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INTERIOR OBSERVATIONS FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT: COMPATIBILITY, IDENTIFIABILITY, AND CONDITIONING

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Abstract. For differential equations with piecewise constant argument on nonuniform intervals of constancy, an interior value is treated as an observation of the unknown preceding partition-point state. For linear systems, we characterize attainable data, derive exact first-order asymptotics for the smallest singular value of nearby weighted observation matrices, and obtain explicit lower bounds at positive observation separations that certify full column rank under simultaneous coefficient and observation-time uncertainty. For nonlinear equations, variational and projected Volterra estimates yield fold criteria, joint identification, a global Lipschitz graph representation of compatible data, quantitative residual–distance bounds, and noise estimates that distinguish components parallel and orthogonal to the linear observation space. A logistic population model illustrates inverse-branch coalescence and exact two-time identification.

1. Introduction

Let $\{\theta_j\}_{j\geq 0}$ be a strictly increasing sequence of partition points. For a differential equation with piecewise constant argument, the deviating argument equals θ_j on $[\theta_j, \theta_{j+1})$. If an observation time t_0 lies strictly inside this interval, the preceding partition-point state $x(\theta_j)$ is not prescribed by the observation $x(t_0) = x_0$. The inverse questions are therefore whether x_0 is attainable, whether it identifies the preceding state, and how stable that reconstruction is.

For a candidate $\xi = x(\theta_j)$, let $u_j(\cdot;\xi)$ denote the interval solution and set

$$P_{j,t_0}(\xi) := u_j(t_0;\xi).$$

Thus attainability is determined by Ran P_{j,t_0}, multiplicity by the preimages of P_{j,t_0}, and conditioning by quantitative inverse estimates. Forward uniqueness for a fixed ξ does not imply injectivity of the observation map.

Differential equations with piecewise constant argument have been studied in existence, oscillation, stability, periodicity, Green-function, and comparison problems; see [4, 12, 13, 15, 16, 19, 21, 26, 27, 29, 34]. Generalized piecewise constant arguments and impulsive variants are treated, for example, in [25, 35]. More specific Green-function and almost-periodic analyses include [5, 6, 7]. Recent periodic-orbit and stability results for nonlinear delayed models appear in [8, 10]. The qualitative influence of deviating arguments has also been investigated through oscillation, periodicity, and asymptotic analysis. Oscillation and periodicity criteria for scalar differential equations with generalized mixed piecewise constant arguments have been established [11]. Further results address oscillation in second-order superlinear neutral Emden–Fowler differential equations [24] and asymptotic behavior in higher-order sublinear delayed Emden–Fowler differential equations [23]. These studies enrich the qualitative theory of differential equations with deviating arguments and provide a broader mathematical context for the present investigation of compatibility, identifiability, and conditioning of interior observations. Wiener [33] studied pointwise initial-value

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problems for delays with intervals of constancy, while Pinto and Robledo [28] studied control-theoretic observability for linear time-varying systems with piecewise constant delay. Multipoint and initial-interior parameter problems appear in [1, 3, 17]. The problem here is different: interior values inside a constancy interval are inverse data for the preceding partition-point state, and we quantify their compatibility, identifiability, and conditioning directly from the interval dynamics.

Classical singular-value perturbation theory [30, 31, 32] and Lyusternik–Graves/global inverse-mapping results [14, 18, 20] provide the ambient tools. The central new contribution is the explicit reduction of the loss of identifiability to the equation coefficients and the observation schedule. To the best of our knowledge, Propositions 4.4 and 4.5 give the first exact first-order smallest-singular-value asymptotics for coalescing interior observations in differential equations with piecewise constant argument, including the weighted formula and centered-placement criterion. The resulting stacked-observation test can recover full column rank even when every individual observation matrix is singular. We then convert the asymptotic coefficient into explicit lower bounds for the smallest singular value at positive observation separations, including simultaneous coefficient and timing uncertainty. For nonlinear systems, the projected Volterra analysis yields a global compatible-data graph and separate parallel/orthogonal noise estimates; to the best of our knowledge, these projected results are also new for this class.

Scalar variational equations detect folds and give two-time identification, while the generalized logistic population equation provides an exact application; related population and epidemic models appear in [2, 9, 22].

Section 2 establishes the framework for forward solutions. Section 3 examines compatibility and identifiability from one or two nonlinear interior observations. Section 4 develops the linear compatibility and conditioning theory. Section 5 presents the nonlinear perturbation analysis and estimates for compatible data. Finally, Section 6 applies the theory to a logistic population model.

2. Preliminaries and forward solutions

Let $0 = \theta_0 < \theta_1 < \dots$, $\theta_j \to \infty$, and define

$$\beta(t) = \theta_j, \quad t \in I_j := [\theta_j, \theta_{j+1}).$$

Only strict increase and $\theta_j \to \infty$ are assumed for the partition; no uniform upper or lower bound on $\theta_{j+1} - \theta_j$ is imposed. Consider

$$x'(t) = f\big(t, x(t), x(\beta(t))\big), \quad t \geq 0, \tag{2.1}$$

where $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Here and throughout, $n \geq 1$ is fixed. Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm and the associated matrix norm, $\operatorname{Ran} M$ denotes the range of a matrix or map, and $\sigma_{\min}(M)$ denotes the smallest singular value of a matrix M. If M is nonsingular, then

$$\|M^{-1}\| = \sigma_{\min}(M)^{-1}.$$

For a map H between Euclidean spaces, $\operatorname{Lip}(H)$ denotes its least global Lipschitz constant whenever it is finite. For an affine subspace $a + V$, write $\operatorname{affdim}(a + V) = \dim V$.

The following assumptions are used in Sections 2 and 3.

(A1) For every $x, y \in \mathbb{R}^n$, the function $t \mapsto f(t, x, y)$ is Lebesgue measurable.

(A2) There exist nonnegative functions $L_x, L_y \in L^1_{\mathrm{loc}}([0,\infty))$ and a null set N such that, for every $t \notin N$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$,

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_x(t)\|x_1 - x_2\| + L_y(t)\|y_1 - y_2\|.$$

(A3) $m(t) := \|f(t, 0, 0)\|$ belongs to $L^1_{\mathrm{loc}}([0,\infty))$.

These hypotheses give the usual Carathéodory measurability and growth conditions. Indeed, for almost every t the map $(x, y) \mapsto f(t, x, y)$ is globally Lipschitz and hence continuous, while $t \mapsto f(t, x, y)$ is measurable for every fixed (x, y). Consequently, if v, w are measurable, then $t \mapsto f(t, v(t), w(t))$ is measurable. Moreover,

$$\|f(t, x, y)\| \leq m(t) + L_x(t)\|x\| + L_y(t)\|y\|$$

for almost every t and all $x, y \in \mathbb{R}^n$.

Let $AC([a,b];\mathbb{R}^n)$ denote the absolutely continuous functions from $[a,b]$ to $\mathbb{R}^n$. A solution on $[\theta_j,T]$ is an element of $AC([\theta_j,T];\mathbb{R}^n)$ satisfying (2.1) almost everywhere. Solutions on $[\theta_j,\infty)$ are understood locally in this sense. If $x(\theta_j)=\xi$, then on I_j equation (2.1) reduces to the Carathéodory initial-value problem

$$u'(t)=f(t,u(t),\xi),\quad u(\theta_j)=\xi. \tag{2.2}$$

Carathéodory theory gives a unique maximal solution. Gronwall's inequality yields, throughout its interval of existence,

$$\|u(t)\|\le\Big[\|\xi\|+\int_{\theta_j}^{t}\{m(s)+L_y(s)\|\xi\|\}\,ds\Big]\exp\Big(\int_{\theta_j}^{t}L_x(s)\,ds\Big).$$

The right-hand side is finite on $[\theta_j,\theta_{j+1}]$. Hence u remains bounded, and the growth estimate implies that $u'\in L^1$ up to any finite endpoint of its maximal interval. Therefore $u(t)$ has a finite limit there and the Carathéodory existence theorem restarts the solution from that limit. The maximal solution consequently reaches θ_{j+1}; its value there is understood as the continuous endpoint value, while the differential equation is required only almost everywhere and is unaffected by the change of β at the single partition point. Denote this interval solution by $u_j(\cdot;\xi)$. Repeating the construction interval by interval yields the following forward well-posedness result.

Proposition 2.1. *Assume* (A1)–(A3). *For every $j\ge 0$ and every $\xi\in\mathbb{R}^n$, there exists a unique locally absolutely continuous solution $x:[\theta_j,\infty)\to\mathbb{R}^n$ of* (2.1) *satisfying $x(\theta_j)=\xi$.*

Proof. The interval problem (2.2) has a unique solution on $[\theta_j,\theta_{j+1}]$. Its endpoint value is used as the initial value on the next interval, so consecutive pieces agree at every partition point. A finite concatenation of absolutely continuous functions that agree at their common endpoints is absolutely continuous. Since only finitely many partition points occur on every bounded time interval, the resulting function is locally absolutely continuous on $[\theta_j,\infty)$ and satisfies (2.1) almost everywhere. Uniqueness on each successive interval gives uniqueness of the forward solution. □

For a datum $x(t_0)=x_0$, with $t_0\in(\theta_j,\theta_{j+1})$, the value $x(\theta_j)$ is not prescribed. The next section formulates the corresponding compatibility, multiplicity, and identifiability questions.

3. Compatibility and identifiability from an interior observation

Fix $j\ge 0$ and $t_0\in(\theta_j,\theta_{j+1})$. The interior-data problem studied here prescribes $x(t_0)=x_0$, while the value $x(\theta_j)$ entering the equation on the current interval is unknown. The solution is sought on $[\theta_j,\infty)$.

For a candidate value $\xi=x(\theta_j)$, let $u_j(\cdot;\xi)$ be the interval solution from Section 2 and define the solution map

$$P_{j,t_0}:\mathbb{R}^n\to\mathbb{R}^n,\quad P_{j,t_0}(\xi):=u_j(t_0;\xi).$$

For $x_0\in\mathbb{R}^n$, define the current-interval solution set

$$\mathcal{S}_j(t_0,x_0):=\big\{x\in AC([\theta_j,\theta_{j+1}];\mathbb{R}^n):x'(t)=f(t,x(t),x(\theta_j))\text{ for a.e. }t\in(\theta_j,\theta_{j+1}),x(t_0)=x_0\big\},$$

and the forward solution set

$$\mathcal{F}_j(t_0,x_0):=\big\{x\in AC_{\mathrm{loc}}([\theta_j,\infty);\mathbb{R}^n):x\text{ solves (2.1) a.e. and }x(t_0)=x_0\big\}.$$

The datum x_0 is called compatible at t_0 if $\mathcal{S}_j(t_0,x_0)\neq\emptyset$. Proposition 2.1 shows that this is equivalent to $\mathcal{F}_j(t_0,x_0)\neq\emptyset$. The basic reduction is the following exact correspondence.

Lemma 3.1. *For every $x_0\in\mathbb{R}^n$, evaluation at the preceding partition point defines bijections*

$$\mathcal{S}_j(t_0,x_0)\longleftrightarrow P_{j,t_0}^{-1}(\{x_0\})\longleftrightarrow\mathcal{F}_j(t_0,x_0).$$

More precisely, both maps $x\longmapsto x(\theta_j)$ from $\mathcal{S}_j(t_0,x_0)$ and from $\mathcal{F}_j(t_0,x_0)$ onto $P_{j,t_0}^{-1}(\{x_0\})$ are bijective. Consequently,

$$\mathcal{S}_j(t_0,x_0)\neq\emptyset\iff\mathcal{F}_j(t_0,x_0)\neq\emptyset\iff x_0\in\operatorname{Ran}P_{j,t_0},$$

and the compatible values of $x(\theta_j)$ are precisely the elements of $P_{j,t_0}^{-1}(\{x_0\})$.

Proof. If $x \in \mathcal{S}_j(t_0, x_0)$ and $\xi = x(\theta_j)$, uniqueness of the parameterized interval problem gives $x = u_j(\cdot;\xi)$, hence $P_{j,t_0}(\xi) = x_0$. Conversely, every $\xi \in P_{j,t_0}^{-1}(\{x_0\})$ generates the element $u_j(\cdot;\xi) \in \mathcal{S}_j(t_0, x_0)$. Thus evaluation at θ_j is a bijection between these two sets. Proposition 2.1 extends each such interval solution uniquely to $[\theta_j, \infty)$, while every forward solution restricts to the corresponding interval solution. This gives the second bijection and all stated equivalences. □

Fix one admissible pair L_x, L_y in (A2). The quantities below are associated with this choice; the criterion may be tested with any admissible pair, and sharper bounds can enlarge the parameter range on which it is verified.

We define

$$\Lambda_j(t) := \int_{\theta_j}^{t} L_x(r)\, dr,$$

$$\mathcal{M}_j(t) := e^{\Lambda_j(t)} + \int_{\theta_j}^{t} e^{\Lambda_j(t)-\Lambda_j(s)} L_y(s)\, ds,$$

$$Q_j(t) := \int_{\theta_j}^{t} e^{\Lambda_j(s)} L_y(s)\, ds.$$

The next theorem gives a global bijectivity criterion for the observation map. A positive lower metric bound yields injectivity and quantitative inverse control, while invariance of domain completes the surjectivity argument.

Theorem 3.2. *Fix $j \geq 0$ and $t_0 \in (\theta_j, \theta_{j+1})$. Assume* (A1)–(A3). *If $Q_j(t_0) < 1$, then P_{j,t_0} is a bi-Lipschitz homeomorphism of $\mathbb{R}^n$ onto itself. More precisely, for all $\xi, \eta \in \mathbb{R}^n$,*

$$c_{j,t_0}\|\xi - \eta\| \leq \|P_{j,t_0}(\xi) - P_{j,t_0}(\eta)\| \leq \mathcal{M}_j(t_0)\|\xi - \eta\|, \tag{3.1}$$

where

$$c_{j,t_0} := e^{-\Lambda_j(t_0)}[1 - Q_j(t_0)] > 0.$$

Consequently,

$$\mathrm{Lip}(P_{j,t_0}^{-1}) \leq c_{j,t_0}^{-1}.$$

Every $x_0 \in \mathbb{R}^n$ therefore determines a unique value of $x(\theta_j)$ and a unique locally absolutely continuous solution on $[\theta_j, \infty)$ satisfying $x(t_0) = x_0$.

Proof. For $w(t) = u_j(t;\xi) - u_j(t;\eta)$, $d = \|\xi - \eta\|$, and $r(t) = \|w(t)\|$, absolute continuity and (A2) imply, almost everywhere,

$$-L_x(t)r(t) - L_y(t)d \leq r'(t) \leq L_x(t)r(t) + L_y(t)d, \quad r(\theta_j) = d.$$

Applying integrating factors to the two inequalities gives

$$e^{-\Lambda_j(t)}[1 - Q_j(t)]d \leq r(t) \leq \mathcal{M}_j(t)d.$$

Since Q_j is nondecreasing and $Q_j(t_0) < 1$, evaluation at t_0 yields (3.1); in particular P_{j,t_0} is continuous, injective, and its inverse on its image is c_{j,t_0}^{-1}-Lipschitz.

Let $E = P_{j,t_0}(\mathbb{R}^n)$. If $y_k = P_{j,t_0}(\xi_k) \to y$, the lower bound in (3.1) makes (ξ_k) Cauchy. Hence $\xi_k \to \xi$ and continuity gives $y = P_{j,t_0}(\xi)$, so E is closed. Invariance of domain makes E open because $P_{j,t_0} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and injective. As E is nonempty and $\mathbb{R}^n$ is connected, $E = \mathbb{R}^n$. The final assertion follows from Lemma 3.1 and Proposition 2.1. □

The lower bound also gives the recovery estimate

$$\|P_{j,t_0}^{-1}(x_0) - P_{j,t_0}^{-1}(\widetilde{x}_0)\| \leq c_{j,t_0}^{-1}\|x_0 - \widetilde{x}_0\|.$$

No differentiability or Jacobian hypothesis is used.

Remark 3.3. If $L_x = L_y = l$ on the current interval and $h = t_0 - \theta_j$, then $Q_j(t_0) = e^{lh} - 1$ and $c_{j,t_0} = 2e^{-lh} - 1$, so the theorem applies for $lh < \log 2$. With the sharp bounds $L_x = 0$, $L_y = 1$, the equations $x' = -x(\theta_j)$ and $x' = x(\theta_j)$ have observation maps $(1-h)\xi$ and $(1+h)\xi$, respectively. Thus the criterion can be exact at a singular time but is not necessary for invertibility; Section 4 gives the exact linear alternatives.

3.1. **A local fold criterion for scalar equations.** The range–preimage correspondence of Lemma 3.1 is not restricted to globally invertible maps. In scalar equations, a failure of local invertibility can be detected directly through the variational equations with respect to the preceding partition-point value. The following result is local: it does not assert that there are no additional preimages elsewhere.

Proposition 3.4. *Let $n = 1$, fix $j \geq 0$, $t_0 \in (\theta_j, \theta_{j+1})$, and let $\xi_* \in \mathbb{R}$. Assume that the interval problem*

$$u'(t) = f(t, u(t), \xi), \quad u(\theta_j) = \xi \tag{3.2}$$

has a unique solution on $[\theta_j, t_0]$ for every ξ in an open neighborhood of ξ_*. Assume that $t \mapsto f(t, x, y)$ is measurable for every fixed (x, y), that $f(t, 0, 0)$ is integrable on $[\theta_j, t_0]$, and that for almost every t the function $(x, y) \mapsto f(t, x, y)$ is C^3. Suppose its partial derivatives of orders one through three are measurable in t at fixed (x, y) and, for every compact $K \subset \mathbb{R}^2$, have an integrable majorant on $[\theta_j, t_0]$ that is uniform over $(x, y) \in K$. The global Lipschitz assumption (A2) *is not required. Let $u_*(t) = u(t; \xi_*)$ and define v, w as solutions of*

$$v' = f_x(t, u_*(t), \xi_*)v + f_y(t, u_*(t), \xi_*), \quad v(\theta_j) = 1, \tag{3.3}$$

$$\begin{gathered} w' = f_x(t, u_*(t), \xi_*)w + f_{xx}(t, u_*(t), \xi_*)v^2 + 2f_{xy}(t, u_*(t), \xi_*)v + f_{yy}(t, u_*(t), \xi_*), \\ w(\theta_j) = 0. \end{gathered} \tag{3.4}$$

If

$$v(t_0) = 0, \quad \kappa := w(t_0) \neq 0, \tag{3.5}$$

then ξ_* is a strict local maximum of the scalar solution map P_{j,t_0} when $\kappa < 0$, and a strict local minimum when $\kappa > 0$. Writing $y_c = P_{j,t_0}(\xi_*)$ and $s = \operatorname{sgn}(\kappa)$, there are neighborhoods of ξ_* and y_c such that, for all sufficiently small $\delta > 0$, the equation

$$P_{j,t_0}(\xi) = y_c + s\delta$$

has exactly two solutions in the chosen neighborhood, while $P_{j,t_0}(\xi) = y_c$ has exactly one and $P_{j,t_0}(\xi) = y_c - s\delta$ has none there. The two local solutions satisfy

$$\xi_\pm(\delta) = \xi_* \pm \sqrt{\frac{2\delta}{|\kappa|}} + O(\delta) \quad (\delta \downarrow 0). \tag{3.6}$$

Their inverse derivatives, taken with respect to the observed level y, obey

$$\Big|\frac{d\xi_\pm}{dy}\Big| = \frac{1 + O(\sqrt{\delta})}{\sqrt{2|\kappa|\delta}}. \tag{3.7}$$

Proof. The assumed integrable majorants for the state derivatives permit parameter differentiation of the integral equation for (3.2), using difference quotients and Gronwall's inequality on the compact time interval. In a sufficiently small neighborhood of ξ_* this yields C^3 dependence of $u(t; \xi)$ on ξ. Differentiating the equation once and twice proves, respectively, (3.3) and (3.4), and gives

$$P'_{j,t_0}(\xi_*) = v(t_0), \quad P''_{j,t_0}(\xi_*) = w(t_0) = \kappa.$$

Taylor's theorem and (3.5) now imply

$$P_{j,t_0}(\xi_* + z) = y_c + \frac{\kappa}{2}z^2 + O(|z|^3), \tag{3.8}$$

and differentiation yields $P'_{j,t_0}(\xi_* + z) = \kappa z + O(z^2)$. Thus P_{j,t_0} is strictly monotone on each of two sufficiently short intervals meeting at ξ_*, with opposite monotonicities on the two sides. This proves the local multiplicity alternatives. For either of the two inverse branches, put $z = \xi_\pm(\delta) - \xi_*$. Then (3.8) gives

$$\delta = \frac{|\kappa|}{2}z^2 + O(|z|^3).$$

It follows first that $|z| = O(\sqrt{\delta})$ and then that $z^2 = 2\delta/|\kappa| + O(\delta^{3/2})$. Taking the respective signs proves (3.6). Finally, on either noncritical branch the inverse function theorem gives $d\xi_\pm/dy = 1/P'_{j,t_0}(\xi_\pm)$. Using $P'_{j,t_0}(\xi_* + z) = \kappa z + O(z^2)$ and (3.6) yields (3.7). □

The criterion is formulated using derivatives of the original differential equation rather than an assumed closed form of the solution map. The non-vanishing second sensitivity $w(t_0)$ is essential: a vanishing first sensitivity alone does not imply a fold or exactly two nearby compatible states. The logistic example in Section 6 provides an explicit realization of (3.5).

3.2. **Two observations for scalar nonlinear equations.** The range-preimage formulation also permits joint observations without an explicit formula for the nonlinear interval map. A monotonicity condition on the dependence on the piecewise constant argument ensures that two observations distinguish all preceding states in a specified invariant interval. The result does not require either one-time map to be invertible.

Theorem 3.5. *Fix $\theta_j < t_1 < t_2 < \theta_{j+1}$, and let J, D be intervals in $\mathbb{R}$, with D open and $J \subset D$. Suppose that, for every $\xi \in J$, the scalar interval problem*

$$u'(t) = f(t, u(t), \xi), \quad u(\theta_j) = \xi$$

has a unique solution through t_2, taking values in D. Assume that for almost every $t \in [t_1, t_2]$, the function $(x, y) \mapsto f(t, x, y)$ is continuously differentiable on an open neighborhood of $D \times J$. Assume that its first partial derivatives are measurable in t and have an integrable majorant, uniform on each compact subset of $D \times J$. Suppose moreover that f_y has a fixed strict sign on $D \times J$: either

$$f_y(t, x, y) > 0 \quad \text{for every } (x, y) \in D \times J$$

for almost every $t \in [t_1, t_2]$, or the same inequality holds with $<$ instead of $>$. Then the joint observation map

$$\mathcal{P}_{j;t_1,t_2} : J \to \mathbb{R}^2, \quad \mathcal{P}_{j;t_1,t_2}(\xi) := \big(u_j(t_1; \xi), u_j(t_2; \xi)\big)$$

is injective.

If, additionally, for some nonnegative $\ell, m \in L^1([t_1, t_2])$ with $\int_{t_1}^{t_2} m(t)\, dt > 0$, one has

$$|f_x(t, x, y)| \le \ell(t), \quad |f_y(t, x, y)| \ge m(t)$$

for every $(x, y) \in D \times J$ and almost every $t \in [t_1, t_2]$, then, with $L = \int_{t_1}^{t_2} \ell(t)\, dt$ and $M = \int_{t_1}^{t_2} m(t)\, dt$, every $\xi, \eta \in J$ satisfy

$$|\xi - \eta| \le \frac{e^{2L}}{M}\Big(|P_{j,t_1}(\xi) - P_{j,t_1}(\eta)| + |P_{j,t_2}(\xi) - P_{j,t_2}(\eta)|\Big). \tag{3.9}$$

In particular the inverse of $\mathcal{P}_{j;t_1,t_2}$ on its image is Lipschitz for the sum metric on the two observations. This theorem makes no assertion that all pairs in $\mathbb{R}^2$ are compatible.

Proof. Fix distinct $\xi, \eta \in J$ and set $w(t) = u_j(t; \xi) - u_j(t; \eta)$. On $[t_1, t_2]$, the fundamental theorem of calculus applied separately to the two state arguments gives, for almost every t,

$$w'(t) = a(t)w(t) + b(t)(\xi - \eta),$$

where

$$a(t) = \int_0^1 f_x\big(t, u_j(t; \eta) + sw(t), \xi\big)\, ds,$$
$$b(t) = \int_0^1 f_y\big(t, u_j(t; \eta), \eta + s(\xi - \eta)\big)\, ds.$$

Because D and J are intervals and both trajectories stay in D, every argument in these integrals remains in $D \times J$. The integrable-majorant assumption makes a, b integrable; the sign condition gives $b(t) > 0$ almost everywhere or $b(t) < 0$ almost everywhere on $[t_1, t_2]$, with the same alternative for all t. Write $A(t) = \int_{t_1}^{t} a(s)\, ds$. The integrating-factor identity is

$$e^{-A(t_2)}w(t_2) - w(t_1) = (\xi - \eta) \int_{t_1}^{t_2} e^{-A(s)} b(s)\, ds. \tag{3.10}$$

The integral on the right is nonzero because its integrand has a strict fixed sign almost everywhere. Therefore $w(t_1)$ and $w(t_2)$ cannot both vanish: the joint map is injective.

For the quantitative statement, $|a(t)| \le \ell(t)$ and $|b(t)| \ge m(t)$, and b still has a fixed sign. Hence $|A(t)| \le L$ and

$$\Big| \int_{t_1}^{t_2} e^{-A(s)} b(s)\, ds \Big| \ge e^{-L} M.$$

Taking absolute values in (3.10) yields

$$|\xi - \eta| e^{-L} M \le e^{L} |w(t_2)| + |w(t_1)|.$$

Since $L \ge 0$, the stated bound (3.9) follows. □

The time separation in (3.9) is essential. For $x' = -x(\beta(t))$, observations at $t_1 = \theta_j + 1$ and $t_2 = \theta_j + 1 + d$ are 0 and $-d\xi$; hence the optimal inverse Lipschitz constant for the sum metric is $1/d$, exactly the value supplied by (3.9) when $\ell = 0$ and $m = 1$.

The fixed-sign hypothesis also cannot be replaced by $f_y \ne 0$ almost everywhere. Take $\theta_j = 0$, $\theta_{j+1} > 3$, and $f(t, x, y) = b(t)y$ with $b = -1, 1, -1$ on $[0, 1), [1, 2), [2, 3)$, respectively. Then $u(t; \xi) = [1 + \int_0^t b(s)\, ds]\xi$, so the observations at $t = 1$ and $t = 3$ are both zero for every ξ, although $|f_y| = 1$ almost everywhere on $[1, 3]$. The sign condition prevents precisely this cancellation; no necessity claim is made.

4. Linear systems: compatibility and singularity

4.1. **Compatibility and multiplicity.** We consider

$$x'(t) = A(t)x(t) + B(t)x(\beta(t)) + g(t),$$

where

$$A, B \in L^1_{\mathrm{loc}}([0, \infty); \mathbb{R}^{n \times n}), \quad g \in L^1_{\mathrm{loc}}([0, \infty); \mathbb{R}^n).$$

This system satisfies the assumptions of Sections 2 and 3 with

$$L_x(t) = \|A(t)\|, \quad L_y(t) = \|B(t)\|, \quad m(t) = \|g(t)\|.$$

Let $\Phi(t, s)$ denote the evolution operator associated with

$$z'(t) = A(t)z(t).$$

For a fixed s, the map $t \mapsto \Phi(t, s)$ is absolutely continuous, $\Phi(s, s) = I$, and

$$\frac{\partial}{\partial t} \Phi(t, s) = A(t)\Phi(t, s)$$

for almost every t. Moreover, $(t, s) \mapsto \Phi(t, s)$ is jointly continuous on every compact triangle $\{(t, s) : a \le s \le t \le b\}$; in particular, it is bounded there.

Fix $j \ge 0$ and set $\xi = x(\theta_j)$. On the current interval,

$$x'(t) = A(t)x(t) + B(t)\xi + g(t), \quad x(\theta_j) = \xi.$$

For a fixed ξ, the forcing $B(\cdot)\xi + g(\cdot)$ belongs to $L^1([\theta_j, \theta_{j+1}]; \mathbb{R}^n)$. Hence variation of constants for the linear system with locally integrable coefficients gives, for $\theta_j \le t \le \theta_{j+1}$,

$$x(t) = G_j(t)\xi + r_j(t),$$

where

$$G_j(t) := \Phi(t, \theta_j) + \int_{\theta_j}^{t} \Phi(t, s)B(s)\, ds,$$

$$r_j(t) := \int_{\theta_j}^{t} \Phi(t, s)g(s)\, ds.$$

Since Φ is bounded on compact triangles and B, g are locally integrable, these integrals are well defined. Standard variation-of-constants arguments give

$$G_j(\theta_j) = I, \quad r_j(\theta_j) = 0,$$

and G_j, r_j are absolutely continuous with

$$G_j'(t) = A(t)G_j(t) + B(t), \quad r_j'(t) = A(t)r_j(t) + g(t)$$

for almost every $t \in I_j$. In particular, the solution map is affine:

$$P_{j,t_0}(\xi) = G_j(t_0)\xi + r_j(t_0). \tag{4.1}$$

Thus prescribing $x(t_0) = x_0$ is equivalent to the finite-dimensional linear equation

$$G_j(t_0)\xi = x_0 - r_j(t_0).$$

The resulting compatibility and multiplicity alternatives are exact.

Theorem 4.1. *Fix $j \geq 0$ and $t_0 \in (\theta_j, \theta_{j+1})$. Then the set of compatible interior data is the affine subspace*

$$\mathcal{C}_{j,t_0} = r_j(t_0) + \operatorname{Ran} G_j(t_0),$$

and hence $\operatorname{affdim} \mathcal{C}_{j,t_0} = \operatorname{rank} G_j(t_0)$. *If $x_0 \in \mathcal{C}_{j,t_0}$ and ξ_p satisfies*

$$G_j(t_0)\xi_p = x_0 - r_j(t_0),$$

then

$$\mathcal{S}_j(t_0, x_0) = \big\{x_z : \ x_z(t) = G_j(t)(\xi_p + z) + r_j(t), \quad z \in \ker G_j(t_0)\big\}.$$

The map $z \mapsto x_z$ is an affine bijection from $\ker G_j(t_0)$ onto $\mathcal{S}_j(t_0, x_0)$. Consequently,

$$\operatorname{affdim} \mathcal{S}_j(t_0, x_0) = \dim \ker G_j(t_0) = n - \operatorname{rank} G_j(t_0).$$

In particular,

$$\operatorname{affdim} \mathcal{C}_{j,t_0} + \operatorname{affdim} \mathcal{S}_j(t_0, x_0) = n.$$

The displayed formula for x_z applies on the current interval; the bijection with globally continued solutions follows from Proposition 2.1. Thus, if $G_j(t_0)$ is singular, the attainable interior data form a proper affine subspace and each compatible datum admits a positive-dimensional affine family of solutions. If $G_j(t_0)$ is nonsingular, then $\mathcal{C}_{j,t_0} = \mathbb{R}^n$ and every $x_0 \in \mathbb{R}^n$ determines the unique value

$$x(\theta_j) = G_j(t_0)^{-1}[x_0 - r_j(t_0)].$$

Proof. By (4.1), compatibility is equivalent to solvability of

$$G_j(t_0)\xi = x_0 - r_j(t_0),$$

which gives $\mathcal{C}_{j,t_0} = r_j(t_0) + \operatorname{Ran} G_j(t_0)$ and $\operatorname{affdim} \mathcal{C}_{j,t_0} = \operatorname{rank} G_j(t_0)$. If ξ_p is one solution, all solutions are $\xi_p + \ker G_j(t_0)$, and variation of constants gives

$$x_z(t) = G_j(t)(\xi_p + z) + r_j(t), \quad z \in \ker G_j(t_0).$$

Lemma 3.1 makes $z \mapsto x_z$ onto $\mathcal{S}_j(t_0, x_0)$. It is injective because $x_{z_1} = x_{z_2}$ implies, upon evaluation at $t = \theta_j$ and use of $G_j(\theta_j) = I$, that $z_1 = z_2$. Hence $\mathcal{S}_j(t_0, x_0)$ is affinely bijective to $\ker G_j(t_0)$, and rank–nullity yields the stated affine dimension and the identity of dimensions. The singular and nonsingular alternatives follow immediately. Proposition 2.1 gives the unique continuation to $[\theta_j, \infty)$. □

Thus the range determines attainable observations, the kernel describes the full sets of compatible preceding states, and their complementary dimensions sum to n. Nonsingularity is precisely the one-to-one case.

For observations affected by error, the same classification has an exact projection form. Let $G^\dagger$ denote the Moore–Penrose inverse of a matrix G.

Corollary 4.2. *Fix $j \geq 0$ and $t_0 \in (\theta_j, \theta_{j+1})$, and write $G := G_j(t_0)$, $r := r_j(t_0)$. For every $y \in \mathbb{R}^n$,*

$$\operatorname{dist}\big(y, r + \operatorname{Ran} G\big) = \big\|(I - GG^\dagger)(y - r)\big\|. \tag{4.2}$$

The orthogonal projection of y onto the compatible-data set is

$$y_{\mathrm{c}} = r + GG^\dagger(y - r).$$

Moreover, $\xi_{\mathrm{ls}} := G^\dagger(y - r)$ is the unique minimum-norm least-squares state for the observation y, and it generates y_{c} through the affine solution map. Consequently, y is compatible if and only if the right-hand side of (4.2) is zero. In that case, ξ_{ls} is the unique compatible preceding partition-point state of minimum Euclidean norm, and all compatible states are $\xi_{\mathrm{ls}} + \ker G$.

Proof. The matrix $GG^\dagger$ is the orthogonal projector onto $\operatorname{Ran} G$. Hence $r + GG^\dagger(y - r)$ is the orthogonal projection of y onto $r+\operatorname{Ran} G$, which proves (4.2). The Moore–Penrose vector $G^\dagger(y-r)$ is the unique minimum-norm minimizer of $\|G\xi - (y - r)\|$, and its image under the affine map is $r + GG^\dagger(y-r) = y_c$. If $y - r \in \operatorname{Ran} G$, the standard Moore–Penrose decomposition gives all exact solutions of $G\xi = y - r$ as $G^\dagger(y - r) + \ker G$, with $G^\dagger(y - r) \perp \ker G$; it is therefore the unique minimum-norm exact solution. □

Corollary 4.2 turns the singular linear alternative into a quantitative consistency test: the residual in (4.2) measures exactly how far an observed interior state lies from the set that the model can generate at t_0. The minimum-norm least-squares reconstruction has its own exact conditioning constant. Indeed, the affine map

$$y \longmapsto G^\dagger(y - r)$$

has Lipschitz constant $\|G^\dagger\|$. If G has rank $r_G > 0$ and positive singular values

$$\sigma_1(G) \geq \cdots \geq \sigma_{r_G}(G) > 0,$$

then

$$\|G^\dagger\| = \frac{1}{\sigma_{r_G}(G)}.$$

Hence the minimum-norm least-squares reconstruction is globally Lipschitz, with exact amplification factor $\|G^\dagger\|$; small positive singular values make this reconstruction increasingly sensitive to perturbations in the observation. If $G = 0$, the minimum-norm least-squares map is identically zero.

If $G_j(t_0)$ is nonsingular, then on the current interval

$$x(t) = G_j(t)G_j(t_0)^{-1}[x_0 - r_j(t_0)] + r_j(t), \quad \theta_j \leq t \leq \theta_{j+1}.$$

This formula describes the solution on the current interval. Its continuation beyond θ_{j+1} is determined recursively by Proposition 2.1.

The affine map from the prescribed interior state to $x(\theta_j)$ is

$$\mathcal{I}_{t_0}(x_0) := G_j(t_0)^{-1}[x_0 - r_j(t_0)].$$

Translations do not affect the Lipschitz constant, and the Lipschitz constant of a linear map is its induced operator norm. Hence

$$\operatorname{Lip}(\mathcal{I}_{t_0}) = \|G_j(t_0)^{-1}\| = \frac{1}{\sigma_{\min}(G_j(t_0))}.$$

For two interior data $x_0, \widetilde{x}_0$, the corresponding current-interval solutions satisfy

$$\|x(t) - \widetilde{x}(t)\| \leq \|G_j(t)G_j(t_0)^{-1}\| \, \|x_0 - \widetilde{x}_0\|, \quad \theta_j \leq t \leq \theta_{j+1}.$$

4.2. **Compatibility of two interior observations.** A single interior state may fail to determine the preceding partition-point state because $G_j(t_0)$ is singular. Additional observations at distinct times on the *same* interval of constancy are subject to a different compatibility condition. Write $t_1, t_2 \in (\theta_j, \theta_{j+1})$ with $t_1 < t_2$ and set

$$\mathcal{G}_j = \begin{pmatrix} G_j(t_1) \\ G_j(t_2) \end{pmatrix}, \quad \mathcal{R}_j = \begin{pmatrix} r_j(t_1) \\ r_j(t_2) \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The matrix $\mathcal{G}_j$ is $2n \times n$ and the observation vector is measured using the Euclidean norm in $\mathbb{R}^{2n}$. No claim is made that an arbitrary pair (y_1, y_2) is attainable.

Proposition 4.3. *Under the linear assumptions of Theorem 4.1, the two interior data $x(t_1) = y_1$ and $x(t_2) = y_2$ arise from a common solution on $[\theta_j, \infty)$ if and only if*

$$\mathcal{Y} - \mathcal{R}_j \in \operatorname{Ran} \mathcal{G}_j. \tag{4.3}$$

If the pair is compatible and ξ_p is one compatible preceding state, then the complete set of such states is

$$\xi_p + \ker \mathcal{G}_j = \xi_p + \big(\ker G_j(t_1) \cap \ker G_j(t_2)\big).$$

Consequently, the preceding state and the associated forward solution are unique exactly when $\mathcal{G}_j$ has full column rank. For an arbitrary pair $\mathcal{Y}$, its exact compatibility residual is

$$\operatorname{dist}(\mathcal{Y}, \mathcal{R}_j + \operatorname{Ran} \mathcal{G}_j) = \|(I - \mathcal{G}_j \mathcal{G}_j^\dagger)(\mathcal{Y} - \mathcal{R}_j)\|, \tag{4.4}$$

and $\mathcal{G}_j^\dagger(\mathcal{Y} - \mathcal{R}_j)$ is the unique minimum-norm least-squares preceding state. If $\mathcal{G}_j$ has full column rank, the exact Lipschitz constant of this reconstruction on $\mathbb{R}^{2n}$ is

$$\|\mathcal{G}_j^\dagger\| = \frac{1}{\sigma_{\min}(\mathcal{G}_j)}.$$

The same constant governs inversion restricted to compatible observation pairs.

Proof. Variation of constants at t_1 and t_2 gives the stacked affine observation equation

$$\mathcal{Y} = \mathcal{G}_j \xi + \mathcal{R}_j.$$

The range condition is therefore equivalent to compatibility. All solutions of the two matrix equations differ by a vector annihilated by both matrices, so their difference space is $\ker G_j(t_1) \cap \ker G_j(t_2)$. The correspondence lemma and unique forward continuation identify each such state with a distinct solution. The orthogonal projection onto $\operatorname{Ran} \mathcal{G}_j$ proves the residual formula, and the Moore–Penrose properties yield the minimum-norm least-squares statement. In the full-column-rank case, $\mathcal{G}_j^\dagger$ has operator norm $1/\sigma_{\min}(\mathcal{G}_j)$; restricting to the range does not lower this optimal constant because a right singular vector corresponding to the smallest singular value has an image in $\operatorname{Ran} \mathcal{G}_j$. □

The combined conditioning satisfies the exact Gram identity

$$\mathcal{G}_j^T \mathcal{G}_j = G_j(t_1)^T G_j(t_1) + G_j(t_2)^T G_j(t_2),$$

so the variational characterization of the smallest eigenvalue gives

$$\sigma_{\min}(\mathcal{G}_j)^2 \geq \sigma_{\min}(G_j(t_1))^2 + \sigma_{\min}(G_j(t_2))^2.$$

The inequality can be strict, including when both one-time observation matrices are singular. The rotating rank-one construction following Corollary 4.7 gives an explicit example in which the stacked observation is nevertheless uniquely invertible.

Proposition 4.4. *Fix $j \geq 0$ and an interior time $t_* \in (\theta_j, \theta_{j+1})$ at which $G_* := G_j(t_*)$ is singular. Suppose the coefficient matrices A and B are continuous in a neighborhood of t_*. Then the following conditions are equivalent:*

(i) *$B(t_*)$ is injective when restricted to $\ker G_*$;*
(ii) *the $2n \times n$ matrix*

$$\mathcal{T}_* := \begin{pmatrix} G_* \\ A(t_*)G_* + B(t_*) \end{pmatrix}$$

has full column rank.

Write $K = \ker G_*$ and $R = \operatorname{Ran} G_*$, and let Q_R denote the orthogonal projector onto R. In all cases define

$$\mu_* := \min_{\substack{k \in K \\ \|k\|=1}} \left(\frac{1}{2} \|Q_R B(t_*) k\|^2 + \|(I - Q_R) B(t_*) k\|^2 \right)^{1/2}.$$

Then the exact first-order asymptotic formula is

$$\lim_{h \downarrow 0} \frac{1}{h} \sigma_{\min}\left(\begin{pmatrix} G_j(t_*) \\ G_j(t_* + h) \end{pmatrix} \right) = \mu_*. \tag{4.5}$$

Moreover, $\mu_* > 0$ if and only if conditions (i)–(ii) *hold. In this case the two observations uniquely determine every compatible preceding state for sufficiently small $h > 0$, and the optimal inverse Lipschitz constant is $(\mu_* h)^{-1}(1 + o(1))$. If these conditions fail, the smallest singular value is $o(h)$; this does not preclude recovery at higher orders in h. The conclusions require $0 < h < \theta_{j+1} - t_*$.*

Proof. Since $G_j'(t) = A(t)G_j(t) + B(t)$ near t_*, write $H := G_j'(t_*)$ and $G_h := G_j(t_* + h) = G_* + hH + o(h)$ in operator norm. Here the remainder is uniform over unit vectors, because continuity of A and B near t_* makes G_j continuously differentiable there. Since G_* is singular, $K = \ker G_*$ is nontrivial and the unit-sphere minimization defining μ_* is over a nonempty compact set. For any $k \in K$, $Hk = B(t_*)k$. The matrix $\mathcal{T}_* = (G_*^T, H^T)^T$ has full column rank exactly when $\ker G_* \cap \ker H = \{0\}$, which is equivalent to (i). Thus (i) and (ii) agree.

We determine the exact leading singular value by minimizing over unit vectors. For $k \in K$, $\|k\| = 1$, and $u \in K^\perp$, the trial vector $v_h = (k + hu)/\|k + hu\|$ gives

$$\lim_{h\downarrow 0} h^{-2} \left\| \begin{pmatrix} G_* v_h \\ G_h v_h \end{pmatrix} \right\|^2 = \|G_* u\|^2 + \|G_* u + Hk\|^2,$$

so the limit superior is bounded by the corresponding infimum. For the reverse inequality, let v_h be a unit minimizer. The trial bound gives $\|G_* v_h\| = O(h)$; hence $v_h = k_h + hu_h$ with $k_h \in K$, $u_h \in K^\perp$, $\{u_h\}$ bounded, and $\|k_h\| \to 1$. From every sequence $h \downarrow 0$ one can therefore extract a subsequence with $k_h \to k$, $u_h \to u$, where $k \in K$, $\|k\| = 1$, $u \in K^\perp$. Operator-norm differentiability then gives the same displayed expression as the subsequential limit. Thus the limit inferior is at least the infimum, proving existence of the full limit and equality with that infimum.

Because $G_* K^\perp = R$, put $z = G_* u \in R$ and minimize $\|z\|^2 + \|z + Hk\|^2$ in z. Its minimizer is $z = -\frac{1}{2} Q_R Hk$, and the minimum equals

$$\tfrac{1}{2}\|Q_R Hk\|^2 + \|(I - Q_R)Hk\|^2 = \tfrac{1}{2}\|Q_R B(t_*)k\|^2 + \|(I - Q_R)B(t_*)k\|^2.$$

Minimizing over unit $k \in K$ proves (4.5). Its coefficient is strictly positive exactly when $B(t_*)$ is injective on K; the inverse-amplification claims follow from Proposition 4.3. □

For multiple observations moving at prescribed rates relative to the singular time, the leading coefficient separates the effect of the observation schedule from that of the coefficient matrix on the kernel directions. General small-singular-value expansions are available in [30, 32]; the new feature here is that the reduced coefficient is evaluated explicitly from the interval equation through $G_j'(t_*)k = B(t_*)k$ on $\ker G_j(t_*)$, while retaining the full dependence on the weighted observation schedule. This explicit reduction is what makes the centered-placement criterion and the subsequent positive-separation robustness bounds possible.

Proposition 4.5. *Under the hypotheses of Proposition 4.4, let $m \geq 2$, choose distinct fixed offsets $c_1 < \cdots < c_m$ and positive weights $w_1, \ldots, w_m$. Write $W = \sum_{i=1}^m w_i$ and*

$$\bar{c}_w = \frac{1}{W}\sum_{i=1}^m w_i c_i, \quad S_w = \sum_{i=1}^m w_i c_i^2, \quad V_w = \sum_{i=1}^m w_i (c_i - \bar{c}_w)^2.$$

For sufficiently small $h > 0$ all $t_* + c_i h$ lie in the same interval of constancy. Define the weighted observation matrix

$$\mathcal{G}_w(h) = \begin{pmatrix} \sqrt{w_1} G_j(t_* + c_1 h) \\ \vdots \\ \sqrt{w_m} G_j(t_* + c_m h) \end{pmatrix}.$$

With G_*, K, R, and Q_R as in Proposition 4.4, one has

$$\lim_{h\downarrow 0} \frac{\sigma_{\min}(\mathcal{G}_w(h))}{h} = \min_{\substack{k \in K \\ \|k\|=1}} \left[V_w \|Q_R B(t_*)k\|^2 + S_w \|(I - Q_R)B(t_*)k\|^2 \right]^{1/2}. \tag{4.6}$$

Since $V_w, S_w > 0$, the coefficient is strictly positive exactly when $B(t_*)$ is injective on K. For fixed weights and fixed $S_w > 0$, its value is no larger than

$$\sqrt{S_w} \min_{k \in K,\, \|k\|=1} \|B(t_*)k\|. \tag{4.7}$$

Every admissible centered configuration $\sum_{i=1}^m w_i c_i = 0$ attains this upper bound. Here the comparison is over unconstrained real offsets with the same weighted squared offset S_w, not over one-sided or otherwise restricted sampling schedules.

Proof. Put $H = G_j'(t_*)$, so $Hk = B(t_*)k$ for $k \in K$. For each fixed i, $G_j(t_* + c_i h) = G_* + c_i hH + o(h)$ in operator norm. For any unit $k \in K$ and $u \in K^\perp$, the trial vectors $v_h = (k + hu)/\|k + hu\|$ yield the upper bound for the squared coefficient by

$$\sum_{i=1}^{m} w_i \|G_* u + c_i Hk\|^2.$$

Conversely, a fixed unit $k \in K$ gives $\sigma_{\min}(\mathcal{G}_w(h)) = O(h)$. For a unit minimizer write $v_h = k_h + hu_h$ with $k_h \in K$, $u_h \in K^\perp$. Positivity of the weights and the lower bound for G_* on $K^\perp$ make $\{u_h\}$ bounded and $\|k_h\| \to 1$. Every sequence $h \downarrow 0$ therefore has a subsequence with $k_h \to k$, $\|k\| = 1$, and $u_h \to u$. The finitely many operator-norm expansions yield

$$h^{-2}\|\mathcal{G}_w(h)v_h\|^2 \to \sum_i w_i \|G_* u + c_i Hk\|^2.$$

Together with the trial-vector bound, this proves existence of the full limit and identifies its square with the minimum of the displayed quadratic expression over unit $k \in K$ and $u \in K^\perp$.

As $G_* K^\perp = R$, substitute $z = G_* u \in R$ and $Hk = Q_R Hk + (I - Q_R)Hk$. Orthogonality gives

$$\begin{aligned}&\sum_{i=1}^{m} w_i \|z + c_i Q_R Hk\|^2 + S_w \|(I - Q_R)Hk\|^2 \\ &= W\|z + \bar{c}_w Q_R Hk\|^2 + V_w \|Q_R Hk\|^2 + S_w \|(I - Q_R)Hk\|^2.\end{aligned}$$

Minimization over $z \in R$ therefore gives (4.6). Distinct offsets and positive weights ensure $V_w > 0$, and they also ensure $S_w > 0$. Finally $V_w = S_w - W\bar{c}_w^2 \le S_w$ and hence, for each k,

$$V_w \|Q_R B(t_*)k\|^2 + S_w \|(I - Q_R)B(t_*)k\|^2 \le S_w \|B(t_*)k\|^2.$$

Taking minima proves (4.7). If $\bar{c}_w = 0$, then $V_w = S_w$, making the bound an equality. □

For $m = 2$, $w_1 = w_2 = 1$, and offsets $a < b$, one has $V_w = (b - a)^2/2$ and $S_w = a^2 + b^2$. Hence

$$\begin{aligned}&\lim_{h \downarrow 0} \frac{1}{h} \sigma_{\min}\left(\begin{pmatrix} G_j(t_* + ah) \\ G_j(t_* + bh) \end{pmatrix}\right) \\ &= \min_{k \in K,\ \|k\| = 1} \Big(\frac{(b - a)^2}{2} \|Q_R B(t_*)k\|^2 + (a^2 + b^2)\|(I - Q_R)B(t_*)k\|^2 \Big)^{1/2}.\end{aligned} \tag{4.8}$$

In particular, symmetric observations ($a = -1$, $b = 1$) satisfy

$$\lim_{h \downarrow 0} \frac{1}{h} \sigma_{\min}\left(\begin{pmatrix} G_j(t_* - h) \\ G_j(t_* + h) \end{pmatrix}\right) = \sqrt{2} \min_{k \in K,\ \|k\| = 1} \|B(t_*)k\|. \tag{4.9}$$

Thus the two-observation formula is a special case of the weighted result, rather than a separate mechanism. Centering provides an optimal first-order guarantee at fixed weighted squared offsets; it does not assert optimality for nonlinear observation noise, observation costs, or higher-order degeneracies when $B(t_*)|_K$ is not injective. A common rescaling $w_i \mapsto \lambda w_i$ multiplies both the weighted observation matrix and the coefficient in (4.6) by $\sqrt{\lambda}$; fixing S_w therefore removes this trivial scaling from the placement comparison.

The coefficient in Proposition 4.5 is an asymptotic quantity. The next consequence gives an explicit range of positive observation separations for which this coefficient already guarantees identification. Its extra regularity assumption is needed for a quadratic, rather than an $o(h)$, remainder.

Corollary 4.6. *Adopt the notation and hypotheses of Proposition 4.5. Suppose additionally that G_j' is Lipschitz with constant $M \ge 0$ on $[t_* - r, t_* + r] \subset (\theta_j, \theta_{j+1})$, for some $r > 0$. Set*

$$H := G_j'(t_*), \quad L := \sqrt{S_w}\,\|H\|, \quad C := \frac{M}{2}\Big(\sum_{i=1}^{m} w_i c_i^4\Big)^{1/2},$$

and let κ denote the right-hand side of (4.6). Put $d := \max_i |c_i| > 0$. If G_* has positive rank, let $\gamma > 0$ be the smallest nonzero singular value of $\mathcal{A}_0 = (\sqrt{w_i} G_*)_{i=1}^m$; thus $\gamma = \sqrt{W}\,\sigma_{\operatorname{rank} G_*}(G_*)$,

where the singular values of G_* are indexed in decreasing order. For every $h > 0$ satisfying $hd < r$ and $hL \le \gamma/2$,

$$h\kappa - h^2\Big(C + \frac{2\kappa L + L^2}{\gamma}\Big) \le \sigma_{\min}(\mathcal{G}_w(h)) \le h\kappa + h^2\Big(C + \frac{L^2}{\gamma}\Big). \tag{4.10}$$

If $G_* = 0$, the same conclusion holds with the simpler bounds

$$h\kappa - Ch^2 \le \sigma_{\min}(\mathcal{G}_w(h)) \le h\kappa + Ch^2, \tag{4.11}$$

provided $hd < r$; in this case $\kappa = \sqrt{S_w}\,\sigma_{\min}(H)$. Consequently, if $\kappa > 0$ and D denotes the coefficient of h^2 subtracted in the applicable lower bound, then $hD \le \kappa/2$ and the corresponding restrictions on h imply

$$\sigma_{\min}(\mathcal{G}_w(h)) \ge \frac{\kappa h}{2} > 0, \quad \|\mathcal{G}_w(h)^\dagger\| \le \frac{2}{\kappa h}.$$

For a linear equation the last inequality is a reconstruction sensitivity bound for compatible weighted observations. If $\kappa = 0$, the upper bound instead implies $\sigma_{\min}(\mathcal{G}_w(h)) \le h^2(C + L^2/\gamma)$ in the positive-rank case (and $\le Ch^2$ in the zero-rank case). Thus any full-column-rank recovery along such spacings has inverse amplification at least of quadratic order, whenever the stated upper coefficient is positive; if that coefficient vanishes, this estimate forces rank deficiency. It does not account for numerical integration or uncertain coefficient data.

Proof. The Lipschitz property of G_j' and integration of its difference from H give, for $|s| \le r$,

$$\|G_j(t_* + s) - G_* - sH\| \le \tfrac{1}{2}Ms^2.$$

Write $\mathcal{A}_1 = (\sqrt{w_i}c_iH)_{i=1}^m$. Hence $\|\mathcal{A}_1\| \le L$ and

$$\mathcal{G}_w(h) = \mathcal{A}_0 + h\mathcal{A}_1 + E(h), \quad \|E(h)\| \le Ch^2.$$

If $G_* = 0$, then $\mathcal{A}_0 = 0$ and $\sigma_{\min}(\mathcal{A}_1) = \kappa$; the singular-value perturbation inequality directly proves (4.11).

Suppose G_* has positive rank. Let $K = \ker G_*$, $U = K^\perp$, $E_0 = \operatorname{Ran}\mathcal{A}_0$, and P_0 be the orthogonal projector onto E_0. The restriction $\mathcal{A}_0|_U : U \to E_0$ is bijective, with inverse norm $1/\gamma$. Define

$$J = (\mathcal{A}_0|_U)^{-1}P_0\mathcal{A}_1|_K, \quad F = (I - P_0)\mathcal{A}_1|_K.$$

Then $\|J\| \le L/\gamma$. For $k \in K$, orthogonal projection onto $E_0^\perp$ gives explicitly

$$\|Fk\|^2 = V_w\|Q_RHk\|^2 + S_w\|(I - Q_R)Hk\|^2.$$

Consequently $\sigma_{\min}(F) = \kappa$ on the domain K, and in particular $\kappa \le \sqrt{S_w}\,\|H\| = L$. On $U \oplus K$ consider the invertible shear $T_h(u,k) = (u - hJk, k)$; it satisfies $\|T_h\| \le 1 + hL/\gamma$. Its image under the first-order approximation can be written as

$$(\mathcal{A}_0 + h\mathcal{A}_1)T_h(u,k) = C_hu + D_hu + hFk - h^2\mathcal{A}_1Jk,$$

where $C_h = (\mathcal{A}_0 + hP_0\mathcal{A}_1)|_U$ takes values in E_0 and $D_h = h(I - P_0)\mathcal{A}_1|_U$ takes values in $E_0^\perp$. Here $\sigma_{\min}(C_h) \ge \gamma - hL$ and $\|D_h\| \le hL$. Applying the invertible row shear which subtracts $D_hC_h^{-1}$ times the E_0 component from the $E_0^\perp$ component gives

$$\|C_hu + D_hu + hFk\| \ge \frac{\min\{\gamma - hL, h\kappa\}}{1 + hL/(\gamma - hL)}\sqrt{\|u\|^2 + \|k\|^2}.$$

For $hL \le \gamma/2$ we have $h\kappa \le hL \le \gamma - hL$. The term $h^2\mathcal{A}_1Jk$ has norm at most $h^2L^2\|k\|/\gamma$. Dividing the resulting lower bound by $\|T_h\|$ yields

$$\sigma_{\min}(\mathcal{A}_0 + h\mathcal{A}_1) \ge h\kappa\frac{\gamma - hL}{\gamma + hL} - \frac{h^2L^2}{\gamma} \ge h\kappa - \frac{h^2(2\kappa L + L^2)}{\gamma}.$$

For the reverse inequality, choose a unit $k \in K$ such that $\|Fk\| = \kappa$, and test on the unit normalization of $T_h(0,k) = k - hJk$, whose norm is at least one. Since

$$(\mathcal{A}_0 + h\mathcal{A}_1)T_h(0,k) = hFk - h^2\mathcal{A}_1Jk,$$

it follows that $\sigma_{\min}(\mathcal{A}_0 + h\mathcal{A}_1) \le h\kappa + h^2L^2/\gamma$. Finally, the singular-value perturbation bound for $E(h)$ proves (4.10). If the lower bound is at least $h\kappa/2 > 0$, the weighted matrix has full column rank, and the Moore–Penrose inverse norm gives the final estimate. □

The additional regularity is checkable from the coefficients. For example, if A and B are Lipschitz near t_* with constants L_A, L_B, and there $\|A\| \le M_A$, $\|B\| \le M_B$, $\|G_j\| \le M_G$, then the identity $G_j' = AG_j + B$ implies that one may take

$$M = L_A M_G + M_A(M_A M_G + M_B) + L_B.$$

The bounds are sharp in the scalar zero-rank case $x'(t) = -x(\theta_j)$ at $t_* = \theta_j + 1$, whenever that time is interior: $G_* = 0$, $H = -1$, and G_j' is constant. Here $C = 0$ and $\sigma_{\min}(\mathcal{G}_w(h)) = h\sqrt{S_w} = h\kappa$ for every admissible spacing. For positive-rank singularities, (4.10) supplies a conservative but fully explicit guarantee at a finite observation separation.

The preceding estimate can be made robust to simultaneous uncertainty in the equation coefficients and in the observation times. The following statement keeps these two error mechanisms separate at the level of each observation, so localized errors are not replaced by a single worst-case allowance over the whole interval.

Corollary 4.7. *Consider a nominal linear system with coefficients A_0, B_0 on the current interval of constancy, and write G_j^0 for its interval solution matrix, initialized by $G_j^0(\theta_j) = I$. Suppose that $G_j^0(t_*)$ is singular and that the nominal system satisfies the hypotheses of Corollary 4.6, with positive weights w_i, offsets c_i, nominal observation times $t_i = t_* + c_i h$, first-order coefficient κ_0, and the nonnegative quadratic coefficient D_0 from the applicable lower bound there. Assume that h satisfies the corresponding nominal spacing and spectral-gap restrictions.*

Let A, B be the actual coefficients, with the same partition points, and let the actual observation times be $s_i = t_i + \tau_i \in (\theta_j, \theta_{j+1})$. On

$$J_+ := [\theta_j, \max_i\{s_i, t_i\}]$$

assume that A, B, A_0, B_0 are measurable and locally bounded and choose nonnegative integrable majorants a, d_A, d_B such that almost everywhere

$$\|A(t)\| \le a(t), \quad \|A(t) - A_0(t)\| \le d_A(t), \quad \|B(t) - B_0(t)\| \le d_B(t).$$

Define

$$\mathcal{E}(t) := \int_{\theta_j}^{t} \exp\Big(\int_s^t a(u)\,du\Big)\big[d_A(s)\|G_j^0(s)\| + d_B(s)\big]\,ds \tag{4.12}$$

and, for each observation,

$$V_i := \int_{\min\{s_i,t_i\}}^{\max\{s_i,t_i\}} \|A_0(u)G_j^0(u) + B_0(u)\|\,du.$$

Put

$$\Delta_{\text{rob}} := \Big(\sum_{i=1}^{m} w_i[\mathcal{E}(s_i) + V_i]^2\Big)^{1/2}. \tag{4.13}$$

Then the actual weighted observation matrix

$$\widetilde{\mathcal{G}}_w = (\sqrt{w_i}G_j(s_i))_{i=1}^m$$

satisfies

$$\sigma_{\min}(\widetilde{\mathcal{G}}_w) \ge h\kappa_0 - h^2 D_0 - \Delta_{\text{rob}}. \tag{4.14}$$

Whenever the right-hand side is strictly positive, $\widetilde{\mathcal{G}}_w$ has full column rank and

$$\|\widetilde{\mathcal{G}}_w^{\dagger}\| \le \frac{1}{h\kappa_0 - h^2 D_0 - \Delta_{\text{rob}}}. \tag{4.15}$$

If the observation times are exact, $s_i = t_i$, then $V_i = 0$ and

$$\Delta_{\text{rob}} = \Delta_{\text{int}} := \Big(\sum_{i=1}^{m} w_i\mathcal{E}(t_i)^2\Big)^{1/2}.$$

For a simpler coefficient-only certificate, suppose on $J = [\theta_j, \max_i t_i]$ that

$$\|A\|, \|A_0\| \le M_A, \quad \|B_0\| \le M_B, \quad \|A - A_0\| \le \varepsilon_A, \quad \|B - B_0\| \le \varepsilon_B$$

almost everywhere. Writing $\ell = \max_i t_i - \theta_j$,

$$M_0 = e^{M_A \ell}(1 + M_B \ell), \quad \Delta_{\mathrm{coef}} = \sqrt{W}\, \ell e^{M_A \ell}(\varepsilon_A M_0 + \varepsilon_B), \quad W = \sum_i w_i,$$

one has $\Delta_{\mathrm{int}} \le \Delta_{\mathrm{coef}}$. The result concerns the exact linear observation matrices; it does not include measurement noise, numerical integration or rounding error, uncertainty in the affine inhomogeneous response, or motion of the partition points.

Proof. Set $E(t) = G_j(t) - G_j^0(t)$. Since both matrices are initialized by the identity at θ_j,

$$E' = AE + (A - A_0)G_j^0 + (B - B_0), \quad E(\theta_j) = 0$$

almost everywhere. Variation of constants with the actual coefficient A gives

$$\|E(t)\| \le \mathcal{E}(t), \quad t \in J_+.$$

Moreover, absolute continuity of the nominal solution matrix and $(G_j^0)' = A_0 G_j^0 + B_0$ imply

$$\|G_j^0(s_i) - G_j^0(t_i)\| \le V_i.$$

Hence

$$\|G_j(s_i) - G_j^0(t_i)\| \le \mathcal{E}(s_i) + V_i.$$

After stacking the weighted blocks, for every unit vector v,

$$\|[\widetilde{\mathcal{G}}_w - \mathcal{G}_w^0(h)]v\|^2 \le \sum_i w_i[\mathcal{E}(s_i) + V_i]^2 = \Delta_{\mathrm{rob}}^2.$$

Taking the supremum over v gives $\|\widetilde{\mathcal{G}}_w - \mathcal{G}_w^0(h)\| \le \Delta_{\mathrm{rob}}$. The singular-value perturbation inequality together with the nominal lower bound from Corollary 4.6 proves (4.14). Strict positivity gives full column rank, and the Moore–Penrose inverse-norm identity gives (4.15).

For the final coefficient-only estimate, variation of constants for $(G_j^0)' = A_0 G_j^0 + B_0$ yields $\|G_j^0(t)\| \le M_0$ on J. Taking $a = M_A$, $d_A = \varepsilon_A$, and $d_B = \varepsilon_B$ in (4.12) gives

$$\mathcal{E}(t_i) \le \ell e^{M_A \ell}(\varepsilon_A M_0 + \varepsilon_B),$$

and therefore $\Delta_{\mathrm{int}} \le \Delta_{\mathrm{coef}}$. □

The strict threshold is essential even without coefficient error. For $G_j(t) = 1 - (t - \theta_j)$, let $t_* = \theta_j + 1$, nominal times $t_1 = t_* - h$ and $t_2 = t_* + h$, and $w_1 = w_2 = 1$. If timing errors shift *both* actual observations to $s_1 = s_2 = t_*$, then $\sigma_{\min}(\widetilde{\mathcal{G}}_w) = 0$ while $h\kappa_0 = \sqrt{2}h$ and $\Delta_{\mathrm{rob}} = \sqrt{2}h$, so equality holds in (4.14) with zero margin. All these times are assumed interior to the same interval of constancy.

The integral error allowance retains the observation-specific accumulation of uncertainty. For example, on a single interval $[0, 2)$ take a scalar equation with $A_0 = A = 0$, $B_0 = -1$, and $B(t) = -1 + \mathbf{1}_{(1.1,1.15)}(t)$. Choose $t_* = 1$, $h = 0.2$, offsets $c_1 = -1$, $c_2 = 1$, and $w_1 = w_2 = 1$. The nominal matrix is $G_j^0(t) = 1 - t$; it satisfies $\kappa_0 = \sqrt{2}$, $D_0 = 0$, and the observation times are 0.8 and 1.2. Taking the exact coefficient-error majorants gives $\mathcal{E}(0.8) = 0$, $\mathcal{E}(1.2) = 0.05$, so that $\Delta_{\mathrm{int}} = 0.05$. The resulting lower bound $0.2\sqrt{2} - 0.05 > 0$ certifies full column rank. In contrast, the uniform bound with $\varepsilon_B = 1$ gives $\Delta_{\mathrm{coef}} = 1.2\sqrt{2}$ and is inconclusive. In fact the actual stacked matrix has entries 0.2 and -0.15 and smallest singular value 0.25. Thus the refinement can distinguish a localized coefficient error from an error acting across the whole observation interval.

Unlike the first-order separation term $h\kappa_0$, the coefficient-error allowance need not vanish as $h \downarrow 0$: with fixed nonzero coefficient uncertainties and $t_* > \theta_j$, even the integral error bound can remain bounded away from zero. Thus decreasing the separation does not necessarily improve the *certified* inverse, and the bound may become inconclusive. This is a limitation of the certificate, not an assertion that the actual matrix loses rank.

The rank test and the exact coefficient depend on the differential-equation coefficients, rather than on separately estimated observation maps. Both individual maps can remain singular. To see this, take $n = 2$, $A = 0$, and an interval of constancy longer than 2. Writing $s = t - \theta_j \geq 0$, put

$$\varphi(s) = s(s-1), \quad d(s) = \begin{cases} (1-s)^2, & 0 \leq s \leq 1, \\ 0, & s \geq 1, \end{cases}$$

and define

$$G(s) := \begin{pmatrix} \cos\varphi(s) & \sin\varphi(s) \\ 0 & d(s) \end{pmatrix}, \quad B(\theta_j + s) := G'(s). \tag{4.16}$$

The matrix G is continuously differentiable, $G(0) = I$, and $G(s) = I + \int_0^s B(\theta_j + r)\, dr$; hence it is precisely the interval solution matrix for this piecewise-constant-argument system. At $s = 1$,

$$G(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B(\theta_j + 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus $B(\theta_j + 1)$ is injective on $\ker G(1)$. For each sufficiently small $h > 0$, both $G(1)$ and $G(1+h)$ have rank one, but their stacked matrix has rank two. Its smallest singular value is exactly

$$\sqrt{1 - \cos(h + h^2)} = \frac{h}{\sqrt{2}} + O(h^2).$$

The coefficient in (4.5) is therefore sharp, including when *both* individual observations are singular. The test is sufficient, not necessary for two-time recovery: keeping the same $d(s)$ but choosing $\varphi(s) = s(s-1)^2$ instead gives $B(\theta_j + 1) = 0$, whereas both observation matrices remain singular and their stacked smallest singular value is $\sqrt{1 - \cos((1+h)h^2)} \sim h^2/\sqrt{2}$. Thus recovery may still occur after failure of the first-order rank test, with a more severe loss of conditioning. In the first rotating rank-one construction, observations placed at $s = 1 - h$ and $s = 1 + h$ provide a direct check of the symmetric special case (4.9): since $d(1-h) = h^2$ and $d(1+h) = 0$, their stacked smallest singular value is $\sqrt{2}\, h + O(h^2)$, in agreement with the coefficient $\sqrt{2}$ in (4.9). This calculation concerns the same single interval of constancy.

4.3. **Inverse conditioning near singularity.** The exact classification in Theorem 4.1 also determines what happens as nonsingular times approach a time at which G_j is singular. At every nonsingular time, the affine map from the prescribed interior datum to $x(\theta_j)$ has the exact Lipschitz constant $\sigma_{\min}(G_j(t_0))^{-1}$. At a singular time this global inverse ceases to exist: depending on the datum, compatibility either fails or the compatible values of $x(\theta_j)$ form a nontrivial affine subspace of $\mathbb{R}^n$. The following consequence quantifies the transition and does not require the singular time to be isolated.

Corollary 4.8. *Fix $j \geq 0$. Let $t_* \in (\theta_j, \theta_{j+1})$ be such that $G_j(t_*)$ is singular. Suppose that $t_k \in (\theta_j, \theta_{j+1})$, $t_k \to t_*$, and $G_j(t_k)$ is nonsingular for every k. Then*

$$\sigma_{\min}(G_j(t_k)) \to 0^+, \quad \|G_j(t_k)^{-1}\| \to \infty.$$

For every $t \in (\theta_j, \theta_{j+1})$ such that $G_j(t)$ is nonsingular, define

$$\mathcal{I}_t(x_0) := G_j(t)^{-1}[x_0 - r_j(t)].$$

Then

$$\mathrm{Lip}(\mathcal{I}_{t_k}) = \|G_j(t_k)^{-1}\| \longrightarrow \infty.$$

Proof. Continuity of G_j and the 1-Lipschitz property

$$|\sigma_{\min}(M) - \sigma_{\min}(N)| \leq \|M - N\|$$

of the smallest singular value give

$$0 < \sigma_{\min}(G_j(t_k)) \leq \|G_j(t_k) - G_j(t_*)\| \longrightarrow 0,$$

because $G_j(t_*)$ is singular. Hence

$$\|G_j(t_k)^{-1}\| = \sigma_{\min}(G_j(t_k))^{-1} \longrightarrow \infty.$$

Since $\mathcal{I}_{t_k}$ is affine with linear part $G_j(t_k)^{-1}$, its exact Lipschitz constant is $\|G_j(t_k)^{-1}\|$, which proves the final assertion. □

Example 4.9. *Assume $\theta_{j+1} - \theta_j > 1$ and consider*

$$x'(t) = Ax(t) + Bx(\theta_j),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

With $h = t - \theta_j$ and $G(h) := G_j(\theta_j + h)$,

$$e^{Ah} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad G(h) = \begin{pmatrix} 1 + h^2 & h \\ 2h & 1 \end{pmatrix},$$

and

$$\det G(h) = 1 - h^2.$$

Since $h > 0$ on the interior of the current interval, the only interior value at which $G(h)$ is singular is $h = 1$. At this time,

$$G(1) = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix},$$

with

$$\operatorname{Ran} G(1) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \ker G(1) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

Thus a prescribed interior state $x_0 = (p, q)^T$ at $t_0 = \theta_j + 1$ is compatible if and only if $p = q$. If $p = q = c$, all compatible values of $x(\theta_j)$ are

$$\xi = \begin{pmatrix} 0 \\ c \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad s \in \mathbb{R}.$$

If $p \neq q$, no compatible value of $x(\theta_j)$ exists.

The same example quantifies the behavior of the affine inverse near the time at which $G(h)$ is singular. Since

$$\sigma_{\min}(G(h))\, \sigma_{\max}(G(h)) = |\det G(h)| = |1 - h^2|$$

and, by continuity of singular values, $\sigma_{\max}(G(h)) \to \sigma_{\max}(G(1)) = \sqrt{10}$ as $h \to 1$,

$$\sigma_{\min}(G(h)) \sim \sqrt{\frac{2}{5}}\, |h - 1|, \quad h \to 1.$$

Consequently,

$$\|G(h)^{-1}\| \sim \sqrt{\frac{5}{2}}\, \frac{1}{|h - 1|}, \quad h \to 1, \quad h \neq 1.$$

Thus the smallest singular value vanishes linearly as the prescribed time approaches the value at which $G(h)$ is singular, and the exact Lipschitz constant of the affine inverse diverges at the reciprocal rate. In this example, every nearby nonsingular observation time gives unique recovery of the preceding state, while at the singular time the range–kernel alternatives give either incompatibility or non-uniqueness.

Figure 1 shows the inverse norm from Example 4.9 and its leading singular asymptotic behavior on both sides of the critical elapsed time. The plotted window $0.55 \leq h \leq 1.45$ is contained in a single interval of constancy whenever that interval has length greater than 1.45; the singular-time calculation itself requires only length greater than 1.

The perturbation analysis below uses $G_j(t_0)$ as the linear part of the solution map and examines whether bijectivity and inverse stability persist in the presence of a Lipschitz nonlinear term.

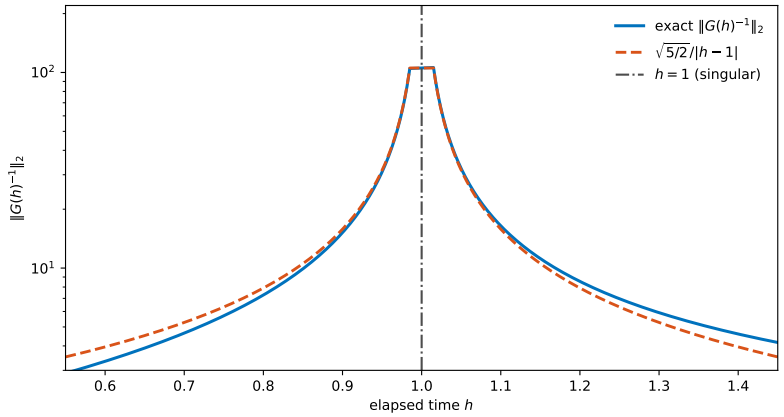


FIGURE 1. Inverse conditioning for the linear example: the exact norm of the inverse and its reciprocal-distance asymptotic approximation. The vertical line is the singular time $h = 1$.

5. Nonlinear perturbations and inverse estimates

Consider

$$x'(t) = A(t)x(t) + B(t)x(\beta(t)) + F\big(t, x(t), x(\beta(t))\big),$$

where $A, B \in L^1_{\mathrm{loc}}([0,\infty); \mathbb{R}^{n\times n})$, $F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Assume that, for every $x, y \in \mathbb{R}^n$, the map $t \mapsto F(t, x, y)$ is Lebesgue measurable, and that there exist nonnegative functions

$$m_F, \ell_x, \ell_y \in L^1_{\mathrm{loc}}([0,\infty))$$

and a null set $N_F \subset [0,\infty)$ such that, for every $t \notin N_F$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, $\|F(t,0,0)\| \le m_F(t)$ and

$$\|F(t,x_1,y_1) - F(t,x_2,y_2)\| \le \ell_x(t)\|x_1 - x_2\| + \ell_y(t)\|y_1 - y_2\|.$$

The full vector field therefore satisfies (A1)–(A3) with

$$L_x(t) = \|A(t)\| + \ell_x(t), \quad L_y(t) = \|B(t)\| + \ell_y(t), \quad m(t) = \|F(t,0,0)\| \le m_F(t).$$

Thus Proposition 2.1 applies to the full vector field, and the solution map P_{j,t_0} is well defined by the construction of Section 3. No differentiability of F with respect to x or y is assumed. The function m_F is used to guarantee the required Carathéodory growth and forward well-posedness; the inverse threshold developed below depends on the linear evolution and on the Lipschitz moduli ℓ_x, ℓ_y, not on the size of $F(t,0,0)$. If Theorem 3.2 is applied directly to this full vector field, the linear piecewise constant term contributes through the modulus

$$L_y(t) = \|B(t)\| + \ell_y(t).$$

The analysis in this section retains the full linear contribution in G_j and estimates only the nonlinear remainder.

Throughout this section, one decomposition

$$A(t)x + B(t)y + F(t,x,y)$$

and one admissible pair of Lipschitz bounds ℓ_x, ℓ_y for F are fixed. The quantities Φ, G_j, and N_{j,t_0} are understood relative to this decomposition, and ρ_j and $\widehat{\nu}_{j,t_0}$ relative to these bounds. The solution map P_{j,t_0} itself is determined by the differential equation. Consequently, the sufficient inequality in Theorem 5.2 depends on the chosen splitting and Lipschitz bounds, although invertibility of P_{j,t_0} does not. Any admissible splitting may be used, so sharper bounds or a more favorable splitting can enlarge the verified parameter range.

5.1. **A Volterra bound for the nonlinear remainder.** Fix $j \geq 0$ and set $\xi = x(\theta_j)$. The corresponding interval solution $u_j(\cdot;\xi)$ satisfies

$$u_j'(t;\xi) = A(t)u_j(t;\xi) + B(t)\xi + F(t, u_j(t;\xi), \xi)$$

for almost every $t \in (\theta_j, \theta_{j+1})$, with $u_j(\theta_j;\xi) = \xi$. By the same Carathéodory argument used in Section 2, the map $t \mapsto F(t, u_j(t;\xi), \xi)$ is measurable. Since $u_j(\cdot;\xi)$ is bounded on the compact interval and

$$\|F(t, u_j(t;\xi), \xi)\| \leq m_F(t) + \ell_x(t)\|u_j(t;\xi)\| + \ell_y(t)\|\xi\|$$

for almost every t, this term belongs to $L^1([\theta_j, \theta_{j+1}]; \mathbb{R}^n)$. Hence variation of constants gives

$$u_j(t;\xi) = G_j(t)\xi + \int_{\theta_j}^{t} \Phi(t,s)F(s, u_j(s;\xi), \xi)\, ds, \quad \theta_j \leq t \leq \theta_{j+1}.$$

For $t_0 \in (\theta_j, \theta_{j+1})$, the same solution map as in Section 3 therefore has the decomposition

$$P_{j,t_0}(\xi) = G_j(t_0)\xi + N_{j,t_0}(\xi). \tag{5.1}$$

The induced nonlinear remainder in (5.1) is defined by

$$N_{j,t_0}(\xi) := \int_{\theta_j}^{t_0} \Phi(t_0, s)F(s, u_j(s;\xi), \xi)\, ds. \tag{5.2}$$

To compare the nonlinear remainder with the linear part represented by G_j, a quantitative Lipschitz estimate for N_{j,t_0} is needed. The required bound is obtained from a scalar Volterra integral equation.

Proposition 5.1. *Fix $j \geq 0$ and $t_0 \in (\theta_j, \theta_{j+1})$, and define*

$$a_j(t) := \|G_j(t)\| + \int_{\theta_j}^{t} \|\Phi(t,s)\|\ell_y(s)\, ds,$$
$$K_j(t,s) := \|\Phi(t,s)\|\ell_x(s), \quad \theta_j \leq s \leq t \leq t_0.$$

Then $a_j \in C([\theta_j, t_0])$. The Volterra equation

$$\rho_j(t) = a_j(t) + \int_{\theta_j}^{t} K_j(t,s)\rho_j(s)\, ds \tag{5.3}$$

has a unique solution ρ_j in the class of bounded measurable functions on $[\theta_j, t_0]$. Moreover, $\rho_j \in C([\theta_j, t_0])$ and $\rho_j \geq 0$. If $t_0 < t_1 < \theta_{j+1}$, uniqueness implies that the solution constructed on $[\theta_j, t_1]$ restricts to the one on $[\theta_j, t_0]$; hence the notation ρ_j is consistent as the interior endpoint varies. We define

$$\widehat{\nu}_{j,t_0} := \int_{\theta_j}^{t_0} \|\Phi(t_0, s)\|[\ell_x(s)\rho_j(s) + \ell_y(s)]\, ds.$$

Then, for all $\xi, \eta \in \mathbb{R}^n$ and all $t \in [\theta_j, t_0]$,

$$\|u_j(t;\xi) - u_j(t;\eta)\| \leq \rho_j(t)\|\xi - \eta\|. \tag{5.4}$$

Moreover, $\widehat{\nu}_{j,t_0} < \infty$ and $\mathrm{Lip}(N_{j,t_0}) \leq \widehat{\nu}_{j,t_0}$.

Proof. Let $\xi, \eta \in \mathbb{R}^n$, put $Z(t) := u_j(t;\xi) - u_j(t;\eta)$ and $d := \|\xi - \eta\|$, and subtract the two variation-of-constants formulas. The Lipschitz estimate for F gives

$$\|Z(t)\| \leq a_j(t)d + \int_{\theta_j}^{t} K_j(t,s)\|Z(s)\|\, ds.$$

Joint continuity of Φ on the compact triangle gives

$$M_\Phi := \sup_{\theta_j \leq s \leq t \leq t_0} \|\Phi(t,s)\| < \infty.$$

Dominated convergence, after extending the relevant integrands by zero to $s > t$, shows that a_j is continuous and that

$$(\mathcal{V}v)(t) := \int_{\theta_j}^{t} K_j(t,s)v(s)\, ds$$

maps bounded measurable functions into continuous functions. Writing $\mathcal{B}_b$ for the space of bounded measurable real-valued functions on $[\theta_j, t_0]$, equipped with the supremum norm, $\mathcal{V}$ is a positive bounded linear operator on $\mathcal{B}_b$, because

$$0 \le K_j(t,s) \le M_\Phi \ell_x(s)$$

and hence

$$\|\mathcal{V}v\|_\infty \le M_\Phi \Big(\int_{\theta_j}^{t_0} \ell_x(s)\, ds \Big) \|v\|_\infty.$$

For $m \ge 1$, successive substitution and Tonelli's theorem give, for each $t \in [\theta_j, t_0]$,

$$|\mathcal{V}^m v(t)| \le \|v\|_\infty M_\Phi^m \int_{\theta_j \le s_m \le \cdots \le s_1 \le t} \prod_{r=1}^{m} \ell_x(s_r)\, ds_m \dots ds_1.$$

Because the integrand is symmetric and nonnegative, the integral over the ordered simplex equals $1/m!$ times the integral over $[\theta_j, t]^m$. Therefore

$$\|\mathcal{V}^m v\|_\infty \le \|v\|_\infty \frac{\big(M_\Phi \int_{\theta_j}^{t_0} \ell_x(s)\, ds\big)^m}{m!}. \tag{5.5}$$

Since a_j is continuous on the compact interval, it is bounded. The Weierstrass M-test and (5.5) therefore show that

$$\rho_j := \sum_{m=0}^{\infty} \mathcal{V}^m a_j$$

converges uniformly. Each term is continuous, and all terms are nonnegative because $a_j \ge 0$ and $\mathcal{V}$ is positive; hence $\rho_j \in C([\theta_j, t_0])$ and $\rho_j \ge 0$. Since $\mathcal{V}$ is bounded in the supremum norm, it may be applied to the uniformly convergent series, which gives

$$\mathcal{V}\rho_j = \sum_{m=1}^{\infty} \mathcal{V}^m a_j = \rho_j - a_j.$$

Thus ρ_j satisfies (5.3). If ρ and $\widetilde{\rho}$ are two bounded measurable solutions, then

$$\rho - \widetilde{\rho} = \mathcal{V}^m(\rho - \widetilde{\rho}) \quad (m \ge 1).$$

Applying (5.5) and letting $m \to \infty$ yields $\|\rho - \widetilde{\rho}\|_\infty = 0$, proving uniqueness.

If $d = 0$, then $Z \equiv 0$. For $d > 0$, set $q(t) := \|Z(t)\|/d$. From $q \le a_j + \mathcal{V}q$, iteration yields

$$q \le \sum_{k=0}^{m} \mathcal{V}^k a_j + \mathcal{V}^{m+1} q.$$

The function q is bounded, and (5.5) makes the last term converge uniformly to zero. Hence

$$\|u_j(t;\xi) - u_j(t;\eta)\| \le \rho_j(t) \|\xi - \eta\|.$$

The definition (5.2), the Lipschitz bound for F, and (5.4) now give

$$\begin{aligned} \|N_{j,t_0}(\xi) - N_{j,t_0}(\eta)\| &\le \int_{\theta_j}^{t_0} \|\Phi(t_0,s)\| \big[\ell_x(s)\|u_j(s;\xi) - u_j(s;\eta)\| + \ell_y(s)\|\xi - \eta\|\big]\, ds \\ &\le \widehat{\nu}_{j,t_0} \|\xi - \eta\|. \end{aligned}$$

Taking the supremum over $\xi \ne \eta$ proves $\mathrm{Lip}(N_{j,t_0}) \le \widehat{\nu}_{j,t_0}$. Finally, $\widehat{\nu}_{j,t_0} < \infty$ because ρ_j is bounded, Φ is bounded on the compact triangle, and $\ell_x, \ell_y \in L^1([\theta_j, t_0])$. □

5.2. **A coefficient-based Lipschitz perturbation criterion.** The Lyusternik–Graves perturbation framework guarantees stability of invertibility under Lipschitz perturbations; broader global inverse theorems are available in [14, 18, 20]. For the present equation, the decisive step is Proposition 5.1: it converts the nonlinear interval dynamics into the explicit equation-dependent constant $\widehat{\nu}_{j,t_0}$, which can be compared directly with the weakest linear observation direction $\sigma_{\min}(G_j(t_0))$. No separate nonsingularity assumption is required: the strict inequality below forces it.

Theorem 5.2. *Fix $j \geq 0$ and $t_0 \in (\theta_j, \theta_{j+1})$, with the decomposition and admissible bounds fixed as above. If*

$$\widehat{\nu}_{j,t_0} < \sigma_{\min}(G_j(t_0)),$$

then $P_{j,t_0} : \mathbb{R}^n \to \mathbb{R}^n$ is a bi-Lipschitz homeomorphism of $\mathbb{R}^n$ onto itself. More precisely, for all $\xi, \eta \in \mathbb{R}^n$,

$$\begin{aligned}[\sigma_{\min}(G_j(t_0)) - \widehat{\nu}_{j,t_0}]\|\xi - \eta\| &\leq \|P_{j,t_0}(\xi) - P_{j,t_0}(\eta)\| \\ &\leq [\|G_j(t_0)\| + \widehat{\nu}_{j,t_0}]\|\xi - \eta\|.\end{aligned} \tag{5.6}$$

Moreover,

$$\mathrm{Lip}(P_{j,t_0}^{-1}) \leq \frac{1}{\sigma_{\min}(G_j(t_0)) - \widehat{\nu}_{j,t_0}}.$$

Consequently, every prescribed interior state $x_0 \in \mathbb{R}^n$ determines a unique value of $x(\theta_j)$ and a unique locally absolutely continuous solution on $[\theta_j, \infty)$ satisfying $x(t_0) = x_0$.

Proof. For $d = \xi - \eta$, (5.1) and Proposition 5.1 give

$$\|G_j(t_0)d\| - \widehat{\nu}_{j,t_0}\|d\| \leq \|P_{j,t_0}(\xi) - P_{j,t_0}(\eta)\| \leq \|G_j(t_0)d\| + \widehat{\nu}_{j,t_0}\|d\|,$$

which proves (5.6) and injectivity. The strict inequality implies that $G_j(t_0)$ is nonsingular. For any $x_0 \in \mathbb{R}^n$ the map

$$\mathcal{T}_{x_0}(\xi) = G_j(t_0)^{-1}[x_0 - N_{j,t_0}(\xi)]$$

is a contraction with factor at most $q = \widehat{\nu}_{j,t_0}/\sigma_{\min}(G_j(t_0)) < 1$. Its unique fixed point satisfies $P_{j,t_0}(\xi) = x_0$, proving surjectivity. The lower bound in (5.6) gives the stated inverse Lipschitz constant, and Lemma 3.1 completes the compatibility and uniqueness conclusion. □

Theorem 5.2 places the interval problem within the Lyusternik–Graves perturbation framework. Its equation-dependent content is the Volterra construction of $\widehat{\nu}_{j,t_0}$ and hence the computable margin $\sigma_{\min}(G_j(t_0)) - \widehat{\nu}_{j,t_0}$.

The strict inequality in Theorem 5.2 cannot in general be replaced by a non-strict one if only $\sigma_{\min}(G_j(t_0))$ and the Lipschitz bound for the remainder are used. Indeed, fix $h = t_0 - \theta_j > 0$ and consider the scalar equation

$$x'(t) = -\frac{1}{h}x(\beta(t)).$$

Use the decomposition $A = B = 0$ and $F(t, x, y) = -y/h$. Then $G_j(t_0) = 1$, one may take $\ell_x = 0$ and $\ell_y = 1/h$, and Proposition 5.1 gives $\widehat{\nu}_{j,t_0} = 1 = \sigma_{\min}(G_j(t_0))$. On the other hand,

$$P_{j,t_0}(\xi) = \xi - \frac{t_0 - \theta_j}{h}\xi = 0$$

for every $\xi \in \mathbb{R}$. Thus the solution map loses injectivity exactly at equality. The threshold is therefore sharp at the level of the metric comparison used in Theorem 5.2; additional structure of a particular equation may, of course, permit invertibility beyond this sufficient regime.

The contraction used in the proof also provides a provably convergent reconstruction procedure for the unknown preceding state.

Corollary 5.3. *Assume the hypotheses of Theorem 5.2, fix $x_0 \in \mathbb{R}^n$, and set*

$$q := \frac{\widehat{\nu}_{j,t_0}}{\sigma_{\min}(G_j(t_0))} \in [0, 1).$$

Starting from an arbitrary $\xi^{(0)} \in \mathbb{R}^n$, define

$$\xi^{(m+1)} := G_j(t_0)^{-1}\big[x_0 - N_{j,t_0}(\xi^{(m)})\big], \quad m \geq 0. \tag{5.7}$$

Then $\xi^{(m)}$ converges to the unique state

$$\xi^* = P_{j,t_0}^{-1}(x_0).$$

For every $m \geq 0$,

$$\|\xi^{(m)} - \xi^*\| \leq \frac{q^m}{1-q}\|\xi^{(1)} - \xi^{(0)}\|, \tag{5.8}$$

and for every $m \geq 1$,

$$\|\xi^{(m)} - \xi^*\| \leq \frac{q}{1-q}\|\xi^{(m)} - \xi^{(m-1)}\|. \tag{5.9}$$

Moreover, the observation residual satisfies

$$\|P_{j,t_0}(\xi^{(m)}) - x_0\| \leq \widehat{\nu}_{j,t_0}\|\xi^{(m)} - \xi^{(m-1)}\|, \quad m \geq 1. \tag{5.10}$$

Proof. This is the Banach iteration for the contraction $\mathcal{T}_{x_0}$ in the proof of Theorem 5.2. Summing the geometric tail of successive differences gives (5.8) and (5.9). Moreover,

$$P_{j,t_0}(\xi^{(m)}) - x_0 = N_{j,t_0}(\xi^{(m)}) - N_{j,t_0}(\xi^{(m-1)}),$$

so Proposition 5.1 yields (5.10). □

Each evaluation of N_{j,t_0} uses the well-posed forward interval problem; no backward integration across the discontinuous argument is required. The bounds concern the exact continuous solution map, so a computer-validated implementation additionally needs forward-solver and arithmetic error bounds.

A coarser certificate avoids solving the scalar Volterra equation. Define

$$M_\Phi := \sup_{\theta_j \leq s \leq t \leq t_0} \|\Phi(t,s)\|, \quad M_G := \sup_{\theta_j \leq t \leq t_0} \|G_j(t)\|,$$
$$\mathcal{L}_x := \int_{\theta_j}^{t_0} \ell_x(s)\,ds, \quad \mathcal{L}_y := \int_{\theta_j}^{t_0} \ell_y(s)\,ds,$$

and set

$$R_* = (M_G + M_\Phi \mathcal{L}_y)e^{M_\Phi \mathcal{L}_x}, \quad \nu_* = M_\Phi(R_*\mathcal{L}_x + \mathcal{L}_y).$$

Gronwall's inequality applied to (5.3) gives $\rho_j \leq R_*$, hence $\widehat{\nu}_{j,t_0} \leq \nu_*$. Therefore $\nu_* < \sigma_{\min}(G_j(t_0))$ is a directly computable sufficient condition for Theorem 5.2, with

$$\mathrm{Lip}(P_{j,t_0}^{-1}) \leq [\sigma_{\min}(G_j(t_0)) - \nu_*]^{-1}.$$

5.3. **Joint nonlinear observations and compatibility.** The linear joint-observation criterion of Proposition 4.3 can persist under nonlinear perturbations even if neither single-time observation map is injective. In contrast with a single observation, the joint map takes values in $\mathbb{R}^{2n}$, so injectivity cannot imply surjectivity onto all observation pairs. This distinction is essential in the following quantitative result.

Fix $\theta_j < t_1 < t_2 < \theta_{j+1}$. Under the assumptions of Section 5, put

$$G_i = G_j(t_i), \quad N_i = N_{j,t_i}, \quad \nu_i = \widehat{\nu}_{j,t_i} \quad (i = 1,2),$$
$$\mathbb{G} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad \mathbb{N}(\xi) = \begin{pmatrix} N_1(\xi) \\ N_2(\xi) \end{pmatrix}, \quad \mathbb{P}(\xi) = \begin{pmatrix} P_{j,t_1}(\xi) \\ P_{j,t_2}(\xi) \end{pmatrix}, \quad \nu_{12} = \sqrt{\nu_1^2 + \nu_2^2}.$$

We use Euclidean norms in both $\mathbb{R}^n$ and $\mathbb{R}^{2n}$. Define

$$\Pi := \mathbb{G}\mathbb{G}^\dagger, \quad \nu_\| := \mathrm{Lip}(\Pi\mathbb{N}), \quad \nu_\perp := \mathrm{Lip}((I - \Pi)\mathbb{N}).$$

These global constants are finite, with $\nu_\|, \nu_\perp \leq \nu_{12}$. The original Volterra bound thus provides a sufficient test, but the projected constants can yield a strictly stronger conclusion. Corollary 5.6 below gives additional coefficient-based bounds for these constants.

Theorem 5.4. *If*

$$\nu_{\|} < \sigma_{\min}(\mathbb{G}), \tag{5.11}$$

then $\sigma_{\min}(\mathbb{G}) > 0$, so $\mathbb{G}$ has full column rank. Moreover, the map $\mathbb{P} : \mathbb{R}^n \to \mathbb{R}^{2n}$ is injective, its image is closed, and for all $\xi, \eta \in \mathbb{R}^n$,

$$[\sigma_{\min}(\mathbb{G}) - \nu_{\|}]\|\xi - \eta\| \le \|\mathbb{P}(\xi) - \mathbb{P}(\eta)\| \le [\|\mathbb{G}\| + \nu_{12}]\|\xi - \eta\|. \tag{5.12}$$

In particular, compatible observation pairs determine a unique preceding state, with inverse Lipschitz constant bounded above by $[\sigma_{\min}(\mathbb{G}) - \nu_{\|}]^{-1}$. The earlier sufficient condition $\nu_{12} < \sigma_{\min}(\mathbb{G})$ remains available because $\nu_{\|} \le \nu_{12}$.

For an arbitrary pair $Y \in \mathbb{R}^{2n}$, consider the iteration

$$\xi^{(m+1)} = \mathbb{G}^{\dagger}[Y - \mathbb{N}(\xi^{(m)})], \quad m \ge 0, \tag{5.13}$$

from any $\xi^{(0)} \in \mathbb{R}^n$. It converges to a unique limit ξ_Y with contraction factor at most $q_{12} := \nu_{\|}/\sigma_{\min}(\mathbb{G}) < 1$. The exact compatibility test for the pair Y is

$$Y \in \mathbb{P}(\mathbb{R}^n) \iff r(Y) := Y - \mathbb{P}(\xi_Y) = 0. \tag{5.14}$$

More precisely,

$$r(Y) = (I - \mathbb{G}\mathbb{G}^{\dagger})[Y - \mathbb{N}(\xi_Y)], \quad r(Y) \perp \operatorname{Ran}\mathbb{G}.$$

For any $Y, \widetilde{Y} \in \mathbb{R}^{2n}$,

$$\|\xi_Y - \xi_{\widetilde{Y}}\| \le \frac{\|\Pi(Y - \widetilde{Y})\|}{\sigma_{\min}(\mathbb{G}) - \nu_{\|}} \le \frac{\|Y - \widetilde{Y}\|}{\sigma_{\min}(\mathbb{G}) - \nu_{\|}}. \tag{5.15}$$

The residual in (5.14) *certifies exact compatibility; without additional assumptions, ξ_Y is not asserted to minimize the nonlinear least-squares error $\|\mathbb{P}(\xi) - Y\|$.*

Proof. Proposition 5.1, applied at the two observation times, gives

$$\|\mathbb{N}(\xi) - \mathbb{N}(\eta)\|^2 \le (\nu_1^2 + \nu_2^2)\|\xi - \eta\|^2.$$

Since $\mathbb{P}(\xi) = \mathbb{G}\xi + \mathbb{N}(\xi)$ and $\Pi\mathbb{G} = \mathbb{G}$, the projected reverse-triangle inequality gives

$$\|\mathbb{P}(\xi) - \mathbb{P}(\eta)\| \ge \|\Pi(\mathbb{P}(\xi) - \mathbb{P}(\eta))\| \ge [\sigma_{\min}(\mathbb{G}) - \nu_{\|}]\|\xi - \eta\|.$$

The ordinary triangle inequality yields the upper estimate in (5.12). The strict positive lower bound gives injectivity. It also makes the image closed: for any convergent sequence of compatible observation pairs, the corresponding preceding states are Cauchy, and the continuity of $\mathbb{P}$ passes their limit to a compatible state. Unlike Theorem 5.2, no surjectivity onto $\mathbb{R}^{2n}$ follows or is claimed.

Because $\nu_{\|} \ge 0$, the strict inequality (5.11) forces $\sigma_{\min}(\mathbb{G}) > 0$ and therefore full column rank. Hence $\mathbb{G}^{\dagger}\mathbb{G} = I_n$ and $\|\mathbb{G}^{\dagger}\| = \sigma_{\min}(\mathbb{G})^{-1}$. For fixed Y, the map $\xi \mapsto \mathbb{G}^{\dagger}[Y - \mathbb{N}(\xi)]$ is a contraction on the complete space $\mathbb{R}^n$ with factor at most q_{12}, since $\mathbb{G}^{\dagger}(I - \Pi) = 0$ and only $\Pi\mathbb{N}$ enters this iteration. The contraction mapping theorem gives its unique fixed point ξ_Y and convergence of (5.13). If $Y = \mathbb{P}(\xi)$ for some ξ, then $\mathbb{G}\xi = Y - \mathbb{N}(\xi)$, and consequently ξ is a fixed point. By uniqueness it equals ξ_Y and $r(Y) = 0$. Conversely, $r(Y) = 0$ means exactly that $Y = \mathbb{P}(\xi_Y)$. At the fixed point,

$$\mathbb{G}\xi_Y = \mathbb{G}\mathbb{G}^{\dagger}[Y - \mathbb{N}(\xi_Y)],$$

which gives the stated formula for $r(Y)$ and its orthogonality. Finally, subtracting the fixed-point equations and using $\mathbb{G}^{\dagger} = \mathbb{G}^{\dagger}\Pi$ gives

$$\|\xi_Y - \xi_{\widetilde{Y}}\| \le \frac{\|\Pi(Y - \widetilde{Y})\| + \nu_{\|}\|\xi_Y - \xi_{\widetilde{Y}}\|}{\sigma_{\min}(\mathbb{G})}.$$

Rearranging proves (5.15). □

The projected joint-observation criterion goes beyond an exact yes-or-no compatibility test: it identifies the geometry of the entire nonlinear compatible-data set and quantifies how incompatible data split relative to the linear observation space. This geometric information is not contained in the generic inverse-mapping principle used to obtain the fixed point.

Corollary 5.5. *Under the assumptions of Theorem 5.4, let*

$$E := \operatorname{Ran}\mathbb{G}, \quad \alpha := \frac{\nu_\perp}{\sigma_{\min}(\mathbb{G}) - \nu_\parallel}.$$

There exists a globally Lipschitz map $\psi : E \to E^\perp$ such that

$$\mathbb{P}(\mathbb{R}^n) = \{z + \psi(z) : z \in E\}, \quad \operatorname{Lip}(\psi) \le \alpha. \tag{5.16}$$

For every observation pair $Y \in \mathbb{R}^{2n}$, the fixed-point limit ξ_Y in (5.13) depends only on ΠY, and

$$\mathbb{P}(\xi_Y) = \Pi Y + \psi(\Pi Y), \quad r(Y) = (I - \Pi)Y - \psi(\Pi Y) \in E^\perp.$$

Its residual has the following two-sided relation to the distance from the actual nonlinear compatible-data set:

$$\frac{\|r(Y)\|}{\sqrt{1+\alpha^2}} \le \operatorname{dist}\big(Y, \mathbb{P}(\mathbb{R}^n)\big) \le \|r(Y)\|. \tag{5.17}$$

The map $\mathcal{R}(Y) := \mathbb{P}(\xi_Y)$ is a globally Lipschitz retraction onto the compatible-data set, though it need not be the nearest-point projection. More precisely,

$$\|\mathcal{R}(Y) - \mathcal{R}(\widetilde{Y})\| \le \sqrt{1+\alpha^2}\,\|\Pi(Y - \widetilde{Y})\| \le \sqrt{1+\alpha^2}\,\|Y - \widetilde{Y}\|. \tag{5.18}$$

Moreover, for $m \ge 1$ let $r_m(Y) := Y - \mathbb{P}(\xi^{(m)})$ and set

$$e_m := (\|\mathbb{G}\| + \nu_{12})\frac{q_{12}}{1 - q_{12}}\|\xi^{(m)} - \xi^{(m-1)}\|.$$

Then the finite-iteration quantities give the rigorous estimate

$$\frac{\max\{0, \|r_m(Y)\| - e_m\}}{\sqrt{1+\alpha^2}} \le \operatorname{dist}\big(Y, \mathbb{P}(\mathbb{R}^n)\big) \le \|r_m(Y)\|. \tag{5.19}$$

In particular, $\|r_m(Y)\| > e_m$ rigorously certifies incompatibility after only m iterations; and $\|r_m(Y)\| \le \epsilon$ certifies that the observation lies within ϵ of the compatible-data set.

Proof. Because $\mathbb{G}$ has full column rank, Π is the orthogonal projector onto E, and $\mathbb{G}^\dagger\Pi = \mathbb{G}^\dagger$. For any $z \in E$, the map

$$\xi \longmapsto \mathbb{G}^\dagger[z - \mathbb{N}(\xi)]$$

is a contraction with factor at most $q_{12} < 1$; denote its unique fixed point by ξ_z. Multiplying the fixed-point identity by $\mathbb{G}$ gives $\mathbb{G}\xi_z = z - \Pi\mathbb{N}(\xi_z)$, and hence $\Pi\mathbb{P}(\xi_z) = z$. Conversely, every $\xi \in \mathbb{R}^n$ is the fixed point associated with $z = \Pi\mathbb{P}(\xi)$. Thus the compatible-data set has the graph representation (5.16), with

$$\psi(z) := (I - \Pi)\mathbb{N}(\xi_z).$$

For $z, z' \in E$, subtracting the fixed-point equations gives

$$\|\xi_z - \xi_{z'}\| \le \frac{\|z - z'\|}{\sigma_{\min}(\mathbb{G}) - \nu_\parallel},$$

so the bound $\|\psi(z) - \psi(z')\| \le \nu_\perp\|\xi_z - \xi_{z'}\| \le \alpha\|z - z'\|$ holds without requiring the full nonlinear Lipschitz constant to be small. Since $\mathbb{G}^\dagger Y = \mathbb{G}^\dagger\Pi Y$, the limit ξ_Y equals $\xi_{\Pi Y}$, proving the claimed residual identity.

Put $z = \Pi Y$ and $w = (I - \Pi)Y$. Given any $z' \in E$, ψ being α-Lipschitz implies

$$\begin{aligned}\|r(Y)\| = \|w - \psi(z)\| &\le \|w - \psi(z')\| + \alpha\|z - z'\| \\ &\le \sqrt{1+\alpha^2}\sqrt{\|w - \psi(z')\|^2 + \|z - z'\|^2}.\end{aligned}$$

Taking the infimum over z' proves the lower bound in (5.17). Choosing $z' = z$ proves the upper bound. For compatible $Y = \mathbb{P}(\xi)$, Theorem 5.4 ensures $\xi_Y = \xi$, so $\mathcal{R}$ is a retraction. Moreover, with $z = \Pi Y$ and $\widetilde{z} = \Pi\widetilde{Y}$, orthogonality of E and $E^\perp$ gives

$$\|\mathcal{R}(Y) - \mathcal{R}(\widetilde{Y})\|^2 = \|z - \widetilde{z}\|^2 + \|\psi(z) - \psi(\widetilde{z})\|^2 \le (1+\alpha^2)\|z - \widetilde{z}\|^2,$$

which proves (5.18) directly.

Finally the contraction estimate yields

$$\|\xi^{(m)} - \xi_Y\| \le \frac{q_{12}}{1-q_{12}}\|\xi^{(m)} - \xi^{(m-1)}\|.$$

The upper bound in (5.12) shows $\|r_m(Y) - r(Y)\| \le e_m$. Combining $\big|\|r_m(Y)\| - \|r(Y)\|\big| \le e_m$ with (5.17) proves the lower bound in (5.19). The upper distance estimate does not involve an iteration-error term: each finite iterate $\xi^{(m)}$ produces an exactly compatible pair under the mathematical observation map. Specifically, $\mathbb{P}(\xi^{(m)})$ is compatible, so $\operatorname{dist}(Y, \mathbb{P}(\mathbb{R}^n)) \le \|Y - \mathbb{P}(\xi^{(m)})\| = \|r_m(Y)\|$. The two stopping criteria follow immediately. □

The projection constants in Theorem 5.4 can be bounded without differentiating the nonlinear observation maps or optimizing over all pairs of preceding states. The following bound expresses the required quantities in terms of the coefficient matrices, Lipschitz moduli, and the scalar Volterra solution already constructed in Proposition 5.1.

Corollary 5.6. *Assume the hypotheses of Section 5, and fix $\theta_j < t_1 < t_2 < \theta_{j+1}$. Let ρ_j be the solution of (5.3) on $[\theta_j, t_2]$, and define the measurable $2n \times n$ matrix kernel*

$$\mathcal{H}_{12}(s) := \begin{cases} \begin{pmatrix} \Phi(t_1, s) \\ \Phi(t_2, s) \end{pmatrix}, & \theta_j \le s \le t_1, \\ \begin{pmatrix} 0 \\ \Phi(t_2, s) \end{pmatrix}, & t_1 < s \le t_2. \end{cases}$$

With $\Pi = \mathbb{G}\mathbb{G}^\dagger$ as in Theorem 5.4, put

$$\overline{\nu}_\| := \int_{\theta_j}^{t_2} \|\Pi \mathcal{H}_{12}(s)\|[\ell_x(s)\rho_j(s) + \ell_y(s)]\, ds,$$

$$\overline{\nu}_\perp := \int_{\theta_j}^{t_2} \|(I-\Pi)\mathcal{H}_{12}(s)\|[\ell_x(s)\rho_j(s) + \ell_y(s)]\, ds.$$

These constants are finite and satisfy

$$\nu_\| \le \overline{\nu}_\|, \quad \nu_\perp \le \overline{\nu}_\perp. \tag{5.20}$$

In particular, the directly testable condition

$$\overline{\nu}_\| < \sigma_{\min}(\mathbb{G}) \tag{5.21}$$

implies all conclusions of Theorem 5.4 and Corollary 5.5. Their quantitative bounds remain valid with the larger, coefficient-based bounds $\overline{\nu}_\|$ and $\overline{\nu}_\perp$ used in place of the respective Lipschitz constants. Thus a fully specified valid choice for the graph slope, contraction factor, and finite-iteration error bound is

$$\overline{\alpha} := \frac{\overline{\nu}_\perp}{\sigma_{\min}(\mathbb{G}) - \overline{\nu}_\|}, \quad \overline{q} := \frac{\overline{\nu}_\|}{\sigma_{\min}(\mathbb{G})},$$

$$\overline{e}_m := (\|\mathbb{G}\| + \nu_{12})\frac{\overline{q}}{1-\overline{q}}\|\xi^{(m)} - \xi^{(m-1)}\|, \quad m \ge 1.$$

The observation maps and iteration are exact mathematical maps; validating a numerical implementation additionally requires bounds on its forward-solution and arithmetic errors.

Proof. For $\xi, \eta \in \mathbb{R}^n$, write $\Delta F(s) = F(s, u_j(s;\xi), \xi) - F(s, u_j(s;\eta), \eta)$. Variation of constants at the two observation times gives

$$\mathbb{N}(\xi) - \mathbb{N}(\eta) = \int_{\theta_j}^{t_2} \mathcal{H}_{12}(s)\Delta F(s)\, ds.$$

By Proposition 5.1 and the Lipschitz assumption on F,

$$\|\Delta F(s)\| \le [\ell_x(s)\rho_j(s) + \ell_y(s)]\|\xi - \eta\|$$

for almost every $s \in [\theta_j, t_2]$. Apply Π and $I - \Pi$ separately under the integral, and then the integral triangle inequality, to obtain (5.20). All integrals are finite since ρ_j is continuous and

bounded, Φ is bounded on the compact time triangle, and ℓ_x, ℓ_y are integrable. Under (5.21), $\nu_{\|} \le \overline{\nu}_{\|} < \sigma_{\min}(\mathbb{G})$, so the earlier theorem and corollary apply. Increasing their nonnegative Lipschitz upper bounds weakens, but preserves, each asserted inequality and the contraction and stopping estimates, as claimed. □

For instance, take $n = 1$, $A = B = 0$, and $F(t, x, y) = cy$, with $c \in \mathbb{R}$, on an interval of constancy longer than 2, and choose $t_1 = \theta_j + 1$, $t_2 = \theta_j + 2$. Then $\mathbb{G} = (1, 1)^T$, $\ell_x = 0$, $\ell_y = |c|$, and $\mathbb{N}(\xi) = c(\xi, 2\xi)^T$. Direct calculations give

$$\nu_{\|} = \overline{\nu}_{\|} = \frac{3|c|}{\sqrt{2}}, \quad \nu_{\perp} = \overline{\nu}_{\perp} = \frac{|c|}{\sqrt{2}}.$$

Thus both projected Volterra estimates are attained in a scalar piecewise constant argument equation. The sufficient condition (5.21) becomes $|c| < 2/3$; it is not claimed to be necessary for injectivity of this example. If instead $F(t, x, y) = c\mathbf{1}_{(\theta_j+1,\theta_j+2)}(t)y$, then $\mathbb{N}(\xi) = c(0, \xi)^T$ and $\overline{\nu}_{\|} = \nu_{\|} = |c|/\sqrt{2}$, whereas the unprojected Volterra bound is $\nu_{12} = |c|$. Thus the explicit projected condition holds for $|c| < 2$, including $\sqrt{2} \le |c| < 2$ where the unprojected condition $\nu_{12} < \sigma_{\min}(\mathbb{G})$ fails. This shows that the coefficient-based projected test can be strictly stronger without evaluation of the nonlinear observation map's global Lipschitz constant.

The residual–distance estimates also give a rigorous interpretation of measurement errors. Unlike exact compatibility, this formulation permits an observed pair to lie outside the range of the observation map. In the decomposition associated with $\operatorname{Ran}\mathbb{G}$, the error components parallel and orthogonal to the linear observation space play different roles: only the parallel component enters the fixed-point iteration.

Corollary 5.7. *Assume the hypotheses of Theorem 5.4, and use α, q_{12}, e_m, and r_m as in Corollary 5.5. Let $\delta \ge 0$ be an upper bound on the combined Euclidean error in the two observations. For $Y \in \mathbb{R}^{2n}$, the following statements hold.*

(i) If $\|r_m(Y)\| \le \delta$ for some $m \ge 1$, there is a compatible observation pair Z with $\|Y - Z\| \le \delta$. One such pair is $Z = \mathbb{P}(\xi^{(m)})$.

(ii) If for some $m \ge 1$,

$$\|r_m(Y)\| - e_m > \sqrt{1 + \alpha^2}\,\delta, \tag{5.22}$$

then $\|Y - Z\| > \delta$ for every compatible observation pair $Z \in \mathbb{P}(\mathbb{R}^n)$. These two tests are sufficient rather than exhaustive: neither inequality need hold for a given Y and m.

(iii) If $Y = \mathbb{P}(\xi_*) + e$ with $\|e\| \le \delta$, then

$$\|r(Y)\| \le \sqrt{1 + \alpha^2}\,\delta, \tag{5.23}$$

and, writing $e_{\|} := \Pi e$ and $e_{\perp} := (I - \Pi)e$, the fixed-point reconstruction satisfies the stronger, direction-sensitive bounds

$$\begin{aligned} \|\xi_Y - \xi_*\| &\le \frac{\|e_{\|}\|}{\sigma_{\min}(\mathbb{G}) - \nu_{\|}}, \\ \max\{0, \|e_{\perp}\| - \alpha\|e_{\|}\|\} &\le \|r(Y)\| \le \|e_{\perp}\| + \alpha\|e_{\|}\|. \end{aligned} \tag{5.24}$$

For every $m \ge 1$,

$$\begin{aligned} \|\xi^{(m)} - \xi_*\| &\le \frac{q_{12}}{1 - q_{12}}\|\xi^{(m)} - \xi^{(m-1)}\| + \frac{\|e_{\|}\|}{\sigma_{\min}(\mathbb{G}) - \nu_{\|}} \\ &\le \frac{q_{12}}{1 - q_{12}}\|\xi^{(m)} - \xi^{(m-1)}\| + \frac{\delta}{\sigma_{\min}(\mathbb{G}) - \nu_{\|}}. \end{aligned} \tag{5.25}$$

In particular, if $\Pi e = 0$ then $\xi_Y = \xi_*$ and $r(Y) = e$: an observation error orthogonal to $\operatorname{Ran}\mathbb{G}$ leaves the reconstructed preceding state unchanged, although the compatibility residual detects that error.

When the coefficient test (5.21) *is available, all the assertions remain valid on replacing α, q_{12}, $\nu_{\|}$ and e_m in these upper bounds and tests by $\overline{\alpha}$, $\overline{q}$, $\overline{\nu}_{\|}$ and $\overline{e}_m$, respectively. These inequalities concern exact evaluation of $\mathbb{P}$ and do not by themselves bound numerical forward-solver or rounding errors.*

Proof. For (i), the finite-iterate pair $\mathbb{P}(\xi^{(m)})$ is in the compatible-data set and has distance $\|Y - \mathbb{P}(\xi^{(m)})\| = \|r_m(Y)\|$ from Y. For (ii), the finite-iteration distance estimate (5.19) yields

$$\operatorname{dist}(Y, \mathbb{P}(\mathbb{R}^n)) \geq \frac{\max\{0, \|r_m(Y)\| - e_m\}}{\sqrt{1+\alpha^2}}.$$

Thus (5.22) excludes every compatible pair within radius δ. To prove (iii), put $Z_* = \mathbb{P}(\xi_*)$; then $\operatorname{dist}(Y, \mathbb{P}(\mathbb{R}^n)) \leq \|Y - Z_*\| \leq \delta$. The lower bound in (5.17) gives (5.23), while the dependence of the fixed point only on ΠY and (5.15) yield the first bound in (5.24). Indeed, $\xi_Y = \xi_{\Pi Y}$, $\xi_* = \xi_{\Pi Z_*}$, and $\Pi(Y - Z_*) = e_\|$. With $z_* := \Pi Z_*$ and the graph map ψ from Corollary 5.5, the residual identity reads

$$r(Y) = e_\perp + \psi(z_*) - \psi(z_* + e_\|).$$

The α-Lipschitz estimate for ψ and the direct and reverse triangle inequalities establish both residual bounds in (5.24). If $e_\| = 0$, the fixed-point identity gives $\xi_Y = \xi_*$ and the last residual identity gives $r(Y) = e_\perp = e$ exactly. Finally, the finite-iteration contraction bound in the proof of Corollary 5.5, followed by the triangle inequality, proves (5.25). Under (5.21), the indicated constants are upper bounds for the original ones, and the denominator in (5.25) remains strictly positive. The replacement therefore preserves every inequality. □

Example 5.8. The criterion can apply when *each nonlinear single-time map* actually fails to be injective. Take $n = 2$, $\theta_{j+1} - \theta_j > 2$, $A = 0$, and the matrix B from the explicit two-observation example after Proposition 4.3. With $h = t - \theta_j$, define

$$q(h) = \begin{cases} 1, & 0 \leq h < 1, \\ -1, & 1 \leq h < 2, \\ 0, & h \geq 2, \end{cases} \qquad F(t, x, y) = \varepsilon q(h) \begin{pmatrix} \tanh(y_1) \\ 0 \end{pmatrix},$$

where $0 < \varepsilon = 1/10$. The nonlinear vector field is globally Lipschitz, with $\ell_x = 0$ and $\ell_y = \varepsilon|q|$. On $0 \leq h \leq 2$, $\Phi = I$ and $\|G_j(\theta_j + h)\| \leq 1$. Proposition 5.1 therefore gives $\nu_1 \leq \varepsilon$ and $\nu_2 \leq 2\varepsilon$ for $t_1 = \theta_j + 1$ and $t_2 = \theta_j + 2$. Since $\sigma_{\min}(\mathbb{G}) = 1$,

$$\nu_{12} \leq \sqrt{5}\,\varepsilon < 1,$$

so Theorem 5.4 applies. Direct integration yields

$$P_{j,t_1}(\xi_1, \xi_2) = \begin{pmatrix} \xi_1 + \varepsilon \tanh \xi_1 \\ 0 \end{pmatrix}, \quad P_{j,t_2}(\xi_1, \xi_2) = \begin{pmatrix} 0 \\ \xi_2 \end{pmatrix}.$$

Each individual nonlinear map is noninjective on $\mathbb{R}^2$, while their joint map is injective and has a globally Lipschitz inverse on its image. This example illustrates a genuinely nonlinear restoration of uniqueness by two observations within one interval of constancy; it does not assert that every pair in $\mathbb{R}^4$ is compatible.

The projected condition in Theorem 5.4 can hold when the unprojected small-remainder test fails by an arbitrarily large margin. For example, take a scalar equation on an interval of constancy of length greater than 2,

$$x'(t) = K q(t - \theta_j) \tanh(x(\theta_j)), \quad K > 0,$$

where $q(h) = 1$ for $0 \leq h < 1$, $q(h) = -2$ for $1 \leq h < 2$, and $q(h) = 0$ afterwards. For the observations at $t_1 = \theta_j + 1$ and $t_2 = \theta_j + 2$, direct integration gives

$$\mathbb{P}(\xi) = \begin{pmatrix} \xi + K \tanh \xi \\ \xi - K \tanh \xi \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbb{N}(\xi) = K \tanh \xi \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Here $E = \operatorname{span}\{(1,1)^T\}$, $\Pi\mathbb{N} = 0$, and hence $\nu_\| = 0$ for *every* $K > 0$. The joint observations recover $\xi = (P_{j,t_1}(\xi) + P_{j,t_2}(\xi))/2$ exactly. In contrast, $\operatorname{Lip}(\mathbb{N}) = \sqrt{2}K$ and $\sigma_{\min}(\mathbb{G}) = \sqrt{2}$, so a test controlling the unprojected remainder by its full Lipschitz constant fails whenever $K \geq 1$. The graph slope is nevertheless bounded by K: large perturbations orthogonal to the linear observation space need not destroy reconstruction. This example also shows that the factor $\sqrt{1+\alpha^2}$ in (5.17) cannot generally be reduced using only the graph Lipschitz constant. Indeed, $\alpha = K$ in this

example: in orthonormal coordinates along E and $E^\perp$, the compatible-data graph has equation $w = \sqrt{2}K\tanh(z/\sqrt{2})$ and has slope K at $z = 0$. For $Y_\delta = \delta(1,-1)^T/\sqrt{2}$ and $\delta \downarrow 0$, its residual norm is δ, whereas the distance to the graph is

$$\operatorname{dist}(Y_\delta, \mathbb{P}(\mathbb{R})) = \frac{\delta}{\sqrt{1+K^2}} + o(\delta).$$

To see the asymptotic identity, put $z = \delta v$ in the squared distance to a graph point; dividing by δ^2 gives $v^2 + (1 - Kv + o(1))^2$, uniformly for bounded v, whose minimum is $1/(1+K^2)$. Minimizers have $z = O(\delta)$ because $z = 0$ already gives distance δ, which justifies restricting to bounded v. Thus the lower residual–distance constant is optimal within the class of Lipschitz graphs covered by the corollary.

5.4. **Comparison of sufficient conditions.** The next scalar example shows why retaining the linear delayed term in G_j can materially enlarge the parameter range covered by the perturbation criterion.

Example 5.9. Consider

$$x'(t) = \lambda x(\theta_j) + \varepsilon \sin x(t), \quad \lambda > 0, \quad \varepsilon \neq 0,$$

and fix $t_0 \in (\theta_j, \theta_{j+1})$. Set $h := t_0 - \theta_j$ and $a := |\varepsilon| > 0$. For the decomposition in this section, $A = 0$, $B = \lambda$, and $F(t,x,y) = \varepsilon \sin x$. Thus $\Phi(t,s) = 1$, $G_j(t_0) = 1 + \lambda h$, and one may take $\ell_x = a$ and $\ell_y = 0$. Using elapsed time $s = t - \theta_j$, set

$$\varrho(s) := \rho_j(\theta_j + s), \quad 0 \le s \le h.$$

Then

$$\varrho(s) = 1 + \lambda s + a\int_0^s \varrho(r)\,dr.$$

This integral equation implies that ϱ is absolutely continuous and satisfies

$$\varrho'(s) = \lambda + a\varrho(s) \quad \text{for a.e. } s \in (0,h), \quad \varrho(0) = 1.$$

Hence

$$\varrho(s) = e^{as} + \frac{\lambda}{a}(e^{as} - 1).$$

Moreover,

$$\widehat{\nu}_{j,t_0} = a\int_0^h \varrho(s)\,ds = \varrho(h) - 1 - \lambda h.$$

Introduce the dimensionless quantities $\chi := ah$ and $\mu := \lambda h$, and let

$$\phi(\chi) := \frac{e^\chi - 1}{\chi}, \quad \chi > 0.$$

Then

$$\widehat{\nu}_{j,t_0} = e^\chi - 1 + \mu[\phi(\chi) - 1], \quad \sigma_{\min}(G_j(t_0)) = 1 + \mu.$$

Hence the hypothesis of Theorem 5.2 is exactly

$$e^\chi - 2 + \mu[\phi(\chi) - 2] < 0.$$

The function ϕ is strictly increasing because

$$\phi'(\chi) = \frac{(\chi - 1)e^\chi + 1}{\chi^2} > 0, \quad \chi > 0.$$

Indeed, the numerator $q(\chi) := (\chi - 1)e^\chi + 1$ satisfies $q(0) = 0$ and $q'(\chi) = \chi e^\chi > 0$ for $\chi > 0$. Since $\phi(\chi) \to 1$ as $\chi \downarrow 0$ and $\phi(\chi) \to \infty$, there is a unique $\chi_* > 0$ such that

$$\phi(\chi_*) = 2, \quad \text{equivalently} \quad e^{\chi_*} = 1 + 2\chi_*.$$

Numerically, $\chi_* \approx 1.2564312086$. Consequently, the hypothesis of Theorem 5.2 can be classified explicitly for this example:

$$0 < \chi \le \log 2 : \text{ it holds for every } \mu > 0,$$

$$\log 2 < \chi < \chi_* : \text{it holds if and only if } \mu > \frac{e^{\chi}-2}{2-\phi(\chi)},$$
$$\chi \geq \chi_* : \text{ it does not hold for any } \mu > 0.$$

For comparison, applying Theorem 3.2 with the constant global Lipschitz moduli $L_x = a$ and $L_y = \lambda$ gives

$$Q_j(t_0) = \frac{\lambda}{a}(e^{ah}-1) = \mu\phi(\chi).$$

Thus the hypothesis of Theorem 3.2 holds if and only if

$$\mu < \frac{1}{\phi(\chi)}.$$

In particular, for every fixed $0 < \chi \leq \log 2$, the hypothesis of Theorem 5.2 holds for all $\mu > 0$, whereas the hypothesis of Theorem 3.2 fails once $\mu \geq 1/\phi(\chi)$. More generally, if $0 < \chi < \chi_*$, define

$$M(\chi) := \begin{cases} \frac{1}{\phi(\chi)}, & 0 < \chi \leq \log 2, \\ \max\left\{\frac{1}{\phi(\chi)}, \frac{e^{\chi}-2}{2-\phi(\chi)}\right\}, & \log 2 < \chi < \chi_*. \end{cases}$$

Then every $\mu > M(\chi)$ satisfies the hypothesis of Theorem 5.2 and fails the hypothesis of Theorem 3.2. Consequently, for fixed a and h with $0 < ah < \chi_*$, the sufficient condition of Theorem 5.2 holds throughout the unbounded interval

$$\lambda > \frac{M(ah)}{h},$$

whereas the sufficient condition of Theorem 3.2 fails throughout the same interval. The distinction comes from retaining the term $B(t)x(\theta_j)$ in the matrix G_j instead of absorbing its norm into the direct Lipschitz bound. Thus the perturbation criterion covers an unbounded parameter range that is not covered by the criterion based directly on the Lipschitz bounds of the full vector field.

6. A logistic population model

6.1. **Attainable levels and inverse branches.** Aruğaslan and Güzel studied the stability of the generalized logistic population model [2]

$$x'(t) = \big(a - bx(\beta(t))\big)x(t), \quad a, b > 0, \tag{6.1}$$

where $x(t)$ is the population density and a/b is the positive equilibrium. The partition points need not be equally spaced. In [2], the partition-point dynamics are reduced to a nonautonomous difference equation and used to study stability, while related asymptotic behavior about the positive equilibrium is investigated in [22].

The role of this application is to test the inverse formulation on a nonlinear population model whose forward interval dynamics are explicit but whose right-hand side is not globally Lipschitz on $\mathbb{R}^2$. It therefore lies outside the hypotheses used for the sufficient criteria of Sections 3 and 5 and provides an independent exact benchmark. For a population level x_0 prescribed or observed at $t_0 \in (\theta_j, \theta_{j+1})$, the inverse question is concrete: does any nonnegative preceding population generate that measurement, is the generating population unique, and how sensitive is its recovery near the boundary of the attainable data?

For each candidate $\xi = x(\theta_j) \geq 0$, equation (6.1) is explicitly solvable on the current interval. Consequently, range, multiplicity and conditioning can be determined exactly on the natural invariant state space $[0, \infty)$. This makes it possible to distinguish a forward solution that is perfectly well defined from an inverse measurement that is impossible, ambiguous or ill conditioned. The linearization at the positive equilibrium then connects this nonlinear benchmark with the exact linear theory of Section 4.

Proposition 6.1. *Fix $j \geq 0$ and $t_0 \in (\theta_j, \theta_{j+1})$, and set $h := t_0 - \theta_j > 0$. For equation (6.1), define*

$$P_h(\xi) := \xi e^{(a-b\xi)h}, \quad \xi \geq 0,$$

$$\xi_c(h) := \frac{1}{bh}, \quad M(h) := \frac{e^{ah-1}}{bh}.$$

Then $P_h([0,\infty)) = [0, M(h)]$. In particular, $x_0 = 0$ has the unique nonnegative compatible value $x(\theta_j) = 0$. For $x_0 > 0$:

(i) *if $0 < x_0 < M(h)$, exactly two positive values of $x(\theta_j)$ are compatible with $x(t_0) = x_0$;*
(ii) *if $x_0 = M(h)$, the unique compatible value is $x(\theta_j) = \xi_c(h)$;*
(iii) *if $x_0 > M(h)$, no positive value of $x(\theta_j)$ is compatible with $x(t_0) = x_0$.*

When $0 < x_0 \le M(h)$, the compatible values are given by

$$\xi_k = -\frac{1}{bh} W_k\big(-bh\, x_0 e^{-ah}\big), \quad k \in \{0, -1\}, \tag{6.2}$$

where W is the Lambert W function defined by $W(z)e^{W(z)} = z$, W_0 is its principal real branch, and W_{-1} is its lower real branch on $[-e^{-1}, 0)$. If $0 < x_0 < M(h)$, the two values are distinct and satisfy $\xi_0 < \xi_c(h) < \xi_{-1}$; at $x_0 = M(h)$ the two branches coincide.

Proof. For fixed $\xi \ge 0$, equation (6.1) becomes

$$u'(t) = (a - b\xi)u(t), \quad u(\theta_j) = \xi,$$

so

$$u(t) = \xi e^{(a-b\xi)(t-\theta_j)}$$

and hence $u(t_0) = P_h(\xi)$. Thus P_h is precisely the map from a nonnegative partition-point population $x(\theta_j) = \xi$ to the population at the prescribed interior time t_0. Moreover,

$$P_h'(\xi) = e^{(a-b\xi)h}(1 - bh\xi).$$

Thus $P_h(0) = 0$, P_h is strictly increasing on $(0, \xi_c(h))$, strictly decreasing on $(\xi_c(h), \infty)$, and has its unique maximum at $\xi_c(h)$. Since

$$\lim_{\xi \downarrow 0} P_h(\xi) = 0, \quad \lim_{\xi \to \infty} P_h(\xi) = 0,$$
$$P_h(\xi_c(h)) = \frac{1}{bh} e^{ah-1} = M(h),$$

the range statement, the zero-level assertion, and the three positive alternatives follow. Finally,

$$x_0 = \xi e^{(a-b\xi)h}$$

is equivalent to

$$-bh\, x_0 e^{-ah} = (-bh\xi)e^{-bh\xi}.$$

For $0 < x_0 \le M(h)$, the argument on the left belongs to $[-e^{-1}, 0)$, so the two real branches W_0 and W_{-1} give (6.2). At $x_0 = M(h)$ both branches equal -1 and give $\xi = \xi_c(h)$. □

The threshold $M(h)$ gives the nonlinear counterpart of the linear range-preimage classification without requiring a global Lipschitz condition for the logistic vector field.

The compatibility boundary also has an exact conditioning interpretation. For $0 < x_0 < M(h)$, let

$$\xi_-(x_0) := -\frac{1}{bh} W_0\big(-bhx_0 e^{-ah}\big), \quad \xi_+(x_0) := -\frac{1}{bh} W_{-1}\big(-bhx_0 e^{-ah}\big).$$

Then $\xi_-(x_0) < \xi_c(h) < \xi_+(x_0)$ are the two positive inverse branches.

Corollary 6.2. *For each fixed $h > 0$, the branches ξ_- and ξ_+ are continuously differentiable on $(0, M(h))$ and satisfy*

$$\frac{d\xi_\pm}{dx_0} = \frac{\xi_\pm(x_0)}{x_0[1 - bh\xi_\pm(x_0)]}. \tag{6.3}$$

Consequently,

$$\Big|\frac{d\xi_\pm}{dx_0}\Big| \to \infty \quad \text{as } x_0 \uparrow M(h).$$

Moreover, with $\xi_c = 1/(bh)$ *and* $\delta = M(h) - x_0 \downarrow 0$,

$$\xi_\pm(x_0) = \xi_c \pm \frac{1}{bh}\sqrt{\frac{2\delta}{M(h)}} + O(\delta). \tag{6.4}$$

Thus the boundary of the attainable population interval is a fold of the observation map: the two reconstructions merge with square-root scaling and the inverse sensitivity diverges from both sides.

Proof. On either monotone branch, $P_h(\xi_\pm(x_0)) = x_0$ and $P_h'(\xi_\pm(x_0)) \neq 0$. Differentiation gives

$$\frac{d\xi_\pm}{dx_0} = \frac{1}{P_h'(\xi_\pm(x_0))} = \frac{\xi_\pm(x_0)}{x_0[1 - bh\xi_\pm(x_0)]},$$

which proves (6.3). As $x_0 \uparrow M(h)$, both branches converge to ξ_c, so the denominator tends to zero while the numerator tends to $\xi_c > 0$; hence the derivative magnitudes diverge.

At ξ_c, one has $P_h'(\xi_c) = 0$ and

$$P_h''(\xi_c) = -bh\, e^{ah-1} = -(bh)^2 M(h) < 0.$$

Taylor's formula therefore yields

$$P_h(\xi) = M(h) - \frac{(bh)^2 M(h)}{2}(\xi - \xi_c)^2 + O(|\xi - \xi_c|^3).$$

Writing $\delta = M(h) - x_0$ and $y := \xi - \xi_c$, the preceding expansion gives

$$\delta = \frac{(bh)^2 M(h)}{2} y^2 + O(|y|^3).$$

Along either local branch, $y \to 0$ as $\delta \downarrow 0$; the last relation first implies $|y| = O(\sqrt{\delta})$. Substitution back into the remainder then yields

$$y^2 = \frac{2\delta}{(bh)^2 M(h)} + O(\delta^{3/2}),$$

and therefore, with the sign chosen according to the increasing or decreasing branch,

$$\xi - \xi_c = \pm\frac{1}{bh}\sqrt{\frac{2\delta}{M(h)}} + O(\delta).$$

This is (6.4). □

6.2. **Two-time identification in the population model.** The loss of uniqueness from a single interior measurement is not inherent to the forward population dynamics: two positive observations made at separate times in the same interval determine the preceding population.

Proposition 6.3. *Let* $a, b > 0$, *and let* $\theta_j < t_1 < t_2 < \theta_{j+1}$. *Put* $h_i = t_i - \theta_j$ *and* $\Delta h = h_2 - h_1 > 0$. *For any two prescribed nonnegative measurements* $x(t_i) = y_i$, *the following exhaustive compatibility alternatives hold.*

(i) *If* $y_1 = y_2 = 0$, *the unique compatible preceding population is* $x(\theta_j) = 0$.
(ii) *If exactly one of* y_1, y_2 *vanishes, there is no compatible preceding population.*
(iii) *If* $y_1, y_2 > 0$, *define*

$$\xi_{12} := \frac{1}{b}\Big(a - \frac{\log(y_2/y_1)}{\Delta h}\Big). \tag{6.5}$$

The measurements are compatible with a nonnegative preceding population if and only if $\xi_{12} > 0$ *and*

$$y_1 = \xi_{12} e^{(a - b\xi_{12})h_1}. \tag{6.6}$$

In that case ξ_{12} *is the unique compatible preceding population.*

For two positive compatible measurement pairs (y_1, y_2) *and* $(\widetilde{y}_1, \widetilde{y}_2)$, *the corresponding preceding populations satisfy the exact identity*

$$|\xi_{12} - \widetilde{\xi}_{12}| = \frac{1}{b\Delta h}\big|\log\frac{y_2}{y_1} - \log\frac{\widetilde{y}_2}{\widetilde{y}_1}\big|. \tag{6.7}$$

Thus recovery is stable in logarithmic observation ratios with sharp factor $1/(b\Delta h)$*, which deteriorates as the two observation times approach each other. This does not give a uniform Lipschitz estimate with respect to untransformed populations approaching zero.*

Proof. The interval formula gives $y_i = \xi e^{(a-b\xi)h_i}$ for $i = 1, 2$ and $\xi \geq 0$. The two values vanish simultaneously exactly when $\xi = 0$. If both measurements are positive, division eliminates the preceding population:

$$\frac{y_2}{y_1} = e^{(a-b\xi)(h_2-h_1)}.$$

Taking logarithms forces $\xi = \xi_{12}$, which is positive for any compatible positive pair. Conversely, if $\xi_{12} > 0$ and (6.6) holds, the ratio identity yields the second observation as well. This proves necessity, sufficiency and uniqueness. Subtracting the explicit reconstruction formulas for two compatible pairs proves (6.7). □

Proposition 6.3 also illustrates the general nonlinear two-observation principle of Theorem 3.5: for $x > 0$ the derivative of the logistic vector field with respect to its piecewise constant argument is $f_y(t, x, y) = -bx < 0$. Positivity is invariant for every positive preceding population, so the joint map is injective on $(0, \infty)$ even without using the explicit logarithmic formula. In addition, for a compact preceding-population range $0 < \alpha \leq \xi, \eta \leq \gamma$ and $h_2 = t_2 - \theta_j$, the two observation pairs satisfy

$$|\xi - \eta| \leq \frac{e^{b\gamma h_2}}{b\alpha(t_2 - t_1)}\big(|P_{h_1}(\xi) - P_{h_1}(\eta)| + |P_{h_2}(\xi) - P_{h_2}(\eta)|\big). \tag{6.8}$$

Indeed, the exact solutions obey $P_{h_i}(\zeta) \geq \alpha e^{-b\gamma h_2}$ for $\zeta \in [\alpha, \gamma]$, so the logarithm is Lipschitz with constant at most $e^{b\gamma h_2}/\alpha$ on the relevant positive observations. Applying this bound to the logarithmic-ratio identity proves (6.8). The compact-range estimate is not uniform as $\alpha \downarrow 0$ or $t_2 \downarrow t_1$.

In particular, the two positive preceding populations that can produce a single observation strictly below $M(h_1)$ cannot also produce the same second observation at $h_2 \neq h_1$. Thus a second interior measurement eliminates the single-time ambiguity, including when one observation is made at a fold, provided both measurements are consistent with the model.

Figure 2 illustrates Proposition 6.1 and Corollary 6.2 for $a = b = 1$. The common preceding value $\xi = 1$ gives the observation $x_0 = 1$ in all three cases. For $h = 0.5$ and $h = 2$, another positive preceding value exists, whereas at $h = 1$ the branches meet in a tangency.

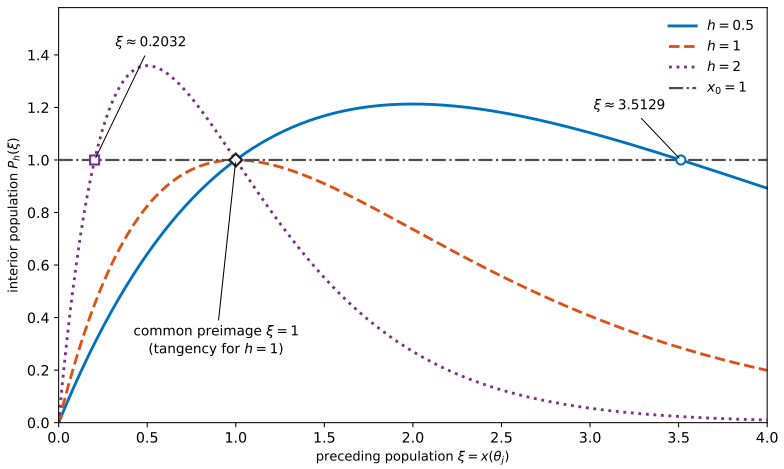


FIGURE 2. Solution map in the logistic population model for $a = b = 1$. The common preimage $\xi = 1$ is a tangency only at $h = 1$; for $h = 0.5$ and $h = 2$ the prescribed level $x_0 = 1$ has a second positive preimage.

6.3. **Critical time and inverse sensitivity.**

Corollary 6.4. *Let $x^* := a/b$ be the positive equilibrium of (6.1). For the elapsed time $h = t_0 - \theta_j$ the following statements hold.*

(i) If $ah \neq 1$, the observation $x(t_0) = x^*$ has exactly two positive compatible values of $x(\theta_j)$; if $ah = 1$, its unique positive compatible value is $x(\theta_j) = x^*$.

(ii) The linearization of (6.1) at x^* is

$$z'(t) = -az(\beta(t)), \tag{6.9}$$

and the matrix $G_j(t_0)$ in Theorem 4.1 reduces to the scalar $1 - ah$. Thus the linearized map is singular precisely at $h_* = 1/a$; this is an interior time of the current interval exactly when $1/a < \theta_{j+1} - \theta_j$.

(iii) If $1/a < \theta_{j+1} - \theta_j$, then at the interior time $h = h_*$ the nonlinear map has a nondegenerate maximum at x^* and

$$P_{h_*}(\xi) = x^* - \frac{b}{2a}(\xi - x^*)^2 + O(|\xi - x^*|^3). \tag{6.10}$$

Consequently, if $x_0 = x^* - \varepsilon$ with $\varepsilon > 0$ sufficiently small, the two positive compatible values satisfy

$$\xi_\pm = x^* \pm \sqrt{\frac{2a}{b}}\,\varepsilon + O(\varepsilon).$$

Proof. Since $P_h(x^*) = x^*$ and, by $\log(ah) \le ah - 1$,

$$\frac{M(h)}{x^*} = \frac{e^{ah-1}}{ah} \ge 1,$$

with equality if and only if $ah = 1$, part (i) follows from Proposition 6.1. For the linearization, writing $x = x^* + z$ gives

$$z'(t) = -az(\beta(t)) - bz(t)z(\beta(t)),$$

so the first-order equation is (6.9). On the current interval its solution is

$$z(t) = \big[1 - a(t - \theta_j)\big]z(\theta_j),$$

and therefore $G_j(t_0) = 1 - ah$. This proves (ii). Moreover, $P_h'(x^*) = 1 - ah$. Assume now that $1/a < \theta_{j+1} - \theta_j$, so that $h_* = 1/a$ is an admissible interior elapsed time. Then $P_{h_*}'(x^*) = 0$ and

$$P_{h_*}''(x^*) = -\frac{b}{a} < 0.$$

Taylor's formula gives (6.10). For $x_0 = x^* - \varepsilon$ and either of the two local inverse branches, set $\delta = \xi - x^*$. Proposition 6.1 implies $\delta \to 0$ as $\varepsilon \downarrow 0$, and (6.10) gives

$$\varepsilon = \frac{b}{2a}\delta^2 + O(|\delta|^3).$$

Hence $|\delta| = O(\sqrt{\varepsilon})$ and then

$$\delta^2 = \frac{2a}{b}\varepsilon + O(\varepsilon^{3/2}).$$

Taking the sign corresponding to the increasing or decreasing branch yields

$$\delta = \pm\sqrt{\frac{2a}{b}}\,\varepsilon + O(\varepsilon),$$

which proves the final formula. □

For $ah \neq 1$, one of the two compatible values in part (i) is the equilibrium value x^* itself. The second value generates a nonconstant interval trajectory that reaches the level x^* at t_0; it is therefore a second preimage of the observation, not a second positive equilibrium.

The relation with the exact linear theory is quantitative. For $h \neq 1/a$, the linearized map sends the preceding deviation $z(\theta_j)$ to

$$z(t_0) = (1 - ah)z(\theta_j).$$

Hence Theorem 4.1 gives the linear inverse

$$\mathcal{I}_h(z_0) = \frac{z_0}{1-ah}, \quad \mathrm{Lip}(\mathcal{I}_h) = \frac{1}{|1-ah|}.$$

For the nonlinear population map, the inverse function theorem gives, when $h \neq 1/a$, a unique local inverse branch Q_h through the equilibrium, characterized by

$$Q_h(x^*) = x^*, \quad Q_h'(x^*) = \frac{1}{P_h'(x^*)} = \frac{1}{1-ah}.$$

Consequently,

$$|Q_h'(x^*)| = \mathrm{Lip}(\mathcal{I}_h) = \frac{1}{|1-ah|}. \tag{6.11}$$

Whenever $1/a < \theta_{j+1} - \theta_j$, both quantities in (6.11) diverge as admissible values of h approach $h_* = 1/a$. Thus the inverse-conditioning blow-up predicted by Corollary 4.8 is inherited exactly by the derivative of the nonlinear inverse branch at the equilibrium.

The local fold can also be certified without using the explicit expression for P_h. In Proposition 3.4, for the logistic vector field $f(t,x,y) = (a-by)x$, one has $f_x = a-by$, $f_y = -bx$, $f_{xx} = f_{yy} = 0$, and $f_{xy} = -b$. Along the equilibrium trajectory $u_*(t) = x^* = a/b$, equations (3.3)–(3.4) give, with $s = t - \theta_j$,

$$v(t) = 1 - as, \quad w(t) = -2bs + abs^2.$$

At $h_* = 1/a$ these satisfy $v(t_0) = 0$ and $w(t_0) = -b/a < 0$. Thus the fold condition is detected directly from the differential equation, and (3.6) agrees with the square-root formula in Corollary 6.4.

The dependence of this fold on the observation time is explicit:

$$\xi_c(h) = \frac{1}{bh}, \quad y_c(h) = M(h) = \frac{e^{ah-1}}{bh}.$$

Direct differentiation at $h_* = 1/a$ gives

$$\xi_c'(h_*) = -\frac{a^2}{b}, \quad y_c'(h_*) = 0, \quad M''(h_*) = \frac{a^3}{b}.$$

Consequently, the critical observation level satisfies

$$M(h) = \frac{a}{b} + \frac{a^3}{2b}\Big(h - \frac{1}{a}\Big)^2 + O\Big(|h - \frac{1}{a}|^3\Big) \quad (h \to 1/a). \tag{6.12}$$

Thus the attainable-level boundary has a quadratic tangency to the constant equilibrium observation as the interior observation time passes through h_*. This is a statement about the moving fold of the solution map, distinct from the square-root splitting of inverse branches at fixed time.

At the critical time itself, the linear and nonlinear conclusions must be distinguished. For the linearized equation, $G_j(t_0) = 0$: a nonzero prescribed deviation is incompatible, whereas $z_0 = 0$ is compatible with every preceding deviation $z(\theta_j)$. For the nonlinear population model, by contrast, $x_0 = x^*$ has the unique positive compatible value $x(\theta_j) = x^*$, while levels slightly below x^* have two compatible positive values that merge according to the square-root law in Corollary 6.4. The singular linearization therefore detects the same critical elapsed time at which the nonlinear solution map loses local invertibility, without identifying the two multiplicity mechanisms.

For a numerical illustration, normalize $a = b = 1$ and consider an interval of constancy of length greater than 2, so that $x^* = 1$ and all three elapsed times in Table 1 are admissible. For the same observed level $x_0 = 1$, Proposition 6.1 gives the compatible partition-point populations listed in Table 1.

The same observed population level can therefore correspond to substantially different preceding population levels. At the critical elapsed time $h = 1$, the two positive inverse branches meet at the equilibrium.

TABLE 1. Compatible preceding population levels for the observation $x(t_0) = 1$ when $a = b = 1$.

h	$M(h)$	positive compatible values of $x(\theta_j)$
0.5	$1.213061\ldots$	$1,\ 3.512862\ldots$
1	1	1
2	$1.359141\ldots$	$0.203188\ldots,\ 1$

7. Conclusions

Interior observations lead to a finite-dimensional compatibility problem for the preceding partition-point state. The analysis separates three questions: whether observed data are compatible, whether they identify the preceding state, and how sensitive that recovery is. For linear systems, we obtain exact first-order smallest-singular-value coefficients for nearby weighted observations and quantitative full-column-rank criteria at positive observation separations under coefficient and observation-time errors. For nonlinear systems, variational and projected Volterra estimates give fold criteria, joint identification, a Lipschitz compatible-data graph, residual–distance estimates, finite-iteration compatibility tests, and noise bounds that distinguish components parallel and orthogonal to the linear observation space. The logistic population model provides explicit single- and two-time formulas and separates nonlinear fold singularity from singular linear conditioning.

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