15th annual Conference of Applied Mathematics, Univ. of Central Oklahoma,
Electronic Journal of Differential Equations, Conference 02, 1999, pp 19–27.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Oscillation of the solution to a singular differential equation *

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Abstract

Let u be a solution to the initial-value problem

$$u''(t) + \tfrac{N-1}{t}u'(t) + u(t) + u(t)|u(t)|^{4/(N-2)} = 0, \quad t \in (0,T]$$
$$u(0) = \tfrac{1}{2}, \quad u'(0) = 0\,.$$

In this paper we show that if $N \leq 6$, then the distance between the two consecutive zeroes of u is "close" to π. The proof is based on an energy analysis and the Sturm comparison theorem.

1 Introduction

We consider the boundary-value problem

$$u''(t) + \tfrac{N-1}{t}u'(t) + u(t) + u(t)|u(t)|^{s} = 0\,, \quad t \in (0,T] \tag{1.1}$$
$$u(0) = d, \quad u'(0) = 0\,,$$

where $s = \frac{4}{N-2}$, and d and T are given real numbers. These initial-value problems arise in the study of radially symmetric solutions of elliptic boundary-value problems of the type

$$\Delta u(x) + u(x) + u(x)|u(x)|^{s} = 0, \quad x \in \Omega, \tag{1.2}$$
$$u(x) = 0, \quad x \in \partial\Omega$$

where Ω is a ball in $\mathbb{R}^N$ of radius T. Since the solutions of (1.1) that satisfy

$$u(T) = 0 \tag{1.3}$$

are radially symmetric solutions to (1.2) the study of (1.2) is reduced to the study of (1.1) subject to the boundary condition (1.3). Furthermore, the non-linearity involved in the equation has the so-called critical Sobolev exponent growth, which raises questions regarding the existence as well as multiplicity of

* *1991 Mathematics Subject Classifications:* 34B15, 34A10, 35J65.
Key words and phrases: critical exponent, singular equation, Sturm comparison theorem.

Published November 24, 1999.

solutions. The study of boundary value problems involving the critical Sobolev exponent $\frac{N+2}{N-2}$, where N is the dimension of the space, has been of great interest because a number of problems in geometry and physics, such as Yambe's problem lead to equations involving critical exponents. It is known that problem (1.2) has infinitely many radially symmetric solutions with arbitrary many nodal curves if $N > 7$ (see [7]). The case $N = 3, 4, 5, 6$ is still partially open. It has been proven (see [5]) that for $N = 3, 4$ there are finitely many solutions. Because of the unresolved problems for $N \leq 6$ we consider that case and concentrate on the analysis of the oscillation of solutions to (1.1). Arguments of the contraction mapping principle (see [8]) show that problem (1.1) has a unique solution $u(\cdot, d)$ on $[0, T]$. In this paper we concentrate on estimating the zeroes of u for the case $N \leq 6$, which generalizes the previously proven result for $N = 6$. We let $d = 1/2$.

Our main result is the following theorem:

Theorem 1.1 *If $u(\cdot, \frac{1}{2})$ is a solution to (1.1) and $N < 6$ then*

$$\begin{aligned} \pi - 0.035 < x_{i+1} - x_i < \pi + 0.065, &\quad \textit{for } i \geq 3, \\ \pi - 0.04 < x_{i+1} - x_i < \pi + 0.1585, &\quad \textit{for } i = 1, 2\,, \end{aligned}$$

where $x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ denote the zeroes of $u(\cdot, \frac{1}{2})$.

Similar estimates are proven for the case $N = 6$ in [6]. The proof of Theorem 1.1 is based on the energy analysis and the Sturm comparison theorem. The framework of the proof is the mathematical induction that requires a delicate energy analysis of the solution u at the first three zeroes. For the sake of clarity, we state some preliminary results obtained in [6] without proof.

2 Some Preliminary Results

By using the Pohozaev's identity in [6] we show that (see also [3, 4, 10]).

Lemma 2.1 *For $0 < t$ we have*

$$t^{N-1} H(t) = \int_0^t r^{N-1} u^2(r)\, dr\,,$$

where

$$\begin{aligned} H(t) &= t\left(\frac{(u'(t))^2}{2} + \frac{|u(t)|^{s+2}}{s+2} + \frac{|u(t)|^2}{2}\right) + \frac{N-2}{2} u(t)u'(t) \\ &= tE(t) + \frac{N-2}{2} u(t) u'(t)\,. \end{aligned} \tag{2.1}$$

The following lemma gives an inequality that is crucial in analyzing energy of a solution at any two consecutive zeroes.

Lemma 2.2 *Let x_i and x_{i+1} denote two consecutive zeroes of $u(\cdot, \frac{1}{2})$. Then*

$$E(x_{i+1}) \leq \frac{E(x_i)}{N} \left\{ (N-1) \left(\frac{x_i}{x_{i+1}}\right)^N + 1 \right\}.$$

3 Analysis at the First Three Zeroes of $u(\cdot, \frac{1}{2})$

The analysis and the proof in the remainder of this paper will be conducted for the case $N = 3, 4, 5$. In this section we estimate the location of the first three zeroes of $u(\cdot, \frac{1}{2})$ numerically using Maple and obtain,
If $N = 3$ then

$$3.1 < x_1 < 3.2, \quad 6.2 < x_2 < 6.3, \quad 9.4 < x_3 < 9.5 \quad 12 < x_4. \tag{3.1}$$

If $N = 4$ then

$$3.6 < x_1 < 3.7, \quad 6.8 < x_2 < 6.9, \quad 10.0 < x_3 < 10.1\,. \tag{3.2}$$

If $N = 5$ then

$$4.2 < x_1 < 4.3, \quad 7.4 < x_2 < 7.5, \quad 10.6 < x_3 < 10.7\,. \tag{3.3}$$

Next, we conduct the energy analysis of u at the first three zeroes and prove the following lemma.

Lemma 3.1 *If x_1, x_2, x_3 denote the first zero of $u(\cdot, \frac{1}{2})$, then we have*

$$\begin{gathered}
|u'(x_1)| \le .3, \quad |u'(x_2)| \le .2, \quad |u'(x_3)| \le .15, \quad \textit{for } N = 3\,,\\
|u'(x_1)| \le .233, \quad |u'(x_2)| \le .131, \quad |u'(x_3)| \le .085, \quad \textit{for } N = 4\,,\\
|u'(x_1)| \le .188, \quad |u'(x_2)| \le .095, \quad |u'(x_3)| \le .056, \quad \textit{for } N = 5\,.
\end{gathered}$$

Proof. From

$$-u'(t) = t^{-(N-1)} \int_0^t r^{N-1}(1 + |u(r)|^s)u(r)\,dr \tag{3.4}$$

on $[0, x_1]$ we see that $u' < 0$ and

$$-u'(t) \ge \frac{t}{N} u(t). \tag{3.5}$$

Hence, integrating (3.5) on $[0, t]$ we obtain

$$u(t) \le \frac{1}{2} e^{-\frac{t^2}{2N}}, \tag{3.6}$$

where we have also used the fact that $u(0) = 1/2$.

Let $N = 3$. From (3.6) we see that $u(2) \le\ .26$. We estimate $E(x_1)$ by using Lemma 2.1 and infer

$$\begin{aligned}
{x_1}^3 E(x_1) &= \int_0^2 r^2 u^2(r) dr + \int_2^{x_1} r^2 u^2(r) dr \\
&\le \frac{2^3}{3 \cdot 2^2} + (0.26)^2 \left(\frac{x_1^3}{3} - \frac{8}{3} \right).
\end{aligned} \tag{3.7}$$

Since $E(x_1) = (u'(x_1))^2/2$, from (3.1) and (3.7) it follows that

$$|u'(x_1)| \leq 0.3\,, \tag{3.8}$$

On the other hand, from Lemma 2.2, (3.1) and (3.8) we have

$$\begin{aligned} (u'(x_2))^2 &\leq \frac{(u'(x_1))^2}{3}\left(2\left(\frac{x_1}{x_2}\right)^3 + 1\right) \\ &\leq \frac{(.3)^2}{3}\left(2\left(\frac{3.2}{6.2}\right)^3 + 1\right). \end{aligned} \tag{3.9}$$

Thus

$$|u'(x_2)| \leq .2. \tag{3.10}$$

Reiterating the argument we obtain

$$\begin{aligned} (u'(x_3))^2 &\leq \frac{(u'(x_2))^2}{3}\left(2\left(\frac{x_2}{x_3}\right)^3 + 1\right) \\ &\leq \frac{(.2)^2}{3}\left(2\left(\frac{6.3}{9.4}\right)^3 + 1\right). \end{aligned} \tag{3.11}$$

Hence

$$|u'(x_3)| \leq .15\,, \tag{3.12}$$

which concludes the proof for case $N = 3$.

If $N = 4$ then from (3.6) we see that $u(2.5) \leq .23$. Therefore, arguing as in (3.7) we obtain

$$x_1{}^4 E(x_1) \quad \leq \quad \frac{2.5^4}{4 \cdot 2^2} + (0.23)^2\left(\frac{(x_1)^4}{4} - \frac{(2.5)^4}{4}\right). \tag{3.13}$$

Thus,

$$|u'(x_1)| \leq .233\,. \tag{3.14}$$

Using Lemma 2.2 and (3.2) from (3.14) we infer

$$\begin{aligned} (u'(x_2))^2 &\leq \frac{(u'(x_1))^2}{4}\left(3\left(\frac{x_1}{x_2}\right)^4 + 1\right) \\ &\leq \frac{(.233)^2}{4}\left(3\left(\frac{3.7}{6.9}\right)^4 + 1\right). \end{aligned} \tag{3.15}$$

Hence,

$$|u'(x_2)| \leq .131\,. \tag{3.16}$$

Reiterating the argument in (3.15) and using the estimates in (3.2) for x_2 and x_3 we obtain

$$|u'(x_3)| \leq .085\,, \tag{3.17}$$

which completes the proof for case $N = 4$.

If $N = 5$ then from (3.6) we see that $u(3) \leq .204$. Therefore,

$$x_1{}^5 E(x_1) \quad \leq \quad \frac{3^5}{5 \cdot 2^2} + (0.204)^2 \left(\frac{(x_1)^5}{5} - \frac{3^5}{5} \right). \tag{3.18}$$

Thus, using the estimate for x_1 in (3.3) we obtain

$$|u'(x_1)| \leq .188\,. \tag{3.19}$$

Furthermore,

$$\begin{aligned} (u'(x_2))^2 \quad &\leq \quad \frac{(u'(x_1))^2}{5} \left(4 \left(\frac{x_1}{x_2} \right)^5 + 1 \right) \\ &\leq \quad \frac{(.188)^2}{5} \left(4 \left(\frac{4.3}{7.4} \right)^5 + 1 \right). \end{aligned} \tag{3.20}$$

Hence,

$$|u'(x_2)| \leq .095\,. \tag{3.21}$$

Reiterating the argument in (3.20) and using (3.3) for x_2 and x_3 we infer

$$|u'(x_3)| \leq .056\,, \tag{3.22}$$

which completes the proof for case $N = 5$.

From Lemma 3.1 and (3.1)-(3.3) we obtain the following results:

$$|u'(x_3)| \quad < \quad \frac{1.5}{x_3^{3/4}} \quad \text{for } N = 3, \tag{3.23}$$

$$|u'(x_3)| \quad < \quad \frac{.9}{x_3} \quad \text{for } N = 4, \tag{3.24}$$

$$|u'(x_3)| \quad < \quad \frac{.6}{x_3} \quad \text{for } N = 5. \tag{3.25}$$

4 Zeroes of u on (x_3, ∞)

To prove the ,main theorem we first show:

Lemma 4.1 *If $N = 3, 4, 5$, and x_i and x_{i+1} are two consecutive zeroes of $u(\cdot, \frac{1}{2})$ on (x_3, ∞), then $x_{i+1} - x_i \leq 3.38$. Moreover,*

$$\frac{x_i}{x_{i+1}} \geq .7$$

Proof. Let θ be a differentiable function such that (see [5])

$$\begin{aligned} u(t) &= -\rho(t)\cos\theta(t) \\ u'(t) &= \rho(t)\sin\theta(t) \\ \theta(0) &= 0, \end{aligned} \tag{4.1}$$

with $\rho(t) = \sqrt{u^2(t) + (u'(t))^2}$. An elementary calculation shows (see [6]) that

$$\theta'(t) = \frac{(u'(t))^2 + \frac{N-1}{t}u(t)u'(t) + (1+|u(t)|^s)u^2(t)}{(u'(t))^2 + u^2(t)}. \tag{4.2}$$

Without loss of generality we can assume that $u'(x_i) > 0$. Let $\bar{x}_i \in (x_i, x_{i+1})$, $i > 3$, be such that $u'(\bar{x}_i) = 0$. Then, from (4.2) we see that $\theta' \geq 1$ on $[x_i, \bar{x}_i]$. Thus

$$\bar{x}_i - x_i \leq \frac{\pi}{2}. \tag{4.3}$$

On the other hand on $[\bar{x}_i, x_{i+1}]$, using the fact that $x_i \geq 10$ for $i > 3$ from (4.2) we infer

$$\theta'(t) \geq 1 - \frac{N-1}{2t}\sin 2\theta \geq 1 - \frac{N-1}{20}\sin 2\theta. \tag{4.4}$$

Hence, by integrating (4.4) we obtain

$$\begin{aligned} x_{i+1} - \bar{x}_i &\leq \int_{\pi/2}^{\pi} \frac{d\theta}{1 - \frac{N-1}{20}\sin 2\theta} \\ &= \frac{1}{\sqrt{1 - \left(\frac{N-1}{20}\right)^2}} \left(\arctan \frac{\frac{N-1}{20}}{\sqrt{1 - \left(\frac{N-1}{20}\right)^2}} + \frac{\pi}{2} \right) \\ &\leq 1.8087\,. \end{aligned} \tag{4.5}$$

Therefore, by combining (4.3) with (4.5) we have

$$x_{i+1} - x_i \leq 3.38\,.$$

Furthermore,

$$\frac{x_i}{x_{i+1}} \geq \frac{x_i}{x_i + 3.455} \geq .7\,,$$

where we have used the fact that $x_i \geq 10$ for $i > 3$. Thus, the lemma is proven.

Lemma 4.2 *Let $x_3 < x_4 < \ldots.$ denote the zeroes of $u(\cdot, \frac{1}{2})$.*
If $|u'(x_3)| \leq \frac{M}{x_3}$ for $N = 4, 5$ then

$$|u'(x_i)| \leq \frac{M}{x_i}. \tag{4.6}$$

Furthermore, if $|u'(x_3)| \leq \frac{M}{x_3^{3/4}}$ for $N = 3$ then

$$|u'(x_i)| \leq \frac{M}{x_i^{3/4}}. \tag{4.7}$$

Proof. We prove the lemma by induction. As shown in section 3 (see (3.23)-(3.25)), Lemma 4.2 holds for $i = 3$. Suppose $|u'(x_i)| \leq \frac{M}{x_i}$ for $i > 3$. From Lemmas 2.2 and 4.1 we have

$$\frac{N-1}{N}\left(\frac{x_i}{x_{i+1}}\right)^N + \frac{1}{N} \leq \frac{x_i^2}{(x_{i+1})^2}, \tag{4.8}$$

because $\left(\frac{x_i}{x_{i+1}}\right)^2 \geq .49$ for $N = 4, 5$. Hence, we obtain $|u'(x_{i+1})| \leq \frac{M}{x_{i+1}}$.

For $N = 3$ from Lemmas 2.2 and 4.2 we see that the following holds

$$\frac{2}{3}\left(\frac{x_i}{x_{i+1}}\right)^3 + \frac{1}{3} \leq \frac{x_i^{3/2}}{(x_{i+1})^{3/2}}. \tag{4.9}$$

This concludes the proof of the Lemma 4.2.

5 Proof of Theorem 1.1

Now, we proceed with the proof of the theorem, by estimating the location of zeroes x_i, with $i > 3$. If $N = 4, 5$ we let $w(t) := t^{\frac{N-1}{2}} u(t)$. It can easily be shown that w satisfies

$$w''(t) + \left(1 - \frac{(N-1)(N-3)}{4t^2} + |u(t)|^s\right) w(t) = 0. \tag{5.1}$$

Since for $t \in [x_i, x_{i+1}]$ we have

$$1 - \frac{(N-1)(N-3)}{4t^2} + |u(t)|^s \geq 1 - \frac{(N-1)(N-3)}{4x_i^2},$$

by the Sturm comparison theorem it follows that

$$\begin{aligned} x_{i+1} &\leq x_i + \frac{\pi}{\sqrt{1 - \frac{(N-1)(N-3)}{4x_i^2}}} \leq x_i + \frac{\pi}{1 - \frac{(N-1)(N-3)}{4x_i^2}} \\ &= x_i + \pi\left(1 + \frac{(N-1)(N-3)}{4x_i^2} + \left(\frac{(N-1)(N-3)}{4x_i^2}\right)^2 + \cdots\right) \\ &= x_i + \pi + \frac{(N-1)(N-3)\pi}{4x_i^2} \cdot \frac{1}{1 - \frac{(N-1)(N-3)}{4x_i^2}}. \end{aligned} \tag{5.2}$$

Since $x_i \geq 10$ for $i > 3$ we obtain

$$x_{i+1} \leq x_i + \pi + 0.065. \tag{5.3}$$

On the other hand using Lemma 4.2 we see that

$$\begin{aligned} 1 - \frac{(N-1)(N-3)}{4t^2} + |u(t)|^s &\leq 1 - \frac{(N-1)(N-3)}{4x_{i+1}^2} + \left(\frac{M}{x_i}\right)^s \\ &\leq 1 + \left(\frac{M}{x_i}\right)^s. \end{aligned} \tag{5.4}$$

Hence, from the Sturm comparison theorem it follows that

$$\begin{aligned} x_{i+1} &\geq x_i + \frac{\pi}{\sqrt{1+(\frac{M}{x_i})^s}} \geq x_i + \frac{\pi}{1+\frac{1}{2}(\frac{M}{x_i})^s} \\ &= x_i + \pi\left(1 - \frac{1}{2}(\frac{M}{x_i})^s + \left(\frac{1}{2}(\frac{M}{x_i})^s\right)^2 - \cdots\right) \\ &= x_i + \pi - \frac{1}{2}\pi(\frac{M}{x_i})^s \cdot \frac{1}{1+\frac{1}{2}(\frac{M}{x_i})^s} \\ &\geq x_i + \pi - 0.035, \end{aligned} \tag{5.5}$$

where we have used the fact that $x_i > x_3$, (3.24) and (3.25).

If $N = 3$ we let $w(t) := tu(t)$. It can easily be shown that w satisfies

$$w''(t) + \left(1 + |u(t)|^4\right) w(t) = 0. \tag{5.6}$$

Since for $t \in [x_i, x_{i+1}]$ we have

$$1 + |u(t)|^4 \geq 1,$$

by the Sturm comparison theorem it follows that

$$x_{i+1} \leq x_i + \pi. \tag{5.7}$$

On the other hand using Lemma 4.2 and the fact that $s = 4$ for $N = 3$ we see that

$$\begin{aligned} 1 + |u(t)|^4 &\leq 1 + \left(\frac{M}{x_i^{3/4}}\right)^4 \\ &\leq 1 + \frac{M^4}{x_i^3}. \end{aligned} \tag{5.8}$$

Hence, from the Sturm comparison theorem it follows that

$$\begin{aligned} x_{i+1} &\geq x_i + \frac{\pi}{\sqrt{1+\frac{M^4}{x_i^3}}} \geq x_i + \frac{\pi}{1+\frac{1}{2}\frac{M^4}{x_i^3}} \\ &= x_i + \pi\left(1 - \frac{1}{2}\frac{M^4}{x_i^3} + \left(\frac{1}{2}\frac{M^4}{x_i^3}\right)^2 - \cdots\right) \\ &= x_i + \pi - \frac{1}{2}\pi\frac{M^4}{x_i^3} \cdot \frac{1}{1+\frac{1}{2}\frac{M^4}{x_i^3}} \\ &\geq x_i + \pi - 0.011\,, \end{aligned} \tag{5.9}$$

where we have used the fact that $x_3 > 9$ and $M = 1.5$. By combining (5.3), (5.5), (5.7), (5.9) with (3.1)-(3.3) we conclude the proof of Theorem 1.1.

References

[1] F. Atkinson, H. Brezis, L. Peletier, *Solutions d'equations elliptiques avec exposant de Sobolev critique qui changent de signe,* C. R. Acad. Sci. Paris, t. 306, Serie I (1988), 711-714.

[2] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents,* Comm. Pure Appl. Math. 36(1983), 437-477.

[3] A. Castro, A. Kurepa, *Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball,* Proc. Amer. Math. Soc. 101(1987), 56-64.

[4] ———, *Radially symmetric solutions to a superlinear Dirichlet problem in a ball with jumping nonlinearities,* Trans. Amer. Math. Soc. 315(1989), 353-372.

[5] ———, *Radially symmetric solutions to a Dirichlet problem involving critical exponents* , Trans. Amer. Math. Soc. 343(1994), 907-926.

[6] ———, *Radially symmetric solutions to a Dirichlet problem involving critical exponents when* $N = 6$, Trans. Amer. Math. Soc. 348(1996), 781-798.

[7] G. Cerami, S. Solimini, M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents,* J. Functional Anal. 69(1986), 289-306.

[8] A. Kurepa, *Existence and uniqueness theorem for singular initial value problems and applications",* Publications de L'Institut Mathmatique 45 (1990), 89-93.

[9] ———, *"Estimating Zeroes to a Bessel-like Equation",* Proc. 12-th annual Conference of Applied Mathematics, University of Central Oklahoma (1996), 85-98.

[10] P. Pucci, J. Serrin, *A general variational identity,* Indiana Univ. Math. J.35(1986), 681-703.

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