A condition on the potential for the existence of doubly periodic solutions of a semi-linear fourth-order partial differential equation *

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Abstract

We study the existence of solutions to the fourth order semi-linear equation

$$\Delta^2 u = g(u) + h(x).$$

We show that there is a positive constant $C_*$ such that if $g(\xi) \xi \geq 0$ for $|\xi| \geq \xi_0$ and $\limsup_{|\xi| \to \infty} 2G(\xi)/\xi^2 < C_*$, then for all $h \in L^2(Q)$ with $\int_Q hdx = 0$, the above equation has a weak solution in $H^2_0$. 

1 Introduction

This paper is motivated by the study of the differential equation

$$u'' + g(u) = h(t) = h(t + 2\pi),$$

where $g$ and $h$ are continuous functions. It is assumed that

$$\int_0^{2\pi} h(t) dt = 0.$$  (1.2)

Indeed, if $\hat{h} = \frac{1}{2\pi} \int_0^{2\pi} h(t) dt$, we may replace $g(u)$ by $g(u) - \hat{h}$ and $h$ by $h - \hat{h}$ in (1.1). We write $g \in \Sigma$ if there exists a constant $\xi_0 \geq 0$ such that

$$g(\xi) \xi \geq 0 \quad \text{for} \quad |\xi| \geq \xi_0.$$  (1.3)

Given $g \in \Sigma$, let $G'(\xi) = g(\xi)$, $G(0) = 0$.

Recently Fernandes and Zanolin [2] proved the existence of $2\pi$-periodic solutions of (1.1). Their work shows that if $g \in \Sigma$, (1.2) holds and either $\liminf_{|\xi| \to \infty} 2G(\xi)/\xi^2 < 1/4$ or $\liminf_{|\xi| \to -\infty} 2G(\xi)/\xi^2 < 1/4$, then there exists a $2\pi$-periodic solution of (1.1). Earlier work of Mawhin and Ward showed that if

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either \( \limsup_{\xi \to \infty} g(\xi)/\xi < 1/4 \) or \( \limsup_{\xi \to -\infty} g(\xi)/\xi < 1/4 \), then (1.1) has a solution.

These results led us to consider a more modest question for the partial differential equation

\[
\Delta u + g(u) = h(x),
\]

where \( x = (x_1, x_2) \) and \( h(x_1 + 2\pi, x_2) = h(x_1, x_2 + 2\pi) = h(x_1, x_2) \). Namely if \( Q = [0, 2\pi] \times [0, 2\pi] \), \( g \in \Sigma \), \( h \in L^2(Q) \), and

\[
\int_Q h \, dx = 0; \tag{1.5}
\]

does there exist a constant \( C_\ast \) such that the condition

\[
\limsup_{|\xi| \to \infty} \frac{2G(\xi)}{\xi^2} < C_\ast \tag{1.6}
\]

implies the existence of a \textit{weak} solution to (1.4) with the “boundary condition” \( u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) \)?

Thus we define a solution to be a member of the function space \( H^1_{2\pi} \) such that

\[
\int_Q \left[ u_{x_1} v_{x_1} + u_{x_2} v_{x_2} - g(u)v - h(x)v \right] \, dx = 0
\]

for all \( v \in C^\infty_{2\pi} \), the space of \( C^\infty \) functions defined on \( \mathbb{R}^2 \) which are \( 2\pi \)-periodic in each variable. The space \( H^1_{2\pi} \) is the completion of this space with respect to the norm

\[
\|u\| = \left[ \int_Q (u_{x_1}^2 + u_{x_2}^2 + u^2) \, dx \right]^{1/2}.
\]

The difficulty with this problem is that if \( g \) is only assumed to be continuous and \( u \in H^1_{2\pi} \), it is not generally true that the function \( g(u(x)) \) is \textit{locally integrable}. Also, unless \( g \) satisfies a suitable growth condition, the functional

\[
f(u) = \int_Q \frac{\nabla u^2}{2} - G(u) + h(x)u \, dx
\]

is not of class \( C^1 \). Thus we abandon this problem and considered the analogous fourth order semi-linear problem

\[
\Delta^2 u = g(u) + h(x) \tag{1.7}
\]

with \( u \in H^2_{2\pi} \), where \( h \) is in \( L^2(Q) \) and \( H^2_{2\pi} \) denotes the completion of \( C^\infty_{2\pi} \) with respect to the norm

\[
\left\{ \int_Q \left[ \sum_{i=1}^2 \sum_{j=1}^2 u_{x_i x_j}^2 + \sum_{i=1}^2 u_{x_i}^2 + u^2 \right] \, dx \right\}^{1/2}.
\]

By a weak solution of (1.7) we mean a \( u \in H^2_{2\pi} \) such that \( \int_Q (\Delta u \Delta v - g(u)v - h(x)v) \, dx = 0 \) for all \( v \) in \( C^\infty_{2\pi} \).
Since it can be shown that $H^2_{2\pi} \subset C_{2\pi}$ (this is essentially the Sobolev embedding theorem), $u \in H^2_{2\pi}$ implies that $g(u(x))$ is continuous. Moreover the compactness of $H^2_{2\pi}$ in $C_{2\pi}$ ensures that the functional $f : H^2_{2\pi} \to \mathbb{R}$ defined by

$$f(u) = \int_{\Omega} \left[ \frac{(\Delta u)^2}{2} - G(u) - h(x)u \right] dx$$

is of class $C^1$. We show that there exists $C_* > 0$ such that if $g \in \Sigma$ and (1.6) holds, then for all $h$ satisfying (1.5), $h \in L^2(Q)$, (1.7) has a weak solution.

We have shown that if

$$C_* = \frac{1}{4\pi^2 a_x^2 + 1}, \quad \text{where} \quad a_x^2 = \frac{1}{\pi^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(i^2 + j^2)^2},$$

then this statement will be true. However, we feel that this is far from the optimal value of $C_*$. It is clear that the optimal value must be less than 1, since it can be shown that if $g(\xi) = \xi$, $h(x_1, x_2) = \sin x_1$, then (1.7) does not have a weak solution, because of resonance.

## 2 Definitions and preliminary lemmas

In this section we state some preliminary lemmas. These results follow more or less from known results (see for example [1]). Full details will be given elsewhere.

Let $Q = \{(x_1, x_2) \mid 0 \leq x_1 \leq 2\pi, 0 \leq x_2 \leq 2\pi\}$. Let $L^2_{2\pi}(\mathbb{R}^2)$ denote the set of real-valued measurable functions defined in $\mathbb{R}^2$ such that if $u \in L^2_{2\pi}(\mathbb{R}^2)$, then $u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) = u(x_1, x_2)$ and such that $u$ restricted to $Q$ is in $L^2(Q)$.

We denote $C_{2\pi}$ and $C_{2\pi}^\infty$ the real-valued functions defined on $\mathbb{R}^2$ which are $2\pi$-periodic in each variable, which are continuous and of class $C^\infty$ respectively.

We denote by $H^2_{2\pi}(\mathbb{R}^2)$ the set of $u \in L^2_{2\pi}(\mathbb{R}^2)$ such that for $p = 1, 2$ there exists $v_p \in L^2_{2\pi}(\mathbb{R}^2)$ such that for all $\phi \in C_{2\pi}^\infty$,

$$-\int_Q (D_p \phi) u dx = \int_Q v_p \phi dx$$

and for $1 \leq p, q \leq 2$ there exists $v_{pq} \in L^2_{2\pi}(\mathbb{R}^2)$ such that for all $\phi \in C_{2\pi}^\infty$,

$$\int_Q (D_p D_q \phi) u dx = \int_Q \phi v_{pq} dx.$$  

Here $D_p = \partial / \partial x_p$, $p = 1, 2$. It is clear that $v_p$, $p = 1, 2$, and $v_{pq}$, $p, q = 1, 2$, are determined uniquely and we write $v_p = D_p u$, $p = 1, 2$, and $v_{pq} = D_p D_q u$, $p, q = 1, 2$.

The space $H^2_{2\pi}(\mathbb{R}^2)$ is a real Hilbert space with inner product given by

$$\langle u, v \rangle = \int_Q \left[ uv + \sum_{p=1}^{2} (D_p u)(D_p v) + \sum_{p,q=1}^{2} (D_p D_q u)(D_p D_q v) \right] dx$$
In the following we denote the Hilbert space $H^2_{2\pi}$ by $\mathbb{E}$ and $\| \cdot \|_{\mathbb{E}}$ will denote the norm given by the inner product defined above.

**Lemma 2.1** If $u \in \mathbb{E}$ then $u$ is equal almost everywhere to a unique function in $C_{2\pi}$. If this function is again denoted by $u$, then there exists a constant $a_0$ such that for all $u \in \mathbb{E}$, $\| u \|_{C_{2\pi}} = \max_{x \in \mathbb{R}^2} |u(x)| \leq a_0 \| u \|_{\mathbb{E}}$. (see [1, p 167]).

We denote by $\hat{\mathbb{E}}$ the set of $u \in \mathbb{E}$ such that $\int_{Q} u \, dx = 0$.

The following result can be proved using multiple Fourier series.

**Lemma 2.2** An inner product on $\hat{\mathbb{E}}$ which is equivalent to the $\mathbb{E}$-inner product is given by

$$\langle u, v \rangle_{\hat{\mathbb{E}}} = \int_{Q} (\Delta u)(\Delta v) \, dx$$

where, as usual $\Delta u = D^2_1 u + D^2_2 u$.

**Lemma 2.3** The best possible constant $a_*$ such that for all $u \in \hat{\mathbb{E}}$,

$$\| u \|_{C_{2\pi}} = \max_{x \in \mathbb{R}^2} |u(x)| \leq a_* \| u \|_{\hat{\mathbb{E}}},$$

where $\| u \|_{\hat{\mathbb{E}}} = \| \Delta u \|_{L^2(Q)}$, is

$$a_* = \frac{1}{2\pi} \left( \sum_{k \neq (0,0)} \frac{1}{|k|^4} \right)^{1/2}$$

(2.1)

it where $Z^2 = Z \times Z$, $Z = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$, and if $k = (k_1, k_2) \in Z^2$, $|k| = \sqrt{k_1^2 + k_2^2}$.

This lemma and the next are proved using multiple Fourier series.

**Lemma 2.4** If $u \in \hat{\mathbb{E}}$, then $\int_{Q} u^2 \, dx \leq \int_{Q} (\Delta u)^2 \, dx$.

The following result is proved using the idea of the proof given in [5, p. 216] except Fourier series are used instead of Fourier transform.

**Lemma 2.5** Let $0 < \alpha < 1$. There exists $M(\alpha)$ such that if $u \in \mathbb{E}$, then for $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$

$$|u(x) - u(y)| \leq M(\alpha) \| u \|_{\mathbb{E}} |x - y|^{\alpha}.$$

Here, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

The final preliminary lemma follows from Lemma 2.1, Lemma 2.5 and Ascoli’s Lemma.

**Lemma 2.6** The injection from $\mathbb{E}$ to $C_{2\pi}$ is compact, that is, if $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{E}$, then there exists a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ such that $\{u_{n_i}\}_{i=1}^{\infty}$ converges uniformly on $\mathbb{R}^2$. 

3 Periodic solutions of a semi-linear elliptic fourth-order partial differential equation

In this section $g$ will always denote a real-valued function defined and continuous on $\mathbb{R}$, and $G$ will denote the function such that $G'(|\xi|) = g(|\xi|)$ for $\xi \in \mathbb{R}$ with $G(0) = 0$. $L^2_{2\pi}$ will denote the closed subspace of $L^2_{2\pi}(\mathbb{R}^2)$ such that for all $h \in L^2_{2\pi}$, $\int_{Q} h(x) \, dx = 0$.

We consider the question of existence of weak solution of the problem

$$\Delta^2 u = g(u) + h(x) \quad u \in H^2_{2\pi}(\mathbb{R}^2)$$

(3.1)

where $h \in \hat{L}^2_{2\pi}$. This is defined to be a function $u \in H^2_{2\pi}(\mathbb{R}^2)$ such that for all $v \in \mathbb{E} (= H^2_{2\pi}(\mathbb{R}^2))$,

$$\int_{Q} [(\Delta u)(\Delta v) - g(u)v - h(x)v] \, dx = 0. \quad (3.2)$$

If $u$ is a function of class $C^4$ which is $2\pi$-periodic in each variable, then (3.1) holds if and only if (3.2) holds.

Let $f: \mathbb{E} \rightarrow \mathbb{R}$ be the function

$$f(u) = \int_{Q} \left[ \frac{|\Delta u|^2}{2} - G(u) - h(x)u \right] \, dx.$$

Since $\mathbb{E} \subset C_{2\pi}$, standard arguments (see, for example, [4]) show that $f \in C^1$. For $v \in \mathbb{E}$,

$$f'(u)(v) = \int_{Q} [(\Delta u)(\Delta v) - g(u)v - h(x)v] \, dx.$$

Therefore, weak solutions of (3.1) coincide with critical points of $f$.

Let $\Sigma$ denote the set of continuous $g: \mathbb{R} \rightarrow \mathbb{R}$ such that there exists some $\xi_0$, depending on $g$, such that

$$g(\xi) \xi \geq 0 \quad \text{for} \quad |\xi| \geq \xi_0. \quad (3.3)$$

**Theorem 3.1** Let $a_*$ be as in (2.1) and let

$$C_* = \frac{1}{4\pi^2 a_*^2 + 1} \quad (3.4)$$

If $g \in \Sigma$ and

$$\limsup_{|\xi| \to \infty} \frac{2G(\xi)}{\xi^2} < C^* \quad (3.5)$$

then, for all $h \in \hat{L}^2_{2\pi}$, there exists a weak solution of (3.1).
Sketch of Proof: The proof is an application of Rabinowitz’s Saddle-Point Theorem [4]. Assume first that $g$ satisfies the stronger condition: There exist $\delta > 0$ and $\xi_0 > 0$ such that

$$|\xi| \geq \xi_0 \text{ implies } \text{sgn}(\xi)g(\xi) \geq \delta.$$  \hfill (3.6)

Assuming that (3.5) holds there exist constants $C_2 \geq 0$ and $C_1$ with

$$C_1 < C_*$$  \hfill (3.6)

such that for all $\xi \in \mathbb{R}$,

$$G(\xi) \leq C_1 \left( \frac{\xi^2}{2} \right) + C_2.$$  \hfill (3.7)

We claim that the functional $f$ defined above satisfies the Palais-Smale condition. To see this let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $E$ such that $\{f(u_n)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ and $f'(u_n) \to 0$ in $E^*$, the topological dual space of $E$.

We first show that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^2(Q)$. Assuming the contrary, we may assume, by considering a subsequence, that $\|u_n\|_{L^2} \to \infty$ as $n \to \infty$.

By assumption, there exists a constant $C_3$ such that $f(u_n) \leq C_3$ for all $n \geq 1$ or

$$\int_Q \left[ \frac{\Delta u_n^2}{2} - G(u_n) - h(x)u_n \right] dx \leq C_3$$

for all $n$. From (3.7) we have that for $n \geq 1$

$$\int_Q |\Delta u_n|^2 dx \leq C_1 \|u_n\|^2_{L^2} + 8\pi^2 C_2 + 2\|h\|_{L^2} \|u_n\|_{L^2} + 2C_3$$

Setting $w_n = u_n/\|u_n\|_{L^2}$ for $n = 1, 2, \ldots$ we obtain

$$\int_Q (\Delta w_n)^2 dx \leq C_1 + \frac{2\|h\|_{L^2}}{\|u_n\|_{L^2}} + \frac{8\pi^2 C_2 + 2C_3}{\|u_n\|^2_{L^2}}$$  \hfill (3.8)

for all $n \geq 1$.

If $\hat{E}$ is defined as in the previous section and if we identify the constant functions with the real numbers $\mathbb{R}$, then

$$E = \hat{E} \oplus \mathbb{R}$$  \hfill (3.9)

For $n \geq 1$, let

$$w_n = z_n + \tau_n,$$  \hfill (3.10)

where $z_n \in \hat{E}$ and $\tau_n \in \mathbb{R}$. Since $\|\Delta z_n\|_{L^2} = \|\Delta w_n\|_{L^2}$, it follows from (3.8) and Lemma 2.2 that the sequence $\{z_n\}_{n=1}^{\infty}$ is bounded in $\hat{E}$. Therefore, since for all $n \geq 1, 4\pi^2 \tau_n^2 \leq \|w_n\|^2_{L^2} = 1$, we infer the existence of a constant $C_4$ such that $\|w_n\|_{\mathbb{R}} < C_4$ for all $n$. 

It follows that there exists a subsequence of \( \{w_n\}_{n=1}^{\infty} \) which converges weakly to \( w \) in \( E \). By considering a subsequence, we may assume, without loss of generality, that the sequence \( \{w_n\}_{n=1}^{\infty} \) itself converges weakly to \( w \).

If \( w = z + \tau \) where \( z \in \overline{E} \) and \( \tau \in \mathbb{R} \), then \( z_n \) converges weakly to \( z \) and \( \tau_n \) converges to \( \tau \) as \( n \to \infty \). From Lemma 2.6, it follows that the sequence \( \{w_n\}_{n=1}^{\infty} \) converges uniformly to \( w \) on \( \mathbb{R}^2 \), and since \( \lim_{n \to \infty} \tau_n = \tau \), we see that \( \{z_n(x)\}_{n=1}^{\infty} \) converges uniformly to \( z(x) \) on \( \mathbb{R}^2 \).

The uniform convergence implies that \( \|w\|_{L^2} = \lim_{n \to \infty} \|w_n\|_{L^2} = 1 \). From the lower semi-continuity of a norm with respect to weak convergence, it follows from (3.8) that

\[
\|\Delta z\|_{L^2}^2 = \|\Delta w\|_{L^2}^2 \leq \liminf_{n \to \infty} \|\Delta w_n\|_{L^2}^2 \leq C_1.
\]

Therefore, \( \|\Delta z\|_{L^2}^2 \leq C_1 \|w\|_{L^2}^2 = C_1 (\|z\|_{L^2}^2 + 4\pi^2 \tau^2) \) and since, according to Lemma 2.4, \( \|z\|_{L^2} \leq \|\Delta z\|_{L^2} \), it follows that

\[
\|\Delta z\|_{L^2}^2 \leq \frac{C_1 4\pi^2 \tau^2}{1 - C_1}.
\]

(That \( C_1 < 1 \) follows from (3.4) and (3.6)). Since \( 1 = \|w\|_{L^2}^2 = \|z\|_{L^2}^2 + 4\pi^2 \tau^2 \), we see that \( \tau \neq 0 \).

According to lemma 2.3

\[
\max_{x \in \mathbb{R}^2} |z(x)|^2 \leq \left( \frac{a^2 C_1 4\pi^2}{1 - C_1} \right) \tau^2
\]

and from (3.4) and (3.6)

\[
\frac{a^2 C_1 4\pi^2}{1 - C_1} < \frac{a^2 C_2 4\pi^2}{1 - C_2} = 1.
\]

Therefore,

\[
\max_{x \in \mathbb{R}^2} |z(x)| < |\tau|.
\]

Since \( \tau \neq 0 \) it follows that either \( w(x) = z(x) + \tau > 0 \) for all \( x \in \mathbb{R}^2 \) or \( w(x) < 0 \) for all \( x \in \mathbb{R}^2 \). Since \( u_n(x) = \|u_n\|_{L^2} w_n \) either \( u_n(x) \to \infty \) uniformly with respect to \( x \in \mathbb{R}^2 \) or \( u_n(x) \to -\infty \) uniformly with respect to \( x \in \mathbb{R}^2 \). From (3.6) it follows that in the first case

\[
\int_Q g(u_n(x)) \, dx \geq 4\pi^2 \delta
\]

for \( n \) sufficiently large, and in the second case

\[
\int_Q g(u_n(x)) \, dx \leq -4\pi^2 \delta
\]

for \( n \) sufficiently large. But since \( h \in \hat{L}^2_{2\pi} \), \( f'(u_n)(1) = \int_Q -[g(u_n(x)) + h(x)] \, dx = -\int_Q g(u_n(x)) \, dx \). Since \( f'(u_n)(1) \to 0 \) as \( n \to \infty \), therefore we have
a contradiction. This contradiction proves the sequence \( \{u_n\}_{n=1}^{\infty} \) is bounded in \( L^2(Q) \).

From the condition \( f(u_n) \leq C_3 \) for all \( n \) and the condition (3.7), it follows from Lemma 2.2, that \( \{u_n\}_{n=1}^{\infty} \) is bounded in \( E \). Therefore, from the form of \( f' \) and Lemma 2.6, standard arguments (see for example [4]) shows that \( f' \) satisfies the Palais-Smale condition.

The existence of a critical point of \( f \) follows from Rabinowitz's Saddle Point Theorem [4] corresponding to the direct sum decomposition \( E = \bar{E} \oplus \mathbb{R} \). Since, according to Lemma 2.4, for all \( z \in \bar{E}, \|z\|_{L^2} \leq \|\Delta z\|_{L^2} \), it follows that for all \( z \in \bar{E} \),

\[
\int_Q \left[ \frac{(\Delta z)^2}{2} - G(z) - h(x)z \right] \, dx \\
\geq \int_Q \left[ \frac{(\Delta z)^2}{2} - \frac{C_1}{2} \xi^2 - C_2 \right] \, dx - \|h\|_{L^2}\|z\|_{L^2} \\
\geq \left( 1 - \frac{C_1}{2} \right) \int_Q \frac{(\Delta z)^2}{2} \, dx - C_24\pi^2 - \|h\|_{L^2}\|\Delta z\|_{L^2}.
\]

Since, as shown above, \( C_1 < 1 \) it follows that

\[
\inf_{z \in \bar{E}} f(z) > -\infty.
\]

The condition \( g(\xi) \) sgn \( \xi \geq \xi_0 \) for \( \xi \geq \xi_0 \) implies that \( G(\xi) \to \infty \) as \( |\xi| \to \infty \). Therefore, since \( h \in \overline{L}^2_{2\pi} \), it follows that for \( \xi \in \mathbb{R} \),

\[
f(\xi) = \int_Q [-G(\xi) - \xi h(x)] \, dx \leq -4\pi^2 G(\xi) \to -\infty
\]

as \( |\xi| \to \infty \). Thus there exists \( b > 0 \) such that

\[
\max\{f(b), f(-b)\} < \inf_{z \in \bar{E}} f(z).
\]

Since \( f \) satisfies the Palais-Smale condition, it follows that if \( \Gamma \) denotes the set of all continuous mappings \( \gamma : [-b, b] \to E \) with \( \gamma(\pm b) = \pm b \),

\[
C_0 = \inf_{\gamma \in \Gamma} \max_{\xi \in [-b, b]} f(\gamma(\xi)),
\]

then there exists \( u_0 \in E \) such that \( f(u_0) = C_0 \) and \( f'(u_0) = 0 \). This \( u_0 \) is a solution of problem (3.1).

To prove that (3.1) has a solution when it is only assumed that \( g(\xi) \) sgn \( \xi \geq 0 \) for \( |\xi| \geq \xi_0 \). We can use a perturbation argument. We define

\[
r(\xi) = \begin{cases} 
-1 & \text{if } \xi \leq -\xi_0, \\
-1 + \frac{2(\xi + \xi_0)}{2\xi_0} & \text{if } |\xi| \leq \xi_0, \\
1 & \text{if } \xi \geq \xi_0,
\end{cases}
\]

Existence of doubly periodic solutions
For $m = 1, 2, 3, \ldots$, set $g_m(\xi) = g(\xi) + \frac{r(\xi)}{m}$. Then $g_m(\xi)|\xi| \geq \frac{1}{m}$ for $|\xi| \geq \xi_0$ and we still have

$$
\limsup_{|\xi| \to \infty} \frac{2G_m(\xi)}{\xi^2} < C^*.
$$

By what has been shown, (3.1) has a solution when $g = g_m$. The conditions of the theorem imply that there is a priori bound on this solution (the one characterized by the Saddle Point Theorem) in $E$, which is independent of $m$. Using a compactness argument this implies the existence of a solution of (3.1). The computational details of this proof will be published somewhere else.

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References


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