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Existence and number of solutions to semilinear equations with applications to boundary-value problems *

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Abstract

We present recent and some new existence results on the number of solutions to nonlinear equations and to (non)resonant semilinear equations involving nonlinear perturbations of Fredholm maps of index zero. We apply our results to semilinear elliptic, and to semilinear parabolic and hyperbolic periodic boundary-value problems.

1 Introduction

Let X and Y be Banach spaces and $T : X \to Y$ be a nonlinear map of A-proper type. Under various conditions on T, we study in Section 2 the surjectivity and the finitness of the solution set of the equation $Tx = f$. In particular, we look at nonresonant semilinear equations of the form $Ax + Nx = f$ where A is a Fredholm map of index zero and the nonlinear map N is such that $A + N$ is (pseudo) A-proper. We say that this equation is not at resonance if A and N are are such that it is solvable for each $f \in Y$. Applications to semi-abstract nonresonant semilinear equations are given in Section 3. Section 4 is devoted to applications of the results of Section 3 to boundary-value problems (BVP) for semilinear elliptic equations. In Section 5, some comments on periodic BVP's for semilinear parabolic and hyperbolic equations assuming nonuniform nonresonance conditions are made. The existence of solutions for such problems has been studied earlier in [12, 13, 14, 22, 6, 7, 9].

2 Number of solutions to operator equations

In this section, we shall study the number of solutions to the equation $Tx = f$. The unique (approximate) solvability of this equation has been studied in detail in [20], using the A-proper mapping approach.

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Definition A map $T : D \subset X \to Y$ is (pseudo) A-proper with respect to a scheme $\Gamma = \{X_n, Y_n, Q_n\}$ with $\dim X_n = \dim Y_n$ on D if whenever $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and such that $Q_{n_k} T x_{n_k} - Q_{n_k} f \to 0$ for some $f \in Y$, then $\{x_n\}$ has a subsequence converging to $x \in D$ (there is $x \in D$) with $Tx = f$.

Next, we shall define A-proper homotopies.

Definition A homotopy $H : [0,1] \times D \to Y$ is A-proper with respect to Γ on D if $Q_n H_t : D \cap X_n \to Y_n$ is continuous for each t and n, and if $\{x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $t_k \in [0,1]$ with $t_k \to t$ are such that $Q_{n_k} H(t_k, x_{n_k}) - Q_{n_k} f \to 0$ as $k \to \infty$ for some $f \in Y$, then a subsequence of $\{x_{n_k}\}$ converges to $x \in D$ and $H(t,x) = f$.

The classes of A-proper and pseudo A-proper maps are very general. For many examples of such maps, we refer the reader to [15]-[19].

Nonlinear equations

We say that a map $T : X \to Y$ satisfies condition (+) if $\{x_n\}$ is bounded whenever $Tx_n \to f$ in Y. Let Σ be the set of all points $x \in X$ where T is not locally invertible and $\mathrm{card} T^{-1}(\{f\})$ be the cardinal number of the set $T^{-1}(\{f\})$.

Theorem 2.1 ([21]) *Let $T : X \to Y$ be continuous, A-proper and satisfy condition (+). Then*

(a) The set $T^{-1}(\{f\})$ is compact (possibly empty) for each $f \in Y$.

(b) The range $R(T)$ of T is closed and connected.

(c) Σ and $T(\Sigma)$ are closed subsets of X and Y, respectively, and $T(X \setminus \Sigma)$ is open in Y.

(d) $\mathrm{card} T^{-1}(\{f\})$ is constant and finite (it may be 0) on each connected component of the open set $Y \setminus T(\Sigma)$.

(e) if $\Sigma = \emptyset$, then T is a homeomorphism from X to Y.

(f) if $\Sigma \neq \emptyset$, then the boundary $\partial T(X \setminus \Sigma)$ of $T(X \setminus \Sigma)$ satisfies $\partial T(X \setminus \Sigma) \subset T(\Sigma)$.

Proof. Since T is proper by Proposition 2.1 in [21], it is a closed map. Since $X \setminus \Sigma$ is an open set, Σ is a closed set. Hence (a)-(c) hold, where $T(X \setminus \Sigma)$ is open since T is locally invertible on $X \setminus \Sigma$. (d) follows from the Ambrosetti theorem [A] and (e) follows from the global inversion theorem. Next, (b) and (c) imply that

$$T(X) = T(\Sigma) \cup T(X \setminus \Sigma) = T(\Sigma) \cup \overline{F(X \setminus \Sigma)} = \overline{T(X)}. \tag{2.1}$$

Moreover, $\partial T(X \setminus \Sigma) = \overline{T(X \setminus \Sigma)} \setminus T(X \setminus \Sigma)$, which together with (2.1) imply (f). $\diamondsuit$

Next, we shall look at another surjectivity result. Let $J : X \to 2^{X^*}$ be the normalized duality map and $G : X \to Y$ be a bounded map such that $Gx \neq 0$ for all x with $\|x\| \geq r_0$ for some $r_0 > 0$ and

$$\text{For each large } r > 0,\ \deg(Q_n G, B(0,r) \cap X_n, 0) \neq 0 \text{ for all large } n. \tag{2.2}$$

Theorem 2.2 *Let $T : X \to Y$ satisfy conditions $(+)$ and (2.2), and let*

(i) For each $f \in Y$ there is an $r_f > 0$ such that

$$Tx \neq \lambda Gx \text{ for } x \in \partial B(0, r_f), \lambda < 0. \tag{2.3}$$

(ii) $H(t,x) = tTx + (1-t)Gx$ is an A-proper with respect to Γ homotopy on $[0,1] \times X$.

Then T is surjective. Moreover, if T is continuous, then $T^{-1}(\{f\})$ is compact for each $f \in Y$ and the cardinal number card$T^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus T(\Sigma)$.

Proof. The surjectivity of T has been established earlier by the author (see, eg [17, 19]). Moreover, T is continuous and proper by Proposition 2.1 in [21]. Hence, the other assertions of the theorem follow from Theorem 2.1.

Corollary 2.1 *Let $F, K : X \to X$ be continuous ball-condensing maps and $T = I - F$ and $G = I - K$ satisfy (2.2)-(2.3). Then the conclusions of Theorem 2.2 hold for T.*

This corollary is also valid for general condensing maps (see [23]). For a map M, define its quasinorm by $|M| = \limsup_{\|x\| \to \infty} \|Mx\|/\|x\|$.

Theorem 2.3 (cf. [19]) *Let $A : D(A) \subset X \to Y$ be a linear densely defined map and $N : X \to Y$ be bounded and of the form $Nx = B(x)x + Mx$ for some linear maps $B(x) : X \to X$. Assume that there is a $c > |M|$ and a positively homogeneous map $C : X \to Y$ such that*

$$\|Ax - (1-t)Cx - tB(x)x\| \geq c\|x\|,\ x \in D(A) \setminus B(0,R). \tag{2.4}$$

(i) $H_t = A - (1-t)C - tN$ is A-proper with respect to $\Gamma = \{X_n, Y_n, Q_n\}$ for $t \in [0,1)$ and $A - N$ is pseudo A-proper

(ii) For all $r > R$, $\deg(Q_n(A - C), B(0,r) \cap X_n, 0) \neq 0$ for each large n.

Then the equation $Ax - Nx = f$ is solvable for each $f \in Y$. If, in addition, $A - N$ is continuous and A-proper, then $(A-N)^{-1}(\{f\})$ is compact for each $f \in Y$ and card$(A-N)^{-1}(\{f\})$ is constant, finite and positive on each connected component of the set $Y \setminus (A-N)(\Sigma)$.

Proof. Regarding the surjectivity of $A-N$, it suffices to solve $Ax-Nx=0$. Define $H(t,x)=Ax-(1-t)Cx-tNx$ on $[0,1]\times D(A)$. Then there is an $r>0$ such that

$$H(t,x)\neq 0 \text{ for } x\in\partial B(0,r)\cap D(A),\ t\in[0,1]. \tag{2.5}$$

If not, then there are $x_n\in H$ and $t_n\in[0,1]$ such that $\|x_n\|\to\infty$ and $H(t_n,x_n)=0$. Let $\epsilon>0$ be small such that $|M|\leq(|M|+\epsilon)\|x\|$ for $\|x\|\geq R_1$ and $|M|+\epsilon<c$. For each x_n with $\|x_n\|\geq R_1$ we have that

$$c\|x_n\|\leq\|Ax_n-(1-t)Cx_n-tB(x_n)x_n\|\leq(|M|+\epsilon)\|x_n\|.$$

Dividing by $\|x_n\|$, this leads to a contradiction and (2.5) holds. Hence, $A-N$ is surjective by the homotopy result in [16, 17]. Next, it is easy to see that $\|(A-N)x\|\to\infty$ as $\|x\|\to\infty$ by (2.4). Hence, the other assertions follow from Theorem 2.1.

3 Semi-abstract nonresonance problems

Let $Q\subset R^n$ be a bounded domain, V be a closed subspace of $W_2^{2m}(Q)$ containing the test functions and $L:V\to L_2$ be a linear map with closed range in $H=L_2(Q)$. Let V_1 be a closed subspace of V and L_1 be the restriction of L to V_1. Assume

(L1) Each eigenvalue λ_j of L_1 has a finite multiplicity and the corresponding eigenfunctions $\{\ldots,w_{-1},w_0,w_1,\ldots\}$ form a complete set in V_1.

Let $A=A_1+L$ for some linear map $A_1:V\to H$. For a fixed integer j, define $B:V\to H$ by $Bu=-Au+\lambda_j u$.

(B1) There is $\lambda\neq\lambda_j$, $j=1,2,\ldots$, such that the map $B-\lambda I=-A_1-L-(\lambda-\lambda_j)I:V\to L_2$ is bijective.

Let $\lambda\neq\lambda_j$ for each $j=1,2,\ldots$ be fixed, $\Gamma=\{Y_n,Q_n\}$ be a projectionally complete scheme for L_2 and $X_n=(B-\lambda I)^{-1}(Y_n)\subset V$ for each n. Then $\Gamma_B=\{X_n,Y_n,Q_n\}$ is an admissible or a projectionally complete scheme for (V,L_2). Since $B-\lambda I:V\to L_2$ is linear, one-to-one and A-proper with respect to Γ_B, there is a constant $c>0$ (depending)only on λ) such that

$$\|(B-\lambda I)u\|\geq c\|u\|_V,\ \ u\in V. \tag{3.1}$$

Consider the following semilinear equation in V

$$Au+g(x,u,Du,\ldots,D^{2m-1}u)u+f(x,u,Du,\ldots,D^{2m}u)=h(x) \tag{3.2}$$

For $u\in H$, set $u^{\pm}=\max(\pm u,0)$. Let $r=\lambda_{j+1}-\lambda_j$. We require that B has the following properties:

Property I B is a closed densely defined map in H with closed range $R(B)$, $(Bu,u)\geq -r^{-1}\|Bu\|^2$ and $R(B)=N(B)^{\perp}$ in H, $N(-L_1+\lambda_j I)\subset N(B)$ and $(Bu,u)=(-Lu+\lambda_j u,u)$ on V.

Property II If $(Bu, u) = -r^{-1}\|Bu\|^2$ for some $u \in V$, then $u \in N(-L_1 + \lambda_j I) \oplus N(-L_1 + \lambda_{j+1} I)$.

Let us note that if B^{-1} is a partial inverse of B and $B^{-1} + r^{-1}I$ is strongly monotone on $R(B)$, i.e. it is a bounded linear map on $R(B)$ and $((B^{-1} + r^{-1}I)u, u) = c_0\|(B^{-1}+r^{-1}I)u\|^2$ on $R(B)$ for some $c_0 > 0$, then ([BF]) Property II holds in the sense that if $(Bu, u) = -r^{-1}\|Bu\|^2$ for some $u \in V$, then $u \in N(B) \oplus N(B + rI)$. If B is selfadjoint or angle bounded in the sense of H. Amann, it is known that $B^{-1} + r^{-1}I$ is strongly monotone. If $B \neq B^*$ and B is a normal map, the strong monotonicity of $B^{-1} + r^{-1}I$ has been discussed in Hetzer [8].

Some properties of B are given next.

Lemma 3.1 *Let B have Properties I and II. Suppose that $p_\pm \in L_\infty(Q)$ are such that $0 \le p_\pm(x) \le r$ for a.e. $x \in Q$ and*

$$\int_Q [p_+(v^+)^2 + p_-(v^-)^2] > 0 \textit{ for all } v \in N(-L_1 + \lambda_j I) \setminus \{0\}$$

and

$$\int_Q [(r - p_+)(w^+)^2 + (r - p_-)(w^-)^2] > 0 \textit{ for all } w \in N(-L_1 + \lambda_{j+1} I) \setminus \{0\}.$$

Then the equation

$$Bu + p_+u^+ - p_-u^- = 0 \tag{3.3}$$

has only the trivial solution.

Proof. Define $p : Q \times R \to R$ by

$$p(x, u) = p_+(x) \text{ if } u \ge 0,$$
$$p(x, u) = p_-(x) \text{ if } u \le 0.$$

Then

$$0 \le p(x, u) \le r \text{ for } (x, u) \in Q \times R \tag{3.4}$$

and, for $u \in H$ and a.e. $x \in Q$,

$$\begin{aligned} p(x, u(x))u(x) &= p(x, u(x))u^+(x) - p(x, u(x))u^-(x) \\ &= p_+(x)u^+(x) - p_-(x)u^-(x). \end{aligned}$$

Define $P : V \subset H \to H$ by $(Pu)(x) = p(x, u(x))u(x)$ for a.e. $x \in Q$. Then (3.3) is equivalent to

$$Bu + Pu = 0, \quad u \in V. \tag{3.5}$$

By (3.4), we have that $\|Pu\|^2 \le r(Pu, u)$ on V. Moreover, for each solution $u \in V$ of (3.5), we get by Property I that

$$-r^{-1}\|Pu\|^2 = -r^{-1}\|Bu\|^2 \le (Bu, u) = (-Pu, u)$$

and so $\|Pu\|^2 \geq r(Pu, u)$. Hence, $\|Pu\|^2 = r(Pu, u)$ and $(Bu, u) = -r^{-1}\|Bu\|^2$. By Property II, we get that $u \in N(-L_1 + \lambda_j I) \oplus N(-L_1 + \lambda_{j+1} I)$. Hence, $u = v + w$ with $v \in N(-L_1 + \lambda_j I)$ and $w \in N(-L_1 + \lambda_{j+1} I)$. Since u is a solution of (3.3), we get that

$$\begin{aligned}
(Bu, u) &= (-Lu + \lambda_j u, u) \\
&= (-Lv + \lambda_j v, v) + (-Lv + \lambda_j v, w) + (-Lw + \lambda_j w, v + w) \\
&= (-Lw + \lambda_{j+1} w - rw, v + w) \\
&= (-rw, w)
\end{aligned}$$

and so $(-rw, w) + (p(., u(.))(v + w), v + w) = 0$. Then

$$\begin{aligned}
&(v - w, -rw + p(., u(.))(v + w)) \\
&\quad = (v + w, -rw + p(., u(.))(v + w)) - 2(w, -rw + p(., u(.))(v + w)) \\
&\quad = -2(w, -rw + p(., u(.))(v + w)) \\
&\quad = -2(v + w, -rw - B(v + w)) + 2(v, -rw - B(v + w)) \\
&\quad = -2(v, rw + B(v + w)) = -2(v, B(v + w)) = 0
\end{aligned}$$

since $v \in N(-L_1 + \lambda_j I) \subset N(B)$ and $R(B) = N(B)^{\perp}$. Since

$$\begin{aligned}
(p(., u(.))(v + w), v - w) &= (p(., u(.))v, v) + ([r - p(., u(.))]w, w) \\
&\quad + ([r - p(., u(.))]w, -v) + (p(., u(.))v, -w) \\
&= (p(., u(.))v, v) + ([r - p(., u(.))]w, w)
\end{aligned}$$

we get that

$$(p(., u(.))v, v) + ([r - p(., u(.))]w, w) = (p(., u(.))(v + w), v - w) + (rw, -v + w) = 0. \tag{3.6}$$

Since each term in (3.6) is nonnegative by (3.4), we get that each term is zero, i.e.,

$$\int_Q p(x, v(x) + w(x))v^2(x)dx = 0 \tag{3.7}$$

$$\int_Q [(r - p(x, v(x) + w(x))]w^2(x)dx = 0. \tag{3.8}$$

Set $Q_v = \{x \in Q \mid v(x) \neq 0\}$ and $Q_w = \{x \in Q \mid w(x) \neq 0\}$.

By (3.7)-(3.8), we get $p(x, v(x) + w(x)) = 0$ for a.e. $x \in Q_v$ and $p(x, v(x) + w(x)) = r$ for a.e. $x \in Q_w$ and so $Q_v \cap Q_w = \emptyset$. If $Q_v = \emptyset$, then $u = w$ and the equation (3.8) becomes

$$0 = \int_Q [(r - p(x, w(x))]w^2(x)dx = \int_Q (r - p_+)(w^+)^2 + (r - p_-)(w^-)^2$$

so that by our hypothesis, $w = 0$ and therefore $u = 0$.

Next, suppose that $Q_v \neq \emptyset$. Then we have that $p(x, v(x) + w(x)) = 0$ on Q_v and, by (3.8), $\int_{Q_v} rw^2(x) = 0$, i.e., $w(x) = 0$ for a.e. $x \in Q_v$. Then by (3.7)

$$0 = \int_{Q_v} p(x, v(x))v^2(x) = \int_{Q_v} (p_+(v^+)^2 + p_-(v^-)^2) = \int_Q (p_+(v^+)^2 + p_-(v^-)^2).$$

By our assumption, this implies that $v = 0$, in contradiction to $Q_v \neq \emptyset$. Hence, $Q_v = \emptyset$ and $u = 0$.

Lemma 3.2 *Let (L1) and (B1) hold and B have Properties I and II. Suppose that $a_\pm$, $b_\pm \in L_\infty(Q)$ are such that $0 \leq a_\pm(x) \leq b_\pm \leq r$ for a.e. $x \in Q$ and*

$$\int_Q [a_+(v^+)^2 + a_-(v^-)^2] > 0 \text{ for all } v \in N(-L_1 + \lambda_j I) \setminus \{0\} \tag{3.9}$$

and

$$\int_Q [(r-b_+)(w^+)^2 + (r-b_-)(w^-)^2] > 0 \text{ for all } w \in N(-L_1 + \lambda_{j+1} I) \setminus \{0\}. \tag{3.10}$$

Then there exists $\epsilon = \epsilon(a_\pm, b_\pm) > 0$ and $\delta = \delta(a_\pm, b_\pm) > 0$ such that for all $p_\pm \in L_\infty(Q)$ with

$$a_+(x) - \epsilon \leq p_+(x) \leq b_+(x) + \epsilon \tag{3.11}$$

$$a_-(x) - \epsilon \leq p_-(x) \leq b_-(x) + \epsilon \tag{3.12}$$

for a.e. $x \in Q$ and for all $u \in V$, one has

$$\|Bu + p_+u^+ - p_-u^-\| \geq \delta \|u\|_V. \tag{3.13}$$

Proof. If this is not the case, then we can find the sequences $\{u_k\} \subset V$, with $\|u_k\|_V = 1$ for each k and $\{p_\pm^k\} \subset L_\infty(Q)$ such that

$$a_\pm(x) - k^{-1} \leq p_\pm(x) \leq b_\pm(x) + k^{-1} \text{ a.e. on } Q \tag{3.14}$$

and

$$Bu_k + p_+^k u_k^- - p_-^k u_k^- = v_k \to 0 \text{ as } k \to \infty. \tag{3.15}$$

Then $p_\pm^k \to p_\pm$ weakly in H with $a_\pm(x) \leq p_\pm(x) \leq b_\pm(x)$ a.e. on Q. Let $\mu \neq \lambda_j$ and consider the identity

$$\begin{aligned} u_k + (B - \mu I)^{-1}[(p_+^k - p_+)u_k^+ - (p_-^k - p_-)u_k^-] \\ = (B - \mu I)^{-1}(-p_+u_k^+ + p_-u_k^- - \mu u_k + v_k). \end{aligned} \tag{3.16}$$

By the compactness of the embedding of V into L_2, we have that $u_k \to u$ in L_2 as well as $u_k^\pm \to u^\pm$ in L_2. Since $(B - \mu I)^{-1}$ is continuous both as a map from L_2 to V and from L_2 to L_2, we get that

$$(B - \mu I)^{-1}(-p_+u_k^+ + p_-u_k^- - \mu u_k + v_k) \to (B - \mu I)^{-1}(-p_+u^+ + p_-u^- - \mu u) \tag{3.17}$$

in L_2 and V. Next, we shall show that $p_\pm^k \to p_\pm u^\pm$ weakly in H. For $\phi \in C_0^\infty(Q)$, we have that

$$\begin{aligned}(p_+^k u_k^+ - p_+u^+, \phi) &= (p_+^k(u_k^+ - u^+), \phi) + ((p_+^k - p_+)u^+, \phi) \\ &\leq c\|u_k^+ - u^+\| + ((p_+^k - p_+)u^+, \phi)\end{aligned}$$

which approaches zero as k approaches ∞. Hence, $p_+^k u_k^+ \to p_+u^+$ weakly in L_2 by the density of $C_0^\infty(Q)$ in L_2, and similarly, $p_-^k u_k^- \to p_-u^-$ weakly in L_2. Hence, (3.16)-(3.17) imply that $u = (B - \mu I)^{-1}(-p_+u^+ + p_-u^- - \mu u)$, i.e., $Bu + p_+u^+ - p_-u^- = 0$. Moreover, for each $v \in N(-L_1 + \lambda_j I) \setminus \{0\}$, we have that

$$\int_Q p_+(v^+)^2 + p_-(v^-)^2 \geq \int_Q a_+(v^+)^2 + a_-(v^-)^2 > 0$$

and, for each $w \in N(-L_1 + \lambda_{j+1} I) \setminus \{0\}$, we have that

$$\int_Q [(r - p_+)(w^+)^2 + (r - p_-)(w^-)^2] \geq \int_Q [(r - b_+)(w^+)^2 + (r - b_-)(w^-)^2] > 0.$$

Hence, by Lemma 3.1, $u = 0$ a.e.on Q. Thus, $u_k \to 0$ in L_2, $\|u_k\|_V = 1$ and $\|p_+^k u_k^+ - p_-^k u_k^- - \mu u_k\| \to 0$. By (3.1), we get that

$$\begin{aligned}\|Bu_k + p_+^k - p_-^k u_k^-\| &\geq \|Bu_k - \mu u_k\| - \|\mu u_k - p_+^k u_k^+ + p_-^k u_k^-\| \\ &\geq c - \|p_+^k u_k^+ - p_-^k u_k^- - \mu u_k\|.\end{aligned}$$

By (3.15), passing to the limit as $k \to \infty$, we get that $0 \geq c > 0$, a contradiction. Hence, the lemma is valid.

Remark 3.1 Modifying suitably the proof of Lemma 3.2, condition (B1) can be replaced by

(B2) $\dim N(B) < \infty$ and the partial inverse of B is compact.

Let R^{s_k} be the vector space whose elements are $\xi = \{\xi_\alpha : |\alpha = (\alpha_1, \ldots, \alpha_n)| \leq k\}$. Each $\xi \in \mathbb{R}^k$ may be written as a pair $\xi = (\eta, \zeta)$ with $\eta \in R^{s_{k-1}}$, $\zeta = \{\xi_\alpha \mid |\alpha| = k\} \in R^{s_k - s_{k-1}} = R^{s_k'}$ and $|\xi| = (\sum_{|\alpha| \leq k} |\xi_\alpha|^2)^{1/2}$. Set $\eta(u) = (Du, \ldots, D^{2m-1}u)$ and $\xi(u) = (u, Du, \ldots, D^{2m}u)$. Define $Nu = g(x, u, \eta(u))u + Fu$, where $Fu = f(x, \xi(u))$. Set $k = s_{2m-1} - 1$.

For our next result, suppose that $\dim N(-L_1 + \lambda_j I) = 1$ and is spanned by a positive function w_j.

Lemma 3.3 *Let B have properties I and II. Suppose that $p_\pm \in L_\infty(Q)$ are such that $0 \leq p_\pm(x) \leq r$ a.e. $x \in Q$ and for $r = \lambda_{j+1} - \lambda_j$*

$$\int [(r - p_+)(w^+)^2 + (r - p_-)(w^-)^2] > 0 \ \textit{for all} \ w \in N(-L_1 + \lambda_{j+1}) \setminus \{0\}.$$

Then, if u is a solution of $Bu + p_+u^+ - p_-u^- = 0$, then $u \in N(-L_1 + \lambda_j I)$.

Proof. If not, then arguing as in Lemma 3.1 we get that $u = 0$ if $Q_v = \emptyset$. Next, suppose that $Q_v \neq \emptyset$. Then it follows from the properties of the eigenfunction w_j and the fact that $v = aw_j(x)$ for some $a \in R$ and therefore $Q_v = Q$. Hence, we must have that $p(x, u(x) = 0$ for a.e. $x \in Q$. By (3.8), we get that $w = 0$ and hence $u = v$. Thus, in both cases, $u = v \in N(-L_1 + \lambda_j I)$.

Theorem 3.1 *Let (L1) and (B1) hold, B have properties I and II and there are functions $\gamma_\pm, \Gamma_\pm \in L_\infty(Q)$ such that for some $j \in J \subset Z$, one has*

$$\lambda_j \le \gamma_\pm(x) \le \Gamma_\pm(x) \le \lambda_{j+1} \text{ for a.e. } x \in Q\,,$$
$$\int_Q [(\gamma_+ - \lambda_j)(v^+)^2 + (\gamma_- - \lambda_j)(v^-)^2] > 0 \tag{3.18}$$

for all $v \in N(L_1 - \lambda_j I) \setminus \{0\}$, and

$$\int_Q [(\lambda_{j+1} - \Gamma_+)(w^+)^2 + (\lambda_{j+1} - \Gamma_-)(w^-)^2] > 0 \tag{3.19}$$

for all $w \in N(L_1 - \lambda_{j+1} I) \setminus \{0\}$. Also suppose that for $\epsilon > 0$ and $\delta > 0$ given in Lemma 3.2,

(G1) there is $\rho > 0$ such that for a.e. $x \in Q$

$$\gamma_+(x) - \epsilon \le g(x, u, \eta(u)) \le \Gamma_+(x) + \epsilon \text{ if } u > \rho, \eta(u) \in \mathbb{R}^k$$
$$\gamma_-(x) - \epsilon \le g(x, u, \eta(u)) \le \Gamma_-(x) + \epsilon \text{ if } u < -\rho, \eta(u) \in \mathbb{R}^k$$

(G2) There are functions $b(x) \in L_\infty(Q)$ and $k_s(x) \in L_2(Q)$ for each $s > 0$ such that

$$|g(x, u, \eta(u))| \le sb(x)(\sum_{|\alpha| \le 2m-1} |D^\alpha u|^2)^{1/2} + k_s(x), u \in V.$$

(F) $\|Fu\| = \|f(x, u, Du, \dots, D^{2m}u)\| \le \beta\|u\|_V + \gamma$ for $\beta \in (0, \delta), \gamma > 0$.

(H) $H_t = A - \lambda_j I - tF : V \to H$ is A-proper with respect to Γ_B for $t \in [0, 1)$ and $H_1 = B - F$ is pseudo A-proper.

Then (3.2) has at least one solution in V for each $h \in H$. If H_1 is A-proper, the set of solutions $S(h)$ of (3.2) is compact for each $h \in L_2$ and card $S(h)$ is constant, finite and positive on each connected component of the set $L_2 \setminus (A - N)(\Sigma)$.

Proof. Let $g_1 : Q \times R \times \mathbb{R}^k \to R$ be given by $g_1(x, u, \eta(u)) = g(x, u, \eta(u)) - \lambda_j$. Then define functions

$$g_+(x, u, \eta(u)) = g_1(x, u, \eta(u)) \text{ for all } (x, \eta(u)) \in Q \times \mathbb{R}^k, u \ge \rho$$
$$g_+(x, u, \eta(u)) = g_1(x, \rho, \eta(u)) \text{ for all } (x, \eta(u)) \in Q \times R^{s_{2m-1}}, 0 \le u \le \rho\,,$$
$$g_-(x, u, \eta(u)) = g_1(x, u, \eta(u)) \text{ for all } (x, \eta(u)) \in Q \times \mathbb{R}^k, u \le -\rho\,,$$
$$g_-(x, u, \eta(u)) = -g_1(x, -\rho, \eta(u)) \text{ for all } (x, \eta(u)) \in Q \times R^{s_{2m-1}}, -\rho \le u \le 0\,,$$
$$q(x, 0, \eta(u)) = g_1(x, 0, \eta(u)) \text{ for all } (x, \eta(u)) \in Q \times \mathbb{R}^k\,,$$
$$q(x, u, \eta(u)) = g_1(x, u, \eta(u))u - g_+(x, u, \eta(u))u \text{ for all } (x, \eta(u)) \in Q \times \mathbb{R}^k$$

and $u > 0$. Also define

$$q(x,u,\eta(u)) = g_1(x,u,\eta(u))u - g_-(x,u,\eta(u))u \text{ for all } (x,\eta(u)) \in Q \times \mathbb{R}^k$$

and $u < 0$. Then q satisfies Caratheodory conditions. Set $a_\pm(x) = \gamma_\pm(x) - \lambda_j$ and $b_\pm(x) = \Gamma_\pm(x) - \lambda_j$. Then

$$a_+(x) - \epsilon \le g_+(x,u,\eta(u)) \le b_+(x) + \epsilon \text{ on } Q \times R_+ \times \mathbb{R}^k$$
$$a_-(x) - \epsilon \le g_-(x,u,\eta(u)) \le b_-(x) + \epsilon \text{ on } Q \times R_- \times \mathbb{R}^k.$$

Then in V, problem (3.2) is equivalent to

$$Bu + g_+(x,u^+,\eta(u))u^+ - g_-(x,-u^-,\eta(u))u^- + f(x,\xi(u)) + q(x,u,\eta(u)) = -h.$$

Then for $u \in H$, set $Q_+(u) = \{x \in Q \mid u(x) > 0\}$, $Q_-(u) = \{x \in Q \mid u(x) < 0\}$ and let $\chi_{Q_\pm}$ be the corresponding characteristic functions. Define the maps $E : H \to L_\infty(Q)$, $F, G, H : V \to H$, respectively, by

$$E(u)(x) = g_+(x,u^+(x),\eta(u))\chi_{Q_+(u)} + g_-(x,-u^-(x),\eta(u))\chi_{Q_-(u)}$$

$G(u)(x) = [E(u)(x)]u(x) = (E(u)u)(x)$ so that

$$G(u)(x) = g_+(x,u^+(x),\eta(u))u^+(x) - g_-(x,-u^-(x),\eta(u))u^-(x),$$

$F(u)(x) = f(x,\xi(u))$ and $H(u)(x) = q(x,u(x),\eta(u))$. Hence (3.2) can be written in the operator form

$$Bu + Gu + Fu + Hu = -h, \quad u \in V. \tag{3.20}$$

We know that G, F and H are well defined, continuous and bounded in H.

Let $C : H \to H$ be defined by $C(u)(x) = b_+(x)u^+(x) - b_-(x)u^-(x)$. Clearly, C is a positively homogeneous map and $C, G, H : V \to H$ are completely continuous maps, i.e. they map weakly convergent sequences in V into strongly convergent sequences in H. Indeed, let us show this, for example, for G. Since V is compactly embedded in H, it follows from the construction of G and (G2) that if $\{u_k\} \subset V$ converges weakly to u_0 in V, then ([K])

$$\|g_+(x,u_k^+,\eta(u_k)) - g_-(x,-u_k^-,\eta(u_k)) - g_+(x,u_0^+,\eta(u_0)) + g_-(x,-u_0^-,\eta(u_0))\|$$

approaches 0. Hence, the map $G : V \to L_p$ is completely continuous since

$$\begin{aligned} \|Gu_k - Gu_0\| &= \|E(u_k)u_k - E(u_0)u_0\| \\ &\le \|E(u_k)(u_k - u_0)\| + \|E(u_k) - E(u_0)\|\|u_0\| \\ &\le \max\{\|a_+\|_\infty + \|a_-\|_\infty + 2\epsilon, \|b_+\|_\infty + \|b_-\|_\infty \\ &\quad +2\epsilon\}\|u_k - u_0\| + \|E(u_k) - E(u_0)\|\,\|u_0\| \to 0\,. \end{aligned}$$

Thus, we have that $H_t = B + (1-t)C + t(F + G + H)$ is A-proper for each $t \in [0,1)$ from $V \to H$ and $H_1 : V \to H$ is pseudo A-proper.

Next, by construction

$$\begin{aligned}(1-t)Cu + tGu &= [(1-t)b_+(x) + tg_+(x, u^+, \eta(u))]u^+(x) \\ &\quad -[(1-t)b_-(x) + tg_-(x, -u^-, \eta(u))]u^-(x)\end{aligned}$$

and, for a.e. $x \in Q$, $\eta(u) \in \mathbb{R}^k$

$$\begin{aligned}a_+(x) - \epsilon \le (1-t)b_+(x) + tg_+(x, u^+(x), \eta(u)) \le b_+(x) + \epsilon \\ a_-(x) - \epsilon \le (1-t)b_-(x) + tg_-(x, -u^-(x), \eta(u)) \le b_(x) + \epsilon\end{aligned}$$

and $|q(x, u, \eta(u))| \le d_\rho(x)$ for a.e. $x \in Q$ and all $(u, \eta(u)) \in R \times \mathbb{R}^k$, where $d_\rho \in L_2(Q)$ is independent of u since g (and hence g_1) grows at most linearly. Hence, by Lemma 3.2 with $p_+(x) = (1-t)b_+(x) + tg_+(x, u^+(x), \eta(u))$ and $p_-(x) = (1-t)b_-(x) + tg_-(x, -u^-(x), \eta(u))$, we get for some $c > 0$

$$\|Bu + (1-t)Cu + tGu\| \ge \delta \|u\|^2 \text{ for all } u \in V.$$

It is left to show that $\deg(Q_n(B+C), B_R \cap V_n, 0) \neq 0$ for all n. Let $\eta \in (0, r)$ be fixed. Then, for each $t \in [0,1]$, and a.e. $x \in Q$, we have that $0 \le (1-t)\eta + tb_\pm \le r$. It is easy to show that $p_+ = (1-t)\eta + tb_+$ and $p_- = (1-t)\eta + tb_-$ satisfy $0 \le p_\pm \le r$ for a.e. $x \in Q$, and

$$\int_Q [p_+(v^+)^2 + p_-(v^-)^2] > 0 \text{ for all } v \in N(-L_1 + \lambda_j I) \setminus \{0\}$$

and

$$\int_Q [r - p_+)(w^+)^2 + (r - p_-)(w^-)^2] > 0 \text{ for all } w \in N(-L_1 + \lambda_{j+1} I) \setminus \{0\}.$$

Hence, one gets that the equation

$$Bu + [(1-t)\eta + tb_+]u^+ + [(1-t)\eta + tb_-]u^- = 0 \tag{3.21}$$

has only the trivial solution for each $t \in [0,1]$. Since the homotopy given by (3.21) is A-proper, there is an $n \ge n_0$ such that for each $R > 0$ and all $n \ge n_0$,

$$\begin{aligned}&\deg(P_n(B + b_+(.)^+ - b_-(.)^-, B(0,R) \cap H_n, 0) \\ &= \ \deg(P_n(B + \eta I), B(0,R) \cap H_n, 0) = \pm 1\,.\end{aligned}$$

Hence, (3.2) is solvable in V by Theorem 2.3. The other assertions also follow from this theorem.

Remark 3.2 Conditions (3.18)-(3.19) hold for a wide class of nonlinearities g. For example, they are implied by $\lambda_j < \lambda_j + \epsilon \le \gamma_+(x) \le \Gamma_+(x) \le \lambda_{j+1}$ and $\lambda_j \le \gamma_-(x) \le \Gamma_-(x) \le \lambda_{j+1} - \epsilon < \lambda_{j+1}$, or $\lambda_j \le \gamma_+(x) \le \Gamma_+(x) \le \lambda_{j+1} - \epsilon < \lambda_{j+1}$ and $\lambda_j < \lambda_j + \epsilon \le \gamma_-(x) \le \Gamma_-(x) \le \lambda_{j+1}$, in the case when the eigenfunctions associated to λ_j and λ_{j+1} change sign in Q.

Next, we shall give some concrete assumptions on f and g that imply (F)-(H) in Theorem 3.1.

Theorem 3.2 *Assume that (L1) and (B1) hold, B have properties I and II and there be functions $\gamma_\pm, \Gamma_\pm \in L_\infty(Q)$ such that for some $j \in J \subset Z$, one has*

$$\lambda_j \leq \gamma_\pm(x) \leq \Gamma_\pm(x) \leq \lambda_{j+1} \ \textit{for a.e. } x \in Q$$

$$\int_Q [(\gamma_+ - \lambda_j)(v^+)^2 + (\gamma_- - \lambda_j)(v^-)^2] > 0 \textit{ for all } v \in N(L_1 - \lambda_j I) \setminus \{0\},$$

$$\int_Q [(\lambda_{j+1} - \Gamma_+)(w^+)^2 + (\lambda_{j+1} - \Gamma_-)(w^-)^2] > 0 \textit{ for all } w \in N(L_1 - \lambda_{j+1} I) \setminus \{0\}.$$

Suppose that for the $\epsilon > 0$ and $\delta > 0$ given in Lemma 2.2,

(G1) there is $\rho > 0$ such that for a.e. $x \in Q$, $\eta(u) \in \mathbb{R}^k$

$$\gamma_+(x) - \epsilon \leq g(x, u, \eta(u)) \leq \Gamma_+(x) + \epsilon \ \textit{ if } \ u > \rho$$
$$\gamma_-(x) - \epsilon \leq g(x, u, \eta(u)) \leq \Gamma_-(x) + \epsilon \ \textit{ if } \ u < -\rho$$

(G2) There are functions $b(x) \in L_\infty(Q)$ and $k_s(x) \in L_2(Q)$ for each $s > 0$ such that

$$|g(x, u, \eta(u))| \leq sb(x)(\sum_{|\alpha| \leq 2m-1} |D^\alpha u|^2)^{1/2} + k_s(x), u \in V.$$

(F1) There are functions $a(x) \in L_\infty(Q)$ and $d_r(x) \in L_2(Q)$ for each $r > 0$ such that

$$|f(x, \xi(u))| \leq ra(x)(\sum_{|\alpha| \leq 2m} |D^\alpha u|^2)^{1/2} + d_r(x), \ \textit{ for all } \ u \in V.$$

(F2) There is a constant $k > 0$ such that $k \leq c$ and

$$|f(x, \eta, \zeta) - f(x, \eta, \zeta')| \leq k \sum_{|\alpha| = 2m} |\zeta_\alpha - \zeta'_\alpha|$$

for a.e. $x \in Q$, all $\eta \in \mathbb{R}^k$ and $\zeta, \zeta' \in R^{s'_{2m}} = R^{s_{2m}} - R^{s_{2m-1}}$, where c is a constant in (3.1).

Then there is a $u \in V$ that satisfies (3.2) for a.e. $x \in Q$. If $k < c$, then all other assertions of Theorem 3.1 also hold.

Proof. It is easy to see that (F) of Theorem 3.1 holds. Hence, it remains to verify (H) of that theorem, i.e. that $H_t = B - tF$ is A-proper with respect to Γ_B for each $t \in [0, 1)$ and H_1 is pseudo A-proper. Since the embedding of V into H is compact, it suffices to show these facts for $F_t = L - tF$. Set $B_\mu = B - \mu I$ for some $\mu \neq \lambda_j$ for each j. Then, for each $t \in [0, 1]$, it follows from (F2), the Holder inequality, and an easy calculation that

$$(F_t u - F_t v, B_\mu(u - v)) \geq (1 - k/c)\|B_\mu(u - v)\|^2 + \phi(u - v) \tag{3.22}$$

where the functional $\phi : V \to R$ is given by

$$\phi(u-v) = t(M(u,v) - M(v,v), B_\mu(u-v)) + \mu(u-v, B_\mu(u-v)),$$

with $M : V \times V \to H$ being the continuous form $M(u,v) = f(x, \eta(u), \zeta(v))$. The functional ϕ is weakly continuous. Indeed, let $u_k \to u$ weakly in V. Then $u_k \to u$ in the W_2^{2m-1}-norm by the Sobolev imbedding theorem and by the results from [10], it is not hard to show that $\phi(u_k - u) \to 0$ as $k \to \infty$. If $k < c$, then (F2) implies that F_t is A-proper with respect to Γ_B (see, e.g., in [16]-[18]). If $k = c$, then F_t is again A-proper for each $t \in [0,1)$ and it is easy to see that F_1 is pseudo L_μ-monotone. Hence, F_1 is pseudo A-proper with respect to Γ_B ([18]) and (H) of Theorem 3.1 holds.

Corollary 3.1 *Let the conditions of Theorem 3.2 hold with (G1) replaced by*

(G1')
$$\gamma_\pm(x) \le \liminf_{u \to \pm\infty} g(x,u,\eta(u)) \le \limsup_{u \to \pm\infty} g(x,u,\eta(u)) \le \Gamma_\pm(t,x)$$

uniformly for a.e. $(x, \eta(u)) \in Q \times \mathbb{R}^k$. *Then there is a* $u \in V$ *that satisfies (3.2) for a.e.* $x \in Q$.

Proof. It is easy to see that (G1') implies (G1).

4 Strong solvability of elliptic BVP's

A. We shall apply the results of Section 3 to strong solvability of elliptic boundary-value problems in V of the form

$$\sum_{|\alpha| \le 2m} A_\alpha(x) D^\alpha u(x) + g(x, u, Du, \dots, D^{2m-1}u)u + f(x, u, Du, \dots, D^{2m}) = h, \tag{4.1}$$

under non-uniform non-resonance conditions. Here $Q \subset R^n$ is a bounded smooth domain, V is a closed subspace of $W_2^{2m}(Q)$ containing the test functions, the linear part is elliptic and $h \in L_2(Q)$. Assume the linear map $L : V \to L_2(Q)$, induced by the linear elliptic operator in (4.1), has closed range in $H = L_2(Q)$ and satisfies conditions (L1), (B1) in Section 3 with $B = -L + \lambda_j I$. Here, $L_1 = L$ and $A_1 = 0$.

Let $\lambda \ne \lambda_j$ for each $j = 1, 2, \dots$ be fixed, $\Gamma = \{Y_n, Q_n\}$ be a projectionally complete scheme for L_2 and $X_n = (B - \lambda I)^{-1}(Y_n) \subset V$ for each n. Then $\Gamma_L = \{X_n, Y_n, Q_n\}$ is an admissible or a projectionally complete scheme for (V, L_2). Since $B - \lambda I : V \to L_2$ is linear, one-to-one and A-proper with respect to Γ_L, there is a constant $c > 0$ (depending) only on λ) such that

$$\|(B - \lambda I)u\| \ge c\|u\|_V, \quad u \in V. \tag{4.2}$$

Theorem 4.1 *Let $B = -L + \lambda_j I$ be a closed densely defined map in H such that $R(B) = N(B)^\perp$, $(Bu, u) \geq -r^{-1}\|Bu\|^2$ on V and if $(Bu, u) = -r^{-1}\|Bu\|^2$ for some $u \in V$, then $u \in N(-L + \lambda_j I) \oplus N(-L + \lambda_{j+1} I)$. Suppose that there are functions $\gamma_\pm, \Gamma_\pm \in L_\infty(Q)$ such that for some $j \in J \subset Z$, one has*

$$\lambda_j \leq \gamma_\pm(x) \leq \Gamma_\pm(x) \leq \lambda_{j+1} \text{ for a.e. } x \in Q$$

and

$$\int_Q [(\gamma_+ - \lambda_j)(v^+)^2 + (\gamma_- - \lambda_j)(v^-)^2] > 0 \text{ for all } v \in N(L - \lambda_j I) \setminus \{0\}$$

and

$$\int_Q [(\lambda_{j+1} - \Gamma_+)(w^+)^2 + (\lambda_{j+1} - \Gamma_-)(w^-)^2] > 0 \text{ for all } w \in N(L - \lambda_{j+1} I) \setminus \{0\}.$$

Suppose that for $\epsilon > 0$ and $\delta > 0$ given in Lemma 3.2,

(G1) there is $\rho > 0$ such that for a.e. $x \in Q$

$$\gamma_+(x) - \epsilon \leq g(x, u, \eta(u)) \leq \Gamma_+(x) + \epsilon \text{ if } u > \rho, \eta(u) \in \mathbb{R}^k$$
$$\gamma_-(x) - \epsilon \leq g(x, u, \eta(u)) \leq \Gamma_-(x) + \epsilon \text{ if } u < -\rho, \eta(u) \in \mathbb{R}^k$$

(G2) There are functions $b(x) \in L_\infty(Q)$ and $k_s(x) \in L_2(Q)$ for each $s > 0$ such that

$$|g(x, u, \eta(u))| \leq sb(x)(\sum_{|\alpha| \leq 2m-1} |D^\alpha u|^2)^{1/2} + k_s(x), u \in V.$$

(F) $\|Fu\| = \|f(x, u, \dots, D^{2m}u)\| \leq \beta\|u\|_V + \gamma$ for some $\beta \in (0, \delta)$, $\gamma > 0$.

(H) $H_t = L - tF : V \to H$ is A-proper with respect to Γ_L for $t \in [0, 1)$ and $L - F$ is pseudo A-proper.

Then (4.1) has a solution $u \in V$ for each $h \in L_2$. If $L - F$ is A-proper, $S(h) = (L - F)^{-1}(\{h\})$ is compact for each $h \in L_2$ and card $S(h)$ is constant, finite and positive on each connected component of the set $L_2 \setminus (L - F)(\Sigma)$.

Proof. It follows from Theorem 3.1 with $L_1 = L$ and $A_1 = 0$. ◊

As before, we give now some concrete conditions on f, g so that (H) holds.

Theorem 4.2 *Let $B = -L + \lambda_j I$ be a closed densely defined map in H such that $R(B) = N(B)^\perp$, $(Bu, u) \geq -r^{-1}\|Bu\|^2$ on V and if $(Bu, u) = -r^{-1}\|Bu\|^2$ for some $u \in V$, then $u \in N(-L + \lambda_j I) \oplus N(-L + \lambda_{j+1} I)$. Suppose that there are functions $\gamma_\pm, \Gamma_\pm \in L_\infty(Q)$ such that for some $j \in J \subset Z$, one has*

$$\lambda_j \leq \gamma_\pm(x) \leq \Gamma_\pm(x) \leq \lambda_{j+1} \text{ for a.e. } x \in Q\,,$$

$$\int_Q [(\gamma_+ - \lambda_j)(v^+)^2 + (\gamma_- - \lambda_j)(v^-)^2] > 0 \text{ for all } v \in N(L - \lambda_j I) \setminus \{0\}\,,$$

$$\int_Q [(\lambda_{j+1} - \Gamma_+)(w^+)^2 + (\lambda_{j+1} - \Gamma_-)(w^-)^2] > 0 \text{ for all } w \in N(L - \lambda_{j+1} I) \setminus \{0\}.$$

Furthermore, suppose that for the $\epsilon > 0$ and $\delta > 0$ given in Lemma 3.2, *(G1) there is $\rho > 0$ such that for a.e. $x \in Q$*

$$\gamma_+(x) - \epsilon \le g(x,u,\eta(u)) \le \Gamma_+(x) + \epsilon \ \text{ if } \ u > \rho, \eta(u) \in \mathbb{R}^k$$
$$\gamma_-(x) - \epsilon \le g(x,u,\eta(u)) \le \Gamma_-(x) + \epsilon \ \text{ if } \ u < -\rho, \eta(u) \in \mathbb{R}^k$$

(G2) There are functions $b(x) \in L_\infty(Q)$ and $k_s(x) \in L_2(Q)$ for each $s > 0$ such that

$$|g(x,u,\eta(u))| \le sb(x)(\sum_{|\alpha|\le 2m-1} |D^\alpha u|^2)^{1/2} + k_s(x), u \in V.$$

(F1) There are functions $a(x) \in L_\infty(Q)$ and $d_r(x) \in L_2(Q)$ for each $r > 0$ such that

$$|f(x,\xi(u))| \le ra(x)(\sum_{|\alpha|\le 2m} |D^\alpha u|^2)^{1/2} + d_r(x), \ \text{ for all } \ u \in V.$$

(F2) There is a constant $k > 0$ such that $k \le c$ and

$$|f(x,\eta,\zeta) - f(x,\eta,\zeta')| \le k \sum_{|\alpha|=2m} |\zeta_\alpha - \zeta'_\alpha|$$

for a.e. $x \in Q$, all $\eta \in \mathbb{R}^k$ and $\zeta, \zeta' \in R^{s'_{2m}} = R^{s_{2m}} - R^{s_{2m-1}}$, where c is a constant in (4.2).

Then there is a $u \in V$ that satisfies (4.1) for a.e. $x \in Q$ and all other assertions of Theorem 4.1 are valid if $k < c$.

Proof. It follows from Theorem 4.1 with $L_1 = L$ and $A_1 = 0$. ◇

For our next result, we assume also

(L2) There is an integer $j \ge 1$ such that $\lambda_j < \lambda_{j+1}$ and $Lw = \lambda_k w$ for $k = j$ and $k = j+1$, has the continuation property, that is if $w(x) = 0$ on a set of positive measure, then $w(x) = 0$ a.e. on Q.

Theorem 4.3 *Let L satisfy (L1)-(L2) and (B1) with $B = -L + \lambda_j I$ and let $\gamma(x), \Gamma(x) \in L_\infty(Q)$ be such that*

(H1) $\lambda_j \le \gamma(x) \le \Gamma(x) \le \lambda_{j+1}$ with $\operatorname{meas}\{x \in Q | \lambda_j \ne \gamma(x)\} > 0$ and $\operatorname{meas}\{x \in Q | \lambda_{j+1} \ne \Gamma(x)\} > 0$.

Suppose that (G1) of Theorem 3.4 holds and for $\epsilon > 0$ and $\delta > 0$ given by Lemma 3.2

(H2) $\gamma(x) - \epsilon \le g(x,\xi) \le \Gamma(x) + \epsilon$ for all $(x,\xi) \in Q \times R^{s_{2m-1}}$

(H3) $\|Fu\| = \|f(x,u,\ldots,D^{2m}u)\| \le \beta\|u\|_V + \gamma$ for some $\beta \in (0,\delta)$, $\gamma > 0$.

(H4) $H_t = L - tF$ is A-proper with respect to Γ_L for $t \in [0,1)$ and $L - F$ is pseudo A-proper.

Then (4.1) has a solution $u \in V$ and all other assertions of Theorem 4.1 are valid.

Proof. Clearly, (L2) and (H1) imply the integral inequalities in Theorem 4.2. Hence, the conclusion follows from this theorem.

Theorem 4.4 *Let L and $\gamma(x)$, $\Gamma(x)$ be as in Theorem 4.3. Let $f : Q \times R^{s_{2m}} \to R$ and $g : Q \times R^{s_{2m-1}} \to R$ be Caratheodory functions such that*

(F1) There are functions $a(x) \in L_\infty(Q)$ and $d_r(x) \in L_p(Q)$ for each $r > 0$ such that

$$|f(x, \xi(u))| \leq ra(x)(\sum_{|\alpha|\leq 2m} |D^\alpha u|^2)^{1/2} + d_r(x), \quad \textit{for all} \ \ u \in V.$$

(F2) There is a constant $k > 0$ such that $k \leq c$ and

$$|f(x, \eta, \zeta) - f(x, \eta, \zeta')| \leq k \sum_{|\alpha|=2m} |\zeta_\alpha - \zeta'_\alpha|$$

for a.e. $x \in Q$, all $\eta \in \mathbb{R}^k$ and $\zeta, \zeta' \in R^{s'_{2m}} = R^{s_{2m}} - R^{s_{2m-1}}$.

(G1) $\lambda_j \leq \gamma(x) \leq liminf_{|u|\to\infty} g(x, u, \eta(u)) \leq limsup_{|u|\to\infty} g(x, u, \eta(u)) \leq \Gamma(x) \leq \lambda_{j+1}$ uniformly for $x \in Q$ and the non-u components $\eta(u)$.

(G2) There are functions $b(x) \in L_\infty(Q)$ and $k_s(x) \in L_p(Q)$ for each $s > 0$ such that

$$|g(x, u, \eta(u))| \leq sb(x)(\sum_{|\alpha|\leq 2m-1} |D^\alpha u|^2)^{1/2} + k_s(x), u \in V.$$

Then there is a $u \in V$ that satisfies Eq. (4.1) for a.e. $x \in Q$ and all other assertions of Theorem 4.1 are valid if $k < c$.

Proof. It follows from Theorem 4.2, as in the case of Corollary 3.1. ♢

Theorem 4.2 extends the existence result of Beresticki-de Figueiredo [3] who assumed $f = 0$ and g to depend only on u. A simplified proof of their results has been given by Mawhin [13]. If f does not depend on derivatives of order $2m$, the existence part of Theorem 4.4 reduces to a result of Mawhin-Ward [14]. Their proofs are based on the Leray-Schauder and the coincidence degree theories respectively.

B. In this subsection we shall look at boundary value problems

$$Lu = \lambda_1 u + g(x, u) = h, \ \text{ in } \ Q, \ \ u|_{\partial Q} = 0 \tag{4.3}$$

where L is either selfadjoint or non-selfadjoint second order elliptic partial differential operator, and λ_1 is the first (resp. principal) eigenvalue of the selfadjoint (resp. nonselfadjoint) operator $-L$, $h \in L_p(Q)$ with $p > n$ and $g : Q \times R \to R$ is a Caratheodory function which grows at most linearly, i.e. there are a constant $c_1 > 0$ and a function $c_2 \in L_p(Q)$, $p > n$, such that

$$|g(x, u)| \leq c_1|u| + c_2(x)$$

for a.e. $x \in Q$ and all $u \in R$. We assume that L is such that the Bony's maximum principal (see eg. [4, 2]) and the abstract Krein-Rutman theorem [11] imply the existence of a real simple eigenvalue $\lambda_1 > 0$ of

$$-Lu = \lambda_1 u, \ \ u|_{\partial Q} = 0$$

of minimal modulus such that there is a corresponding smooth eigenfunction $w_1 > 0$ in Q and $\partial w_1/\partial\eta < 0$ on ∂Q, where $\partial/\partial\eta$ stands for the outward normal derivative. Moreover, if L is nonseladjoint then λ_1 is also an eigenvalue for the adjoint problem

$$-L^* u = \lambda_1 u, \ \ u|_{\partial Q} = 0,$$

such that there is a corresponding smooth eigenfunction $w_1^* > 0$ in Q and $\partial w_1^*/\partial\eta < 0$ on ∂Q.

Now, using Lemma 3.3, we shall prove the following existence result for (4.3) when the nonlinearity $f(x,u) = \lambda_1 u + g(x,u)$ "lies" between the first two eigenvalues λ_1 and λ_2. We assume, without loss of generality, that the following upper bounds are nonnegative

$$g_+(x) = \limsup_{u\to\infty} g(x,u)/u \le \Gamma_+(x), \ \ \text{a.e. on } \ Q \tag{4.4}$$

$$g_-(x) = \limsup_{u\to-\infty} g(x,u)/u \le \Gamma_-(x), \ \ \text{a.e. on } \ Q. \tag{4.5}$$

Since g grows linearly, we can suppose, without loss of generality, that $\Gamma_\pm \in L_p(Q)$, $p > n$.

Theorem 4.5 *Let $g : Q \times R \to R$ be a Caratheodory function that grows linearly, $g_+(x)$ and $g_-(x)$ are different from zero on a set of nonzero measure, and*

$$g(x,u)u \ge 0 \tag{4.6}$$

for a.e. $x \in Q$ and all $u \in R$. Suppose that (4.4)-(4.5) hold and

$$0 \le \Gamma_\pm(x) \le r (= \lambda_2 - \lambda_1), \quad \textit{for a.e.} x \in Q, \tag{4.7}$$

$$\int_{w>0} [r - \Gamma_+] w^2 dx + \int_{w<0} [r - \Gamma_-] w^2 dx > 0, \quad \textit{for all}, \ \ w \in N(L + \lambda_2 I) \setminus \{0\}. \tag{4.8}$$

Then Eq. (4.3) has at least one solution $u \in W_p^2(Q) \cap H_0^1(Q)$, $p > n$, for each $h \in L_p(Q)$. Moreover, $u \in C^{1,\mu}(\bar{Q})$.

Proof. Let γ be a fixed constant with $0 < \gamma < r$ and define the operator $E : W_p^2(Q) \cap H_0^1(Q) \subset C^1(\bar{Q}) \to L_p(Q)$ by

$$Eu = Lu + \lambda_1 u + ru\,.$$

We shall show that there exists a constant $C > 0$ independent of t such that $\|u\|_{C^1} \leq C$ for all possible solutions $u \in W_p^2(Q) \cap H_0^1(Q)$ of the homotopy

$$H(t,u) = Lu + \lambda_1 u + (1-t)\gamma u + tg(x,u) = th, t \in [0,1). \tag{4.9}$$

Clearly, (4.9) has only the trivial solution for $t = 0$. If such a C does not exist, then there exist $t_k \in (0,1)$ and $u_k \in W_p^2(Q)$ such $\|u_k\| \to \infty$ and

$$Eu_k = t_k[\gamma u_k - g(t_k, u_k) + h(x)], \;\; u|_{\partial Q} = 0. \tag{4.10}$$

Set $v_k = u_k/\|u_k\|_{C^1}$. Then, (4.10) becomes

$$Ev_k = t_k[\gamma v_k - g(x,u_k)/\|u_k\|_{C^1} + h/\|u_k\|_{C^1}], \;\; v_k|_{\partial Q} = 0. \tag{4.11}$$

We may assume that $t_k \to t$ and $g(x,u_k)/\|u_k\|_{C^1} \rightharpoonup K(x)$ in $L_p(Q)$ since g has a linear growth. Since $g(x,u_k)/\|u_k\|_{C^1} = g(x,u_k)/u_k(x).v_k(x) \rightharpoonup G(x)v(x)$ with $G(x) \neq 0$ on a set of positive measure, we get that $K(x) = G(x)v(x) \neq 0$ on a set of positive measure. Using L_p-estimate and the compact embedding of $W_p^2(Q)$ into $C^1(\bar{Q})$, we can deduce from (4.11) that $v_k \to v$ in $C^1(\bar{Q})$, $\|v\|_{C^1} = 1$ and $v|_{\partial Q} = 0$. Moreover, $\{Lv_k\}$ is also bounded in $L_p(Q)$ by (4.11). Hence, by the reflexivity of L_p and the weak closedness of L, we may assume that $Lv_k \rightharpoonup Lv$ in L_p with $v \in W_p^2(Q) \cap H_0^1(Q)$ and v solves the equation

$$Ev = t[\gamma v - K(x)], \;\; v|_{\partial Q} = 0. \tag{4.12}$$

As in [9], Eq. (4.12) is equivalent to

$$Lv + \lambda_1 v + p_+(x)v^+ - p_-(x)v^- = 0, \; v|_{\partial Q} = 0 \tag{4.13}$$

where $p_+(x) = (1-t)\gamma + tk_v^+(x)$ and $p_-(x) = (1-t)\gamma + tk_v^-(x)$ and $k_v(x) = K(x)/v(x)$ if $v(x) \neq 0$ and $k_v = 0$ if $v(x) = 0$ since $0 \leq k_v(x) \leq \Gamma_+(x)$ if $v(x) > 0$ and $0 \leq k_v(x) \leq \Gamma_-(x)$ if $v(x) < 0$. Hence, by Lemma 3.3 (or Lemma 1 in [9]), we get that $v \in N(L + \lambda_1 I) \setminus \{0\}$.

Next, passing to the limit in

$$(Lv_k + \lambda_1 v_k + (1-t)\gamma v_k + t_k g(x,u_k)/\|u_k\|_{C^1}, v_k) = (t_k h/\|u_k\|_{C^1}, v_k)$$

we get

$$((1-t)\gamma v + tK, v) = 0.$$

Note that $t \neq 1$, for otherwise (4.12)-(4.13) imply that $p_+(x)v^+ - p_-(x)v^- = 0$ which leads to $K(x) = 0$ a.e. on Q, a contradiction. Hence, $(K,v) < 0$ since $(1-t)\gamma\|v\|^2 = -t(K,v)$. This contradicts the fact that $0 \leq t_k(g(x,u_k)/\|u_k\|_{C^1}, v_k) \to t(K,v)$. Hence, we have shown that all solutions of (4.9) are bounded and, by the Leray-Schauder homotopy theorem, Eq (4.3) has a solution in $W_p^2(Q) \cap H_0^1(Q)$ for each $p \in L_p(Q)$. $\diamondsuit$

Theorem 4.5 extends Theorem 1 in Iannacci-Nkashama-Ward [9] who showed the solvability of Eq (4.3) only for $h \in L_p(Q)$ that are orthogonal to w_1 but without assuming that $g_+(x)$ and $g_-(x)$ are not zero on a set of positive measure. On the other hand, their result extends some earlier ones of de Figueiredo and Ni [6], Gupta [7] and others. As in Theorems 4.4, it will be shown elsewhere that Theorem 4.5 can be extended to include nonlinearities depending on derivatives up to the second order.

5 Time periodic solutions of BVP's for nonlinear parabolic and hyperbolic equations

The semi-abstract results in Section 3 have been used in [21] to prove the existence and the number of solutions of generalized periodic solutions (GPS), under nonuniform nonresonance conditions, for the nonlinear parabolic equation

$$u_t + A_0 u + g(t, x, u, u_t, D_x u, \ldots, D_x^{2m-1} u)u + f(t, x, u, u_t, D_x u, \ldots, D_x^{2m} u) = h$$

in $H = L_2(\Omega)$, where $\Omega = [0, 2\pi] \times Q$ with $Q \subset R^n$, A_0 is a uniformly strongly elliptic operator of order $2m$ in $x \in Q$ for each $t \in [0, 2\pi]$, and the nonlinear hyperbolic equations with damping

$$\begin{aligned} \sigma u_t + u_{tt} + A_0 u &= g(t, x, u, u_t, u_{tt}, D_x u, \ldots, D_x^{2m-1} u)u \\ &\quad + f(t, x, u, u_t, u_{tt}, D_x u, \ldots, D_x^{2m} u) + h \end{aligned}$$

with h in H, $\sigma \neq 0$, boundary conditions

$$u(t, .) \in H_0^m(Q) \text{ for all } t \in (0, 2\pi)\,,$$

and periodicity conditions

$$u(0, x) = u(2\pi, x) \text{ for all } x \in Q\,.$$

These results extend the corresponding existence results in Nkashama-Willem [22], who assumed only the u dependence in g and $f = 0$ and used the coincidence degree theory.

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