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A two dimensional Hammerstein problem: The linear case *

Jun Hua & James L. Moseley

Abstract

Nonlinear equations of the form $L[u] = \lambda g(u)$ where L is a linear operator on a function space and g maps u to the composition function $g \circ u$ arise in the theory of spontaneous combustion. We assume L is invertible so that such an equation can be written as a Hammerstein equation, $u = B[u]$ where $B[u] = \lambda L^{-1}[g(u)]$. To investigate the importance of the growth rate of g and the sign and magnitude of λ on the number of solutions of such problems, in a previous paper we considered the one-dimensional problem $L(x) = \lambda g(x)$ where $L(x) = ax$. This paper extends these results to two dimensions for the linear case.

1 introduction

We wish to investigate the number of solutions (and their computation) to problems of the form

$$L[u] = \lambda g(u) \tag{1.1}$$

where $L : V \to W$ is a linear operator and V and W are function spaces whose domains are the same set, say D, and whose codomains are the real numbers $\mathbb{R}$. If $u \in V$ and $x \in D$, then the value of the function $g(u)$ at x is $g(u(x))$ where $g : \mathbb{R} \to \mathbb{R}$. Thus we use the symbol g for a real valued function of a real variable as well as for the (nonlinear Nemytskii) operator from V to W that this function defines by the composition $g \circ u$. An example is

$$\begin{gathered} -\Delta u = \lambda e^u \quad \vec{x} = [x, y, z]^T \in \Omega \subseteq \mathbb{R}^3 \\ u(\vec{x}) = 0 \quad \vec{x} \in \partial\Omega \end{gathered} \tag{1.2}$$

Here L is the negative of the Laplacian operator in three spacial dimensions with homogeneous Dirichlet boundary conditions, $\vec{x}$ is a point in $\mathbb{R}^3$, Ω is an open connected region in $\mathbb{R}^3$, $\partial\Omega$ is its boundary, $V = \{u \in V_1 : u(\vec{x}) = 0 \ \forall \vec{x} \in \partial\Omega\}$ where $V_1 = C^2(\Omega, \mathbb{R}) \cap C(\bar{\Omega}, \mathbb{R})$, and $W = C(\bar{\Omega}, \mathbb{R})$. Hence $D = \bar{\Omega}$. Such problems arise in the theory of combustion [1, 3, 4, 5, 6]. For this problem in

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$\mathbb{R}^2$, it is known that for $\lambda < 0$, there exists a unique solution. However, for $\lambda > 0$ and large, there is no solution. But for $\lambda > 0$ and small, there are at least two solutions. If $\lambda = 0$, the solution set is the null space of L and since L is invertible, the problem has only the trivial solution $u = 0$. If $g(u) = u$, (1.1) is a spectral problem for L. Hence (1.1) is sometimes referred to as a nonlinear eigenvalue problem.

We assume L is invertible so that (1.1) can be written as the Hammerstein equation

$$u - \lambda L^{-1}[g(u)] = 0. \tag{1.3}$$

A solution of (1.3) is a fixed point of the combined operator $B = \lambda(L^{-1} \circ g)$. The level of difficulty of problems of type (1.1) or (1.3) varies greatly depending on the number of elements in D, the value of n, and the operator L. We list several categories, starting with the easiest.

1. One dimensional problems (i.e., D contains only one element).
2. Multidimensional problems (i.e., D is finite, but contains two or more elements).
3. Infinite dimensional problems with $D \subseteq \mathbb{R}(n = 1)$ and L a first, second, or higher order differential operator.
4. Infinite dimensional problems with $D \subseteq \mathbb{R}^n$, $n = 2, 3, 4, \ldots$ and L a first, second, or higher order partial differential operator.

Since L is linear, we at most have linear coupling and often this coupling is weak. The coupling of L^{-1} may be stronger than the coupling of L and is a reason to examine (1.1) directly even when L is invertible. To investigate the fundamental importance of the growth rate of g and the sign and magnitude of λ on the number of solutions to problems of this type, in a previous paper [2] we considered the one dimensional nonlinear eigenvalue problem

$$ax = \lambda g(x). \tag{1.4}$$

Here $L : \mathbb{R} \to \mathbb{R}$ is $L(x) = ax$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and a and λ are parameters. (If $a \neq 0$, L is invertible.) To this end, we first considered two types of behavior for a continuous function $f : \mathbb{R} \to \mathbb{R}$ (i.e., $f \in C(\mathbb{R}, \mathbb{R})$ consisting of continuous $\forall x \in \mathbb{R}\}$), linear and quadratic. For L invertible ($a \neq 0$), we let $f(x) = x - kg(x)$ where $k = \lambda/a$. Although less restrictive conditions on f can be obtained, for convenience we assumed that f has a continuous second derivative for all x in $\mathbb{R}$; that is, $f \in C^2(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f''(x)$ exists and is continuous $\forall x \in \mathbb{R}\}$. We were interested in sufficient conditions on f that will determine the number of solutions of the equation

$$f(x) = 0. \tag{1.5}$$

The obvious advantage of considering the scalar equation (one dimensional problem) (1.4) or (1.5) over an abstract Hammerstein equation or a Hammerstein equation of the type (1.1) where D is finite or infinite dimensional is that much more (often everything) can be said for many functions $g(x)$ (and classes of functions). However, the techniques investigated here are quite different from the standard fixed point theorems (e.g., contraction mapping theorem and the Brouwer and Schauder fixed point theorems) and are expected to reveal distinctive new results when extended to higher dimensions including methods for solving multidimensional problems. In this paper we obtain results for the linear case of a two dimensional problem by reducing it to a scalar problem of the form (1.5).

2 Two dimensional problem

We consider the system of two equations:

$$ax + by = \lambda g(x) \tag{2.1}$$

$$cx + dy = \lambda g(y) \tag{2.2}$$

where $a, b, c, d, \lambda \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. These scalar equations can be written as the vector or matrix equation

$$A\vec{x} = \lambda \vec{g}(\vec{x}) \tag{2.3}$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{g}\,(\vec{x}) = \begin{bmatrix} g(x) \\ g(y) \end{bmatrix}.$$

If A is invertible, (2.3) can be written as $\lambda A^{-1}\vec{g}(\vec{x}) = \vec{x}$, $\lambda A^{-1}\vec{g}(\vec{x}) - \vec{x} = \vec{0}$, or

$$\vec{f}\,(\vec{x}) = \vec{0} \tag{2.4}$$

where $\vec{f}\,(\vec{x}) = \lambda \vec{g}\,(\vec{x}) - A\vec{x}$ or $\vec{f}\,(\vec{x}) = \lambda A^{-1}\vec{g}(\vec{x}) - \vec{x}$ if A is invertible. The solution set for (2.3) and (2.4) is $S = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{f}(\vec{x}) = \vec{0} \right\}$.

Our analysis proceeds in two steps. Since we only have linear coupling, we first use case analysis to establish that the process of algebraic elimination on the equations (2.1) and (2.2) can always be used to obtain a single equation in one variable whose solutions are one component of a solution to (2.3). Having found one component, we can then substitute into one of the equations (2.1) and (2.2) to obtain a single equation in the other variable whose solutions are the other component of a solution to (2.3). As our second step in the solution process, for each such scalar equation of type (1.5), we then consider sufficient conditions that establish f as being in the linear case (or f having the linear property). We focus mainly on the case where $b = c \neq 0$, $a = d \neq 0$ and $\lambda \neq 0$.

3 Linear case

For convenience in the linear case, we assume $f \in C^1(\mathbb{R}, \mathbb{R})$ and some of the following hypotheses:

H1 $\lim_{x\to\infty} f(x) = +\infty$

H2 $\lim_{x\to\infty} f(x) = -\infty$

H3 $\lim_{x\to-\infty} f(x) = -\infty$

H4 $\lim_{x\to-\infty} f(x) = +\infty$

H5 $f'(x) > 0$ (so that $f$is strictly increasing)

H6 $f'(x) < 0$ (so that f is strictly decreasing).

We consider properties that $f : \mathbb{R} \to \mathbb{R}$ and $F(x) = ax + b$ $(a \neq 0)$ may have in common.

Definition 3.1. If $f(x)$ satisfies H1 and H3, we say that it is **mainly increasing**. If it satisfies H1, H3, and H5, we say that it is **consistently increasing**. On the other hand, if $f(x)$ satisfies H2 and H4, we say it is **mainly decreasing**. If it satisfies H2, H4, and H6, we say that it is **consistently decreasing**. If $f(x)$ is mainly increasing or decreasing, then we say $f(x)$ is **mainly monotonic**. If $f(x)$ is consistently increasing or decreasing, then we say f(x) is **consistently monotonic**.

For completeness we review a theorem that establishes the existence and uniqueness of solutions to (1.5) when $f(x)$ is mainly or consistently monotonic. Like the Jordan Curve Theorem, it is geometrically obvious and an analytic proof can be given [9]:

Theorem 3.1 *If a function f is mainly monotonic, then for any c in $\mathbb{R}$, there exists at least one x in $\mathbb{R}$ such that $f(x) = c$. If the function is consistently monotonic, then for any c in $\mathbb{R}$, there exists exactly one x in $\mathbb{R}$ such that $f(x) = c$.*

If $f(x)$ is consistently monotonic it is similar to the linear function $F(x) = ax + b$ $(a \neq 0)$ in that (1.5), like $F(x) = 0$, has exactly one solution. We say that (1.5) is in the **linear case** and that $f(x)$ has the **linear property**.

4 Reduction to a scalar equation

In the nonlinear equations (2.1) and (2.2) the coupling between the equations is restricted to the linear operator. In this section, we show that these equations can always be decoupled so that we always wish to first solve a nonlinear scalar equation, say in the variable x. Having obtained all solutions x, we may then substitute these into a second equation (e.g., (2.1) if $b \neq 0$) and solve for y. We first consider the easy cases where specific parameters are zero. We then focus on the case where $\lambda \neq 0$, $b \neq 0$, $c \neq 0$, $b = c$, $a = d$, $k = \lambda/b$, and $m = a/b$. If $\lambda = 0$, the problem reduces to finding the null space N_A of the matrix A. If

$\det(A) = ad - bc \neq 0$, then A^{-1} exists and $N_A = \{\vec{0}\}$. If $ad - bc = 0$, then N_A is one dimensional unless $a = b = c = d = 0$. If A is the zero matrix, the solution set S is just $\mathbb{R}^2$. Unless specifically noted, for the rest of this paper we assume that $\lambda \neq 0$.

We now consider the cases where either b or c is zero. Either $b = 0$ or $c = 0$ provides an uncoupling (or one way coupling) of the equations. If $b = 0$, (2.1) is uncoupled from (2.2) (or one-way coupled since (2.2) still depends on x) and we have $ax = \lambda g(x)$. This equation was previously discussed [2] and conditions for the linear (and quadratic) property obtained. If a is also zero, we have that x must satisfy $g(x) = 0$. If $a \neq 0$, depending on the properties of $g(x)$ and the sign of $k_1 (k_1 = \lambda/a)$, cases were determined where there are zero, one, or two solutions. Assuming we have solved $x - k_1 g(x) = 0$ for a value of x, say $x = x_0$, we can substitute $x = x_0$ into (2.2) to obtain $cx_0 + dy = \lambda g(y)$. This equation is similar to $ax = \lambda g(x)$ but with a shift. Conditions for the linear and quadratic property can be obtained using the techniques given in Hua and Moseley [2]. If $c = 0$, (2.2) is uncoupled (or one-way coupled) and the procedure is similar to the case $b = 0$ except that we now solve for y in (2.2) first and then substitute into (2.1). If both b and c are zero, the system is completely decoupled and the equations can be solved separately. Interestingly, in the completely decoupled case, if each equation has two solutions, the system has four solutions.

For the rest of this paper we assume $b \neq 0$ and $c \neq 0$. Solving for b in (2.1), we have

$$y = (\lambda g(x) - ax)/b = (\lambda/b)g(x) - (a/b)x = k_1 g(x) - mx \tag{4.1}$$

where $k_1 = \lambda/b$ and $m = a/b$. Now, letting

$$\phi_1(x) = k_1 g(x) - mx \tag{4.2}$$

and substituting $y = \phi_1(x)$ into (2.2), we get

$$cx + d(\phi_1(x)) = \lambda g(\phi_1(x))\,. \tag{4.3}$$

Since c is non-zero, we divide both sides of (4.3) by c and let $n = d/c$, $k_2 = \lambda/c$ and

$$\phi_2(x) = k_2 g(x) - nx \tag{4.4}$$

to obtain

$$x - \phi_2(\phi_1(x)) = 0\,. \tag{4.5}$$

Now let

$$\begin{aligned} f(x) =& \phi_2(\phi_1(x)) - x \\ =& (\lambda/c)g((\lambda/b)g(x) - (a/b)x) - (\lambda/b)(d/c)g(x) + [(ad - bc)/(bc)]x\,. \end{aligned} \tag{4.6}$$

The solution set of

$$f(x) = 0 \tag{4.7}$$

depends on the parameters a, b, c, d, λ (or on m, n, k_1, and k_2) and the properties of the function $g(x)$ (or on the properties of the functions ϕ_1 and ϕ_2).

To establish uniqueness in the linear case we are interested in the monotonicity of $f(x)$. Although less restrictive conditions can be stated to achieve the results, for convenience we consider only the case where $g(x)$, and hence $\phi_1(x)$, $\phi_2(x)$ and $f(x)$, are in $C^1(\mathbb{R},\mathbb{R})$, the set of function such that $f'(x)$ exists and is continuous. This allows the use of simpler conditions on $f'(x)$. From (4.6), we have

$$\begin{aligned} f'(x) =&\phi_2'(\phi_1(x))\phi_1'(x) - 1 \\ =&k_1k_2[g'(\phi_1(x))][g'(x)] - k_2m[g'(\phi_1(x))] - k_1n[g'(x)] + (mn-1)\,. \end{aligned} \tag{4.8}$$

5 Reformulation when $b \neq 0$ and $c \neq 0$

In the remainder of this paper, we assume b and c are both nonzero. We may then rewrite (2.1) and (2.2) as

$$mx + y = k_1 g(x) \tag{5.1}$$

$$x + ny = k_2 g(y) \tag{5.2}$$

where $k_1 = \lambda/b$, $k_2 = \lambda/c$, $m = a/b$ and $n = d/b$. Then (2.3) can be rewritten as

$$B\vec{x} = \vec{k}\vec{g}(\vec{x}) \tag{5.3}$$

where:

$$B = \begin{bmatrix} m & 1 \\ 1 & n \end{bmatrix}, \quad \vec{k}\vec{g}(\vec{x}) = \begin{bmatrix} k_1g(x) \\ k_2g(x) \end{bmatrix}$$

If A is symmetric, $b = c$ so that $k_1 = k_2 = k$. Hence:

$$\vec{k}\vec{g}(\vec{x}) = \begin{bmatrix} kg(x) \\ kg(x) \end{bmatrix} = k\vec{g}(\vec{x})$$

and (5.3) becomes

$$k\vec{g}(\vec{x}) - B\vec{x} = \vec{0}. \tag{5.4}$$

If $\det B = mn - 1 \neq 0$, we have

$$kB^{-1}\vec{g}(\vec{x}) - \vec{x} = \vec{0} \tag{5.5}$$

or $\vec{f}(\vec{x}) = \vec{0}$, where $\vec{f}(\vec{x}) = k\vec{g}(\vec{x}) - B\vec{x}$ or if $\det B = mn - 1 \neq 0$, $\vec{f}(\vec{x}) = kB^{-1}\vec{g}(\vec{x}) - \vec{x}$. If, in addition, $a = d$ so that $m = n = a/b$, we have

$$B = \begin{bmatrix} m & 1 \\ 1 & m \end{bmatrix}, \phi_1(x) = \phi_2(x) = \phi(x) = kg(x) - mx,$$

so that

$$f(x) = \phi(\phi(x)) - x = kg(kg(x) - mx) - kmg(x) + (m^2-1)x \tag{5.6}$$

$$\begin{aligned} f'(x) =&\phi'(\phi(x))\phi'(x) - 1. \\ =&k^2[g'(\phi(x))][g'(x)] - km[g'(\phi(x))] - km[g'(x)] + (m^2-1) \end{aligned} \tag{5.7}$$

6 Sufficient conditions for infinite limits

In this section we assume $b = c \neq 0$ and $a = d$ so that $k = \lambda/b$, $m = n = a/b$, and

$$f(x) = \phi(\phi(x)) - x = kg(kg(x) - mx) - mkg(x) + (m^2 - 1)x \tag{6.1}$$

where

$$\phi(x) = kg(x) - mx. \tag{6.2}$$

We now establish conditions where the limit of $f(x)$ as x goes to $\pm\infty$ is $\pm\infty$. First note that for $x \neq 0$ and $\phi(x) \neq 0$, we have

$$f(x) = \phi(\phi(x)) - x = ((\phi(\phi(x))/\phi(x))\phi(x)) - x = x((\phi(\phi(x))/\phi(x))(\phi(x)/x) - 1)\,.$$

Hence if all limits exist, we have

$$\begin{aligned}
f_0 &= \lim_{x\to x_0} f(x) = \lim_{x\to x_0} (\phi(\phi(x))/\phi(x)) \lim_{x\to x_0} \phi(x) - \lim_{x\to x_0} x \\
&= (\lim_{x\to x_0} x)(\lim_{x\to x_0} (\phi(\phi(x))/\phi(x)) \lim_{x\to x_0} (\phi(x)/x - 1) \\
&= \lim_{x\to x_0} (\phi(\phi(x))/\phi(x)) \lim_{x\to x_0} \phi(x) - x_0 \\
&= x_0 \lim_{x\to x_0} ((\phi(\phi(x))/\phi(x))) \lim_{x\to x_0} (\phi(x)/(x) - 1)
\end{aligned}$$

Letting $\phi_0 = \lim_{x\to x_0} \phi(x)$, $\phi_1 = \lim_{x\to x_0} \phi(x)/x$, and $\phi_2 = \lim_{y\to \phi_0} \phi(y)/y$, we obtain

$$\begin{aligned}
f_0 &= \lim_{x\to x_0} f(x) = \lim_{y\to \phi_0} (\phi(y)/y)\ \phi_0 - x_0 = x_0(\lim_{y\to \phi_0} (\phi(y)/y)\phi_1 - 1) \\
&= \phi_2\phi_0 - x_0 = x_0(\phi_2\phi_1 - 1)
\end{aligned}$$

Now since

$$\phi(x) = kg(x) - mx = x(k(g(x)/x) - m,$$

$$\phi(x)/x = (kg(x) - mx)/x = k(g(x)/x) - m$$

we have

$$\phi_0 = kg_0 - mx_0 = x_0 = x_0(kg_1 - m) = x_0\phi_1.$$

$$\phi_1 = \lim_{x\to x_0} (\phi(x)/x) = k(\lim_{x\to x_0} g(x)/x) - m = kg_1 - m$$

$$\phi_2 = \lim_{y\to \phi_0} \phi(y)/y = k(\lim_{y\to \phi_0} g(y)/y) - m = kg_2 - m$$

where $g_0 = \lim_{x\to x_0} g(x)$, $g_1 = \lim_{x\to x_0}(g(x)/x)$ and $g_2 = \lim_{y\to\phi_0}(g(y)/y)$. Hence

$$\begin{aligned}
f_0 &= \phi_2\phi_0 - x_0 = x_0(\phi_2\phi_1 - 1) \\
&= (kg_2 - m)(kg_0 - mx_0) - x_0 = x_0[(kg_2 - m)(kg_1 - m) - 1]
\end{aligned}$$

We first consider the case $x_0 = \infty$ for two examples. Suppose $g(x) = e^x$ so that $g_0 = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} e^x = \infty$ and $g_1 = \lim_{x\to\infty} e^x/x =$

$\lim_{x\to\infty} e^x = \infty$. If $k > 0$, then $\phi_1 = kg_1 - m = k(\infty) - m = \infty$. Using the second expression for ϕ_0, we see that $\phi_0 = x_0\phi_1 = (\infty)(\infty) = \infty$. Hence $g_2 = \lim_{y\to\infty} g(y)/y = \lim_{y\to\infty} e^y/y = g_1 = \infty$ so that $\phi_2 = kg_2 - m = k(\infty) - m = \infty$. Hence

$$f_0 = x_0(\phi_2\phi_1 - 1) = (\infty)[(\infty)(\infty) - 1] = \infty.$$

If $k < 0$, then $\phi_1 = kg_1 - m = k(\infty) - m = -\infty$ so that $\phi_0 = x_0\phi_1 = (\infty)(-\infty) = -\infty$. In this case $g_2 = \lim_{y\to-\infty} g(y)/y = \lim_{x\to-\infty} e^x/x = 0$ so that $\phi_2 = kg_2 - m = k(0) - m = -m$. We see that if $m > 0$, then

$$f_0 = x_0(\phi_2\phi_1 - 1) = (\infty)[(-m)(-\infty) - 1] = \infty$$

and if $m < 0$ then $f_0 = -\infty$.

Now suppose $\mathbf{g(x)} = \sinh(\mathbf{x})$so that $g_0 = \infty$ and $g_1 = \infty$. If $k > 0$, then $\phi_1 = kg_1 - m = k(\infty) - m = \infty$ and $\phi_0 = x_0\phi_1 = (\infty)(\infty) = \infty$. Hence $g_2 = g_1 = \infty$ and $\phi_2 = kg_2 - m = k(\infty) - m = \infty$. Hence

$$f_0 = x_0(\phi_2\phi_1 - 1) = (\infty)[(\infty)(\infty) - 1] = \infty.$$

If $k < 0$, then $\phi_1 = kg_1 - m = k(\infty) - m = -\infty$ and $\phi_0 = x_0\phi_1 = (\infty)(-\infty) = -\infty$. Hence $g_2 = \lim_{y\to-\infty} g(y)/y = \lim_{y\to-\infty} \sinh(y)/y = \lim_{y\to-\infty} \cosh(y) = \infty$ and $\phi_2 = kg_2 - m = k(\infty) - m = -\infty$. Hence

$$f_0 = x_0(\phi_2\phi_1 - 1) = (\infty)[(-\infty)(-\infty) - 1] = \infty.$$

We summarize our results in Table 6.1.

$g(x)$	k	m	g_0	g_1	ϕ_1	ϕ_0	g_2	ϕ_2	$f_0 = \lim_{x\to\infty} f(x)$
e^x	$+$	$\pm, 0$	∞	∞	∞	∞	∞	∞	∞
e^x	$-$	$+$	∞	∞	$-\infty$	$-\infty$	0	$-m$	∞
e^x	$-$	$-$	∞	∞	$-\infty$	$-\infty$	0	$-m$	$-\infty$
$\sinh(x)$	$+$	$\pm, 0$	∞	∞	∞	∞	∞	∞	∞
$\sinh(x)$	$-$	$\pm, 0$	∞	∞	$-\infty$	$-\infty$	∞	$-\infty$	∞

Table 6.1: $\lim_{x\to\infty} f(x)$ for two examples with $(x_0 = \infty)$

We now consider the case $x_0 = -\infty$ for the same two examples. First let $g(x) = e^x$ so that $g_0 = \lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} e^x = 0$ and $g_1 = \lim_{x\to-\infty} g(x)/x = \lim_{x\to-\infty} e^x/x = \lim_{x\to-\infty} e^x = 0$. Then $\phi_1 = k(g_1) - m = k(0) - m = -m$. If $m > 0$, then $\phi_0 = x_0\phi_1 = (-\infty) - m) = \infty$ and $g_2 = \lim_{x\to\phi_0} g(y)/y = \lim_{y\to\infty} e^y y = \lim_{y\to\infty} e^y = \infty$. Now if $k > 0$, then $\phi_2 = k(g_2) - m = k(\infty) - m = \infty$ and

$$f_0 = x_0(\phi_2\phi_1 - 1) = (-\infty)((\infty)(-m) - 1) = \infty.$$

If $k < 0$, then $\phi_2 = k(g_2) - m = k(\infty) - m = -\infty$ and

$$f_0 = x_0(\phi_2\phi_1 - 1) = (-\infty)((-\infty)(-m) - 1) = -\infty.$$

On the other hand, if $m < 0$, then $\phi_0 = x_0\phi_1 = (-\infty)(-m) = -\infty$ and $g_2 = \lim_{y\to\phi_0} g(y)/y = \lim_{y\to-\infty} e^y/y = \lim_{y\to-\infty} e^y = 0$. Then $\phi_2 = k(g_2) - m = k(0) - m = -m$. Hence $\phi_2\phi_1 - 1 = m^2 - 1$ and we must also know the sign of this term. If $m^2 - 1 > 0$, then

$$f_0 = x_0(\phi_2\phi_1 - 1) = (-\infty)(m^2 - 1) = -\infty.$$

If $m^2 - 1 < 0$, then $f_0 = (-\infty)(m^2 - 1) = \infty$.
Now suppose $\mathbf{g}(\mathbf{x}) = \sinh(\mathbf{x})$so that now $g_0 = \lim_{x\to-\infty} g(x) = \lim_{x\to-\infty}\sinh(x)$ $(= \lim_{x\to-\infty} x^{2n+1}) = -\infty$ and $g_1 = \lim_{x\to-\infty} g(x)/x = \lim_{x\to-\infty}\sinh(x)/x = \lim_{x\to-\infty}\cosh(x) = \infty$. If $k > 0$, then we have $\phi_1 = kg_1 - m = k(\infty) - m = \infty$ and $\phi_0 = x_0\phi_1 = (-\infty)(\infty) = -\infty$. Hence $g_2 = \lim_{y\to\phi_0} g(y)/y = \lim_{y\to-\infty}\sinh(y)/y = g_1 = \infty$ and $\phi_2 = k(g_2) - m = k(\infty) - m = \infty$. Hence

$$f_0 = x_0(\phi_2\phi_1 - 1) = (-\infty)((\infty)(\infty) - 1) = -\infty$$

If $k < 0$, then $\phi_1 = kg_1 - m = k(\infty) - m = -\infty$ and $\phi_0 = x_0\phi_1 = (-\infty)(-\infty) = \infty$. Hence $g_2 = \lim_{y\to\phi_0} g(y)/y = \lim_{y\to\infty}\sinh(y)/y = \lim_{y\to\infty}\cosh(y) = \infty$ and $\phi_2 = k(g_2) - m = k(\infty) - m = \infty$. Hence $g_2 = \lim_{y\to\phi_0} g(y)/y = \lim_{y\to\infty}\sinh(y)/y = \lim_{y\to\infty}\cosh(y) = \infty$ and $\phi_2 = k(g_2) - m = k(\infty) - m = -\infty$. Hence

$$f_0 = x_0(\phi_2\phi_1 - 1) = (-\infty)((\infty)(-\infty) - 1) = -\infty.$$

We summarize our results in Table 6.2.

$g(x)$	k	m	m^2-1	g_0	g_1	ϕ_1	ϕ_0	g_2	ϕ_2	$\lim\limits_{x\to-\infty} f(x)$
e^x	$+$	$+$	$\pm$	0	0	$-m$	∞	∞	∞	∞
e^x	$-$	$+$	$\pm$	0	0	$-m$	∞	∞	$-\infty$	$-\infty$
e^x	$\pm$	$-$	$+$	0	0	$-m$	$-\infty$	0	$-m$	$-\infty$
e^x	$\pm$	$-$	$-$	0	0	$-m$	$-\infty$	0	$-m$	∞
$\sinh(x)$	$+$	$\pm, 0$	$\pm, 0$	$-\infty$	∞	∞	$-\infty$	∞	∞	$-\infty$
$\sinh(x)$	$-$	$\pm, 0$	$\pm, 0$	$-\infty$	∞	$-\infty$	∞	∞	$-\infty$	$-\infty$

Table 6.2: $\lim_{x\to-\infty} f(x)$ for two examples with $x_0 = -\infty$

7 Development of sufficient conditions for mainly monotonic

To obtain general conditions for mainly monotonic, we summarize the behavior of $f(x)$ for the two examples considered in the previous section. Since we wish to consider behavior for both $x_0 = \infty$ and $x_0 = -\infty$, we add $+$ or $-$ to the subscript for x_0, g_0, g_1, g_2, ϕ_0, ϕ_1, ϕ_2, and f_0. Once an example is selected, the values of g_{0-}, g_{1-}, g_{0+}, and g_{1+} are set. However, it is the values of g_{1-} and g_{1+} and the signs of k, m, and $m^2 - 1$ that determine ϕ_{1-}, ϕ_{0-}, g_{2-}, ϕ_{2-}, f_{0-},

ϕ_{1+}, ϕ_{0+}, g_{2+}, ϕ_{2+}, and f_{0+}. To determinef_{0-}, and f_{0+},from g_{1-} and $g_{1+}$for each of these examples, we partition the $k-m$ plane. For $g(x)=e^x$, the sign of m^2-1 is important only when m is negative. Hence for $g(x)=e^x$ we consider the six cases:

$$\begin{array}{lll} k=+ & m=+ & m^2-1=\pm \\ & m=- & m^2-1=+ \\ & & m^2-1=- \\ k=- & m=+ & m^2-1=\pm \\ & m=- & m^2-1=+ \\ & & m^2-1=- \end{array}$$

For $g(x)=\sinh(x)$ only the sign of k is important and we consider only the two cases

$$\begin{array}{lll} k=+ & m=\pm,0 & m^2-1=\pm,0 \\ k=- & m=\pm,0 & m^2-1=\pm,0 \end{array}$$

We now combine the results of Tables 6.1 and 6.2 in Table 7.1 using an appropriate partition of the $k-m$ plane for each example.

$g(x)$	k	m	m^2-1	g_{1-}	g_{1+}	f_{0-}	f_{0+}	Behavior
e^x	$+$	$+$	$\pm$	0	∞	∞	∞	not mainly monot.
e^x	$+$	$-$	$+$	0	∞	$-\infty$	∞	mainly increasing
e^x	$+$	$-$	$-$	0	∞	∞	∞	not mainly monot.
e^x	$-$	$+$	$\pm$	0	∞	$-\infty$	∞	mainly increasing
e^x	$-$	$-$	$+$	0	∞	$-\infty$	$-\infty$	not mainly monot.
e^x	$-$	$-$	$-$	0	∞	∞	$-\infty$	mainly decreasing
$\sinh x$	$+$	$\pm,0$	$\pm,0$	∞	∞	$-\infty$	∞	mainly increasing
$\sinh x$	$-$	$\pm,0$	$\pm,0$	∞	∞	$-\infty$	∞	mainly increasing

Table 7.1: Summary of behavior of two examples

Again, the values of g_{1-} and g_{1+} and the signs of k, m, and m^2-1 determine the values of f_{0-} and f_{0+} That is, we need not worry about what example we are using, (except to note that there is one) and may classify our results for any example where $g_{1-}=0$ and $g_{1+}=\infty$ (including $g(x)=e^x$) according to the six cases:

1) $k>0$ and $m>0$ (or $k\in(0,\infty)$ and $m\in(0,\infty)$),
2) $k>0$ and $-1<m<0$, (or $k\in(0,\infty)$ and $m\in(-1,0)$)
3) $k>0$ and $m<-1$, (or $k\in(0,\infty)$ and $m\in(-\infty,-1)$)
4) $k<0$ and $m>0$, (or $k\in(-\infty,0)$ and $m\in(0,\infty)$)
5) $k<0$ and $-1<m<0$, (or $k\in(0,\infty)$ and $m\in(-1,0)$)
6) $k<0$ and $m<-1$. (or $k\in(0,\infty)$ and $m\in(-\infty,-1)$)

Also, we may classify our results for any example where $g_{1-}=\infty$ and $g_{1+}=\infty$ (including $g(x)=\sinh(x)$ and $g(x)=x^{2n+1}$ with $n\in\mathbb{N}=\{1,2,3,...\}$) according to the two cases:

1) $k > 0$ (or $k \in (0, \infty)$)
2) $k < 0$ $k \in (-\infty, 0)$)

We reproduce the information in Table 7.1 as Table 7.2 using this classification except that we eliminate cases which are not mainly monotonic.

k	m	g_{1-}	g_{1+}	f_{0-}	f_{0+}	Behavior	Examples
$(0, \infty)$	$(-\infty, -1)$	0	∞	$-\infty$	∞	mainly incr.	$g(x) = e^x$
$(-\infty, 0)$	$(0, \infty)$	0	∞	$-\infty$	∞	mainly incr.	$g(x) = e^x$
$(-\infty, 0)$	$(-1, 0)$	0	∞	∞	$-\infty$	mainly decr.	$g(x) = e^x$
$(0, \infty)$	$(-\infty, \infty)$	∞	∞	$-\infty$	∞	mainly incr.	$g(x) = \sinh x$ or x^{2n+1}
$(-\infty, 0)$	$(-\infty, \infty)$	∞	∞	$-\infty$	∞	mainly incr.	$g(x) = \sinh x$ or x^{2n+1}

Table 7.2: Summary of sufficient conditions for mainly monotonic

8 Proofs of sufficient conditions for mainly monotonic

We now provide formal proofs for the cases in Table 7.2.

Theorem 8.1 *Suppose f is given by (6.1) and one of the following conditions holds:*

MI1 $k > 0$, $-\infty < m < -1$, $\lim_{x\to\infty} g(x)/x = 0$ *and* $\lim_{x\to\infty} g(x)/x = \infty$,

MI2 $k < 0$, $m > 0$, $\lim_{x\to\infty} g(x)/x = 0$ *and* $\lim_{x\to\infty} g(x)/x = \infty$,

MI3 $k \neq 0, \lim_{x\to-\infty} g(x)/x = \infty$ *and* $\lim_{x\to\infty} g(x)/x = \infty$,

Then $\lim_{x\to-\infty} f(x) = -\infty$ *and* $\lim_{x\to\infty} f(x) = \infty$ *so that f is mainly increasing.*

Proof. (MI1) Assume $k > 0$, $-\infty < m < -1$, $g_{1-} = \lim_{x\to-\infty} g(x)/x = 0$, $g_{1+} = \lim_{x\to\infty} g(x)/x = \infty$. Then if $x_0 = -\infty$ we have $\phi_{1-} = \lim_{x\to-\infty} \phi(x)/x = k \lim_{x\to-\infty}(g(x)/x) - m = k(g_{1-}) - m = k(0) - m = -m$.
$\phi_{0-} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to-\infty} x)(k \lim_{x\to-\infty}(g(x)/x) - m) = x_0(kg_1 - m) = x_0\phi_1 = (-\infty)(k(0) - m) = (-\infty)(-m) = -\infty$.
$g_{2-} = \lim_{y\to\phi_0-}(g(y)/y) = \lim_{x\to-\infty}(g(x)/x) = g_{1-} = 0$.
$\phi_{2-} = \lim_{y\to-\phi_0-}(\phi(y)/y) = k(\lim_{y\to-\infty} g(y)/y) - m = k(g_{1-}) - m = k(0) - m = -m$.
$f_{0-} = \lim_{x\to\infty} f(x) = x_{0-}(\phi_{2-}\phi_{1-} - 1) = x_{0-}[(-m)(-m) - 1] = (-\infty)(m^2 - 1) = -\infty$.
And if $x_0 = \infty$ we have

$\phi_{1+} = \lim_{x\to\infty} \phi(x)/x = k \lim_{x\to\infty}(g(x)/x) - m = k(g_{1+}) - m = k(\infty) - m = \infty$.
$\phi_{0+} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to\infty} x)(k \lim_{x\to\infty}(g(x)/x) - m) = x_{0+}(kg_{1+} - m) = x_{0+}\phi_{1+} = (\infty)(k(\infty) - m) = (\infty)(\infty) = \infty$.
$g_{2+} = \lim_{y\to\phi_{0+}}(g(y)/y) = \lim_{x\to\infty}(g(x)/x) = g_{1+} = \infty$.
$\phi_{2+} = \lim_{y\to\phi_{0+}}(\phi(y)/y) = k(\lim_{y\to\infty} g(y)/y) - m = k(g_{1+}) - m = k(\infty) - m = \infty$.
$f_{0+} = \lim_{x\to\infty} f(x) = x_{0+}(\phi_{2+}\phi_{1+} - 1) = (\infty)((\infty)(\infty) - 1) = \infty$.

(MI2) Assume $k < 0$, $,m > 0$, $g_{1-} = \lim_{x\to-\infty} g(x)/x = 0$, $g_{1+} = \lim_{x\to\infty} g(x)/x = \infty$. Then if $x_0 = -\infty$ we have
$\phi_{1-} = \lim_{x\to-\infty} \phi_1(x)/x = k(\lim_{x\to-\infty}(g(x)/x)) - m = k(g_{1-}) - m = k(0) - m = -m$.
$\phi_{0-} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to-\infty} x)(k \lim_{x\to-\infty}(g(x)/x) - m) = x_{0+}(kg_1 - m) = x_0\phi_1 = (-\infty)(k(0) - m) = (-\infty)(-m) = \infty$.
$g_{2-} = \lim_{y\to\phi_{0-}}(g(y)/y) = \lim_{x\to\infty}(g(x)/x) = g_{1+} = \infty$.
$\phi_{2-} = \lim_{y\to\phi_{0-}}(\phi(y)/y) = k(\lim_{y\to\infty} g(y)/y) - m = k(g_{1+}) - m = k(\infty) - m = -\infty$.
$f_{0-} = \lim_{x\to\infty} f(x) = x_{0-}(\phi_{2-}\phi_{1-} - 1) = x_{0-}[(-\infty)(-m) - 1] = (-\infty)(\infty - 1) = -\infty$.
And if $x_0 = \infty$ we have
$\phi_{1+} = \lim_{x\to\infty} \phi(x)/x = k(\lim_{x\to\infty}(g(x)/x)) - m = k(g_{1+}) - m = k(\infty) - m = -\infty$.
$\phi_{0+} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to\infty} x)(k \lim_{x\to\infty}(g(x)/x) - m) = x_{0+}(kg_{1+} - m) = x_{0+}\phi_{1+} = (\infty)(k(\infty) - m) = (\infty)(-\infty) = -\infty$.
$g_{2+} = \lim_{y\to\phi_{0+}}(g(y)/y) = \lim_{x\to-\infty}(g(x)/x) = g_{1-} = 0$.
$\phi_{2+} = \lim_{y\to\phi_{0+}}(\phi(y)/y) = k(\lim_{y\to-\infty} g(y)/y) - m = k(g_{1-}) - m = k(0) - m = -m$.
$f_{0+} = \lim_{x\to\infty} f(x) = x_{0+}(\phi_{2+}\phi_{1+} - 1) = (\infty)((-m)(-\infty) - 1) = \infty$.

(MI3) Assume $k \neq 0$, $g_{1-} = \lim_{x\to-\infty} g(x)/x = \infty$, $g_{1+} = \lim_{x\to\infty} g(x)/x = \infty$. We consider two cases. First we assume $k > 0$. Then if $x_0 = -\infty$ we have:
$\phi_{1-} = \lim_{x\to-\infty} \phi(x)/x = k(\lim_{x\to-\infty}(g(x)/x)) - m = k(g_{1-}) - m = k(\infty) - m = \infty$.
$\phi_{0-} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to-\infty} x)(k \lim_{x\to-\infty}(g(x)/x) - m) = x_0(kg_1 - m) = x_0(\phi_1 = (-\infty)(k(\infty) - m) = (-\infty)(\infty) = -\infty$.
$g_{2-} = \lim_{y\to\phi_{0+}}(g(y)/y) = \lim_{x\to-\infty}(g(x)/x) = g_{1-} = \infty$.
$\phi_{2-} = \lim_{y\to\phi_{0-}}(\phi(y)/y) = k(\lim_{y\to-\infty} g(y)/y) - m = k(g_{1+}) - m = k(\infty) - m = \infty$.
$f_{0-} = \lim_{x\to\infty} f(x) = x_{0-}(\phi_{2-}\phi_{1-} - 1) = x_{0-}[(\infty)(\infty) - 1] = (-\infty)(\infty - 1) = -\infty$.
And if $x_0 = \infty$ we have
$\phi_{1+} = \lim_{x\to\infty} \phi(x)/x = k(\lim_{x\to\infty}(g(x)/x)) - m = k(g_{1+}) - m = k(\infty) - m = \infty$.
$\phi_{0+} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to\infty} x)(k \lim_{x\to\infty}(g(x)/x) - m) = x_{0+}(kg_{1+} - m) = x_{0+}\phi_{1+} = (\infty)(k(\infty) - m) = (\infty)(\infty) = \infty$.
$g_{2+} = \lim_{y\to\phi_{0+}}(g(y)/y) = \lim_{x\to\infty}(g(x)/x) = g_{1+} = \infty$.

$\phi_{2+} = \lim_{y\to\phi_{0+}}(\phi(y)/y) = k(\lim_{y\to\infty} g(y)/y) - m = k(g_{1+}) - m = k(\infty) - m = \infty$.
$f_{0+} = \lim_{x\to\infty} f(x) = x_{0+}(\phi_{2+}\phi_{1+} - 1) = (\infty)((\infty)(\infty) - 1) = \infty$.
On the other hand, let us assume $k < 0$. Then if $x_0 = -\infty$ we have:
$\phi_{1-} = \lim_{x\to-\infty} \phi(x)/x = k(\lim_{x\to-\infty}(g(x)/x)) - m = k(g_{1-}) - m = k(\infty) - m = -\infty$.
$\phi_{0-} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to-\infty} x)(k \lim_{x\to-\infty}(g(x)/x) - m) = x_0(kg_1 - m) = x_0\phi_1 = (-\infty)(k(\infty) - m) = (-\infty)(-\infty) = \infty$.
$g_{2-} = \lim_{y\to\phi_0-}(g(y)/y) = \lim_{x\to\infty}(g(x)/x) = g_{1+} = \infty$.
$\phi_{2-} = \lim_{y\to\phi_0-}(\phi(y)/y) = k(\lim_{y\to\infty} g(y)/y) - m = k(g_{1+}) - m = k(\infty) - m = -\infty$.
$f_{0-} = \lim_{x\to\infty} f(x) = x_{0-}(\phi_{2-}\phi_{1-} - 1) = x_{0-}[(-\infty)(-\infty) - 1] = (-\infty)(\infty - 1) = -\infty$.
And if $x_0 = \infty$ we have
$\phi_{1+} = \lim_{x\to\infty} \phi(x)/x = k(\lim_{x\to\infty}(g(x)/x)) - m = k(g_{1+}) - m = k(\infty) - m = -\infty$.
$\phi_{0+} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to\infty} x)(k \lim_{x\to\infty}(g(x)/x) - m) = x_{0+}(kg_{1+} - m) = x_{0+}\phi_{1+} = (\infty)(k(\infty) - m) = (\infty)(-\infty) = -\infty$.
$g_{2+} = \lim_{y\to\phi_0+}(g(y)/y) = \lim_{x\to-\infty}(g(x)/x) = g_{1-} = \infty$.
$\phi_{2+} = \lim_{y\to\phi_0+}(\phi(y)/y) = k(\lim_{y\to-\infty} g(y)/y) - m = k(g_{1-}) - m = k(\infty) - m = -\infty$.
$f_{0+} = \lim_{x\to\infty} f(x) = x_{0+}(\phi_{2+}\phi_{1+} - 1) = (\infty)((-\infty)(-\infty) - 1) = \infty$.
Hence under the hypotheses MI1, MI2, and MI3, we have that f is mainly increasing. $\square$.

Theorem 8.2 *Suppose f is given by (6.1) and the following condition holds:*

MD1 $k < 0, -1 < m < 0, \lim_{x\to-\infty} g(x)/x = 0, \lim_{x\to\infty} g(x)/x = \infty$.

Then, $\lim_{x\to-\infty} f(x) = \infty$ *and* $\lim_{x\to\infty} f(x) = -\infty$ *so that f is mainly decreasing.*

Proof. Assume $k < 0$, $-1 < m < 0$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, Then if $x_0 = -\infty$ we have:
$\phi_{1-} = \lim_{x\to-\infty} \phi(x)/x = k \lim_{x\to-\infty}(g(x)/x) - m = k(g_{1-}) - m = k(0) - m = -m$.
$\phi_{0-} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to-\infty} x)(k \lim_{x\to-\infty}(g(x)/x) - m) = x_0(kg_1 - m) = x_0\phi_1 = (-\infty)(k(0) - m) = (-\infty)(-m) = -\infty$.
$g_{2-} = \lim_{y\to\phi_{0-}}(g(y)/y) = \lim_{x\to-\infty}(g(x)/x) = g_{1-} = 0$.
$\phi_{2-} = \lim_{y\to\phi_{0-}}(\phi(y)/y) = k(\lim_{y\to-\infty} g(y)/y) - m = k(g_{1-}) - m = k(0) - m = -m$.
$f_{0-} = \lim_{x\to\infty} f(x) = x_{0-}(\phi_{2-}\phi_{1-} - 1) = x_{0-}[(-m)(-m) - 1] = (-\infty)(m^2 - 1) = \infty$.
And if $x_0 = \infty$ we have
$\phi_{1+} = \lim_{x\to\infty} \phi(x)/x = k \lim_{x\to\infty}(g(x)/x)) - m = k(g_{1+}) - m = k(\infty) - m = -\infty$.

$\phi_{0+} = \lim_{x\to\infty} \phi(x) = (\lim_{x\to\infty} x)(k \lim_{x\to\infty}(g(x)/x) - m) = x_{0+}(kg_{1+} - m) = x_{0+}\phi_{1+} = (\infty)(k(\infty) - m) = (\infty)(-\infty) = -\infty$.
$g_{2+} = \lim_{y\to\phi_{0+}}(g(y)/y) = \lim_{x\to-\infty}(g(x)/x) = g_{1-} = 0$.
$\phi_{2+} = \lim_{y\to\phi_{0+}}(\phi(y)/y) = k(\lim_{y\to-\infty} g(y)/y) - m = k(g_{1-}) - m = k(0) - m = -m$.
$f_{0+} = \lim_{x\to\infty} f(x) = x_{0+}(\phi_{2+}\phi_{1+} - 1) = (\infty)((-m)(-\infty) - 1) = -\infty$.
Hence under the hypotheses MD1, we have that f is mainly decreasing. □

9 Sufficient conditions for consistently monotonic

We now investigate the cases given previously for f given by (6.1) to be mainly monotonic and determine sufficient additional conditions needed for f to be consistently monotonic. We will require sufficient conditions for the derivative of f to be either positive or negative for all $x \in \mathbb{R}$. Recall that

$$\begin{aligned} f(x) =& \phi(\phi(x)) - x \\ =& kg(kg(x) - mx) - kmg(x) + (m^2 - 1) \end{aligned} \tag{9.1}$$

$$f'(x) = \phi'(\phi(x))\phi'(x) - 1 \tag{9.2}$$
$$= k^2 g'(\phi(x))g'(x) - km(g'(\phi(x)) + g'(x)) + m^2 - 1$$
$$= kg'(\phi(x))[kg'(x) - m] - kmg'(x) + m^2 - 1$$
$$= kg'(x)[kg'(\phi(x)) - m] - kmg'(\phi(x)) + m^2 - 1 \tag{9.3}$$

We can show that f is consistently monotonic if we show that for all $x \in \mathbb{R}$ each of their terms $k^2(g'\phi(x))g'(x)), -km(g'(\phi(x)) + g'(x))$, and $m^2 - 1$ are of the same sign. Alternatively, the last form of f' is useful when we can show that there exist k and m such that for all $x \in \mathbb{R}$, $kg'(\phi(x)) - m$ is of one sign.

Theorem 9.1 *Suppose f is given by (6.1) and one of the following conditions holds:*

CI1 $k > 0$, $-\infty < m < -1$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$.

CI2 $k < 0$, $m < -1$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$.

CI3 $km < 0$, $m^2 > 1$, $\lim_{x\to-\infty} g(x)/x = \infty$, $\lim_{x\to-\infty} g(x)/x = \infty$, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$.

Then $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, *and for all* $x \in \mathbb{R}$, $f'(x) > 0$ *so that f is consistently increasing.*

Proof. By Theorem 8.1 each of these conditions is sufficient for f to be mainly increasing. It remains to show that for all $x \in \mathbb{R}$, $f'(x) > 0$. In each case, $g'(x) > 0$ for all $x \in \mathbb{R}$ is sufficient.
(CI1) Assume $k > 0, -\infty < m < -1, g_{1-} = \lim_{x\to-\infty} g(x)/x = 0$, $g_{1+} = \lim_{x\to\infty} g(x)/x = \infty$ and for all $x \in \mathbb{R}$, $g'(x) > 0$. Then for all $x \in \mathbb{R}$, $k^2g'(\phi(x))g'(x) > 0$, $-km(g'(\phi(x)) + g'(x)) > 0$, and $m^2 - 1 > 0$. Hence $f'(x) = k^2g'(\phi(x))g'(x) - km(g'(\phi(x)) + g'(x)) + m^2 - 1 > 0$ so that f is consistently increasing.
(CI2) Assume $k < 0$, $m > 1$, $g_{1-} = \lim_{x\to-\infty} g(x)/x = 0$,
$g_{1+} = \lim_{x\to\infty} g(x)/x = \infty$, and for all $x \in \mathbb{R}$, $g'(x) > 0$. As before for all $x \in \mathbb{R}$, we have $k^2g'(\phi(x))g'(x) > 0$, $-km(g(\phi(x))+g'(x)) > 0$, and $m^2-1 > 0$. Hence $f'(x) = k^2g'(\phi(x))g'(x) - km(g'(\phi(x)) + g'(x)) + m^2 - 1 > 0$, so that f is consistently increasing.
(CI3) Assume $km < 0$, $m^2 > 1$, $\lim_{x\to-\infty} g(x)/x = \infty$, $\lim_{x\to-\infty} g(x)/x = \infty$, and for all $x \in \mathbb{R}$, $g'(x) > 0$.As before for all $x \in \mathbb{R}$, we have $k^2g'(\phi(x))g'(x) > 0$, $-km(g(\phi(x)) + g'(x)) > 0$, and $m^2 - 1 > 0$. Hence $f'(x) = k^2g'(\phi(x))g'(x) - km(g'(\phi(x)) + g'(x)) + m^2 - 1 > 0$, so that f is consistently increasing. □

Theorem 9.2 *Suppose f is given by (6.1) and the following condition holds:*

CD1 $k < 0$, $-1 < m < 0$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, and for all $x \in \mathbb{R}$, $g'(x) > 0$ and $g'(\phi(x)) < m/k$, where $\phi(x) = kg(x) - mx$.

Then $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$ and for all $x \in \mathbb{R}$, $f'(x) < 0$ so that f is consistently decreasing.

Proof. By Theorem 8.2 these conditions are sufficient for f to be mainly decreasing. It remains to show that for all $x \in \mathbb{R}$, $f'(x) > 0$.
(CD1) Assume $k < 0$, $-1 < m < 0$, $g_{1-} = \lim_{x\to-\infty} g(x)/x = 0$, $g_{1+} = \lim_{x\to-\infty} g(x)/x = \infty$ and for all $x \in \mathbb{R}$, $g'(x) > 0$ and $g'(\phi(x)) < m/k$, where $\phi(x) = kg(x) - mx$. We have for all $x \in \mathbb{R}$, that $-kmg'(\phi(x)) < 0$ and $m^2 - 1 < 0$. Also since $g'(\phi(x)) < m/k$ we have that $kg'(\phi(x)) - m > 0$. Hence $k^2(g'(\phi(x)) - km(g'(x)) = k(g'(x))(kg'(\phi(x)) - m) < 0$ so that $f'(x) = k^2g'(\phi(x))g'(x) - km(g'(\phi(x)) + g'(x)) + m^2 - 1 > 0$ and hence f is consistently decreasing. □

We show that our example does indeed satisfy the condition $g'(\phi(x)) < m/k$ given in CD1 for a nonempty set of m and k's. Let $g(x) = e^x$ so that $g'(x) = e^x$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, and $\phi(x) = ke^x - mx$.

If $k < 0$, and $-1 < m < 0$, then $\phi(-\infty) = -\infty$ and $\phi(\infty) = -\infty$. Since $\phi'(x) = ke^x - m$, the maximum value of $\phi(x)$ occurs at $x = x_m = \ln(m/k)$. Hence the maximum value of $g'(\phi(x))$ is $g_m = g'(\phi(x_m)) = \exp\{ke^{\ln(m/k)} - m\ln(m/k)\} = \exp\{k(m/k) - m\ln(m/k)\} = e^m(m/k)^{-m}$. Hence $g'(\phi(x)) \le g_m = e^m(m/k)^{-m} < m/k$ provided $e^m < (m/k)^{m+1}$, $e^{m/(m+1)} < (m/k)$, $k/m < e^{(m+1)/m}$, or $k > mee^{l/m}$. We summarize our results in a theorem.

Theorem 9.3 *Let $g(x) = e^x$ and $\phi(x) = ke^x - mx$. If $k < 0$, $-1 < m < 0$,and $k > mee^{l/m}$ then $\forall x \in \mathbb{R}$, $g'(\phi(x)) < m/k$.*

For example, if $m = -1/2$, then $mee^{l/m} = (-1/2)ee^{-2} = -e^{-1}/2 = -1/(2e) \approx -0.18397$ and $g'(\phi(x)) < m/k$ if $-0.18397 < k < 0$.

10 Existence and uniqueness results for the linear case

In this section we assume $b = c \neq 0$ and $a = d$ so that $k = \lambda/b$, and $m = n = a/b$. In Section 8, we gave sufficient conditions for f given by (6.1) to be mainly monotonic. In Section 9, we gave sufficient conditions for f to be consistently monotonic. In this section we state corollaries to these results which give sufficient conditions for the scalar equation (4.7) or the vector equation (5.5) to have at least one or exactly one solution.

Corollary 10.1 *Suppose f is given by (6.1) and one of the following conditions holds:*

CI1 $k > 0$, $-\infty < m < -1$, $\lim_{x\to-\infty} g(x)/x = 0$, *and* $\lim_{x\to\infty} g(x)/x = \infty$.

CI2 $k < 0$,$m > 1$, $\lim_{x\to-\infty} g(x)/x = 0$,*and* $\lim_{x\to\infty} g(x)/x = \infty$.

CI3 $km < 0$, $m^2 > 1$, $\lim_{x\to-\infty} g(x)/x = \infty$,*and* $\lim_{x\to-\infty} g(x)/x = \infty$

CDI $k < 0$, $-1 < m < 0$, $\lim_{x\to\infty} g(x)/x = \infty$, *and* $\lim_{x\to-\infty} g(x)/x = 0$.

Then the scalar equation (4.7) and the vector equation (5.5) have at least one solution.

Corollary 10.2 *Suppose f is given by (6.1) and one of the following conditions holds:*

CI1 $k > 0$, $-\infty < m < -1$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$.

CI2 $k < 0$, $m > 1$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$.

CI3 $km < 0$, $m^2 > 1$, $\lim_{x\to-\infty} g(x)/x = \infty$, $\lim g(x)/x = \infty$, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$.

CDI $k < 0$, $-1 < m < 0$, $\lim_{x\to-\infty} g(x)/x = 0$, $\lim_{x\to\infty} g(x)/x = \infty$,, *and for all* $x \in \mathbb{R}$, $g'(x) > 0$ *and* $g'(\phi(x)) < m/k$.

Then the scalar equation (4.7) and the vector equation (5.5) have exactly one solution.

If f is given by (6.1) and any of the conditions of Corollary 10.1 hold, a solution of the scalar equation (4.7) exists and can be computed using the method of bisection after values of x have been found where f has opposite signs. The value of y can then be found using (4.1) to obtain a solution of the vector equation (5.5). If any of the conditions of Corollary 10.2 hold, the solution obtained is unique.

Summary

Solving the scalar system (2.1) and (2.2) can be reduced to solving a single scalar equation. For the case $b = c \neq 0$ and $a = d$ so that $k = \lambda/b$ and $m = n = a/b$, and $g \in C(\mathbb{R}, \mathbb{R})$, sufficient conditions can be given to ensure that at least one solution exists. If $g \in C^1(\mathbb{R}, \mathbb{R})$, additional conditions can be given to ensure uniqueness. Solutions can be obtained using the method of bisection.

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James L. Moseley & Jun Hua
West Virginia University
Morgantown, West Virginia 26506 USA
e-mail: moseley@math.wvu.edu Telephone: 304-293-2011