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A BIFURCATION RESULT FOR STURM-LIOUVILLE PROBLEMS WITH A SET-VALUED TERM

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Abstract

It is established in this note that $-(ku')' + g(\cdot, u) \in \mu F(\cdot, u)$, $u'(0) = 0 = u'(1)$, has a multiple bifurcation point at $(0, \mathbf{0})$ in the sense that infinitely many continua meet at $(0, \mathbf{0})$. F is a "set-valued representation" of a function with jump discontinuities along the line segment $[0, 1] \times \{0\}$. The proof relies on a Sturm-Liouville version of Rabinowitz's bifurcation theorem and an approximation procedure.

1. Introduction

Of concern is

$$\begin{gathered} -(ku')'(x) + g(x, u(x)) \in \mu F(x, u(x)) \quad x \in (0, 1) \text{ a.e.} \\ u'(0) = 0, \quad u'(1) = 0\,, \end{gathered} \tag{1}$$

under hypotheses motivated by the situation found for Budyko-North type energy balance climate models (cf. [5, 6, 7, 8, 9] and the references therein). In particular, the set-valued right hand side arises from a jump discontinuity of the albedo at the ice-edge in these models. By filling in such a gap (this is the solution concept we adapt), one arrives at the set-valued problem (1). We are here interested in a considerably simplified version as compared to the situation from climate modeling, e.g. a one-dimensional regular Sturm-Liouville differential operator substitutes for a two-dimensional Laplace-Beltrami operator or a singular Legendre-type operator, and the jump discontinuity is transformed to $u = 0$ in a way, which resembles only locally the climatological problem. The latter suffices for our purposes, since the global structure has already been investigated in [8] and [3], where the existence of an S-shaped principal solution branch is established. Computer simulations actually suggest that the branch structure of the original problem is very different from what we obtain for (1) in this paper. Here the effect of the discontinuity at zero is a solution branch which consists of infinitely many subbranches all meeting in $(0, \mathbf{0})$. Two subbranches are distinguished by the number of zeroes of the respective solutions. Thus, one would expect such a "multiple" bifurcation to occur on the relevant segment of the principal solution branch in the climatological setting, too. Unlike in our case, the subbranches would however be bounded (mushrooms in the sense of [13]). An example in Appendix B of [16], a Sturm-Liouville problem of

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Budyko type with infinitely many solutions, seemingly supports such a conjecture. But, numerical computations do not reveal any bifurcation point on the principal branch, rather the branch structure resembles that for Sellers-type models (continuous setting, cf. [10, 12] and [11] for a survey). Thus, our results are to be understood as an incentive for further investigations in order to clarify this issue.

Let us illustrate what is going on by means of the following simple example

$$\begin{aligned} -u'' + bu &\in \mu \operatorname{sgn} u \quad \text{on } [0,\pi]\text{a.e.} \\ u'(0) = 0 \quad & u'(\pi) = 0\,, \end{aligned} \tag{2}$$

with $b > 0$ and

$$\operatorname{sgn}(y) = \begin{cases} \{y/|y|\} & y \in \mathbb{R} \setminus \{0\} \\ [-1,1] & y = 0. \end{cases}$$

Assume that one approximates sgn by a sequence of functions $h_j \in C^\infty(\mathbb{R})$ satisfying $h_j(0) = 0$, $h_j{}'(0) = j$, $h_j|_{(-\infty,-1/j]} \equiv -1$, $h_j|_{[1/j,\infty)} \equiv 1$, $h_j{}'(y) > 0$ for $y \in (-1/j, 1/j)$ and $(h_j)_{j\in\mathbb{N}}$ uniformly convergent on $(-\infty,-r) \cup (r,\infty)$ for $r > 0$ to $y \mapsto y/|y|$. Then results of Rabinowitz and Crandall and Rabinowitz ensure that $\mu_{j,n} := (n^2 + b)/j$ is a bifurcation point for $n \in \mathbb{Z}_+$ of

$$\begin{aligned} -u'' + bu &= \mu h_j \circ u \quad \text{on } [0,\pi]\text{a.e.} \\ u'(0) = 0 \quad & u'(\pi) = 0, \end{aligned}$$

that the branch $\mathcal{S}_{j,n}$ emanating from $(\mu_{j,n}, 0)$ is unbounded in $\mathbf{R}_+ \times C([0,\pi])$ and that $(\mu, w) \in \mathcal{S}_{j,n}$ implies that w has exactly n zeroes, which are all simple. Letting $j \to \infty$ yields $\mu_{j,n} \to 0$ uniformly for n in any finite subset of $\mathbf{Z}_+$, hence any finite number of bifurcation points comes arbitrarily close to 0 for a sufficiently large $j \in \mathbb{N}$, thus one expects infinitely many curves of solutions of (2) bifurcating from $(0,0)$. In fact, these subbranches can be described explicitly as curves $\mu \mapsto (|\mu|, u_{\mu,j})$ $(\mu \in \mathbb{R})$, where $u_{\mu,0} \equiv \mu/b$ (no zeroes),

$$u_{\mu,1}(x) := \begin{cases} \frac{\mu}{b}\left[1 - \frac{\cosh(\sqrt{b}x)}{\cosh(\sqrt{b}\pi/2)}\right] & \text{if } x \in [0,\pi/2] \\ -\frac{\mu}{b}\left[1 - \frac{\cosh(\sqrt{b}(x-\pi))}{\cosh(\sqrt{b}\pi/2)}\right] & \text{if } x \in (\pi/2,\pi] \end{cases}$$

(one zero) or

$$u_{\mu,2}(x) := \begin{cases} \frac{\mu}{b}\left[1 - \frac{\cosh(\sqrt{b}x)}{\cosh(\sqrt{b}\pi/4)}\right] & \text{if } x \in [0,\pi/4] \\ -\frac{\mu}{b}\left[1 - \frac{\cosh(\sqrt{b}(x-\pi/2))}{\cosh(\sqrt{b}\pi/4)}\right] & \text{if } x \in (\pi/4,3\pi/4] \\ \frac{\mu}{b}\left[1 - \frac{\cosh(\sqrt{b}(x-\pi))}{\cosh(\sqrt{b}\pi/4)}\right] & \text{if } x \in (3\pi/4,\pi] \end{cases}$$

(two zeroes) and so on.

The same structure has been observed in another context by [2; Theorem 2.1.] for a similar reason. The paper deals with a quasilinear Sturm-Liouville problem; the differential operator is the radial part of a p-Laplacian and the right hand side vanishes at zero with a slower rate than $|x|^{p-1}$, thus the "derivative" is infinite relative to the intrinsic scaling associated with the nonlinear diffusion.

Of course, the use of approximating branches is standard in numerical bifurcation theory, and limiting processes for sets go back at least to the work of Kuratowski. Nowadays, the technical tool for the latter is a so-called Whyburn Lemma, which will be stated in Section 3. It should be mentioned that the approach outlined above has been used in [2].

Our findings are also related to results in [4, 15], where bifurcation from intervals was investigated. In a sense, here the real line is the bifurcation interval, and therefore the (extended) Rabinowitz alternative established in these papers becomes meaningless.

We have not attempted in this note to strive for generality, but shall present the method for the case we are interested in. Obtaining other versions is then a matter of routine. The precise result is stated in the next section, proofs are given in Section 3, which is followed by a concluding remark.

2. The Bifurcation Result

Throughout we assume:

(H1) $k \in C^1([0,1])$, $\inf k > 0$;

(H2) $g \in C([0,1] \times \mathbb{R})$, $g(x,\cdot)$ strictly increasing for $x \in [0,1]$, $g_1(x) := \lim_{y \to 0} \dfrac{g(x,y)}{y}$ exists uniformly for $x \in [0,1]$;

(H3) $f_+ \in C([0,1] \times \mathbb{R}_+, (0,\infty))$, $\inf f_+ > 0$, $f_- \in C([0,1] \times \mathbb{R}_-, (-\infty,0))$, $\sup f_- < 0$.

Let F in (1) be given by

$$F(x,y) := \begin{cases} \{f_+(x,y)\} & x \in [0,1],\ y > 0, \\ [f_-(x,0), f_+(x,0)] & x \in [0,1], \\ \{f_-(x,y)\} & x \in [0,1],\ y < 0 \end{cases}$$

and set $\mathcal{S} := \{(\mu, w) \in \mathbb{R} \times W^{2,\infty}([0,1]), (\mu, w) \text{ solves } (1)\}$. Throughout $\mathcal{S}$ will be considered as subset of the Banach space $Y := \mathbb{R} \times C^1([0,1])$ under the norm $\|\cdot\|_Y : (\mu, w) \mapsto \max\{|\mu|, \|w\|_\infty, \|w'\|_\infty\}$. It is useful to note:

Remark 1. The hypotheses (H1)–(H3) imply that

$$\mathcal{S} \cap \Big((-\infty, 0] \times C^1([0,1])\Big) = (-\infty, 0] \times \{(0, \mathbf{0})\}\,.$$

In fact, denoting by u^+ and u^- the positive and negative parts of u, respectively, one gets for $(\mu, u) \in \mathcal{S}$ with $\mu \leq 0$ that

$$\begin{aligned} &\int_0^1 [k(x)u'(x)^2 + g(x,u(x))u(x)]dx \\ &\qquad = \mu \int_0^1 [f_+(x,u(x))u^+(x) + |f_-(x,u(x))|u^-(x)]dx \leq 0, \end{aligned}$$

i.e. $u \equiv 0$ in view of $g(x\cdot)$ strictly increasing and $g(\cdot, 0) \equiv 0$.

Our main result is.

Theorem. *Let* (H1)–(H3) *be fulfilled. Then there exist sequences* $\left(C_n^{\pm}\right)_{n\in\mathbb{Z}_+}$ *of unbounded, closed, connected subsets of* $\mathcal{S}$ *with* $(0,\mathbf{0}) \in C_n^{\pm}$ *and the property that* u *has exactly* n *zeroes, which are all simple, if* $(\mu,u) \in C_n^{\pm} \setminus \{(0,\mathbf{0})\}$*. Moreover,* u *is positive (negative) on an interval* $(0,\tilde{x})$ *for some* $\tilde{x} \in (0,1]$*, if* $(\mu,u) \in C_n^{+}$ $((\mu,u) \in C_n^{-})$ *and* $u \not\equiv 0$*.*

Actually, such continua can be obtained as upper limits in the sense of Kuratowski of sequences of solution continua from associated continuous problems. To this end one sets $d_f := \min\{\inf f_+, \inf|f_-|\}$ and selects an approximation sequence $(f_l) \in C([0,1]\times\mathbb{R},\mathbb{R})^{\mathbb{N}}$ of F satisfying

(A1) $f_l(x,y) = ly$ for $x \in [0,1]$ and $y \in [-\frac{d_f}{2l}, \frac{d_f}{2l}]$;

(A2) $f_l(x,y) \times \operatorname{sgn}(y) \geq \frac{d_f}{2}$ for $x \in [0,1]$ and $|y| \geq \frac{d_f}{2l}$, $f_l \leq f_+$ on $[0,1]\times[\frac{d_f}{2l}, \frac{d_f}{l}]$, $f_l \geq f_-$ on $[0,1]\times[-\frac{d_f}{l}, -\frac{d_f}{2l}]$;

(A3) $f_l(x,y) = f_+(x,y)$ for $x \in [0,1]$ and $y \geq \frac{d_f}{l}$; $f_l(x,y) = f_-(x,y)$ for $x \in [0,1]$ and $y \leq -\frac{d_f}{l}$;

(A4) $(f_l(x,y))_{l\in\mathbb{N}}$ nondecreasing for $(x,y) \in [0,1]\times(0,\infty)$ and nonincreasing for $(x,y) \in [0,1]\times(-\infty,0)$.

It is easy to see thanks to (H2) and (A1) that

$$\begin{gathered} -(kv_l')'(x) + g(x,v_l(x)) = \mu f_l(x,v_l(x)) \quad x \in [0,1] \\ v_l'(0) = 0 \quad v_l'(1) = 0\,. \end{gathered} \tag{3_l}$$

falls into the scope of the Sturm-Liouville version of the celebrated Rabinowitz bifurcation theorem (cf. [14] for a more general, but somewhat different setting).

Indeed, denote the strictly increasing sequence of simple eigenvalues of

$$\begin{gathered} -(k\psi')' + g_1\psi = \lambda\psi \quad \text{on } [0,1] \\ \psi'(0) = 0 \quad \psi'(1) = 0 \end{gathered} \tag{4}$$

by $(\lambda_n)_{n\in\mathbb{Z}_+}$ and set $\mu_{n,l} := \lambda_n/l$. Then $(\mu_{n,l},\mathbf{0})$ is a bifurcation point of the solution set of (3_l) for every $n \in \mathbb{Z}_+$, and for each $(n,l) \in \mathbb{Z}_+\times\mathbb{N}$ there exist two unbounded closed connected subsets $C_{n,l}^{\pm}$ of the solution set of (3_l) with

- $C_{n,l}^{+} \cap C_{n,l}^{-} = \{(\mu_{n,l},\mathbf{0})\}$. Moreover, $(\mu_{n,l},\mathbf{0})$ is the only bifurcation point contained in $C_{n,l}^{\pm}$;
- If $(\mu,\vartheta) \in C_{n,l}^{+}$ and $\vartheta \not\equiv 0$, then ϑ possesses exactly n simple zeroes (and no multiple zeroes) in $(0,1)$ and is positive on $(0,\delta)$ for some $\delta > 0$;
- $C_{n,l}^{-}$ fulfills the previous statement in case that positive on $(0,\delta)$ is replaced by negative on $(0,\delta)$.

Utilizing a so-called Whyburn Lemma [17] we establish.

Lemma 1. *Let* (H1) - (H3) *and* (A1)–(A4) *be satisfied and* $\mathcal{B}_r$ *denote the ball with center* $(0,\mathbf{0})$ *and radius* r *in* Y*. Then continua* $C_n^{\pm}$ *fulfilling the properties stated in the Theorem can be constructed as* $C_n^{\pm} = \bigcup_{j\in\mathbb{N}}\Big(\limsup_{l\to\infty}\big(C_{n,l}^{\pm}\cap\mathcal{B}_j\big)\Big)$*.*

3. Proof of Main Result

The proof is given for C_n^+. Recall Kuratowski's notion of lower and upper limits of sequences of sets.

Definition 1. Let X be a metric space and $(Z_l)_{l\in\mathbf{N}}$ be a sequence of subsets of X. $\limsup\limits_{l\to\infty} Z_l := \{x \in X : \liminf\limits_{l\to\infty} \operatorname{dist}(x, Z_l) = 0\}$ is called the upper limit of the sequence (Z_l), whereas

$$\liminf_{l\to\infty} Z_l := \{x \in X : \lim_{l\to\infty} \operatorname{dist}(x, Z_l) = 0\}$$

is called the lower limit of the sequence(Z_l).

Whyburn's result tells us that

Lemma 2. *$\limsup\limits_{l\to\infty} Z_l$ is nonempty, compact and connected provided that X is complete, $\liminf\limits_{l\to\infty} Z_l \neq \emptyset$ and $\bigcup\limits_{l\in\mathbb{N}} Z_l$ is relatively compact.*

Fixing $n \in \mathbb{Z}_+$ and $r \in (0,\infty)$ we are going to apply this lemma to $(C_{n,l}^+ \cap \mathcal{B}_r)_{l\in\mathbb{N}}$. Since $\varphi \mapsto -(k\varphi')' + (g_1+1)\varphi$ for $\varphi \in C^2([0,1])$ with $\varphi'(0) = 0 = \varphi'(1)$ has a completely continuous inverse from $C([0,1])$ into $C^1([0,1])$, the relative compactness of $\cup_{l\in\mathbb{N}}(C_{n,l}^+ \cap \mathcal{B}_r)$ in Y follows by standard arguments. Moreover, $(0,\mathbf{0}) \in \liminf\limits_{\mathbf{l}\to\infty}(\mathbf{C}_{\mathbf{n},\mathbf{l}}^+ \cap \mathcal{B}_{\mathbf{r}})$ holds because of $\lim_{l\to\infty} \mu_{n,l} = 0$. Thus, $C_n^+(r) := \limsup_{l\to\infty}(C_{n,l}^+ \cap \mathcal{B}_r)$ is nonempty, compact and connected in Y. Setting $C_n^+ := \bigcup_{j\in\mathbb{N}} C_n^+(j)$ we claim that C_n^+ has the properties as stated in the Theorem.

Lemma 3. *C_n^+ is unbounded, closed and connected in Y and contains $(0,\mathbf{0})$.*

The proof or the above lemma is an easy exercise!

Lemma 4. *If $(\mu,u) \in C_n^+$, then (μ,u) is a solution of (1) and $u \in W^{2,\infty}([0,1])$.*

Proof. By definition there exist sequences $(\kappa_l) \in \mathbb{N}^{\mathbb{N}}$ strictly increasing, and $\big((\nu_{\kappa_l}, v_{\kappa_l})\big) \in Y^{\mathbb{N}}$ with $(\nu_{\kappa_l}, v_{\kappa_l}) \in C_{n,\kappa_l}^+$ for $l \in \mathbb{N}$ and $(\nu_{\kappa_l}, v_{\kappa_l}) \to (\mu, u)$. Since $\Big(f_{\kappa_l}(\cdot, v_{\kappa_l}(\cdot))\Big)_{l\in\mathbb{N}}$ is uniformly bounded, we can assume after passing to a subsequence, if necessary, that it converges weakly in $L_2([0,1])$ to some ϕ. We claim that $\phi(x) \in F(x, u(x))$ a.e. on $(0,1)$.

Let $x_0 \in (0,1)$ with $u(x_0) > 0$. Then there exist $\rho > 0$ and $\delta \in (0, \min\{x_0, 1-x_0\})$ with $u(x) > \rho$ for all $x \in (x_0-\delta, x_0+\delta)$, hence there is an $l_0 \in \mathbb{N}$ with $v_{\kappa_l}(x) > \frac{\rho}{2}$ for all $l \geq l_0$ and $x \in (x_0-\delta, x_0+\delta)$. Choose $l_1 > l_0$ with $\frac{d_f}{\kappa_{l_1}} < \frac{\rho}{2}$. Then $f_{\kappa_l}(x, v_{\kappa_l}(x)) = f_+(x, v_{\kappa_l}(x))$ for all $l \geq l_1$ and all $x \in (x_0-\delta, x_0+\delta)$, which yields $\phi(x) = f_+(x, u(x))$ for $x \in (x_0-\delta, x_0+\delta)$ a.e.. Likewise, the case $u(x_0) < 0$ is treated.

Next, let $\Xi := \{x \in (0,1)\colon \phi(x) > f_+(x,0)\}$. Suppose that $\operatorname{meas}(\Xi) > 0$. Then $\varepsilon := \int_\Xi [\phi(x) - f_+(x,0)]dx > 0$, and one finds $\eta \in (0,\infty)$ with $\operatorname{meas}(\Xi)|f_+(x,y) -$

$f_+(x,0)| \le \frac{\varepsilon}{2}$ for $x \in [0,1]$ and $y \in [0,\eta]$. Choosing $l_2 \in \mathbb{N}$ with $\|v_{\kappa_l} - u\|_\infty < \eta$ for $l \ge l_2$ and observing that $f_{\kappa_l}(\cdot,y) < 0$ if $y < 0$, one obtains for $l \ge l_2$:

$$\begin{aligned}\int_\Xi [\phi(x) - f_{\kappa_l}(x, v_{\kappa_l}(x))]dx =& \int_\Xi [\phi(x) - f_+(x,0)]dx \\ &+ \int_\Xi [f_+(x,0) - f_{\kappa_l}(x, v_{\kappa_l}(x))]dx \\ \ge& \varepsilon + \int_\Xi [f_+(x,0) - f_{\kappa_l}(x, v^+_{\kappa_l}(x))]dx \\ \ge& \frac{\varepsilon}{2},\end{aligned}$$

which contradicts $f_{\kappa_l}(\cdot, v_{\kappa_l}(\cdot)) \overset{L_2}{\rightharpoonup} \phi$. Thus, $\mathrm{meas}(\Xi) = 0$.

Likewise, one derives $\phi(x) \ge f_-(x,0)$ for almost every $x \in [0,1]$ with $u(x) = 0$, hence $\phi(x) \in F(x, u(x))$ holds for $x \in [0,1]$ a.e..

Now, let A be the closed linear operator in $L_2([0,1])$ defined by $\mathrm{dom}(A) := \{\varphi \in W^{2,2}([0,1]) : \varphi'(0) = 0 = \varphi'(1)\}$ and $A\varphi := -(k\varphi')'$. Clearly, $\nu_{\kappa_l} f_{\kappa_l}(\cdot, v_{\kappa_l}(\cdot)) - g(\cdot, v_{\kappa_l}(\cdot) \overset{L_2}{\rightharpoonup} \mu\phi - g(\cdot, u(\cdot))$, hence $v_{\kappa_l} \to u$ and the fact that A is weakly closed yield $Au = \mu\phi - g(\cdot, u(\cdot))$, i.e. $Au + g(\cdot, u(\cdot)) \in F(\cdot, u(\cdot))$ a.e.. Finally, note that $u \in W^{2,\infty}([0,1])$ thanks to the uniform boundedness of u and assumptions (H2) and (H3).

Let $Z_u := \{x \in [0,1] : u(x) = 0\}$ denote the set of zeroes of u for $u \in C([0,1])$ and $SZ_u := \{x \in Z_u : u'(x) \neq 0\}$ the set of simple zeroes of u for $u \in C^1([0,1])$.

Lemma 5. *Let $(\mu, u) \in \mathcal{S}$, $u \not\equiv 0$ and $Z_u \setminus SZ_u \neq \emptyset$. Then SZ_u is an infinite set.*

Proof. Note that $u \not\equiv 0$, hence $\mu > 0$ by Remark 1. Let $\hat{z} \in Z_u \setminus SZ_u$ be a boundary point of Z_u. Consider the case $\hat{z} > 0$ and assume that there is an $\varepsilon \in (0, \hat{z})$ with $(\hat{z} - \varepsilon, \hat{z}) \cap Z_u = \emptyset$. Then either $Z_u \cap [0, \hat{z}) = \emptyset$, and one sets $\underline{z} = 0$ or there exists a $\underline{z} \in [0, \hat{z} - \varepsilon]$ with $u(\underline{z}) = 0$ and $(\underline{z}, \hat{z}) \cap Z_u = \emptyset$. Let $\sigma := \mathrm{sgn}(u|_{(\underline{z},\hat{z})})$, $v := \sigma u$ and $\bar{y} := \inf\{|y| : y \in \mathbb{R}, |g(x,y)| \ge d_f\}$, where d_f has the same meaning as in Section 2. It follows from (H2) and (H3) that $\bar{y} > 0$. Setting

$$\varsigma := \begin{cases} + & \text{if } \sigma = 1 \\ - & \text{if } \sigma = -1 \end{cases}$$

and $\rho := \|g(\cdot, u)\|_\infty / \bar{y}$, one obtains

$$\mu\sigma f_\varsigma(x, \sigma v(x)) - \sigma g(x, \sigma v(x)) + \rho v(x) \ge 0 \quad \text{for } x \in [\underline{z}, \hat{z}] \text{ a.e..}$$

Hence v satisfies $-(kv')' + \rho v \ge 0$ on $(\underline{z}, \hat{z})$ and either $v'(\underline{z}) = 0 = v(\hat{z})$ or $v(\underline{z}) = 0 = v(\hat{z})$. Moreover, the fact that $v(x) > 0$ for $x \in (\underline{z}, \hat{z})$ allows us to assume that $v \in C^2((\underline{z}, \hat{z}))$, thus the strong maximum principle yields $v'(\hat{z}) < 0$, a contradiction. Consequently, $\hat{z}$ is a accumulation point of $(0, \hat{z}) \cap Z_u$, and one finds a strictly increasing sequence $(z_j)_{j \in \mathbb{N}}$ such that $z_j \to \hat{z}$, $(z_{2j-1}, z_{2j}) \cap Z_u = \emptyset$ and $u(z_j) = 0$. The same argument as above shows $u'(z_j) \neq 0$, and SZ_u is an infinite set. The other cases are treated likewise.

Remark 2. Let $(\mu, u) \in C_n^{\pm}$. By construction, one finds a sequence $(v_j)_{j\in\mathbb{N}}$ with $Z_{v_j} = SZ_{v_j}$ for $j \in \mathbb{N}$, SZ_{v_j} contains n elements and $v_j \overset{C^1}{\to} u$, hence u cannot have more than n simple zeroes, consequently Lemma 5 implies $u \equiv 0$ in case that $Z_u \setminus SZ_u \neq \emptyset$.

The following lemma excludes this latter possibility if $\mu > 0$. In order to state it, we introduce the following notation. The principal eigenvalue of

$$-\|k\|_\infty \psi'' + \|g_1\|_\infty \psi = \nu\psi \quad \text{on } (0, \frac{1}{n+1}) \tag{5}$$
$$\psi(0) = 0 \quad \psi(\frac{1}{n+1}) = 0$$

will be denoted by $\nu^\star(n)$.

Lemma 6. *Let* (H1)–(H3) *and* (A1)–(A4) *be fulfilled,* $n \in \mathbb{Z}_+$, $\underline{\mu} \in (0,\infty)$ *and* $l_0 \in \mathbb{N}$ *with* $\underline{\mu} l_0 > \nu^\star(n)$. *Then there exists a* $\delta > 0$ *such that* $\|u\|_\infty \geq \delta$ *for all* $(\mu, u) \in C_{n,l}^+$ *with* $\mu \geq \underline{\mu}$ *and* $l \geq l_0$.

Proof. (cf. [2] for a similar reasoning) Let $\varepsilon \in (0, \frac{1}{2}(\underline{\mu} l_0 - \nu^\star(n)))$ and choose $\delta \in (0, \frac{d_f}{2l_0})]$ with $|g(x,y) - g_1(x)y| \leq \varepsilon|y|$ for all $x \in [0,1]$ and $|y| \leq \delta$. Let $l \in \mathbb{N}$ with $l \geq l_0$ and $(\mu, u) \in C_{n,l}^+$ with $\mu \geq \underline{\mu}$. Let us fix ideas by considering the case, where one can find $\underline{x}$, $\bar{x} \in (0,1)$ with $\bar{x} - \underline{x} \geq \frac{1}{n+1}$, $u(\underline{x}) = 0 = u(\bar{x})$ and $u(x) > 0$ for $x \in (\underline{x}, \bar{x})$.

Assume that $u(x) \leq \delta$ for $x \in (\underline{x}, \bar{x})$. Noting that (A1) and the first part of (A2) guarantee that $|f_l(x,y)| \geq l_0|y|$ for all $x \in [0,1]$, $|y| \leq \frac{d_f}{2l_0}$ and $l \geq l_0$, one obtains:

$$\begin{aligned} -(ku')'(x) + g_1(x)u(x) &= \mu f_l(x, u(x)) - \big(g(x,u(x)) - g_1(x)u(x)\big) \\ &\geq \underline{\mu} l_0 u(x) - \varepsilon u(x) \\ &> \frac{1}{2}(\underline{\mu} l_0 + \nu^\star(n))u(x) \\ &> \nu^\star(n)u(x) \end{aligned}$$

for $x \in (\underline{x}, \bar{x})$. The principal eigenvalue ν_0 of

$$-(k\psi')' + g_1(\cdot)\psi = \nu\psi \quad \text{on } (\underline{x}, \bar{x}) \tag{6}$$
$$\psi(\underline{x}) = 0 \quad \psi(\bar{x}) = 0\,.$$

satisfies $\nu_0 \leq \nu^\star(n)$ in view of $\bar{x} - \underline{x} \geq \frac{1}{n+1}$, $k \leq \|k\|_\infty$ and $g_1 \leq \|g_1\|_\infty$, hence

$$\begin{aligned} \nu_0 \int_{\underline{x}}^{\bar{x}} y(x)u(x)\,dx &= \int_{\underline{x}}^{\bar{x}} [-(ky')'(x) + g_1(x)y(x)]u(x)]\,dx \\ &= \int_{\underline{x}}^{\bar{x}} [-(ku')'(x) + g_1(x)u(x)]y(x)]\,dx \\ &> \nu_0 \int_{\underline{x}}^{\bar{x}} u(x)y(x)\,dx \end{aligned}$$

for any nonnegative principal eigenfunction y of (6) which is a contradiction, and $\|u\|_\infty > \delta$ is established in that case.

The case $u(x) < 0$ on $(\underline{x}, \bar{x})$ can be handled by dealing with $-u$ as above. If there is no such interval $(\underline{x}, \bar{x})$ of length at least $1/(n+1)$, one finds either an $\bar{x} \geq 1/(n+1)$ such that u has no zero in $(0, \bar{x})$ or an $\underline{x} \leq 1 - \frac{1}{n+1}$ such that u has no zero in $(\underline{x}, 1)$. Noting that the principal eigenvalue of (6) becomes smaller, if one replaces $\psi(\underline{x}) = 0$ by $\psi'(0) = 0$ or $\psi(\bar{x}) = 0$ by $\psi'(1) = 0$, one can use the same reasoning for these two cases, too.

Now, we can finish the proof of the theorem. It follows from Lemma 4 that $C_n^+ \subseteq \mathcal{S}$. Moreover, Lemma 6 shows that $(\mu, \mathbf{0}) \in C_n^+$ implies $\mu = 0$. Remark 2 therefore implies that given $(\mu, u) \in C_n^+ \setminus \{(0, \mathbf{0})\}$, u has only finitely many zeroes which are all simple. The construction of C_n^+ and the fact that the zero number is constant in a C^1-neighborhood of a C^1-function with finitely many simple zeroes imply therefore that u has exactly n simple zeroes.

4. Concluding Remark

We have focused in this paper on a class of Sturm-Liouville problems with some special features that were suggested by Budyko-North type energy balance climate models. Of course, one can also deal by this technique with other boundary conditions and nonlinearities and can allow nontrivial solutions for negative parameter values. Furthermore, it is possible to consider radial solvability of second order elliptic boundary value problems including degenerate cases similar to [2]; roughly speaking, the approach works whenever one has sufficient regularity of the solutions and nodal properties as in the Sturm-Liouville case. One challenge ahead is to come up with substitutes for this nodal properties (e.g. Morse indices etc.) in order to address general second order elliptic boundary value problems. The other central issue is to analyze the more complex situation arising from the climate model.

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