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A remark on some nonlinear elliptic problems *

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Abstract

We shall prove an existence result of $W_0^{1,p}(\Omega)$ solutions for the boundary value problem

$$\begin{gathered} -\operatorname{div} a(x,u,\nabla u) = F \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{0.1}$$

with right hand side in $W^{-1,p'}(\Omega)$. The features of the equation are that no restrictions on the growth of the function $a(x,s,\xi)$ with respect to s are assumed and that $a(x,s,\xi)$ with respect to ξ is monotone, but not strictly monotone. We overcome the difficulty of the uncontrolled growth of a thanks to a suitable definition of solution (similar to the one introduced in [1] for the study of the Dirichlet problem in L^1) and the difficulty of the not strict monotonicity thanks to a technique (the L^1-version of Minty's Lemma) similar to the one used in [5].

1 Introduction and assumptions

We deal with boundary value problems with differential operators A defined as

$$A(v) = -\operatorname{div}(a(x,v,\nabla v))$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω). We assume that there exist a real positive constant α, a continuous function $\beta(s)$ and a nonnegative function k in $L^{p'}(\Omega)$, where $1 < p$, such that for almost every x in Ω, for every s in $\mathbb{R}$, for every ξ and η in $\mathbb{R}^N$

$$a(x,s,\xi)\cdot\xi \geq \alpha|\xi|^p\,, \tag{1.1}$$

$$[a(x,s,\xi) - a(x,s,\eta)]\cdot[\xi-\eta] \geq 0\,, \tag{1.2}$$

$$|a(x,s,\xi)| \leq \big(k(x) + [\beta(s)|\xi|]^{p-1}\big)\,. \tag{1.3}$$

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Thus, A is not well defined on the whole $W_0^{1,p}(\Omega)$, but only in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Note that if we assume

$$[a(x,s,\xi)-a(x,s,\eta)]\cdot[\xi-\eta]>0, \quad \xi\neq\eta, \tag{1.4}$$

$$|a(x,s,\xi)|\leq \big(k(x)+|s|^{p-1}+|\xi|^{p-1}\big), \tag{1.5}$$

instead of (1.2), (1.3), A turns out to be pseudomonotone, coercive and is hence surjective on $W_0^{1,p}(\Omega)$ (see [6, 7, 8, 10]). A model operator for our setting is

$$-\sum_i \frac{\partial}{\partial x_i}\Big((1+|v|^{\gamma_i}\chi_{E_i})\frac{\partial v}{\partial x_i}\Big)$$

where $\gamma_i\geq 0$ and χ_{E_i} is the characteristic function of the measurable subset $E_i\subset\Omega$. Concerning the right hand side of (0.1), we assume that

$$F\in W^{-1,p'}(\Omega). \tag{1.6}$$

The aim of this note is to prove existence of solutions for (0.1) under the weaker assumption (1.2), without using the almost everywhere convergence of the gradients of the approximate equations, since this is impossible to prove in our setting. The main tools of our proof are a version of Minty's Lemma, similar to that one used in [5] to study nonlinear boundary value problems in L^1, and a definition of solution, similar to the one introduced in [1] for the Dirichlet problem in L^1. Other existence results of finite energy solutions, similar to the one of Theorem 2.2, can be found [3] (see also [9, 13, 2, 4] for existence results and [12, 14] for uniqueness results).

2 Existence

We recall that, for $k>0$ and s in $\mathbb{R}$, the truncating function is defined as

$$T_k(s)=\min\big\{k,\max\{-k,s\}\big\}.$$

The composition of functions in $W_0^{1,p}(\Omega)$ with T_k will play an important role in our approach to the existence of solutions of (0.1). More precisely, we will use the following definition of solution, which is similar to the one introduced in [1] for the Dirichlet problem in L^1.

Definition 2.1 A function $u\in W_0^{1,p}(\Omega)$ is a T-solution of (0.1) if

$$\begin{aligned}&u\in W_0^{1,p}(\Omega),\quad \forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega):\\ &\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi]=\langle F,T_k[u-\varphi]\rangle\end{aligned}\tag{2.1}$$

Theorem 2.2 *Under the assumptions* (1.1), (1.2), (1.3), (1.6) *there exists a* T*-solution* u *of* (2.1).

Remark 2.3 Note that, even if $a(x,s,\xi)$ is unbounded with respect to s, the integral in (2.1) is well defined, since $\nabla T_k[u-\varphi]$ is not zero on the subset $\{x\in\Omega : |u(x)-\varphi(x)|\le k\}$, that is in subsets where u is bounded.

Remark 2.4 If in (2.1) we take $\varphi = 0$, we have

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u) = \langle F, T_k(u)\rangle,$$

so that Lebesgue and Fatou Theorems imply that $a(x,u,\nabla u)\nabla u \in L^1(\Omega)$, but we are not able to prove that $a(x,u,\nabla u)\in L^1(\Omega)$, which would imply that u is a solution in $\mathcal{D}'(\Omega)$, that is

$$u\in W_0^{1,p}(\Omega) : \int_\Omega a(x,u,\nabla u)\nabla\phi = \langle F,\phi\rangle, \quad \forall\phi\in\mathcal{D}(\Omega).$$

Proof of Theorem 2.2. Consider the approximate problems

$$u_n \in W_0^{1,p}(\Omega) : -\operatorname{div}(a(x,T_n(u_n),\nabla u_n) = F. \tag{2.2}$$

The solutions u_n exist thanks to the Leray-Lions existence theorem (see [10]). Moreover, the use of u_n as test function in (2.2) and the assumption (1.1) imply that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Thus, there exists a function $u\in W_0^{1,p}(\Omega)$ and a subsequence $\{u_{n_j}\}$ such that u_{n_j} converges weakly to u in $W_0^{1,p}(\Omega)$ and almost everywhere. Now we use an idea of G.J. Minty ([11]), in the framework of [5]: thanks to the monotonicity of $a(x,s,\xi)$ with respect to ξ, if $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $n > k+\|\varphi\|_{L^\infty(\Omega)}$, we have that

$$\int_\Omega a(x,u_{n_j},\nabla\varphi)\nabla T_k[u_{n_j}-\varphi] \le \langle F, T_k[u_{n_j}-\varphi]\rangle.$$

The weak convergence of the sequence $\{u_{n_j}\}$ in $W_0^{1,p}(\Omega)$ and the remark that $\nabla T_k[u_{n_j}-\varphi]$ is not zero on the subset $\{x\in\Omega : |u_{n_j}(x)-\varphi(x)|\le k\}$ (subset of $\{x\in\Omega : |u_{n_j}(x)|\le k+\|\varphi\|_{L^\infty(\Omega)}\}$) allow to pass to the limit in the previous inequality, so that

$$\int_\Omega a(x,u,\nabla\varphi)\nabla T_k[u-\varphi] \le \langle F, T_k[u-\varphi]\rangle.$$

Let h and k be positive real numbers, let t belong to $(-1,1)$, and let ψ be a function in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Choose $\varphi = T_h(u) + t\,T_k[u-\psi]$ in the previous inequality. Setting $G_k(s) = s - T_k(s)$, we obtain

$$\begin{aligned} I &= \int_\Omega a(x,u,\nabla T_h(u) + t\nabla T_k[u-\psi])\nabla T_k(G_h(u) - t\,T_k[u-\psi]) \\ &\le \langle F, T_k(G_h(u)-t, T_k[u-\psi])\rangle = J\,. \end{aligned}$$

We then have

$$\begin{aligned} I &= \int_{\{|G_h(u)-t\,T_k[u-\psi]|\le k\}} a(x,u,\nabla T_h(u)+t\,\nabla T_k[u-\psi])\nabla G_h(u) \\ &\quad -t\int_{\{|G_h(u)-t\,T_k[u-\psi]|\le k\}} a(x,u,\nabla T_h(u)+t\,\nabla T_k[u-\psi])\nabla T_k[u-\psi] \\ &= H+L\,. \end{aligned}$$

Choosing $h \geq k + \|\psi\|_{L^\infty(\Omega)}$, we have $|T_k[u-\psi]| = k$ on the set $\{|u| \geq h\}$; on the same set, we have $\nabla T_h(u) = 0$. Since $\nabla G_h(u)$ is different from zero only on $\{|u| \geq h\}$, we obtain

$$H = \int_{\{|G_h(u)-t\,T_k[u-\psi]|\le k\}} a(x,u,0)\nabla G_h(u) = 0\,,$$

being $a(x,s,0) = 0$ as a consequence of (1.1). Since $\nabla T_k[u-\psi]$ is different from zero only on the set $\{x \in \Omega : |u(x)-\psi(x)| \leq k\}$, and on this set $|u| \leq k + \|\psi\|_{L^\infty(\Omega)} \leq h$, then

$$\begin{aligned} &\{|G_h(u)-t\,T_k[u-\psi]|\le k\}\cap\{|u-\psi|\le k\} \\ &\quad = \{|-t\,T_k[u-\psi]|\le k\}\cap\{|u-\psi|\le k\} \\ &\quad = \{|u-\psi|\le k\}\,, \end{aligned}$$

where the last passage is due to the fact that $|t| < 1$. Hence,

$$L = -t\int_\Omega a(x,u,\nabla u + t\,T_k[u-\psi])\nabla T_k[u-\psi]\,,$$

and so, for $h \geq k + \|\psi\|_{L^\infty(\Omega)}$, we have

$$I = -t\int_\Omega a(x,u,\nabla u + t\,T_k[u-\psi])\nabla T_k[u-\psi]\,.$$

On the other hand, we have, since $|t| < 1$, $T_k(s)$ is odd and $u \in W_0^{1,p}(\Omega)$,

$$\lim_{h\to+\infty}\langle F, T_k(G_h(u)-t\,T_k[u-\psi])\rangle = -t\langle F, T_k[u-\psi]\rangle$$

We thus have proved that

$$-t\int_\Omega a(x,u,\nabla u + t\,T_k[u-\psi])\nabla T_k[u-\psi] \le -t\langle F,\, T_k[u-\psi]\rangle\,,$$

for every $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$. Choosing $t > 0$, dividing by t, and then letting t tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\psi] \ge \langle F,\, T_k[u-\psi]\rangle\,,$$

while the reverse inequality is obtained choosing $t < 0$, dividing by $-t$, and then letting t tend to zero. This completes the proof of Theorem 2.2. $\square$

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