2001-Luminy conference on Quasilinear Elliptic and Parabolic Equations and Systems,
Electronic Journal of Differential Equations, Conference 08, 2002, pp 23–34.
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Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian *

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Abstract

We study the nonnegative solutions to equation

$$-\Delta_p u = u^q + \lambda\nu,$$

in a bounded domain Ω of $\mathbb{R}^N$, where $1 < p < N$, $q > p-1$, ν is a nonnegative Radon measure on Ω, and $\lambda > 0$ is a parameter. We give necessary conditions on ν for existence, with λ small enough, in terms of capacity. We also give a priori estimates of the solutions.

1 Introduction

Let Ω be a bounded regular domain in $\mathbb{R}^N$. We denote by $\mathcal{M}(\Omega)$ the set of Radon measures on Ω, $\mathcal{M}^+(\Omega)$ the set of nonnegative ones, and by $\mathcal{M}_b(\Omega), \mathcal{M}_b^+(\Omega)$ the subsets of bounded ones. We consider the quasilinear elliptic problem with a source term:

$$\begin{gathered} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u + \mu, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

with $1 < p < N$, $q > p-1$, and $\mu \in \mathcal{M}_b^+(\Omega)$. We look for conditions on the measure μ ensuring that the problem admits a nonnegative solution, and essentially in terms of capacity. In order to take account of the size of the measure, we will study the problem with

$$\mu = \lambda\nu, \quad \lambda \geq 0,$$

where $\nu \in \mathcal{M}_b^+(\Omega)$ is fixed and λ is a parameter. Recall a result of [3] in case $p = 2$, $N \geq 3$, which gives a necessary and sufficient condition for existence:

Mathematics Subject Classifications: 35J70, 35B45, 35D05.
Key words: Degenerate quasilinear equations, measure data, capacities, a priori estimates.

Published October 21, 2002.

Theorem 1.1 ([3]) *The following problem:*

$$\begin{aligned} -\Delta u &= u^q + \lambda\nu, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $\nu \in \mathcal{M}_b^+(\Omega)$, $\nu \neq 0$, has a nonnegative solution (in the integral sense) if and only if

$$\lambda \int_\Omega \varphi d\nu \leq \frac{q-1}{q^{q'}} \int_\Omega \varphi^{1-q'}(-\Delta\varphi)^{q'} dx, \tag{1.3}$$

for any $\varphi \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$ such that $-\Delta\varphi \geq 0$, with compact support in Ω.

Thus if q is subcritical, that means $q < N/(N-2)$, problem (1.2) always admits a solution for λ small enough. In case $q \geq N/(N-2)$, in order to obtain existence, the measure $\mu = \lambda\nu$ has to be small enough, and also not to charge some small sets, in particular the point sets (this was first observed in [15]). More precisely, if the measure is compactly supported, from [3], condition (1.3) implies that

$$\int_K d\nu \leq C \operatorname{cap}_{2,q'}(K, \mathbb{R}^N), \quad \text{for every compact set } K \subset \Omega, \tag{1.4}$$

where for any domain Ω and any $m \in \mathbb{N}^*$ and $r > 1$, $\operatorname{cap}_{m,r}$ is the capacity associated to the Sobolev space $W_0^{m,r}(\Omega)$, defined by

$$\operatorname{cap}_{m,r}(K,\Omega) = \inf\left\{ \|\psi\|^r_{W_0^{m,r}(\Omega)} : \psi \in \mathcal{D}(\Omega), 0 \leq \psi \leq 1, \psi = 1 \text{ on } K \right\}.$$

In fact it was proved in [2] that (1.4) is also sufficient:

Theorem 1.2 ([2]) *Assume that ν has a compact support in Ω. Then problem (1.2) has a solution for any $\lambda \geq 0$ small enough if and only if there exists $C > 0$ such that (1.4) holds.*

Condition (1.4) implies that μ does not charge the sets with $2, q'$- capacity zero. But it is stronger: if $q > N/(N-2)$ (resp. $q = N/(N-2)$), there exists a function $\nu \in L^s(\Omega)$ with $1 \leq s < N/2q'$ (resp. $s = 1$) such that problem (1.2) admits no solution, for any $\lambda > 0$.

Concerning problem (1.1) with $p \neq 2$, the question is much harder, because the full duality argument used in [3] cannot be used for the p-Laplacian. The first thing is to define a notion of solution, as it is the case for the problem without reaction term. In Section 2 we recall the usual notions of entropy solutions, which suppose that the measure is bounded; this leads to assume that $u^q \in L^1(\Omega)$. We denote by

$$\overline{P} = \frac{N(p-1)}{N-p}$$

the critical exponent linked to the p-Laplacian, and we set

$$q^* = q/(q-p+1),$$

(hence $q^* = q'$ if $p = 2$). In Section 3 we prove our main result:

Theorem 1.3 *Let $\nu \in \mathcal{M}_b^+(\Omega)$ and $\lambda \geq 0$. Assume that problem*

$$\begin{aligned} -\Delta_p u = u^q + \lambda\nu, \quad &\text{in } \Omega, \\ u = 0, \quad &\text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

has a nonnegative entropy solution (hence $u^q \in L^1(\Omega)$). Then for any $R > pq^*$, there exists $C = C(N,p,q,R,\Omega) > 0$ such that

$$\lambda \int_\Omega \varphi d\nu + \int_\Omega u^q \varphi \, dx \leq C\Big(\int_\Omega \varphi^{1-R} |\nabla\varphi|^R dx\Big)^{pq^*/R}, \tag{1.6}$$

for any $\varphi \in W_0^{1,p}(\Omega) \cap W^{1,s}(\Omega)$ $(s > N)$ such that $0 \leq \varphi \leq 1$ in Ω. And for any $\alpha < 0$, there exists $C = C(\alpha,N,p,q,R,\Omega) > 0$ such that

$$\int_\Omega (u+1)^{\alpha-1} |\nabla u|^p \varphi \, dx \leq C\Big(1 + \int_\Omega u^q \varphi \, dx\Big)\Big(\int_\Omega \varphi^{1-R} |\nabla\varphi|^R dx\Big)^{p/R}. \tag{1.7}$$

This Theorem gives a priori estimate not only of the size of the measure, but also of the integral $\int_\Omega u^q \varphi \, dx$, independently on u. In the case $p = 2$, this was first remarked by [12] when $\mu = 0$; it was the starting point for proving L^∞ universal estimates. It was also used in [7] and [8] for obtaining a priori estimates with a general measure μ. As a consequence we deduce the following:

Theorem 1.4 *If problem (1.5) has a solution, then, for any $R > pq^*$, there exists $C = C(N,p,q,R,\Omega) > 0$ such that*

$$\lambda \int_K d\nu \leq C \, (\text{cap}_{\,1,R}(K,\Omega))^{pq^*/R}, \quad \textit{for every compact set } K \subset \Omega. \tag{1.8}$$

and if ν has a compact support in Ω, there exists $C = C(N,p,q,R,\mu) > 0$ such that

$$\lambda \int_K d\nu \leq C \, (\text{cap}_{\,1,R}(K,\mathbb{R}^N))^{pq^*/R}, \quad \textit{for every compact set } K \subset \Omega. \tag{1.9}$$

In particular, if $q > \overline{P}$, then ν does not charge the point sets. Moreover for any $1 \leq s < N/pq^*$, there exists a function $\nu \in L^s(\Omega)$ such that for any $\lambda > 0$, problem (1.5) admits no solution.

In Section 4, we mention some partially or fully open problems linked to this study. We refer to [5] for more complete results for problem (1.1) with possible signed measure μ, and for the problem with an absorption term

$$\begin{aligned} -\Delta_p u + |u|^{q-1} u = \mu, \quad &\text{in } \Omega, \\ u = 0, \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.10}$$

2 Entropy solutions

First recall some well-known results concerning the problem

$$\begin{aligned} -\Delta_p u &= \mu, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

with $\mu \in \mathcal{M}_b(\Omega)$. We set

$$P_0 = \frac{2N}{N+1}, \quad P_1 = 2 - \frac{1}{N},$$

so that $1 < P_0 < P_1$, and $P > P_0 \iff \overline{P} > 1$. When $p > P_1$, problem (2.1) admits at least a solution u in the sense of distributions, such that $u \in W_0^{1,r}(\Omega)$ for any $1 \leq r < \overline{P}$. In the general case, one can define a notion of entropy or renormalized solutions in four equivalent ways, see [11], which allow to give a sense to the gradient in any case: they are solutions such that $\nabla T_k(u) \in L^1_{loc}(\Omega)$ for any $k > 0$, where

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\,\mathrm{sign}(s), & \text{if } |s| > k, \end{cases} \tag{2.2}$$

and the gradient of u, denoted by $y = \nabla u$ is defined by

$$\nabla(T_k(u)) = y \times 1_{\{|u| \leq k\}} \quad \text{a.e. in } \Omega. \tag{2.3}$$

For any $p > 1$ there exists at least an entropy solution of (2.1), and it is unique if $\mu \in L^1(\Omega)$. Moreover any entropy solution satisfies the equation in the sense of distributions. The role of P_0 and P_1 is shown by the estimates

$$\begin{aligned} u^{p-1} &\in L^s(\Omega), \quad \text{for any } 1 \leq s < N/(N-p), \\ |\nabla u|^{p-1} &\in L^r(\Omega), \quad \text{for any } 1 \leq r < N/(N-1). \end{aligned}$$

Thus the gradient is well defined in $L^1(\Omega)$ if and only if $p > P_1$ and u itself is in $L^1(\Omega)$ if and only if $p > P_0$.

Recall that any measure $\mu \in \mathcal{M}_b(\Omega)$ can be decomposed as

$$\mu = \mu_0 + \mu_s^+ - \mu_s,$$

where $\mu_0 \in \mathcal{M}_{0,b}(\Omega)$, set of bounded measures such that

$$\mu_0(B) = 0 \quad \text{for any Borel set } B \subset \Omega \text{ such that } \operatorname{cap}_{1,p}(B, \Omega) = 0; \tag{2.4}$$

and μ_s^+, μ_s^- are nonnegative and concentrated on a set E with $\operatorname{cap}_{1,p}(E, \Omega) = 0$. If $\mu \in \mathcal{M}_b^+(\Omega)$, then μ_0 is nonnegative, and $\mu = \mu_0 + \mu_s^+$.

We will use one of the four equivalent definitions of solution: u is an entropy solution if u is measurable and finite *a.e.* in Ω, and

$$T_k(u) \in W_0^{1,p}(\Omega) \quad \text{for every } k > 0, \tag{2.5}$$

and the gradient defined by (2.3) satisfies

$$|\nabla u|^{p-1} \in L^r(\Omega), \quad \text{for any } 1 \leq r < N/(N-1), \tag{2.6}$$

and u satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla(h(u)\varphi)dx = \int_\Omega h(u)\varphi d\mu_0 \\ + h(+\infty)\int_\Omega \varphi d\mu_s^+ - h(-\infty)\int_\Omega \varphi d\mu_s^-,$$

for any $h \in W^{1,\infty}(\mathbb{R})$ and h' has a compact support, and any $\varphi \in W^{1,s}(\Omega)$ for some $s > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$.

In the same way, for given $\mu = \mu_0 + \mu_s^+ \in \mathcal{M}_b^+(\Omega)$, a nonnegative entropy solution u of problem (1.1) will be a measurable function u such that $u^q \in L^1(\Omega)$ and u is an entropy solution of problem

$$-\Delta_p u = \mu - u^q \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega.$$

In particular

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla(h(u)\varphi)dx + \int_\Omega u^q h(u)\varphi\, dx = \int_\Omega h(u)\varphi d\mu_0 + h(+\infty)\int_\Omega \varphi d\mu_s^+,$$

for any h and φ as above.

3 Proofs and comments

Proof of Theorem 1.3 Let $\mu = \lambda\nu = \mu_0 + \mu_s^+$, where $\mu_0 \in \mathcal{M}_{0,b}(\Omega)$ and μ_s^+ is singular, and let $\alpha \in (1-p, 0)$ be a parameter. For any $k > 0$, we set $u_k = T_k(u)$, and, for any $\varepsilon \in (0, k)$,

$$h_{\alpha,k,\varepsilon}(r) = (T_k(r^+) + \varepsilon)^\alpha = \begin{cases} \varepsilon^\alpha, & \text{if } r \leq 0, \\ (r+\varepsilon)^\alpha, & \text{if } 0 \leq r \leq k, \\ (k+\varepsilon)^\alpha, & \text{if } r \geq k. \end{cases}$$

We choose in (2) the test functions $h = h_{\alpha,k,\varepsilon}$, and $\varphi \in W_0^{1,p}(\Omega) \cap W^{1,s}(\Omega)$, with $s > N$ and $\varphi \geq 0$ in Ω, and obtain

$$\begin{aligned}
&\int_\Omega (u_k+\varepsilon)^\alpha \varphi d\mu_0 + (k+\varepsilon)^\alpha \int_\Omega \varphi d\mu_s^+ + \int_\Omega (u_k+\varepsilon)^\alpha u^q \varphi\, dx \\
&+|\alpha| \int_\Omega \int_\Omega (u_k+\varepsilon)^{\alpha-1} |\nabla u_k|^p \varphi\, dx \\
&\quad = \int_\Omega (u_k+\varepsilon)^\alpha |\nabla u|^{p-2} \nabla u . \nabla \varphi\, dx \\
&\quad \leq \int_\Omega (u_k+\varepsilon)^\alpha |\nabla u_k|^{p-1} |\nabla \varphi| dx + \int_{\{u \geq k\}} (u_k+\varepsilon)^\alpha |\nabla u|^{p-1} |\nabla \varphi| dx \\
&\quad \leq \frac{|\alpha|}{2} \int_\Omega (u_k+\varepsilon)^{\alpha-1} |\nabla u_k|^p \ \varphi\, dx + C \int_\Omega (u_k+\varepsilon)^{\alpha+p-1} \varphi^{1-p} |\nabla \varphi|^p \ dx \\
&\qquad + (k+\varepsilon)^\alpha \int_{\{u \geq k\}} |\nabla u|^{p-1} |\nabla \varphi| dx,
\end{aligned}$$

where $C = C(\alpha) > 0$.

Now from Hölder inequality, setting $\theta = q/(p-1+\alpha) > 1$,

$$\begin{aligned}
&\int_\Omega (u_k+\varepsilon)^{\alpha+p-1} \varphi^{1-p} |\nabla \varphi|^p \ dx \\
&\qquad \leq \Big(\int_\Omega (u_k+\varepsilon)^q \varphi\, dx \Big)^{1/\theta} \Big(\int_\Omega \varphi^{1-p\theta'} |\nabla \varphi|^{p\theta'} dx \Big)^{1/\theta'}.
\end{aligned}$$

In particular for any $k > 1$,

$$\begin{aligned}
&\frac{|\alpha|}{2} \int_\Omega \int_\Omega (u_k+\varepsilon)^{\alpha-1} |\nabla u_k|^p \varphi\, dx \\
&\leq C \Big(\int_\Omega (u_k+\varepsilon)^q \varphi\, dx \Big)^{1/\theta} \Big(\int_\Omega \varphi^{1-p\theta'} |\nabla \varphi|^{p\theta'} dx \Big)^{1/\theta'} + \int_{\{u \geq k\}} |\nabla u|^{p-1} |\nabla \varphi| dx.
\end{aligned} \tag{3.1}$$

Letting ε tend to 0, we get

$$\begin{aligned}
\frac{|\alpha|}{2} \int_\Omega u_k^{\alpha-1} |\nabla u_k|^p \varphi\, dx \leq & C \Big(\int_\Omega u_k^q \varphi\, dx \Big)^{1/\theta} \Big(\int_\Omega \varphi^{1-p\theta'} |\nabla \varphi|^{p\theta'} dx \Big)^{1/\theta'} \\
& + \int_{\{u \geq k\}} |\nabla u|^{p-1} |\nabla \varphi| dx.
\end{aligned} \tag{3.2}$$

Choosing now $h(u)=1$ in (2), with the same φ, we find

$$\begin{aligned}
\int_\Omega \varphi d\mu_0 + \int_\Omega \varphi d\mu_s^+ + \int_\Omega u^q\varphi\,dx &= \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi\,dx \\
&\leq \int_\Omega u_k^{(\alpha-1)/p'}|\nabla u|^{p-1}u_k^{(1-\alpha)/p'}|\nabla\varphi|dx + \int_{\{u\geq k\}}|\nabla u|^{p-1}|\nabla\varphi|dx \\
&\leq \Big(\int_\Omega u_k^{\alpha-1}|\nabla u_k|^p\ \varphi\,dx\Big)^{1/p'}\Big(\int_\Omega u_k^{(1-\alpha)(p-1)}\varphi^{1-p}|\nabla\varphi|^p dx\Big)^{1/p} \\
&\quad + \int_{\{u\geq k\}}|\nabla u|^{p-1}|\nabla\varphi|dx. \qquad (3.3)
\end{aligned}$$

Since $q>p-1$, we can fix $\alpha\in(1-p,0)$ such that $\tau=q/(1-\alpha)(p-1)>1$. From (3.2) and (3.3), we derive

$$\begin{aligned}
&\int_\Omega \varphi d\mu + \int_\Omega u^q\varphi\,dx \\
&\leq \Big(\int_\Omega u_k^{\alpha-1}|\nabla u_k|^p\ \varphi\,dx\Big)^{1/p'}\Big(\int_\Omega u_k^q\varphi\,dx\Big)^{1/\tau p}\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{1/\tau'p} \\
&\quad + \int_{\{u\geq k\}}|\nabla u|^{p-1}|\nabla\varphi|dx \\
&\leq \Big(C\Big(\int_\Omega u_k^q\varphi\,dx\Big)^{1/\theta}\Big(\int_\Omega \varphi^{1-p\theta'}|\nabla\varphi|^{p\theta'}dx\Big)^{1/\theta'} + \int_{\{u\geq k\}}|\nabla u|^{p-1}|\nabla\varphi|dx\Big)^{1/p'} \\
&\quad \times\Big(\int_\Omega u_k^q\varphi\,dx\Big)^{1/\tau p}\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{1/\tau'p} + \int_{\{u\geq k\}}|\nabla u|^{p-1}|\nabla\varphi|dx.
\end{aligned}$$

Now we can let k tend to ∞, since $u^q+|\nabla u|^{p-1}\in L^1(\Omega)$. It follows that

$$\begin{aligned}
\int_\Omega \varphi d\mu + \int_\Omega u^q\varphi\,dx &\leq C\Big(\int_\Omega u^q\varphi\,dx\Big)^{1/p'\theta+1/\tau p} \qquad (3.4)\\
&\quad\times\Big(\int_\Omega \varphi^{1-p\theta'}|\nabla\varphi|^{p\theta'}dx\Big)^{1/p'\theta'}\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{1/\tau'p},
\end{aligned}$$

with a new $C=C(\alpha,N,p,q)$. Since $1/\theta'p'+1/\tau'p=1/q^*=1-(1/\theta p'+1/\tau p)$, we find in particular

$$\begin{aligned}
&\Big(\int_\Omega u^q\varphi\,dx\Big)^{1-(p-1)/q} \\
&=\Big(\int_\Omega u^q\varphi\,dx\Big)^{(1/p'\theta'+1/\tau'p)} \\
&\leq C\Big(\int_\Omega \varphi^{1-p\theta'}|\nabla\varphi|^{p\theta'}dx\Big)^{1/p'\theta'}\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{1/\tau'p}.
\end{aligned}$$

Consequently

$$\begin{aligned}&\int_\Omega u^q\varphi\,dx\\ \le& C\Big(\int_\Omega \varphi^{1-p\theta'}|\nabla\varphi|^{p\theta'}dx\Big)^{\tau'p/(\tau'p+p'\theta')}\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{p'\theta'/(\tau'p+p'\theta')}.\end{aligned}$$

Notice that $\tau < q/(p-1) < \theta$, then from Hölder inequality,

$$\begin{aligned}\int_\Omega \varphi^{1-p\theta'}|\nabla\varphi|^{p\theta'}dx &\le \Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{\theta'/\tau'}\Big(\int_\Omega \varphi\,dx\Big)^{1-\theta'/\tau'}\\ &\le C\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{\theta'/\tau'},\end{aligned}$$

with a new constant $C = C(N, p, q, \alpha, \Omega)$, since $0 \le \varphi \le 1$. Therefore

$$\int_\Omega u^q\varphi\,dx \le C\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{q^*/\tau'}, \tag{3.5}$$

with a new constant $C > 0$. Moreover, from (3.4) and (3.5),

$$\int_\Omega \varphi d\mu \le C\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{(q^*-1+1/p'+1/p)/\tau'},$$

then

$$\int_\Omega \varphi d\mu + \int_\Omega u^q\varphi\,dx \le C\Big(\int_\Omega \varphi^{1-\tau'p}|\nabla\varphi|^{\tau'p}dx\Big)^{q^*/\tau'}.$$

We can choose $|\alpha|$ sufficiently small, such that

$$pq^* < p\tau' = q/(q-p+1-|\alpha|(p-1)) \le R;$$

thus we deduce (1.6) from Hölder inequality. Also, for any $\alpha < 0$, with $|\alpha|$ small enough, from (3.1), taking $\varepsilon = 1$ and letting k tend to ∞, we obtain

$$\begin{aligned}&\frac{|\alpha|}{2}\int_\Omega\int_\Omega (u+1)^{\alpha-1}|\nabla u|^p\varphi\,dx\\ \le& C\Big(\int_\Omega (u+1)^q\varphi, dx\Big)^{1/\theta}\Big(\int_\Omega \varphi^{1-p\theta'}|\nabla\varphi|^{p\theta'}dx\Big)^{1/\theta'}\\ \le& C\Big(1+\int_\Omega u^q\varphi\,dx\Big)\Big(\int_\Omega \varphi^{1-R}|\nabla\varphi|^{R}dx\Big)^{p/R}.\end{aligned}$$

Then (1.7) follows for any $\alpha < 0$. □

When $p = 2$, Theorem 1.1 naturally gives a stronger result, since any set with $1, R$ - capacity zero for some $R > 2q'$ has also a $2, q'$- capacity zero, see [1]. The capacity involved in Theorem 1.3 is of order 1 instead of 2, because we cannot use the full duality argument of the linear case. However, observe that a point set has a $1, 2q'$- capacity zero if and only if $q > N/(N-2)$, that means if and only if it has a $2, q'$- capacity zero.

Proof of Theorem 1.4 Let $\psi_n \in \mathcal{D}(\Omega)$ such that $0 \le \psi_n \le 1$ and $\psi_n \ge \chi_K$ and $\|\psi_n\|^R_{W^{1,R}(\Omega)}$ tends to $\operatorname{cap}_{1,R}(K,\Omega)$ as n tends to ∞. Choosing $\varphi = \psi_n^R$ in (1.6), we deduce that

$$\lambda \int_K d\nu \le C\Big(\int_\Omega |\nabla\psi_n|^R dx\Big)^{pq^*/R} \le C\,\|\psi_n\|^R_{W^{1,R}(\Omega)},$$

with new constants $C = C(N,p,q,R,\Omega)$, and (1.8) follows. If ν has a compact support X in Ω, then (1.9) holds after localization on a neighborhood of X. Assume moreover that $q > \overline{P}$, then we can choose R such that $pq^* < R < N$. Thus any point set $\{a\}$ of Ω has a $1,R$ - capacity zero, hence $\nu(\{a\}) = 0$. Moreover taking $K = \overline{B(x_0,r)}$ with $r > 0$ small enough, we derive

$$\lambda \int_{B(x_0,r)} d\nu \le Cr^{N-R}, \tag{3.6}$$

with $C = C(N,p,q,R,x_0,\Omega)$. For any $1 \le s < N/pq^*$, we can construct a function $\nu \in L^s(\Omega)$ with a singularity in $|x-x_0|^{-k}$ with $pq^* < k < N/s$, and with compact support in Ω, such that for any $\lambda > 0$, $\lambda\nu$ does not satisfy (3.6) for $pq^* < R < k$. Then there exists no solution of problem (1.5). □

4 Open problems

Problem 1: Can we obtain sufficient conditions of existence?
In the subcritical case $q < \overline{P}$, at least when $p > P_0$, the existence of solutions of problem (1.1), with possibly signed measure μ, is shown in [13]. In the supercritical case, the problem is entirely open, even for L^s functions. In particular it would be interesting to extend to the case $p \ne 2$ a consequence of Theorem 1.1:

Theorem 4.1 ([3]) *Assume that $N \ge 3$, and $\nu \in L^s(\Omega)$, for some $s \ge 1$. If $q > N/(N-2)$ and $s \ge N/2q'$, or $q = N/(N-2)$ and $s > N/2q'$, then problem (1.2) has a solution for λ small enough.*

Problem 2: Can we solve problems (2.1) and (1.5) if μ is not bounded?
Let us begin by the case without reaction term. For any $x \in \Omega$, denote by $\rho(x)$ the distance from x to $\partial\Omega$. When $p = 2$, problem (2.1) is well posed in fact for any measure μ, possibly unbounded, such that $\int_\Omega \rho d|\mu| < \infty$: it admits a unique integral solution

$$u(x) = G(\mu) = \int_\Omega \mathcal{G}(x,y)d\mu(y), \tag{4.1}$$

where $\mathcal{G}$ is the Green kernel. And u is also the weak solution of the problem in the sense that $u \in L^1(\Omega)$ and

$$\int_\Omega u(-\Delta\xi)dx = \int_\Omega \xi d\mu, \tag{4.2}$$

for any $\xi \in C^1(\overline{\Omega})$ vanishing on $\partial\Omega$ with $\nabla\xi$ is Lipschitz continuous, see [7]. The case where μ is a function f, such that $\int_\Omega \rho f dx < \infty$, was first considered by Brézis, see [17]. The problem is open when $p \neq 2$: up to now we have no existence results concerning equation (2.1) when μ may be unbounded, even in the case $p > P_1$, where the definition of the gradient does not cause any problem.

Now let us consider the problem with source term. When $p = 2$, it was studied in [14] and specified in [9]:

Theorem 4.2 ([14]) *Let $\nu \in \mathcal{M}^+(\Omega)$, $\nu \neq 0$ such that $\int_\Omega \rho d\nu < \infty$. Then problem (1.2) has a solution such that $G(u^q) < \infty$, a.e. in Ω, for any $\lambda \geq 0$ small enough, if and only if there exists $C > 0$ such that*

$$G(G^q(\nu)) \leq CG(\nu), \quad \textit{a.e. in } \Omega. \tag{4.3}$$

Notice that condition $G(u^q) < \infty$ *a.e.* in Ω, is satisfied as soon as $\int_\Omega \rho f u^q dx < \infty$, and the solutions are taken in the integral sense. More recently new existence results and a priori estimates were given in [8], covering the case of measures μ such that $\int_\Omega \rho^\gamma d\mu < \infty$ for some $0 \leq \gamma \leq 1$. Condition (4.3) allows to obtain a supersolution, and then a solution by using an iterative scheme. It is available for much more general linear operators, see [14] and [16]. It seems to be difficult to extend to nonlinear ones, since it is based on a representation formula. However Kalton and Verbitski [14] also gave necessary and sufficient in terms of capacity with weights, extending the result of [2] to general measures:

Theorem 4.3 ([14]) *Let $\nu \neq 0$ be a nonnegative Radon measure on Ω. Then problem (1.2) has a solution for any $\lambda \geq 0$ small enough if and only if there exists $C > 0$ such that*

$$\int_K d\nu \leq C \operatorname{cap}_{2,q',\rho}(K), \qquad \textit{for every compact set } K \subset \Omega,$$

where

$$\operatorname{cap}_{2,q',\rho}(K) = \inf \Big\{ \int_\Omega w^{q'} \rho^{1-q'} dx : w \geq 0, \quad Gw \geq \rho\chi_K \quad \textit{a.e. in } \Omega \Big\}.$$

One can ask if results of this type can be obtained for the p-Laplacian, using capacities of order 1 with suitable weights.

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