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The spin-statistics connection: Some pedagogical remarks in response to Neuenschwander's question *

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In 1994, D.E. Neuenschwander posed a question (American Journal of Physics **62** (1994) 972): "Has anyone made any progress toward an 'elementary' argument for the spin-statistics theorem?" The theorem in question was proved in 1940 by W. Pauli and says that particles of integer spin cannot be described by fields that anti-commute at space-like separated points, i.e., cannot be fermions, while half-odd-integer spin particles cannot be described by fields that commute at space-like separated points, i.e., cannot be bosons.

It seems to be characteristic of many of the attempts at an elementary proof that the authors want to use only quantum mechanics and the transformation law of the wave function under the Euclidean group or, equivalently, under translations and rotations of space. My first point is that under these hypotheses there is no Spin-Statistics Connection. To prove this statement, pick a positive real number, s, that is either an integer or half an odd integer and construct operators $\Psi(\vec{x}, M), M = -s, -s+1, \dots (s-1), s$, satisfying

$$V(\vec{a}, A)\Psi(x, M)V(\vec{a}, A)^{-1} = \sum_L \mathcal{D}^{(s)}(A^{-1})_{ML}\Psi(R(A)\vec{x} + \vec{a}, L) \tag{1}$$

where $\{\vec{a}, A\} \to V(\vec{a}, A)$ is the unitary representation of the Euclidean group giving the transformation law of the wave function. $R(A)$ is the rotation determined by A and $A \to \mathcal{D}^{(s)}(A)$ is the spin s representation of $SU(2)$. Using the formalism of second quantization, one can make a theory of fermions by requiring

$$\begin{aligned} [\Psi(\vec{x}, M), \Psi(\vec{y}, N)]_+ &= 0 \\ [\Psi(\vec{x}, M), \Psi^*(\vec{y}, N)]_+ &= \delta_{MN}\delta(\vec{x} - \vec{y}) \end{aligned} \tag{2}$$

or, alternatively, a theory of bosons by requiring

$$\begin{aligned} [\Psi(\vec{x}, M), \Psi(\vec{y}, N)]_- &= 0 \\ [\Psi(\vec{x}, M), \Psi^*(\vec{y}, N)]_- &= \delta_{MN}\delta(\vec{x} - \vec{y}). \end{aligned} \tag{3}$$

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The Fock space construction required to realize (1) and (2) or (1) and (3) are easy generalizations of what one finds for spin $\frac{1}{2}$ in the text books of condensed matter physics. (See, for example, Fetter and Walecka, *Quantum Theory of Many Particle Systems*, McGraw-Hill, New York 1971.) It is clear here that spin does not determine statistics and that any positive response to Neuenschwander's question must make assumptions that go beyond Euclidean invariance. For example, in 1940 Pauli assumed Lorentz invariance and also that he was dealing with a quantum field theory of non-interacting particles. Under these hypotheses, he proved the Spin-Statistics Theorem.

It took more than twenty years before the extension of Pauli's result to interacting fields saw the light of day in the work of Lüders and Zumino and of Burgoyne. The extension exploited then recent developments in the general theory of quantized fields. This brings me to my second point which is a response to the question: How does invariance under Lorentz transformations bind statistics to spin, and, in particular, what properties of the Lorentz group are essential?

It is here that I advocate ignoring Henry David Thoreau's advice (Walden Chapter II) "Simplify, simplify!" Instead:

complexify, complexify!

I illustrate the application of this slogan with Lorentz transformations. They are defined as 4×4 matrices satisfying

$$\Lambda^T G \Lambda = G$$

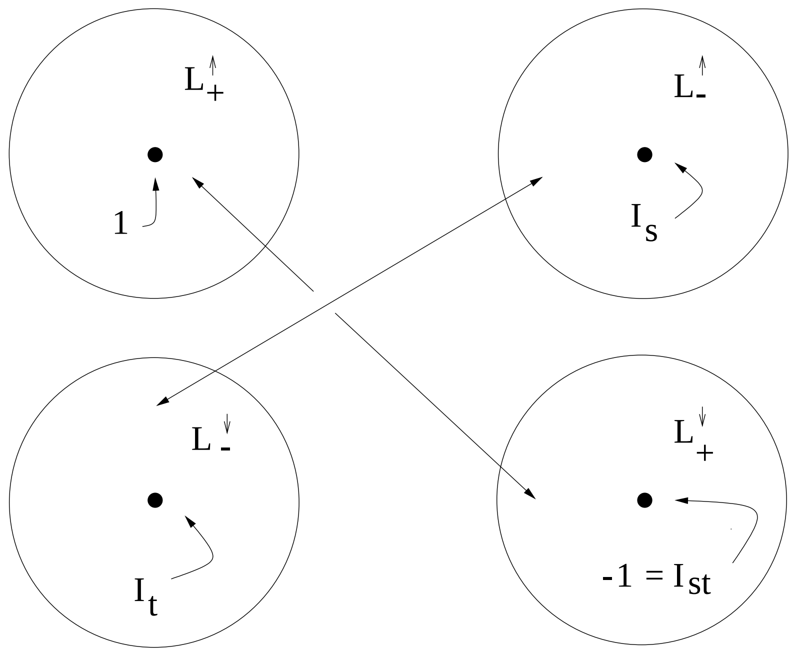
where

$$G = \left\{ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\}.$$

For the direct application to physics, Λ is real, but one can also consider complex Λ satisfying the definition. For both cases $(\det \Lambda)^2 = 1$, so $\det \lambda = \pm 1$. If $\det \Lambda = +1$, we call Λ a *proper* Lorentz transformation; if $\det \Lambda = -1$, *improper*. For real Lorentz transformations, there is another distinction according to the sign of Λ^o_o, sgn $\Lambda^o{}_o$. If sgn $\Lambda^o_o = +1$, we say that Λ is *orthochronous*; if sgn $\Lambda^o{}_o = -1$, that Λ inverts time.

The real Lorentz group is composed of four components, $L_+^\uparrow$, $L_+^\downarrow$, $L_-^\uparrow$, and $L_-^\downarrow$, disconnected from each other:
The arrow superscript points up for orthochronous transformations and down for those inverting time. The subscript is + for proper transformations and - for improper. The particular Lorentz transformations I_s, I_t, and I_{st} are defined as follows:

$$\text{Space inversion } I_s : \{x^o, \vec{x}\} \to \{x^o, -\vec{x}\}$$
$$\text{Time inversion } I_t : \{x^o, \vec{x}\} \to \{-x^o, \vec{x}\}$$
$$\text{Space-time inversion } I_{st} : \{x^o, \vec{x}\} \to \{-x^o, -\vec{x}\}$$



On the other hand, the Complex Lorentz group, which is denoted $L(\mathbf{C})$, has two connected components, the proper complex Lorentz transformations $L_+(\mathbf{C})$ and the improper $L_-(\mathbf{C})$. One can pass from $L_+^\uparrow$ to $L_+^\downarrow$ through $L_+(\mathbf{C})$. In particular the curve of complex Lorentz transformations

$$\Lambda_t = \left\{ \begin{matrix} \cosh it & 0 & 0 & \sinh it \\ 0 & cost & -\sin t & 0 \\ 0 & \sin t & \cos t & 0 \\ \sinh it & 0 & 0 & \cosh it \end{matrix} \right\} \quad , \qquad 0 \le t \le \pi$$

connects $\mathbf{1}= \Lambda_o$ to $\mathbf{-1}= I_{st} = \Lambda_\pi$. Similarly one can pass from $L_-^\uparrow$ to $L_-^\downarrow$ via $L_-(\mathbf{C})$. The existence of these connections is indicated by the diagonal arrows in the diagram. How can this phenomenon possibly have anything to do with physics?

The answer is that in the general theory of quantized fields, there is a sequence of functions (or better, generalized functions) associated with the fields, the vacuum expectation values of monomials in the fields. For example, to a scalar field ϕ, there are associated the vacuum expectation values

$$F_n(x_1, \dots x_n) = (\Psi_o, \phi(x_1) \dots \phi(x_n)\Psi_o).$$

If we take the transformation law of the states under the restricted Poincaré group as the unitary representation: $\{a, \Lambda\} \to U(a, \Lambda)$ (a is a translation of space-time, Λ a Lorentz transformation in $L_+^\uparrow$), then the scalar field has a transformation law

$$U(a,\Lambda)\phi(x)U(a,\Lambda)^{-1} = \phi(\Lambda x + a),$$

and in view of the invariance

$$U(a,\Lambda)\Psi_o = \Psi_o$$

of the vacuum state Ψ_o, we have

$$F_n(\Lambda x_1 + a, \Lambda x_2 + a, \dots \Lambda x_n + a) = F_n(x_1, \dots x_n)\ .$$

This relation implies that F_n depends only on the difference variables

$$\xi_1 = x_2 - x_1,\ \xi_2 = x_3 - x_2,\ \dots,\ \xi_{n-1} = x_n - x_{n-1}$$

and so, in a change of notation, we write

$$F_n(\Lambda\xi_1, \dots \Lambda\xi_{n-1}) = F_n(\xi_1, \dots \xi_{n-1}).$$

The next step in this line of argument is to recognize that the spectral condition, i.e., the physical requirement that the energy momentum vector of the state of the system lies in or on the cone $\overline{V}_+$, implies that F_n is the Fourier transform of a distribution G_n which vanishes unless each of its $n-1$ arguments lies in $\overline{V}_+$. In formulae,

$$F_n(\xi_1, \dots \xi_{n-1}) = \int d^4p, \dots d^4p_{n-1} \exp i \sum_{j=1}^{n-1} p_j \cdot \xi_j\ G_n(p_1, \dots p_{n-1}).$$

Here $G_n(p_1, \dots p_{n-1}) = 0$ unless $p_1, \dots p_{n-1}$ all lie in $\overline{V}_+$, where

$$\overline{V}_+ = \{p; p \cdot p = (p^o)^2 - \vec{p}^2 \geq 0,\ p^o \geq 0\}.$$

Because of the support properties of G_n, this formula can be extended from a Fourier transform to a Laplace transform,

$$\begin{aligned} &F_n(\xi_1 + i\eta_1, \dots \xi_{n-1} + i\eta_{n-1}) \\ &\quad = \int d^4p_1 \dots d^4p_{n-1} \exp i \sum_{j=1}^{n-1} p_j \cdot (\xi_j + i\eta_j) G_n(p_1, \dots p_{n-1}), \end{aligned}$$

and so extended defines an analytic function for all $\xi_1 + i\eta_1, \dots \xi_{n-1} + i\eta_{n-1}$ lying in the tube

$$\xi_j \in \mathbb{R}^4,\ j = 1, \dots n-1 \ \text{ and } \ \eta_j \in V_+,\ \ j = 1, \dots n-1$$

(V_+ is the interior of $\overline{V}_+$). So extended, F_n remains Lorentz invariant:

$$F_n(\Lambda\zeta_1, \Lambda\zeta_2, \dots \Lambda\zeta_{n-1}) = F_n(\zeta_1, \dots \zeta_{n-1}) \ \text{ for all } \ \Lambda \in L_+^\uparrow \tag{4}$$

where $\zeta_1 = \xi_1 + i\eta_1, \dots \zeta_{n-1} = \xi_{n-1} + i\eta_{n-1}$ is any point of the tube.

For a fixed point $\zeta_1 \ldots \zeta_{n-1}$ of the tube, the left-hand side of this equation (4) is an analytic function of Λ. This suggests that F_n can be extended by analytic continuation to points of the form $\Lambda\zeta_1, \ldots \Lambda\zeta_{n-1}$, where Λ is an arbitrary proper complex Lorentz transformation; the set of all such points form the *extended tube*. The BHW Theorem (1957) says that, so extended, F_n is analytic and single-valued in its argument $\zeta_1 \ldots \zeta_{n-1}$ and is invariant under $L_+(\mathbf{C})$, i.e., equation (4) is valid for all $\Lambda \in L_+(\mathbf{C})$.

It is an important and perhaps somewhat surprising property of the extended tube that it contains real points; they are called Jost points because R. Jost first characterized them as follows.

Theorem $\xi_1, \ldots \xi_{n-1}$ is a real point of the extended tube if and only if for every sequence $\lambda_1, \ldots \lambda_{n-1}$ of non-negative real numbers satisfying $\sum_{j=1}^{n-1} \lambda_j > 0$, $\sum_{j=1}^{n-1} \lambda_j \xi_j$ is space-like.

An immediate corollary of this theorem of Jost is the existence of an open set in the real vector variables $x_1, \ldots x_n$ wherein the vacuum expectation value $(\Psi_o, \phi(x_1) \ldots \phi(x_n)\Psi_o)$ is analytic in the $x_1 \ldots x_n$. Furthermore, at such Jost points, application of (4) and its analogues for vacuum expectation values of monomials in the components of tensor or spinor fields or their adjoints yield identities at the core of the "modern" proofs of CPT Symmetry (Jost 1957) and the Spin-Statistics Theorem (Luders and Zumino 1958, Burgoyne 1958). I want to emphasize that no assumption of invariance under $I_{st} = -\,\mathbf{1}$ has been made in these arguments.

My third remark involves a little more detail from Burgoyne's proof, substantiating the remarks I have just made. I treat first the case of a scalar field, ϕ, for which Burgoyne wants to show that

$$[\phi(x), \phi^*(y)]_+ = 0 \quad \text{for} \quad (x-y)^2 < 0$$

leads to the conclusion that ϕ is zero. Take the vacuum expectation value of this relation to obtain

$$(\Psi_o, \phi(x)\phi^*(y)\Psi_o) + (\Psi_o, \phi^*(y)\phi(x)\Psi_o) = 0 \quad \text{for } (x-y)^2 < 0. \qquad (5)$$

Each of the two terms is the boundary value of a function analytic in the *tube*:

$$\begin{aligned} (\Psi_o, \phi(x)\phi^*(y)\Psi_o) &= \lim_{\substack{\eta\to 0\\ \eta\in V_+}} F_{\phi\phi^*}\ (y - x + i\eta) \\ (\Psi_o, \phi^*(y)\phi(x)\Psi_o) &= \lim_{\substack{\eta\to 0\\ \eta\in V_+}} F_{\phi^*\phi}\ (x - y + i\eta) \end{aligned}$$

Both $F_{\phi\phi^*}$ and $F_{\phi^*\phi}$ are invariant under $L_+^\uparrow$ and therefore by the previous argument analytic in the extended tube and invariant under $L_+(\mathbf{C})$. The real points of the extended tube are here all pairs x, y for which $x - y$ is space-like, so the relation (5) implies that

$$F_{\phi\phi^*}(\zeta) + F_{\phi^*\phi}(-\zeta) = 0 \qquad (6)$$

throughout the extended tube, or, using the invariance of $F_{\phi^*\phi}$ under $\Lambda = -\mathbf{1} = I_{st}$,

$$F_{\phi\phi^*}(\zeta) + F_{\phi^*\phi}(\zeta) = 0. \tag{7}$$

If we pass to the limit $\eta \to 0$ with $\eta \in V_+$ in this relation, we have for all x and y the relation

$$(\Psi_o, \phi(x)\phi^*(y)\Psi_o) + (\Psi_o, \phi^*(-y)\phi(-x)\Psi_o) = 0$$

between distributions. Smearing in x with the test function $f(x)$ and in y with $\overline{f(y)}$, we get for the first term $||\phi(f)^*\Psi_o||^2$ and for the second $||\phi(\hat{f})\Psi_o||^2 = 0$ for all test functions f. Here $\hat{f}(x) = f(-x)$. So $\phi(f)^*$ and $\phi(f)$ both annihilate the vacuum. To conclude from these results that ϕ and ϕ^* are actually zero requires some knowledge of the relations of ϕ and ϕ^* with the rest of the fields of the field theory; I will not say more about that. However, I will add that the extension of the above discussion to treat the case of arbitrary spinor and tensor fields basically only involves the replacement of (4), which expresses invariance under $L_+(\mathbf{C})$, by a transformation law involving the representation which expresses the nature of the tensors or spinors under the action of $L_+(\mathbf{C})$. Explicitly, for ϕ a component of a tensor field, the argument goes just as in the scalar case, because the tensor transformation law reduces to (6) and (7). On the other hand, for ϕ a component of a spinor field, Burgoyne undertakes to show that the vanishing of the commutator of $\phi(x)$ and $\phi^*(y)$ for space-like $(x-y)$ implies

$$\|\phi(f)^*\Psi_o\|^2 \ = \ \|\phi(\hat{f})\Psi_o\|^2 = 0.$$

He has to overcome the apparent difficulty that here (6) is replaced by

$$F_{\phi\phi^*}(\zeta) - F_{\phi^*\phi}(-\zeta) = 0$$

throughout the extended tube. But now the spinor character of ϕ implies that

$$F_{\phi^*\psi}(-\zeta) = -F_{\phi^*\phi}(\zeta).$$

The minus signs compensate and the rest of the proof goes the same way as for a scalar or tensor field.

Finally, I should mention that every result that I have talked about presupposes a space of states that is a Hilbert space with positive metric; Faddeev-Popov ghosts are thereby excluded; they violate the spin-statistics connection.

Discussion

E. Wichmann: How is your assertion about the non-existence of a spin-statistics connection in Euclidean invariant theory related to the paper of Michael Berry (Proc. Roy. Soc. Lond. A **453** (1997) 1771-1790) in which under some assumptions he outlined a proof. What are those assumptions?

A. Wightman: I have not seen the paper of Michael Berry.
(with some delay) I want to thank Dave Jackson for providing me with a photocopy of Berry's paper and the organizers for accepting a delayed response to Eyvind's question.

Berry bases his proof on the assumption of the existence of what he calls a transportable spin-basis. He constructs such a basis for the special case of two particles, $N = 2$, but leaves open the case of general N.

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