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BOUNDARY CONTROL FOR SMALL VIBRATIONS OF HOLED MEMBRANES

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Abstract. In this work we study the exact boundary control for a vibrating membrane with nonsmooth boundary and a hole inside. The edge of the hole moves freely in the direction perpendicular to the rest position of the membrane, or it remains attached to some fixed flat arc. We show that it is possible to act on the outer edge of the membrane causing the vibration to cease after a time $T > 0$ has elapsed.

1. Introduction

Let $B \subset \mathbb{R}^2$ be an open and bounded convex set with the origin inside and boundary ∂B of class C^∞. Let $U \subset \mathbb{R}^2$ be an open, bounded, simply connected set with piecewise smooth boundary ∂U without self intersections and cusps. We assume that the boundary of U is formed by finite number of C^∞ arcs, each of which can be extended beyond its vertices;

$$\partial U = \gamma_1 \cup \cdots \cup \gamma_l \tag{1.1}$$

where $\gamma_i \cap \gamma_j$ is a single point or is empty, for $i \neq j$. We assume that $\overline{B}$, the closure of B, is contained in U. We define

$$\Omega := U - \overline{B} \tag{1.2}$$

and observe that $\partial\Omega = \partial U \cup \partial B$ with

$$\partial U \cap \partial B = \emptyset \tag{1.3}$$

The membrane at rest is represented by the domain Ω and its transverse displacement is described by $u = u(x,t)$, $x \in \Omega$, $t > 0$. The hole in the membrane is represented by B. We assume that along $\partial B \subset \mathbb{R}^2$ the membrane vibrates freely perpendicularly to the membrane's rest position, or it is held fixed on ∂B. We denote ν the outward unit normal vector to $\partial\Omega$ and u_ν $(= \frac{\partial u}{\partial \nu})$ the outward normal derivative of the function u along $\partial\Omega$. So we must have $u_\nu = 0$, or $u = 0$, along ∂B for $t > 0$. Among other simplifying hypotheses, we assume that the membrane undergoes small free vibrations around its resting position, so that the function u satisfies the linear normalized homogeneous wave equation $u_{tt} - \Delta u = 0$, for $x \in \Omega$ and $t \in \mathbb{R}$.

In the literature, the problem studied here is referred as exact boundary control problem for the wave equation in Ω. It consists of finding some time $T > 0$ and establishing a boundary condition along the outer edge ∂U of the membrane, so that for any initial state $(u(\cdot,0), u_t(\cdot,0))$ with finite energy, the membrane's vibration is eliminated at the end of the time interval $[0,T]$, i.e., $u(\cdot,T) = 0$ and $u_t(\cdot,T) = 0$ in Ω. It is worth noting that linearity and time-reversibility imply that the aforementioned controllability allows us to steer any initial state to a desired, and not necessarily zero final state.

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2. Main results

Let us present some of the notation to be used. For an arbitrary non empty open set $\Xi \subset \mathbb{R}^k$, $k = 2, 3$, we denote $C_0^\infty(\Xi)$ the class of infinitely differentiable functions with compact support in Ξ and denote $\mathcal{D}(\Xi)$ the topological vector space constructed from $C_0^\infty(\Xi)$ by endowing it with the inductive limit topology (see [22, 24, 25]). A distribution on Ξ is a continuous linear functional on $\mathcal{D}(\Xi)$. We denote $\mathcal{D}'(\Xi)$ the class of all distributions on Ξ. Equipped with the weak* topology, $\mathcal{D}'(\Xi)$ is a locally convex space.

Let $s, \alpha_1, \dots, \alpha_k$ be non-negative integers, $\alpha = (\alpha_1, \dots, \alpha_k)$ a multi-index and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_k^{\alpha_k}$ ($\partial_j = \frac{\partial}{\partial x_j}$) partial derivative of order $|\alpha| = \sum \alpha_i$. We recall that the L^2 - based Sobolev space of order s, $s = 1, 2, \dots$, is

$$H^s(\Xi) = \{f \in L^2(\Xi) : \partial^\alpha f \in L^2(\Xi),\ |\alpha| \le s\} \tag{2.1}$$

where the derivatives $\partial^\alpha f$ are in $\mathcal{D}'(\Xi)$ sense and $H^0 = L^2$. The usual norm of $H^s(\Xi)$ is

$$\|f\|^2_{H^s(\Xi)} = \sum_{|\alpha| \le s} \|\partial^\alpha f\|^2_{L^2(\Xi)} \tag{2.2}$$

As usual, $H_0^s(\Xi)$ is the subspace of $H^s(\Xi)$ of the functions vanishing on the boundary of Ξ.

For a surface $\Sigma \subset \mathbb{R}^k$, $L^2(\Sigma)$ is the space of square integrable functions on Σ (see [3, 5, 10]). We also recall that the local, L^2-based Sobolev space of order s, given by

$$H^s_{\text{loc}}(\Xi) := \{f \in \mathcal{D}'(\Xi); \phi f \in H^s(\Xi),\ \forall \phi \in \mathcal{D}(\Xi)\} \tag{2.3}$$

and it is a Frechet space with topology defined by a countable family of semi-norms and L^2-convergence in compact sets in Ξ, (see [24]).

Now we present the main results of this work.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be the holed domain, $\Omega = U - \overline{B}$ (as in the introduction), ν its outward unit normal vector along $\partial\Omega$ and α, β are non-negative real numbers such that $\alpha + \beta > 0$. Then, there exists $T > 0$ such that, for each initial state $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$, $u_0 = 0$ on $\partial\Omega$, there exists a control function $g \in L^2(\partial U \times (0, T))$ so that the the solution u of the (mixed Dirichlet - Robin) initial and boundary value problem*

$$\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{in } \Omega \times (0, T) \\ u(\cdot, 0) &= u_0 \quad \text{on } \Omega \\ u_t(\cdot, 0) &= u_1 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial B \times (0, T) \\ \alpha u + \beta u_\nu &= g \quad \text{on } \partial U \times (0, T) \end{aligned} \tag{2.4}$$

satisfies

$$u(\cdot, T) = u_t(\cdot, T) = 0 \quad \text{on } \Omega.$$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be the holed domain, $\Omega = U - \overline{B}$ (as in the introduction), ν its outward unit normal vector along $\partial\Omega$ and α, β are non-negative real numbers such that $\alpha + \beta > 0$. Then, there exists $T > 0$ such that, for each initial state $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ there exists a control function $g \in L^2(\partial U \times (0, T))$ so that the the solution u of the (mixed Neuman- Robin) initial and boundary value problem*

$$\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{in } \Omega \times (0, T) \\ u(\cdot, 0) &= u_0 \quad \text{on } \Omega \\ u_t(\cdot, 0) &= u_1 \quad \text{on } \Omega \\ u_\nu &= 0 \quad \text{on } \partial B \times (0, T) \\ \alpha u + \beta u_\nu &= g \quad \text{on } \partial U \times (0, T) \end{aligned} \tag{2.5}$$

satisfies

$$u(\cdot, T) = u_t(\cdot, T) = 0 \quad \text{on } \Omega.$$

In both theorems we consider controlling the membrane vibration via a mechanism acting like an elastic ring (or an elastic frame) made of springs distributed continuously along ∂U. This a Robin-type boundary control problem. In addition to considering Robin type, our result also accounts for a case of non-smooth boundary. To the best of our understanding, the results presented here are new. The reference [11] is an excellent treatise on the subject, presenting a wide variety of relevant and inspiring results of this nature for smooth domains. For a non-smooth domain with special geometry see [1] and the references therein.

The proofs of Theorems 2.1 and 2.2 are based on the principle "controllability via stabilization" introduced by David Russell in [17, 18] and widely used in control problems for time reversible hyperbolic equations, see [1, 2, 16] for example. Two important points in the proof of the theorems are:

(1) the local energy decay for the Dirichlet and Neuman exterior boundary value problem for the wave equation in $\mathbb{R}^2 - \overline{B}$, and
(2) the local existence of L^2 trace for the normal derivative of the solution to the aforementioned exterior problem on surfaces distant from the obstacle boundary.

It is worth noticing that the method used here lends itself very well to numerical computations since only exterior boundary value problems need to be computed.

The rest of this work is organized as follows: in section 3 we discuss energy decay for the exterior problems for a single convex and smooth obstacle in $\mathbb{R}^2$ under Dirichlet's or Neuman's homogeneous boundary condition. In section 4 we follow [21] to discuss the regularity of the solution to the wave equation along non-characteristic surfaces. Section 5 is dedicated to the proof of the main results.

3. Decay of local energy for the exterior problem

If u solves the homogeneous wave equation $u_{tt} - \Delta u = 0$ in some region of the space including a bounded set $\mathcal{O}$, the total energy inside $\mathcal{O}$, at the time t is

$$E(t, \mathcal{O}) = \frac{1}{2} \int_{\mathcal{O}} [u_t(x,t)^2 + |\nabla u(x,t)|^2] dx \tag{3.1}$$

the sum of kinetic and potential energies within $\mathcal{O}$. The behavior of $E(t, \mathcal{O})$ as $t \to \infty$ has long been the subject of extensive study. The decay of this quantity is referred as local energy decay and have great practical relevance. The literature on this subject is vast, and many important contributions have been made by Morawetz (see for instance [12, 13, 14]). For a quick overview on local energy decay to the wave equation in exterior domains we mention [6] for classical results and [20] for up dated results.

For the problem proposed here, we are interested in the local decay of the solution to the exterior problem involving a single obstacle. To solve our problem, we need a norm on the finite energy space $H^1 \times L^2$ that incorporates energy decay. The total energy (3.1) defines a seminorm on $H^1 \times L^2$, and - as noted in [17], using it would require considering a space of equivalence classes of $H^1 \times L^2$ modulo zero energy. Adding the term $|u(x,t)|^2$ inside the integral of (3.1) and still achieving decay is not an easy task, since the Poincare's inequality is not appropriate to be used, because of the nature of wave propagation in two spatial dimensions. In the case of the wave equation on the whole space, as demonstrated in [18], the explicit formula for the solution was crucial for this purpose; however, the same does not hold for the solution to the exterior problem. For three spatial dimensions and a smooth, star-shaped obstacle, [7, Chapter 4] employs elliptic regularity and multipliers, as in [12, 13], to show that the term $|u(x,t)|^2$ can be incorporated into the total energy expression, thereby allowing for decay to be established. For two spatial dimensions, two results found in the literature have proven useful in overcoming this difficulty. They are briefly presented below: the first, due to W. Dan [4], establishes the decay of $\int_{\mathcal{O}} [|u(x,t)|^2 + u_t(x,t)^2 + |\nabla u(x,t)|^2] dx$ for the exterior problem for the wave equation with homogeneous Dirichlet boundary conditions; the second result, due to Secchi and Shibata [19], addresses the Neuman condition. In this section, we present these results in the form applicable to our control problem.

Lemma 3.1. *Let $B \subset \mathbb{R}^2$ be an open and bounded convex set, with the origin inside and boundary ∂B of class C^1. Consider the exterior boundary value problem in $\mathbb{R}^2 - \overline{B}$*

$$\begin{gathered} w_{tt} - \Delta w = 0 \quad \text{in } (\mathbb{R}^2 - \overline{B}) \times \mathbb{R} \\ w(\cdot, 0) = w_0 \quad \text{on } \mathbb{R}^2 - \overline{B} \\ w_t(\cdot, 0) = w_1 \quad \text{on } \mathbb{R}^2 - \overline{B} \\ w = 0 \quad \text{on } \partial B \times \mathbb{R} \end{gathered} \tag{3.2}$$

with initial state $(w_0, w_1) \in H_0^1(\mathbb{R}^2 - \overline{B}) \times L^2(\mathbb{R}^2 - \overline{B})$ satisfying

$$w_0(x) = w_1(x) = 0, \quad \text{for } |x| > R,$$

for some $R > 0$ sufficiently large. Then, there exists a constant $C_1 > 0$ depending on B and R such that

$$\int_{\Xi_R} \left(|w(x,t)|^2 + |w_t(x,t)|^2 + |\nabla w(x,t)|^2 \right) dx \leq \frac{C_1}{|t|(\log t)^2} \left(\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2 \right) \tag{3.3}$$

as $t \to \infty$ and $\Xi_R = \{x \in \mathbb{R}^2 - \overline{B} : |x| < R\}$. (See [4, Theorem 5.2]*).*

Lemma 3.2. *Let $B \subset \mathbb{R}^2$ be an open and bounded convex set with the origin inside and boundary ∂B of class C^1. Consider the exterior boundary value problem in $\mathbb{R}^2 - \overline{B}$,*

$$\begin{gathered} w_{tt} - \Delta w = 0 \quad \text{in } (\mathbb{R}^2 - \overline{B}) \times \mathbb{R} \\ w(\cdot, 0) = w_0 \quad \text{on } \mathbb{R}^2 - \overline{B} \\ w_t(\cdot, 0) = w_1 \quad \text{on } \mathbb{R}^2 - \overline{B} \\ w_\nu = 0 \quad \text{on } \partial B \times \mathbb{R} \end{gathered} \tag{3.4}$$

with initial state $(w_0, w_1) \in H^1(\mathbb{R}^2 - \overline{B}) \times L^2(\mathbb{R}^2 - \overline{B})$ satisfying

$$w_0(x) = w_1(x) = 0, \text{ for } |x| > R,$$

for some $R > 0$ sufficiently large.

Then, there exists a constant $C_2 > 0$ depending on B and R such that

$$\int_{\Xi_R} \left(|w(x,t)|^2 + |w_t(x,t)|^2 + |\nabla w(x,t)|^2 \right) dx \leq \frac{C_2}{(1 + t)^2} \left(\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2 \right) \tag{3.5}$$

as $t \to \infty$. (See [19, Lema 2.1]*).*

The estimate (3.3) is obtained in [4], for smooth solutions by using results of [23]. For solutions with finite energy, a density argument allows us to extend it to those solutions of (3.2). The estimate (3.5) is obtained in [19] by using results from [4, 8, 14, 23].

4. Trace of the normal derivative

The regularity of the solution $w \in H^1_{\text{loc}}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ to the exterior problem (3.2) or (3.4), and the classical theory of traces does not guarantee the existence of square-integrable derivatives of w along surfaces in $(\mathbb{R}^2 - \overline{B}) \times \mathbb{R}$. However, being a solution to the wave equation allows us take advantage of a result of D. Tataru [21] to read off the normal derivative of w along compact portions of same surfaces immersed in the open set $(\mathbb{R}^2 - \overline{B}) \times \mathbb{R}$. Let $P(\xi, D)$ be a second order hyperbolic partial differential operator with C^∞coefficients depending on $\xi \in \Xi$ for some open set $\Xi \subset \mathbb{R}^k$, $k = 3, 4, \ldots$. Let $\Sigma \subset \Xi$ be an oriented C^∞ surface, time-like with respect to the operator P. Let $\nu = (\nu_1, \ldots, \nu_k)$ be the unit vector normal to Σ. If $\sum a^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j}$ is the principal part of $P(\xi, D)$ then $\sum a^{ij} \frac{\partial w}{\partial \xi_i} \nu_j$ is the conormal derivative of w with respect to $P(\xi, D)$ along Σ. In [21, Theorem 2] establishes that if $w \in H^1_{\text{loc}}(\Xi)$ is such that $P(\xi, D)u \in L^2_{\text{loc}}(\Xi)$ then $\sum a^{ij} \frac{\partial w}{\partial \xi_i} \nu_j \in L^2_{\text{loc}}(\Sigma)$. In order to use this result to the problem studied here we set $\Xi = (\mathbb{R}^2 - \overline{B}) \times \mathbb{R}$ and $\Sigma = \gamma \times \mathbb{R}$ where $\gamma \subset \mathbb{R}^2 - \overline{B}$ is a compact, oriented C^∞ arc with no self intersection and unit normal vector (ν_1, ν_2). Now set $\xi = (x_1, x_2, t)$ and $P(\xi, D) = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$. Observe that $(\nu_1, \nu_2, 0)$ is the unit vector normal to the cylindrical surface $\Sigma = \gamma \times \mathbb{R}$ so that the conormal derivative of w along Σ

is the is $-\frac{\partial w}{\partial \nu}$ where $\frac{\partial w}{\partial \nu}$ is the usual normal derivative of w along Σ. More precisely, for the wave equation we have the following result.

Theorem 4.1. *Let $w \in H^1_{\rm loc}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ be such that $w_{tt} - \Delta w \in L^2_{\rm loc}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ and Σ be a smooth surface, cylindrical parallel to the $t-$axis, immersed in the open set $(\mathbb{R}^2 - \overline{B}) \times \mathbb{R}$ with normal vector ν. Then its normal derivative $\frac{\partial w}{\partial \nu}$ along Σ is locally square integrable on Σ, i.e. $\frac{\partial w}{\partial \nu} \in L^2_{\rm loc}(\Sigma)$ (see* [21, Theorem 2]*).*

Now let $\widetilde{u} \in H^1_{\rm loc}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ be solution to the exterior problem (3.2) or (3.4). Let F_j be a lateral face of the cylinder $\overline{\Omega} \times [0,T]$ corresponding to the arc $\gamma_j \subset \partial U$. We have $F_j \subset \Sigma := \gamma_j \times \mathbb{R}$ with Σ time-like and F_j compact. Let $\phi \in C_0^\infty((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$, $0 \le \phi \le 1$, be a cutt-of function such that $\phi = 1$ in some neighborhood Θ of F_j. Then $\phi\widetilde{u} \in H^1((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ and satisfies $(\phi\widetilde{u})_{tt} - \Delta(\phi\widetilde{u}) = \Psi \in L^2_{\rm loc}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$, and from Theorem 4.1, it follows that $\frac{\partial(\phi\widetilde{u})}{\partial \nu} \in L^2_{\rm loc}(\Sigma)$. Hence $\frac{\partial(\phi\widetilde{u})}{\partial \nu} \in L^2(F_j)$. Since $\phi = 1$ in Θ it follows that $\frac{\partial \widetilde{u}}{\partial \nu} \in L^2(F_j)$. Observing that $\partial U \times [0,T] = F_1 \cup \cdots \cup F_j$, by pasting $\frac{\partial \widetilde{u}}{\partial \nu} \in L^2(F_j)$, $j = 1 \ldots l$, we conclude that $\frac{\partial \widetilde{u}}{\partial \nu} \in L^2(\partial U \times [0,T])$.

5. Proof of Theorems 2.1 and 2.2

We first prove Theorem 2.2 where we have $u_\nu = 0$ on $\partial B \times \mathbb{R}$ and after we highlight the changes necessary for the case where $u = 0$ on $\partial B \times \mathbb{R}$ concluding the proof of Theorem 2.1. To use Russell's approach, we need to introduce two linear operators; the solution operators S_T and S_T^* for the exterior problems in $(\mathbb{R}^2 - \overline{B}) \times \mathbb{R}$, and an appropriate extension operator E. We describe then now. For an arbitrary initial state $(v_0, v_1) \in H^1(\mathbb{R}^2 - \overline{B}) \times L^2(\mathbb{R}^2 - \overline{B})$, let us consider the exterior problem

$$\begin{aligned} v_{tt} - \Delta v = 0 \quad &\text{in } (\mathbb{R}^2 - \overline{B}) \times \mathbb{R} \\ v(\cdot,0) = v_0 \quad &\text{on } \mathbb{R}^2 - \overline{B} \\ v_t(\cdot,0) = v_1 \quad &\text{on } \mathbb{R}^2 - \overline{B} \\ v_\nu = 0 \quad &\text{on } \partial B \times \mathbb{R}\,. \end{aligned} \tag{5.1}$$

For $T > 0$ we set

$$S_T(v_0, v_1) =: (v(\cdot,T), v_t(\cdot,T))$$

where $v \in H^1_{\rm loc}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ solves (5.1). This defines the bounded linear operator

$$\begin{aligned} S_T : H^1(\mathbb{R}^2 \setminus \overline{B}) \times L^2(\mathbb{R}^2 \setminus \overline{B}) &\longrightarrow H^1(\mathbb{R}^2 \setminus \overline{B}) \times L^2(\mathbb{R}^2 \setminus \overline{B}) \\ (v_0, v_1) &\longmapsto S_T(v_0, v_1) = \big(v(\cdot,T), v_t(\cdot,T)\big) \end{aligned}$$

that associates each arbitrary initial data (v_0, v_1) in $H^1(\mathbb{R}^2 - \overline{B}) \times L^2(\mathbb{R}^2 - \overline{B})$ with the state of the system (5.1) at the time $t = T$.

The operator S_T^* is associated with the (reverse) exterior problem

$$\begin{aligned} z_{tt} - \Delta z = 0 \quad &\text{in } (\mathbb{R}^2 - \overline{B}) \times \mathbb{R} \\ z(\cdot,T) = z_0 \quad &\text{on } \mathbb{R}^2 - \overline{B} \\ z_t(\cdot,T) = z_1 \quad &\text{on } \mathbb{R}^2 - \overline{B} \\ z_\nu = 0 \quad &\text{on } \partial B \times \mathbb{R} \end{aligned} \tag{5.2}$$

and associates each arbitrary initial data (z_0, z_1) in $H^1(\mathbb{R}^2 - \overline{B}) \times L^2(\mathbb{R}^2 - \overline{B})$ at time $t = T$, with the state of the system (5.2) at time $t = 0$, i. e.,

$$S_T^*(z_0, z_1) =: (z(\cdot,0), z_t(\cdot,0))$$

where $z \in H^1_{\rm loc}((\mathbb{R}^2 - \overline{B}) \times \mathbb{R})$ solves (5.2). Clearly S_T^* is also a bounded linear operator in $H^1(\mathbb{R}^2 - \overline{B}) \times L^2(\mathbb{R}^2 - \overline{B})$.

Let U_δ be a δ-neighborhood of U with $\delta > 0$ chosen in such way that $U_\delta \subset \{x; |x| \le R\}$. It is convenient to introduce the set $\Omega_\delta =: U_\delta - \overline{B}$, and similarly $\Omega_{\delta/2} =: U_{\delta/2} - \overline{B}$. Hence

$$\Omega \subset \Omega_{\delta/2} \subset \Omega_\delta \subset \Xi_R = \{x \in \mathbb{R}^2 - \overline{B} : |x| < R\} \tag{5.3}$$

Since ∂U, is compact and Lipschitsian, it is possible to construct an extension operator

$$E\colon H^1(\Omega)\times L^2(\Omega)\to H^1(\mathbb{R}^2-\overline{B})\times L^2(\mathbb{R}^2-\overline{B})$$
$$(u_0,u_1)\to E(u_0,u_1)=(v_0,v_1)$$

which is linear, bounded and extends an arbitrary initial data $(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega)$ to a pair $(v_0,v_1)=E(u_0,u_1)\in H^1(\mathbb{R}^2-\overline{B})\times L^2(\mathbb{R}^2-\overline{B})$ in such a way that $v_0=v_1=0$ outside the $\delta/2$-neighborhood $\Omega_{\delta/2}$ of the set Ω. (See [15]).

Now that the operators S_T, S_T^*, and E have been introduced, let's begin the actual proof of the theorem. Let $(\widetilde{u}_0,\widetilde{u}_1)\in H^1(\Omega)\times L^2(\Omega)$ be arbitrary state. We extend $\widetilde{u}_0$ and $\widetilde{u}_1$ to $\mathbb{R}^2-\overline{B}$ by using the operator E obtaining $(v_0,v_1)=E(\widetilde{u}_0,\widetilde{u}_1)$. Next, we take v_0 and v_1 as the initial state at $t=0$ for the exterior problem (5.1) and solve it, obtaining its solution $v\in H^1_{\rm loc}((\mathbb{R}^2-\overline{B})\times\mathbb{R})$. For a time $T>0$ to be determined later, we read the state of the solution to (5.1) at time T; that is, we extract $(v(\cdot,T),\ v_t(\cdot,T))$ and observe that

$$(v(\cdot,T),\ v_t(\cdot,T))=S_T(v_0,v_1)=S_TE(\widetilde{u}_0,\widetilde{u}_1). \tag{5.4}$$

Next we choose a "cut off" function $\varphi\in C_0^\infty(\mathbb{R}^2)$ such that $0\le\varphi\le 1$, $\varphi=1$ on U, $\varphi=0$ outside $U_{\delta/2}$ and localize $(v(\cdot,T),\ v_t(\cdot,T))$ in U_δ by multiplying it by φ. After we use

$$(\varphi v(\cdot,T),\ \varphi v_t(\cdot,T))=\varphi S_TE(\widetilde{u}_0,\widetilde{u}_1).$$

as initial state at time $t=T$ for the exterior problem (5.2) and solve it, obtaining $z\in H^1_{\rm loc}((\mathbb{R}^2-\overline{B})\times\mathbb{R})$.

Now, let us define

$$\widetilde{u}=v-z$$

and observe some special features of this function. Note that $\widetilde{u}\in H^1_{\rm loc}((\mathbb{R}^2-\overline{B})\times\mathbb{R})$ and satisfies

$$\begin{aligned}
\widetilde{u}_{tt}-\Delta\widetilde{u}&=0\quad\text{in }(\mathbb{R}^2-\overline{B})\times\mathbb{R}\\
\widetilde{u}(\cdot,0)&=v_0-z(\cdot,0)\quad\text{on }\mathbb{R}^2-\overline{B}\\
\widetilde{u}_t(\cdot,0)&=v_1-z_t(\cdot,0)\quad\text{on }\mathbb{R}^2-\overline{B}\\
\widetilde{u}_\nu&=0\quad\text{on }\partial B\times\mathbb{R}\\
\widetilde{u}(\cdot,T)=\widetilde{u}_t(\cdot,T)&=0\quad\text{on }\Omega
\end{aligned} \tag{5.5}$$

The last line in above expression follows from

$$\widetilde{u}(\cdot,T)=v(\cdot,T)-z(\cdot,T)=v(\cdot,T)-\varphi v(\cdot,T)=(1-\varphi)v(\cdot,T)$$
$$\widetilde{u}_t(\cdot,T)=v_t(\cdot,T)-z_t(\cdot,T)=v_t(\cdot,T)-\varphi v_t(\cdot,T)=(1-\varphi)v_t(\cdot,T)$$

and observing that $\varphi=1$ on Ω.

The central idea of the approach used here is to find a value T such that, given an initial data $(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega)$, we are able to find $(\widetilde{u}_0,\widetilde{u}_1)\in H^1(\Omega)\times L^2(\Omega)$ so that the construction presented guarantees the validity of

$$\widetilde{u}(\cdot,0)=v_0-z(\cdot,0)=u_0\quad\text{and}\quad\widetilde{u}_t(\cdot,0)=v_1-z_t(\cdot,0)=u_1\text{ in }\Omega \tag{5.6}$$

If this is the case, the restriction of $\widetilde{u}$ to the cylinder $\Omega\times[0,T]$ solves the control problem by taking $g=:\alpha\widetilde{u}+\beta\widetilde{u}_\nu$ along $\partial U\times(0,T)$. Note that, by the Theorem 4.1 , $\widetilde{u}_\nu$ has locally square integrable trace on $\partial U\times[0,T]$. At this point it is important to observe that the assumption $\partial U\cap\partial B=\emptyset$ guarantees that compact parts of $\partial U\times]0,T]$ are entirely in the open set $(\mathbb{R}^2-\overline{B})\times\mathbb{R}$.

Observe that, in terms of the operators S_T, S_T^* and E, we have

$$(z(\cdot,0),z_t(\cdot,0))=S_T^*((\varphi v(\cdot,T),\ \varphi v_t(\cdot,T)))=S_T^*\,\varphi S_TE(\widetilde{u}_0,\widetilde{u}_1),$$

where φ denotes the operator multiplication by φ. By introducing the identity operator I and the operator R of restriction to Ω, the equations (5.6) can be summarized as

$$(I-R\,S_T^*\varphi\,S_TE)(\widetilde{u}_0,\widetilde{u}_1)=(u_0,u_1).$$

Now, setting $K_T =: R\,S_T^*\varphi\,S_T E$, we see that K_T is a bounded linear operator in $H^1(\Omega)\times L^2(\Omega)$ and (5.6) can be written as

$$(I-K_T)(\widetilde{u}_0,\widetilde{u}_1)=(u_0,u_1). \tag{5.7}$$

Hence, solving the control problem reduces to finding $T>0$ for which $I-K_T$ is invertible. This is the point where de energy decay comes in. Now observing that the time reversibility for the wave equation allows us to use estimate (3.5) of Lemma 3.2 for solutions of the backward exterior problem (5.2) and considering the definitions of S_T and S_T^*, as well as the localization φ and the choice of δ we can carry out the successive estimates below.

$$\begin{aligned}
\|K_T(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega)\times L^2(\Omega)} &\le \|(z(\cdot,0),z_t(\cdot,0))\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{\text{const}}{(1+T)^2}\|(z(\cdot,T),z_t(\cdot,T))\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{\text{const}}{(1+T)^2}\|(\varphi v(\cdot,T),\varphi v_t(\cdot,T))\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{\text{const}}{(1+T)^2}\|(v(\cdot,T),v_t(\cdot,T))\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&= \frac{\text{const}}{(1+T)^2}\|S_T E(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega_\delta)\times L^2(\Omega_\delta)}\\
&\le \frac{\text{const}}{(1+T)^4}\|(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega)\times L^2(\Omega)}
\end{aligned}$$

or else,

$$\|K_T(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \frac{const}{(1\ +\ T)^4}\|(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega)\times L^2(\Omega).} \tag{5.8}$$

where *const* is a constant that does not depend on T and the initial state $(\widetilde{u}_0,\widetilde{u}_1)$.

At this point we choose and fix $T>0$ so that $\frac{const}{(1+T)^4}<1$, which makes $I-K_T$ invertible.

To conclude the proof Theorem 2.2 we fix this value of T, solve equation (5.7) and find the correct pair $(\widetilde{u}_0,\widetilde{u}_1)$ to begin the process of constructing $\widetilde{u}$ satisfying (5.5) and (5.6). After we define $u =: \widetilde{u}|_{\Omega\times(0,T)}$ and observe u satisfy (2.5) with $g =: \alpha u+\beta u_\nu$. The function u_ν is constructed by using Theorem 4.1 to determine the trace of $\widetilde{u}_\nu$ along each compact, smooth face of $\partial U\times[0,T]$ as a square-integrable function, and then by piecing them together to define u_ν. Notice that the trace of u as an L^2 function on the faces of $\partial U\times[0,T]$ follows from standard trace theorems as in [10] (see also [3, 5]). This completes the proof of Theorem 2.2.

Now we point out the changes needed to the proof of Theorem 2.1. The first to observe is that the domain of the extension operator E is

$$\{(u_0,u_1)\in H^1(\Omega)\times L^2(\Omega) : u_0=0 \text{ on } \partial B\}$$

and range $H_0^1(\mathbb{R}^2-\overline{B})\times L^2(\mathbb{R}^2-\overline{B})$.

The localization φ remains the same. In the exterior boundary value problems (5.1) and (5.2) we change the Neuman homogeneous boundary condition to the Dirichlet homogeneous boundary condition on ∂B and the operators S_T, S_T^* are now both defined in $H_0^1(\mathbb{R}^2-\overline{B})\times L^2(\mathbb{R}^2-\overline{B})$. The energy decay to be used now is given by the estimate (3.3) of Lemma 3.1. The same procedure will lead us to the equation (5.7). After using estimate (3.3), as we used (3.5) to obtain equation (5.8), we obtain

$$\|K_T(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \frac{const}{|t|^2(\log t)^4}\|(\widetilde{u}_0,\widetilde{u}_1)\|^2_{H^1(\Omega)\times L^2(\Omega).} \tag{5.9}$$

and the rest of the proof follows the same pattern as the previous case. □

Remark 5.1. The controllability time T obtained in Theorem 1 is not optimal. One possibility for improving it and achieving controllability close to the value $T=diam(\Omega)$ would be to analytically extend the mapping $t\to K_t$ to an appropriate sector of the complex plane, as in [9, 16], and then using spectral properties of the family $\{K_t\}$. This could be an interesting challenge.

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References

[1] W. D. Bastos, A. Spezamiglio; A note on the controllability for the wave equation on non-smooth plane domains, *Systems Control Lett.*, 55 (2006), pp 17-20.
[2] W. D. Bastos, A. Spezamiglio, C. A. Raposo; On exact boundary controllability for linealy coupled wave equations, *J. Math. Anal. Appl.*, 381 (2011), pp 557-564.
[3] H. Brezis; *Analyse fonctionnelle – Théorie et applications*, 2e tirage, Masson, Paris, 1987.
[4] W. Dan, On the low-frequency asymptotic expansion for some second-order elliptic systems in a two-dimensional exterior domain, *Math. Methods Appl. Sci.* 19 (1996) pp 1073-1090.
[5] R. Dautray, J.-L. Lions; *Mathematical Analysis and Numerical Methods for Sciences and Technology. Vol. 2 Functional Analysis and Variational Methods*. Springer-Verlag, Berlin, Heidelberg 1988.
[6] De-Fu Liu; Local energy decay for hyperbolic systems in exterior domains. *J. Math. Anal. Appl.*, 128, (1987) pp 312-331.
[7] M. Ikawa; *Hyperbolic Partial Differential Equations and Wave Phonomena*, Translations of Mathematical Monographs, Vol. 189, American Mathematical Society, Providence, RI, 2000.
[8] R. Kleinman, B. Vainberg; Full-low frequency asymptotic expansion for second-order elliptic equations in two dimensions, *Math. Methods Appl. Sci.* 17 (1994) 989–1004.
[9] J. Lagnese; Boundary value control of a class of hyperbolic equations in a general region. *SIAM J. Control and Optimization*, Vol. 15, No 5 (1977) 973-983.
[10] J.-L. Lions, E. Magenes; *Non-Homogeneous Boundary Value Problems and Applicantions*, V. 1, Springer-Verlag, Berlin Heidelberg, 1972.
[11] J.-L. Lions; *Contrôlabilité Exacte Perturbations et Stabilisation de Systèmes Distribués, Tome 1, Controlabilité Exacte*, Mason, Paris, 1988.
[12] C. S. Morawetz; The Decay of solutions of the exterior initial-boundary problem for the wave equation, *Comm. Pure Appl. Math.* 14 (1961) 561-568.
[13] C. S. Morawetz; Exponential decay of solutions of the wave equation. *Comm. Pure Appl. Math.* 19 (1966) 439–444.
[14] C. S. Morawetz; Decay for solutions of the exterior problem for the wave equation, *Comm. Pure and Appl. Math.*, 28 (1975) 229–264.
[15] J. Necas; *Les Méthodes Directes en Theorie des Équations Élliptiques*, Masson, Paris, 1967.
[16] R. S. O. Nunes, W. D. Bastos; Analyticity and near optmal time boundary controllability for the linear Kein-Gordon equation. *J. Math. Anal. Appl.*, 445 (2017), 394-406.
[17] D. Russell; Boundary Value Control ofthe Higher-dimensional Wave Equation, SIAM J, Control, 9 (1971), pp 29-42.
[18] D. Russell; A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.*, v.52 (1973) 189-211.
[19] P. Secchi, Y. Shibata; On the decay of solutions to the 2D Neumann exterior problem for the wave equation. *J. Differential Equations*, 194 (2003) pp 221-236.
[20] A. Soffer; A new paradigm for scattering theory of linear and nonlinear waves: review and open problems. *Adv Cont Discr Mod* 2024, 34 (2024). DOI: 10.1186/s13662-024-03831-6
[21] D. Tataru; On the regularity of boundary traces for the wave equation. *Ann. Scuola Norm. Sup. Pisa CL. Sci.*, (4), v.26 (1998) 185-206.
[22] F. Treves; *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York 1975.
[23] B. Vainberg; On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \to \infty$ of solutions of non-stationary problems, *Russian Math. Surveys*, 30, (1975), 1–58.
[24] R. J. Zimmer; *Essential Results of Functional Analysis*, University Of Chicago Press, 1990.
[25] K. Yosida; *Functional Analysis*, 6th ed. Springer-Verlag, Berlin, Heidelberg 1980.

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