ON THE SOLVABILITY OF DEGENERATED QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. In this article, we study the quasilinear elliptic problem

\[ Au = -\text{div}(a(x,u,\nabla u)) = f(x,u,\nabla u) \quad \text{in} \ D'(\Omega) \]
\[ u = 0 \quad \text{on} \ \partial \Omega, \]

where \( A \) is a Leray-Lions operator from \( W^{1,p}_0(\Omega,w) \) to its dual \( W^{-1,p'}(\Omega,w^*) \).

We show that there exists a solution in \( W^{1,p}_0(\Omega,w) \) provided that

\[ |f(x,r,\xi)| \leq \sigma^{1/q}[g(x) + |r|\eta + \sum_{i=1}^N w_i^{b/p}(x)|\xi_i|]^{\delta}, \]

where \( g(x) \) is a positive function in \( L^q(\Omega) \) and \( \sigma(x) \) is weight function and

\[ 0 \leq \eta < \min(p-1,q-1), \quad 0 \leq \delta < (p-1)/q'. \]

1. Introduction

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^N \), \( N \geq 2 \), and \( p \) be a real number such that

\[ 1 < p < \infty. \]

Let \( w = \{w_i(x)\}, \ 0 \leq i \leq N \} \) be a vector weight functions on \( \Omega \); i.e., each \( w_i(x) \) is a measurable a.e. strictly positive function on \( \Omega \), satisfying some integrability conditions (see section 2). Let us consider the problem

\[ Au = f(x,u,\nabla u) \quad \text{in} \ D'(\Omega) \]
\[ u = 0 \quad \text{on} \ \partial \Omega, \quad (1.1) \]

where \( A \) is a Leray-Lions operator \( Au = -\text{div}(a(x,u,\nabla u)) \) and \( f(x,r,\xi): \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) is a Carathéodory function. Boccardo, Murat and Puel in [3] studied the problem (1.1) in the non weighted case, with \( f \) satisfying the condition

\[ |f(x,r,\xi)| \leq h(|r|)(1 + |\xi|^p), \]

where \( h \) is increasing function from \( \mathbb{R}^+ \) into \( \mathbb{R}^+ \). The existence result is proved assuming the existence of the subsolution and supersolution in \( W^{1,\infty}(\Omega) \), which play an important roll in their work. Further in [2] the author’s studied the problem...
with $f$ satisfies the hypotheses

$$|f(x, r, \xi)| \leq \beta [g(x) + |r|^{p-1} + |\xi|^{p-1}],$$  

(1.2)

$$f(x, r, \xi) r \geq \alpha |r|^p. $$  

(1.3)

Recently, Tsang-Hai Kuo and Chiung-Chion Tsai [8] proved an existence result under the assumption

$$|f(x, r, \xi)| \leq c (1 + |r|^\delta + |\xi|^\eta).$$

Our objective in this paper, is to study the problem (1.1) in weighted Sobolev spaces where $f$ satisfying only the growth condition

$$|f(x, r, \xi)| \leq \sigma^1 q g(x) + |r|^\eta \sum_{i=1}^{N} w_i^{q/p}(x)|\xi_i|^\delta,$$

(2.1)

where $g(x)$ is a positive function in $L^q(\Omega)$, $\sigma$ is a weight function, and

$$0 \leq \eta < \min(p - 1, q - 1), \quad 0 \leq \delta < \frac{p - 1}{q'}.$$  

Note that we obtain the existence result without assuming the condition (1.3) and without knowing a priori the existence of subsolutions and supersolutions. Let us point out that this work can be seen as a generalization of the work in [2] and [8].

2. Preliminaries and Basic Assumptions

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $p$ be a real number such that $1 < p < \infty$, and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions; i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in $\Omega$. Further, we suppose in all our considerations that

$$w_i \in L^1_{\text{loc}}(\Omega),$$  

(2.1)

$$w_i^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega),$$  

(2.2)

for any $0 \leq i \leq N$. We denote by $W^{1,p}(\Omega, w)$ the space of real-valued functions $u \in L^p(\Omega, w_0)$ such that their derivatives in the sense of distributions satisfies

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for } i = 1, \ldots, N.$$  

Which is a Banach space under the norm

$$\|u\|_{1,p,w} = \left[ \int_{\Omega} |u(x)|^p w_0(x) \, dx + \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^p w_i(x) \, dx \right]^{1/p}. $$  

(2.3)

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^{*} = w_i^{-1/p'}, i = 1, \ldots, N\}$ and $p'$ is the conjugate of $p$, i.e. $p' = \frac{p}{p-1}$. For more details we refer the reader to [5]. We start by stating the following assumptions:
(H1) The expression
\[ ||u|| = \left( \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p} w_i(x) \, dx \right)^{1/p} \tag{2.4} \]
is a norm defined on \( W_{0}^{1,p}(\Omega, w) \) and its equivalent to the norm \( 2.3 \). And there exist a weight function \( \sigma \) on \( \Omega \) and a parameter \( 0 < q < \infty \), such that the Hardy inequality
\[ \left( \int_{\Omega} |u(x)|^{q} \sigma(x) \, dx \right)^{1/q} \leq c \left( \sum_{i=1}^{N} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p} w_i(x) \, dx \right)^{1/p}, \tag{2.5} \]
holds for every \( u \in W_{0}^{1,p}(\Omega, w) \) with a constant \( c > 0 \). Moreover, the imbedding
\[ W_{0}^{1,p}(\Omega, w) \hookrightarrow L^{q}(\Omega, \sigma), \tag{2.6} \]
is compact.

Let \( A \) be a nonlinear operator from \( W_{0}^{1,p}(\Omega, w) \) into its dual \( W^{-1,p'}(\Omega, w^*) \) defined by
\[ A(u) = -\text{div}(a(x, u, \nabla u)), \]
where \( a(x, r, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N} \) is a Carathéodory vector-valued function that satisfies the following assumption:

(H2) For \( i = 1, \ldots, N, \)
\[ |a_i(x, r, \xi)| \leq \beta w_i^{1/p}(x)[k(x) + \sigma \frac{1}{q} |r|^{q/p'} + \sum_{j=1}^{N} w_j^{\frac{p}{p'}} |\xi_j|^{p-1}] \tag{2.7} \]
\[ |a(x, r, \xi) - a(x, r, \eta)| (\xi - \eta) > 0 \quad \text{for all } \xi \neq \eta \in \mathbb{R}^{N}; \tag{2.8} \]
\[ a(x, r, \xi) \xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^{p}, \tag{2.9} \]
where \( k(x) \) is a positive function in \( L^{p'}(\Omega) \) and \( \alpha, \beta \) are strictly positive constants.

Let \( f(x, r, \xi) \) is a Carathéodory function satisfying the following assumptions:

(H3)
\[ |f(x, r, \xi)| \leq \sigma^{1/q} [g(x) + |r|^{q} \sigma^{\frac{q}{q'}} + \sum_{i=1}^{N} w_i^{\delta/p}(x)|\xi_i|^{\delta}], \tag{2.10} \]
where \( g(x) \) is a positive function in \( L^{q'}(\Omega) \), and
\[ 0 \leq \eta < \min(p - 1, q - 1), \quad 0 \leq \delta < \frac{p-1}{q'}. \tag{2.11} \]

3. Main result

Consider the problem
\[ -\text{div} a(x, u, \nabla u) = f(x, u, \nabla u) \quad \text{in } D'(\Omega) \]
\[ u = 0 \quad \text{on } \partial \Omega. \tag{3.1} \]

**Theorem 3.1.** Under hypotheses (H1)-(H3), there exist at least one solution to \( 3.1 \).
We first give some definition and some lemmas that will be used in the proof of this theorem.

**Definition** Let $Y$ be a separable reflexive Banach space, the operator $B$ from $Y$ to its dual $Y^*$ is called of the calculus of variations type, if $B$ is bounded and is of the form,

$$B(u) = B(u, u),$$  \hspace{1cm} (3.2)

where $(u, v) \to B(u, v)$ is an operator $Y \times Y$ into $Y^*$ satisfying the following properties:

For $u \in Y$, the mapping $v \mapsto B(u, v)$ is bounded and hemicontinuous from $Y$ to $Y^*$ and $(B(u, u) - B(u, v), u - v) \geq 0$; for $v \in Y$, the mapping $u \mapsto B(u, v)$ is bounded and hemicontinuous from $Y$ to $Y^*$;

If $u_n \rightharpoonup u$ weakly in $Y$ and if $(B(u_n, u_n) - B(u_n, u), u_n - u) \to 0$,

then $B(u_n, v) \to B(u, v)$ weakly in $Y^*$, for all $v \in Y$;

If $u_n \rightharpoonup u$ weakly in $Y$ and if $B(u_n, v) \rightharpoonup \psi$ weakly in $Y^*$,

then $(B(u_n, v), u_n) \to (\psi, u)$.

**Lemma 3.2** (II). Let $g \in L^q(\Omega, \gamma)$, $g_n \in L^q(\Omega, \gamma)$, and $\|g_n\|_{q, \gamma} \leq c \ (1 < q < \infty)$. If $g_n(x) \to g(x)$ a.e. in $\Omega$, then $g_n \rightharpoonup g$ weakly in $L^q(\Omega, \gamma)$, where $\gamma$ is a weight function on $\Omega$.

**Lemma 3.3.** If $u_n \rightharpoonup u$ in $W^{1,p}_0(\Omega, w)$ and $v \in W^{1,p}_0(\Omega, w)$, then $a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v)$ in $L^p(\Omega, w_i^*)$.

**Proof.** From (H2), it follows that

$$|a_i(x, u_n, \nabla v)|^p w_i^{\frac{p'}{p}} \leq \beta |k(x) + |u_n|^{q^*} \sigma^{\frac{p}{q^*}} + \sum_{j=1}^N \frac{\partial v}{\partial x_j} |w_j^{p-1} w_j^\frac{p}{p} |^p$$

$$\leq \gamma |k(x)|^{p'} + |u_n|^{q^*} \sigma + \sum_{j=1}^N \frac{\partial v}{\partial x_j} |w_j^p|,$$

where $\beta$ and $\gamma$ are positive constants. Since $u_n \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega, w)$ and $W^{1,p}_0(\Omega, w) \hookrightarrow L^q(\Omega, \sigma)$, it follows that $u_n \to u$ strongly in $L^q(\Omega, \sigma)$ and $u_n \to u$ a.e. in $\Omega$; hence

$$|a_i(x, u_n, \nabla v)|^p w_i^{\frac{p'}{p}} \to |a_i(x, u, \nabla v)|^p w_i^{\frac{p'}{p}} \quad \text{a.e. in } \Omega,$$

and

$$\gamma |k(x)|^{p'} + |u_n|^{q^*} \sigma + \sum_{j=1}^N \frac{\partial v}{\partial x_j} |w_j^p| \to \gamma |k(x)|^{p'} + |u|^{q^*} \sigma + \sum_{j=1}^N \frac{\partial v}{\partial x_j} |w_j^p|$$

a.e. in $\Omega$. Then, By Vitali’s theorem,

$$a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \quad \text{strongly in } L^p(\Omega, w_i^*), \text{ as } n \to +\infty.$$  \hspace{1cm} (3.8)

**Lemma 3.4** (II). Assume that (H1)–(H2) are satisfied, and let $(u_n)$ be a sequence in $W^{1,p}_0(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}_0(\Omega, w)$ and

$$\int_{\Omega} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \nabla (u_n - u) \, dx \to 0.$$
Since

Then, \( u_n \rightarrow u \) in \( W^{1,p}_0(\Omega, w) \).

For \( v \in W^{1,p}_0(\Omega, w) \), we associate the Nemytskii operator \( F \) with respect to \( f \),

\[
F(v, \nabla v)(x) = f(x, v, \nabla v) \text{ a.e., } x \in \Omega.
\]

**Lemma 3.5.** The mapping \( v \mapsto F(v, \nabla v) \) is continuous from the space \( W^{1,p}_0(\Omega, w) \) to \( L^q(\Omega, \sigma^{1-q}) \).

**Proof.** By hypothesis (H3), we have

\[
|f(x, r, \xi)| \leq \sigma^{1/q}|g(x)| + |r|^s \sigma^s + \sum_{i=1}^N w_i^{\delta/p}(x)|\xi_i|^\delta.
\]

Thanks to Young’s inequality,

\[
|r|^\eta \sigma^{\eta/q} \leq \left( \frac{\eta}{q - 1} \right) |r|^{q-1} \sigma^{(q-1)/q} + 1 \leq |r|^{q-1} \sigma^{q' + 1},
\]

which implies

\[
|f(x, r, \xi)| \leq \sigma^{1/q}|(N + 2) + g(x)| + |r|^{q-1} \sigma^{1/q'} + \sum_{i=1}^N w_i^{1/q'}|\xi_i|^{p/q'}.
\]

Then

\[
|f(x, r, \xi)|^{q'} \sigma^{-q'/q} \leq c_2 |c_1 + g(x)|^{q'} + |r|^{(q-1)q'} \sigma^{q'} + \sum_{i=1}^N w_i |\xi_i|^p.
\]

Since \( f \) is a Carathéodory, and for all subset \( E \) measurable, such that \( |E| < \eta \), we have

\[
\int_E |f(x, v, \nabla v)|^{q'} \sigma^{-q'/q} \, dx \leq c_2 [c_1 + \int_E |v|^q \sigma \, dx] + \int_E \sum_{i=1}^N w_i \frac{\partial v}{\partial x_i} |v|^{p} \, dx.
\]

Then by Vitali’s theorem, we deduce the continuous of the operator \( F \). Moreover,

\[
\left( \int_\Omega |f(x, v, \nabla v)|^{q'} \sigma^{-q'/q} \, dx \right)^{1/q} \leq c_2 [c + \|v\|^{q'/q} + \|v\|^{p/q'}]. \tag{3.9}
\]

**Proof of Theorem 3.1.**

**Step (1)** We will show that the operator \( B : W^{1,p}_0(\Omega, w) \rightarrow W^{1,p'}(\Omega, w^*) \) defined by \( B(v) = A(v) - f(x, v, \nabla v) \) is a calculus of variational.

**Assertion 1.** Let

\[
B(u, v) = -\sum_{i=1}^N \frac{\partial a_i(x, u, \nabla v)}{\partial x_i} - f(x, u, \nabla u).
\]

Then \( B(v, v) = B(v) \) for all \( v \in W^{1,p}_0(\Omega, w) \).

**Assertion 2.** We claim that the operator \( v \mapsto B(u, v) \) is bounded for all \( u \in W^{1,p}_0(\Omega, w) \). Let \( \psi \in W^{1,p}_0(\Omega, w) \), we have

\[
\langle B(u, v), \psi \rangle = \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla v) \frac{\partial \psi}{\partial x_i} - \int_\Omega f(x, u, \nabla u) \psi \, dx.
\]
From Hölder’s inequality, the growth condition (2.7) and the compact imbedding (2.6), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla v) \frac{\partial \psi}{\partial x_i} \leq \sum_{i=1}^{N} \left( \int_{\Omega} |a_i(x, u, \nabla v)|^{r_i} w_i^{-\frac{q_i}{p_i}} dx \right)^{1/r_i} \left( \int_{\Omega} |\frac{\partial \psi}{\partial x_i}|^{p_i} w_i dx \right)^{1/p_i}
\]

(3.10)

\[
\leq c_4 \|\psi\| \sum_{i=1}^{N} \left( \int_{\Omega} (k(x))^{p_i} + |u|^q \sigma + \sum_{j=1}^{N} |\frac{\partial u}{\partial x_j}|^{p_j} w_j dx \right)^{1/p_i}
\]

\[
\leq c_5 \|\psi\| \left[ c_6 + \|u\|^{\frac{q}{p}} + \|v\|^{p-1} \right].
\]

Similarly,

\[
\int_{\Omega} f(x, u, \nabla u) \psi dx \leq \left( \int_{\Omega} |f(x, u, \nabla u)|^{q} \sigma \frac{w_i^q}{w_i^{p_i}} dx \right)^{1/q} \left( \int_{\Omega} |\psi|^q \sigma dx \right)^{1/q},
\]

by (2.5) and (3.9), we have,

\[
\int_{\Omega} f(x, u, \nabla u) \psi dx \leq c\|\psi\| (c_7 + \|u\|^{q-1} + \|u\|^{p/q}).
\]

(3.11)

Since \( u \) and \( v \) belong to \( W_0^{1,p}(\Omega, w) \) and in view of (3.10) and (3.11), we deduce that \( \langle B(u, v), \psi \rangle \) is bounded in \( W_0^{1,p}(\Omega, w) \times W_0^{1,p}(\Omega, w) \).

We claim that the operator \( v \rightarrow B(u, v) \) is hemicontinuous for all \( u \in W_0^{1,p}(\Omega, w) \), i.e., the operator \( \lambda \rightarrow \langle B(u, v_1 + \lambda v_2), \psi \rangle \) is continuous for all \( v_1, v_2, \psi \in W_0^{1,p}(\Omega, w) \).

Since \( a_i \) is a Carathéodory function,

\[
a_i(x, u, \nabla (v_1 + \lambda v_2)) \rightarrow a_i(x, u, \nabla v_1) \quad \text{a.e. in } \Omega \text{ as } \lambda \rightarrow 0.
\]

Further, we know from (2.7) that \( a_i(x, u, \nabla (v_1 + \lambda v_2))_\lambda \) is bounded in \( L^{p_i}(\Omega, w_i^*) \); thus, by Lemma 3.2 we conclude

\[
a_i(x, u, \nabla (v_1 + \lambda v_2)) \rightarrow a_i(x, u, \nabla v_1) \quad \text{weakly in } L^{p_i}(\Omega, w_i^*), \text{ as } \lambda \rightarrow 0.
\]

(3.12)

Hence,

\[
\lim_{\lambda \rightarrow 0} \langle B(u, v_1 + \lambda v_2), \psi \rangle
\]

\[
= \lim_{\lambda \rightarrow 0} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla (v_1 + \lambda v_2)) \frac{\partial \psi}{\partial x_i} dx - \int_{\Omega} f(x, u, \nabla u) \psi dx
\]

(3.13)

\[
= \sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla v_1) \frac{\partial \psi}{\partial x_i} dx - \int_{\Omega} f(x, u, \nabla u) \psi dx
\]

\[
= \langle B(u, v_1), \psi \rangle \quad \text{for all } v_1, v_2, \psi \in W_0^{1,p}(\Omega, w).
\]

Similarly, we show that \( u \rightarrow \langle B(u, v), \psi \rangle \) is bounded and hemicontinuous for all \( v \in W_0^{1,p}(\Omega, w) \). Indeed, By (3.9), we have \( f((x, u_1 + \lambda u_2, \nabla (u_1 + \lambda u_2)))_\lambda \) is bounded in \( L^{q_i}(\Omega, \sigma^{1-q_i}) \) and as \( f \) is a Carathéodory function then

\[
f(x, u_1 + \lambda u_2, \nabla (u_1 + \lambda u_2)) \rightarrow f(x, u_1, \nabla u_1) \quad \text{a.e. in } \Omega.
\]
Hence, Lemma \textbf{3.2} gives,

\begin{equation}
    f(x, u_1 + \lambda u_2, \nabla (u_1 + \lambda u_2)) \to f(x, u_1, \nabla u_2) \quad \text{weakly in } L^{q'}(\Omega, \sigma^{1-q'}) \quad \text{as } \lambda \to 0, \tag{3.14}
\end{equation}

On the other hand, as in (3.12), we have

\begin{equation}
    a_i(x, u_1 + \lambda u_2, \nabla v) \to a_i(x, u_1, \nabla v) \quad \text{in } L^{p'}(\Omega, w_1^*), \quad \text{as } \lambda \to 0. \tag{3.15}
\end{equation}

Combining (3.14) and (3.15), we conclude that, \( u \to B(u, v) \) is bounded and hemi-continuous.

\textbf{Assertion 3.} From \textbf{(2.8)}, we have,

\[ \langle B(u, u) - B(u, v), u - v \rangle = \sum_{i=1}^{N} \int_{\Omega} (a_i(x, u, \nabla u) - a_i(x, u_1, \nabla v)) \left( \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \geq 0 \]

\textbf{Assertion 4.} Assume that \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega, w) \) and \( \langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle \to 0 \), we claim that \( B(u_n, v) \rightharpoonup B(u, v) \) weakly in \( W^{-1,p'}(\Omega, w^*) \).

We can write

\[ \langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle = \sum_{i=1}^{N} \int_{\Omega} [a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)] \frac{\partial}{\partial x_i}(u_n - u) \, dx \to 0 \]

Then, by Lemma \textbf{3.4} we have \( u_n \to u \) strongly in \( W_0^{1,p}(\Omega, w) \) and it follows from Lemma \textbf{3.5} that

\begin{equation}
    f(x, u_n, \nabla u_n) \to f(x, u, \nabla u) \quad \text{in } L^{q'}(\Omega, \sigma^{1-q'}). \tag{3.16}
\end{equation}

Since \( u_n \to u \) in \( L^p(\Omega, w) \) and by \textbf{(2.7)} and \( W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma) \), we can obtain from Lemma \textbf{3.3} that

\begin{equation}
    a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \quad \text{in } L^{p'}(\Omega, w_1^*). \tag{3.17}
\end{equation}

This implies

\begin{equation}
    \int_{\Omega} a_i(x, u_n, \nabla v) \frac{\partial \psi}{\partial x_i} \, dx \to \int_{\Omega} a_i(x, u, \nabla v) \frac{\partial \psi}{\partial x_i} \, dx. \tag{3.18}
\end{equation}

On the other hand, by H"{o}lder's inequality,

\[ \int_{\Omega} |f(x, u_n, \nabla u_n)| \psi \, dx \leq \left( \int_{\Omega} |f(x, u_n, \nabla u_n)|^{q'} \sigma^{1-q'} \, dx \right)^{1/q'} \left( \int_{\Omega} \psi^q \sigma \, dx \right)^{1/q}. \]

Thanks to (3.16), (2.5), and Lebesgue's dominated convergence theorem, we obtain

\begin{equation}
    \int_{\Omega} f(x, u_n, \nabla u_n) \psi \, dx \to \int_{\Omega} f(x, u, \nabla u) \psi \, dx. \tag{3.19}
\end{equation}
Then, we have

\[
\lim_{n \to \infty} \langle B(u_n, \psi), \psi \rangle = \lim_{n \to \infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla v) \frac{\partial u_n}{\partial x_i} \, dx - \int_{\Omega} f(x, u_n, \nabla u_n) \psi \, dx
\]

\[
= \sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla v) \frac{\partial \psi}{\partial x_i} \, dx - \int_{\Omega} f(x, u, \nabla u) \psi \, dx
\]

\[
= \langle B(u, \psi), \psi \rangle, \quad \text{for all } \psi \in W_0^{1,p}(\Omega, w).
\]

**Assertion 5.** Assume \( u_n \to u \) weakly in \( W_0^{1,p}(\Omega, w) \) and \( B(u_n, v) \to \psi \) weakly in \( W^{-1,p'}(\Omega, w) \). We claim that \( \langle B(u_n, v), u_n \rangle \to \langle \psi, u \rangle \). Thanks to \( u_n \to u \) in \( W_0^{1,p}(\Omega, w) \), we obtain by Lemma 3.3,

\[
a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \quad \text{strongly in } L^{p'}(\Omega, w^*) \text{ as } n \to +\infty. \quad (3.20)
\]

And so

\[
\int_{\Omega} a_i(x, u_n, \nabla v) \frac{\partial u_n}{\partial x_i} \, dx \to \int_{\Omega} a_i(x, u, \nabla v) \frac{\partial u}{\partial x_i} \, dx. \quad (3.21)
\]

Hence together with

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla v) \frac{\partial u_n}{\partial x_i} \, dx - \int_{\Omega} f(x, u_n, \nabla u_n) u_n \, dx \to \langle \psi, u \rangle, \quad (3.22)
\]

we have

\[
\langle B(u_n, v), u_n \rangle = \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla v) \frac{\partial u_n}{\partial x_i} \, dx - \int_{\Omega} f(x, u_n, \nabla u_n) u_n \, dx
\]

\[
= \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla v) \left( \frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \, dx + \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla v) \frac{\partial u}{\partial x_i} \, dx
\]

\[
- \int_{\Omega} f(x, u_n, \nabla u_n) u \, dx - \int_{\Omega} f(x, u_n, \nabla u_n)(u_n - u) \, dx.
\]

But in view of (3.20) and (3.21), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla v)(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}) \, dx \to 0. \quad (3.23)
\]

On the other hand, by Hölder’s inequality,

\[
\int_{\Omega} |f(x, u_n, \nabla u_n)(u_n - u)| \, dx
\]

\[
\leq \left( \int_{\Omega} |f(x, u_n, \nabla u_n)|^{q'} \sigma^{q} \, dx \right)^{1/q'} \left( \int_{\Omega} |u_n - u|^{q} \sigma \, dx \right)^{1/q}
\]

\[
\leq c \|u_n - u\|_{L^q(\Omega, \sigma)} \to \text{ as } n \to \infty
\]

i.e.,

\[
\int_{\Omega} f(x, u_n, \nabla u_n)(u_n - u) \, dx \to 0 \quad \text{as } n \to \infty. \quad (3.24)
\]

Thanks to (3.22), (3.23) and (3.24), we obtain

\[
\langle B(u_n, v), u_n \rangle \to \langle \psi, u \rangle.
\]
Step 2. We claim that the operator $B$ satisfies the coercivity condition
\[
\lim_{\|v\| \to +\infty} \frac{\langle B(v), v \rangle}{\|v\|} = \infty.
\] (3.25)

Since
\[
\langle Bv, v \rangle = \sum_{i=1}^{N} \int_{\Omega} a_i(x, v, \nabla v) \frac{\partial v}{\partial x_i} \, dx - \int_{\Omega} f(x, v, \nabla v) v \, dx.
\]
Then, using (2.9), we have
\[
\langle Bv, v \rangle \geq \alpha \sum_{i=1}^{N} w_i \left| \frac{\partial v}{\partial x_i} \right|^p - \int_{\Omega} f(x, v, \nabla v) v \, dx. \tag{3.26}
\]
Moreover,
\[
\int_{\Omega} f(x, v, \nabla v) v \, dx \leq \int_{\Omega} \sigma^{1/q} g(x) v \, dx + \int_{\Omega} |v|^{q+1} \sigma^{(q+1)/q} \, dx + \int_{\Omega} \sum_{i=1}^{N} w_i^{\delta/p} |\frac{\partial v}{\partial x_i}|^{\delta^{1/q}} |v| \, dx. \tag{3.27}
\]
Thanks to Hölder’s inequality and (2.5), we have
\[
\int_{\Omega} \sigma^{1/q} g(x) v \, dx \leq \left( \int_{\Omega} |g(x)|^{q'} \, dx \right)^{1/q'} \left( \int_{\Omega} |v|^q \sigma \, dx \right)^{1/q} \leq c \|v\|. \tag{3.28}
\]
On the other hand, by Hölder’s inequality,
\[
\sum_{i=1}^{N} w_i^{\delta/p} |\frac{\partial v}{\partial x_i}|^{\delta^{1/q}} |v| \leq c \sum_{i=1}^{N} \left( \int_{\Omega} w_i^{q'} |\frac{\partial v}{\partial x_i}|^{q'} \, dx \right)^{1/q'} \left( \int_{\Omega} |v|^q \sigma \, dx \right)^{1/q}. \tag{3.29}
\]
In view of (2.5), we have
\[
\sum_{i=1}^{N} w_i^{\delta/p} |\frac{\partial v}{\partial x_i}|^{\delta^{1/q}} |v| \leq c \sum_{i=1}^{N} \left( \int_{\Omega} w_i^{q'} |\frac{\partial v}{\partial x_i}|^{q'} \, dx \right)^{1/q'} \left( \sum_{i=1}^{N} \int_{\Omega} w_i |\frac{\partial v}{\partial x_i}|^p \, dx \right)^{\delta/p} \leq c \|v\| \|v\|^\delta. \tag{3.30}
\]
Since $0 \leq \frac{\delta q}{p} < 1$, hence by Hölder’s inequality, we deduce
\[
\left( \int_{\Omega} w_i^{\delta/q} |\frac{\partial v}{\partial x_i}|^{\delta q} \, dx \right)^{1/q'} \leq \left( \int_{\Omega} w_i^{\delta/p} |\frac{\partial v}{\partial x_i}|^{p} \, dx \right)^{\delta/p}, \tag{3.31}
\]
remark that,
\[
(a + b)^r \geq c(a^r + b^r) \quad \text{if } 0 \leq r < 1. \tag{3.32}
\]
Combining (3.29), (3.30) and (3.31), we conclude that
\[
\sum_{i=1}^{N} w_i^{\delta/p} |\frac{\partial v}{\partial x_i}|^{\delta^{1/q}} |v| \leq c \|v\| \left( \sum_{i=1}^{N} \int_{\Omega} w_i |\frac{\partial v}{\partial x_i}|^p \, dx \right)^{\delta/p} \leq c \|v\| \|v\|^\delta. \tag{3.32}
\]
Further, $0 \leq \frac{q+1}{q} < 1$, then by Hölder’s inequality and (2.6), we deduce
\[
\int_{\Omega} |v|^{q+1} \sigma^{(q+1)/q} \, dx \leq c \|v\|^{q+1}. \tag{3.33}
\]
Then from (3.26), (3.28), (3.32) and (3.33), we deduce that
\[
\langle Bv, v \rangle \geq \alpha \|v\|^{p-1} - c_1 - c_2 \|v\|^q - c_3 \|v\|^\delta-1
\]
and since \( p - 1 > \eta \) and \( p > \delta \), we conclude that \( \frac{\langle B_\nu, v \rangle}{\|v\|} \to +\infty \). Finally, the proof of Theorem is complete thanks to the classical Theorem in [7]. \( \square \)

4. Examples

Let us consider the Carathéodory functions

\[
a_i(x, r, \xi) = w_i|\xi_i|^{p-1} \text{sgn}(\xi_i)
\]

where \( w_i(x)(i = 1, \ldots, N) \) are given weight functions strictly positive almost everywhere in \( \Omega \). We shall assume that the weight function satisfies \( w_i(x) = w(x) \), \( x \in \Omega \) for \( i = 0, \ldots, N \). It is easy to show that the \( a_i(x, s, \xi) \) are Carathéodory functions satisfying the growth condition (2.7) and the coercivity (2.9). On the other side, the monotonicity condition (2.8) is verified. In fact,

\[
\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \hat{\xi})) (\xi_i - \hat{\xi}_i)
= w(x) \sum_{i=1}^{N-1} (|\xi_i|^{p-1} \text{sgn}(\xi_i) - |\hat{\xi}_i|^{p-1} \text{sgn}(\hat{\xi}_i)) (\xi_i - \hat{\xi}_i) > 0
\]

for almost all \( x \in \Omega \) and for all \( \xi, \hat{\xi} \in \mathbb{R}^N \) with \( \xi \neq \hat{\xi} \), since \( w > 0 \) a.e. in \( \Omega \). We consider the Hardy inequality in the form

\[
\left( \int_{\Omega} |u(x)|^q \sigma(x) \, dx \right)^{1/q} \leq c \left( \int_{\Omega} |\nabla u(x)|^p w(x) \, dx \right)^{1/p},
\]

where \( \sigma \) and \( q \) are defined in (2.5). In particular, let us use a special weight functions \( w \) and \( \sigma \) expressed in terms of the distance to the bounded \( \partial \Omega \). Denote \( d(x) = \text{dist}(x, \partial \Omega) \) and set

\[
w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x).
\]

In this case, the Hardy inequality reads

\[
\left( \int_{\Omega} |u(x)|^q d^\mu(x) \, dx \right)^{1/q} \leq c \left( \int_{\Omega} |\nabla u(x)|^p d^\lambda(x) \, dx \right)^{1/p}.
\]

The corresponding imbedding is compact if:

(i) For, \( 1 < p \leq q < \infty \),

\[
\lambda < p - 1, \quad \frac{N}{q} - \frac{N}{p} + 1 \geq 0, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0.
\]

(ii) For \( 1 \leq q < p < \infty \),

\[
\lambda < p - 1, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0.
\]

Remarks. 1. Condition (4.1) or Condition (4.2) is sufficient for the compact imbedding (2.6) to hold; see for example [4, example 1], [5, example 1.5], and [6, Theorems 19.17, 19.22].

Let us consider the Carathéodory function

\[
f(x, r, \xi) = d^{\frac{\mu}{q}}(x) \left( d^{\frac{\lambda}{q}}(x)|r|^\eta + \sum_{i=1}^{N} d^{\frac{\lambda}{p}}(x)|\xi_i|^\delta + g(x) \right),
\]
with \( g \in L^{q'}(\Omega) \), \( \sigma(x) \) is weight function and \( 0 \leq \eta < \min(p-1,q-1) \), \( 0 \leq \delta < \frac{p-1}{q} \).

Because of its definition, \( f(x,r,\xi) \) satisfies the growth condition (2.10). Also the hypotheses of Theorem 3.1 are satisfied. Therefore, the problem

\[
\sum_{i=1}^{N} \int_{\Omega} \left( d^{\lambda}(x) \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx
\]

\[
= \int_{\Omega} d^{\mu/q}(x) \left( d^{\nu/q}(x) |u|^\eta + \sum_{i=1}^{N} d^{\lambda_i/\delta}(x) |\xi_i|^\delta + g(x) \right) v dx,
\]

for all \( v \in W^{1,p}_0(\Omega, w) \), has at least one solution.

References


