+t)^{2\lambda}\left(\|z_t\|^2+\|z_x\|^2\right)+\int_0^t(1+s)^{\lambda}\left(\|z_x\|^2+\|z_t\|^2\right)\,\mathrm{d}s\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\tag{5.79}$$

Integrating $(1+t)^{1-\lambda}\cdot$(5.78) over $[0,t]$ yields

$$\begin{aligned}
&(1+t)^{1-\lambda}\|z\|+(1+t)^{\lambda+1}\left(\|z_t\|^2+\|z_x\|^2\right)+\int_0^t(1+s)\left(\|z_x\|^2+\|z_t\|^2\right)\,\mathrm{d}s\\
&\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}\tag{5.80}$$

Integrating $(1+t)^{2-2\lambda}\cdot$(5.78) over $[0,t]$ yields

$$\begin{aligned}
&(1+t)^{2-2\lambda}\|z\|^2+(1+t)^{2}\left(\|z_t\|^2+\|z_x\|^2\right)+\int_0^t(1+s)^{2-\lambda}\left(\|z_x\|^2+\|z_t\|^2\right)\,\mathrm{d}s\\
&\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).
\end{aligned}\tag{5.81}$$

Integrating $(1+t)^2\cdot(5.78)$ over $[0,t]$ yields

$$\begin{aligned}&(1+t)^2\|z\|^2+(1+t)^{2\lambda+2}\left(\|z_t\|^2+\|z_x\|^2\right)+\int_0^t(1+s)^{\lambda+2}\left(\|z_x\|^2+\|z_t\|^2\right)\mathrm{d}s\\&\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\end{aligned}\tag{5.82}$$

Integrating $(1+t)^{3-\lambda}\cdot(5.78)$ over $[0,t]$ yields

$$(1+t)^{\lambda+3}\left(\|z_t\|^2+\|z_x\|^2\right)+\int_0^t(1+s)^3\|z_t\|^2\,\mathrm{d}s\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\tag{5.83}$$

This and (5.82) give (5.74) and completes the proof. □

Lemma 5.9. *Under the assumptions of Theorem 5.1, when $1/7<\lambda<1/3$, it holds that*

$$\begin{aligned}&((1+t)^{\frac{5}{2}-\frac{7\lambda}{2}}\|z\|^2+(1+t)^{\frac{7}{2}-\frac{5\lambda}{2}}\left(\|z_x\|^2+\|z_t\|^2\right)\le C\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right),\\&\int_0^t\left((1+s)^{\beta+2}\|z_x\|^2+(1+s)^{\beta-\lambda+3}\|z_t\|^2\right)\mathrm{d}s\\&\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right),\quad \frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda.\end{aligned}\tag{5.84}$$

Proof. Multiplying (5.78) by $(1+t)^2$ yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[(1+t)^{\beta+2}zz_t+\frac{\alpha}{2}(1+t)^{\beta-\lambda+2}z^2+\frac{h}{2}(1+t)^{\beta+\lambda+2}z_t^2\\&-\frac{h}{2}(1+t)^{\beta+\lambda+2}p'(\bar v)z_x^2\Big]\mathrm{d}x+\frac{C_0}{2}\int(1+t)^{\beta+2}z_x^2\,\mathrm{d}x+\Big(\frac{\alpha h}{2}-2\Big)\int(1+t)^{\beta+2}z_t^2\,\mathrm{d}x\\&\le\frac{h}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda+2}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)z_x^2\,\mathrm{d}x+C\int(1+t)^{\beta}z^2\,\mathrm{d}x\\&+C\int(1+t)^{\beta+\lambda+1}\left(z_x^2+z_t^2\right)\mathrm{d}x+C\delta^2\int(1+t)^{\beta}w_x^2\,\mathrm{d}x\\&+C\delta^2\int(1+t)^{\beta+2\lambda}w_{xx}^2\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}.\end{aligned}\tag{5.85}$$

We define

$$H_4(t)\sim(1+t)^{\beta-\lambda+2}\|z\|^2+(1+t)^{\beta+\lambda+2}\left(\|z_x\|^2+\|z_t\|^2\right).$$

Taking $\frac{1}{2}-\frac{5\lambda}{2}<\beta<\lambda$ and $h=\frac{8}{\alpha}$, we choose $T_4\gg1$ satisfying that for all $t\ge T_4$, it holds

$$C(1+t)^{\lambda-1}\le min\{\frac{C_0}{4},\frac{1}{2}(\frac{\alpha h}{2}-1)\}.$$

After integrating (5.85) over $[T_4,t]$ and combining (5.27), (5.32), (5.33) and (5.66), we obtain

$$\begin{aligned}&H_4(t)+\int_{T_4}^t(1+s)^{\beta+2}\left(\|z_x\|^2+\|z_t\|^2\right)\mathrm{d}s\\&\le H_4\left(T_4\right)+C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\end{aligned}\tag{5.86}$$

Now, for $0<t\le T_4$, by applying (5.85), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}H_4(t)+c\int(1+t)^{\beta+2}z_x^2\,\mathrm{d}x+c\int(1+t)^{\beta+2}z_t^2\,\mathrm{d}x\\&\le\frac{\mathrm{d}}{\mathrm{d}t}\int(1+t)^{\beta+\lambda+2}\left(p'\left(w_x+\bar v\right)-p'(\bar v)\right)z_x^2\,\mathrm{d}x+C(1+t)^{-1}H_4(t)\\&+C\delta^2\int(1+t)^{\beta}w_x^2\,\mathrm{d}x+C\delta^2\int(1+t)^{\beta+2\lambda}w_{xx}^2\,\mathrm{d}x+C\delta(1+t)^{\beta+\frac{5\lambda}{2}-\frac{3}{2}}.\end{aligned}\tag{5.87}$$

Applying Gronwall's inequality to (5.87), we obtain

$$H_4(t)+\int_0^t(1+s)^{\beta+2}\left(\|z_x\|^2+\|z_t\|^2\right)\mathrm{d}s\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}}\left(\|w_0\|_2^2+\|z_0\|_1^2+\delta\right).\tag{5.88}$$

By applying (5.86) and (5.88), for any $t > 0$ we obtain

$$\begin{aligned} &H_4(t) + \int_0^t (1+s)^{\beta+2} \left(\|z_x\|^2 + \|z_t\|^2\right) \mathrm{d}s \\ &\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}} \left(\|w_0\|_2^2 + \|z_0\|_1^2 + \delta\right). \end{aligned} \tag{5.89}$$

By integrating $(1+t)^{3-\lambda} \cdot (5.76)$ over $[0,t]$, we obtain

$$\begin{aligned} &(1+t)^{\beta+3} \left(\|z_x\|^2 + \|z_t\|^2\right) + \int_0^t (1+s)^{\beta-\lambda+3} \|z_t\|^2 \,\mathrm{d}s \\ &\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}} \left(\|w_0\|_2^2 + \|z_0\|_1^2 + \delta\right). \end{aligned} \tag{5.90}$$

This and (5.89) give (5.84) which complets the proof. □

By performing calculations similar to those in Lemma 5.9, and integrating (2.9), $(1+t)^{\beta} z_t \cdot (2.9)_{tt}$ and $(1+t)^{\beta+\lambda} z_{tt} \cdot (2.9)_{tt}$ over $[0,t]$, we obtain the following estimates.

Lemma 5.10. *Under the assumptions of Theorem 5.1, when $0 < \lambda < \frac{1}{7}$, it holds that*

$$\begin{aligned} &(1+t)^{2\lambda+4} \left(\|z_{xx}\|^2 + \|z_{xt}\|^2\right) + \int_0^t \left((1+s)^{2\lambda+3} \|z_{xx}\|^2 + (1+s)^{\lambda+4} \|z_{xt}\|^2\right) \mathrm{d}s \\ &\le C \left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right), \end{aligned} \tag{5.91}$$

when $1/7 < \lambda < 1/3$, we have

$$\begin{aligned} &(1+t)^{\frac{9}{2}-\frac{3\lambda}{2}} \left(\|z_{xx}\|^2 + \|z_{xt}\|^2\right) \le C \left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right), \\ &\int_0^t \left((1+s)^{\beta+\lambda+3} \|z_{xx}\|^2 + (1+s)^{\beta+4} \|z_{xt}\|^2\right) \mathrm{d}s \\ &\le C(1+t)^{\beta+\frac{5\lambda}{2}-\frac{1}{2}} \left(\|w_0\|_3^2 + \|z_0\|_2^2 + \delta\right), \quad \frac{1}{2} - \frac{5\lambda}{2} < \beta < \lambda. \end{aligned} \tag{5.92}$$

Using Lemma 5.2-5.7, we have completed the proof of (5.2), (5.3) and (5.4).

Choosing $\beta = \lambda = 1/7$, $\nu = 1$ and $\kappa = 1$ in Lemmas 5.2-5.10, the proof of (5.5) is analogous to that of (5.2), (5.3) and (5.4), hence, we omit it.

Finally, we need to show that (5.1) can be closed. Under this assumption, we have proved that (5.2)-(5.5) hold provided ε_1 is sufficiently small. Therefore, by the continuity argument and Sobolev's inequality, the assumption (5.1) can always be satisfied if $\|w_0\|_3^2 + \|z_0\|_2^2 + \delta$ is sufficiently small. This completes the proof of Theorem 5.1.

By combining the local existence result with the a priori estimates established in Theorem 5.1, and applying the method of solution continuation, we can establish the global-in-time existence of the solution and derive its decay rates. Thus, Theorem 2.1 is proved.

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