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FAST AND HETEROCLINIC SOLUTIONS FOR A SECOND ORDER ODE

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ABSTRACT. We present some results on the existence of fast and heteroclinic solutions of an ODE connected with travelling wave solutions of a Fisher-Kolmogorov's equation. In particular, we present a variational characterization of the minimum speed of propagation.

1. INTRODUCTION

Some chemical and biological systems can be modelled by an autocatalytic process (see, e.g. [9, 10]). In many of these process the system can support propagating wavefronts due to a combination of a reaction effect and a molecular diffusion. The pioneering model in this framework is due to Fisher, [5], who suggested the equation

$$u_t = u_{xx} + u(1-u)$$

for studying the spatial spread of a favoured gene in a population. The simplest generalization of that equation is the so called Fisher-Kolmogorov's equation

$$u_t = u_{xx} + f(u), \tag{1.1}$$

where f is a given function with two zeroes, say $u = 0$ and $u = 1$, and positive on $]0, 1[$ so that $u = 0$ and $u = 1$ are the only two stationary states of (1.1). Equations like (1.1) arises in many problems suggested, for instance, by the classical theory of population genetics or by certain flame propagation problems in chemical reactor theory (see, e.g. [3]).

A *travelling wavefront* or *travelling wave solution* (t.w.s., in short) of (1.1) is a solution $u(t, x)$ having a constant profile, that is, such that

$$u(t, x) = \varphi(x - ct),$$

for some fixed $\varphi(\xi)$ (called *the wave shape*) and a constant c (called *the wave speed*). Specially important for the applications are t.w.s. connecting the two stationary states, $u = 0$ and $u = 1$.

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A simple calculation shows that if $u(t,x) = \varphi(x - ct)$ is a t.w.s. of (1.1), then the wave shape φ is a solution of the ODE

$$u'' + cu' + f(u) = 0. \tag{1.2}$$

When a t.w.s. connects the stationary states, its corresponding wave shape is a positive heteroclinic solution of (1.2) that connects the equilibria 1 and 0, that is, a solution of (1.2) defined on $\mathbb{R}$ and satisfying

$$u(t) \in]0,1[, \quad \forall t \in \mathbb{R}, \quad \lim_{t\to-\infty} u(t) = 1, \quad \lim_{t\to+\infty} u(t) = 0.$$

There is a vast and rich body of literature dealing with the existence of t.w.s. of (1.1) connecting the stationary states, going from the pioneering work of Kolmogorov, Petrovski and Piskounoff [6], through the remarkable paper of Aronson and Weinberger [4] up to more recent approaches (see [1, 7, 8, 12]).

It is well known (see, e.g. [4, 7]) that there exists a positive number, c^*, such that equation (1.2) has a heteroclinic solution connecting 1 and 0 if and only if $c \geq c^*$.

In terms of the Fisher-Kolmogorov's equation, that result says that none t.w.s. of (1.1) starting from the stationary state $u = 1$ and moving with speed less than c^* reaches the stationary state $u = 0$. c^* is called the *minimum propagation speed*.

It is clear that the heteroclinic solution, if there exists, is strictly decreasing. When f is differentiable in $[0,1]$, then $c^* \geq 2\sqrt{f'(0)}$ since otherwise the origin cannot acts as an attractor for positive solutions of equation (1.2). It is also proved (see [4, 7, 1, 12]) that

$$c^* \leq 2\sqrt{\sup_{0<u<1} f(u)/u},$$

with equality if f is concave in $[0,1]$.

On the other hand, the way that positive solutions of (1.2) approach zero in the phase plane, at least when $f \in C^1$, mimics the phase plane picture for the corresponding linearization at the origin: There exists an extremal trajectory, T_c, in the lower half-plane $u' < 0$, that connects some point $(1,a)$, $a \leq 0$, with $(0,0)$ whose slope at the origin is

$$\lambda_1 = \frac{-c - \sqrt{c^2 - 4f'(0)}}{2},$$

while every other trajectory in the region $u > 0$ approaching $(0,0)$ has slope at the origin

$$\lambda_2 = \frac{-c + \sqrt{c^2 - 4f'(0)}}{2}.$$

Moreover, T_c is extremal in the sense that trajectories below it stays bounded away from the origin. Aronson and Weinberger (see [4], theorem 4.1) proved that whenever $c^* > 2\sqrt{f'(0)}$, the extremal trajectory T_{c^*} is an heteroclinic solution between 1 and 0.

This note is a brief summary of the conference given by the author on the "Colloque International d'Analyse Non linéaire d'Oujda", about some recent results obtained in collaboration with J. Campos, A.M. Robles-Pérez and L. Sanchez dealing with some variational problems whose solutions are in correspondence with T_c and that, in particular, let us give a variational characterization of c^*. All the presented results with their proofs can be found in [2].

2. A variational characterization of fast solutions

We say that a solution $u(t)$ of equation (1.2) is *a fast solution* if its corresponding trajectory is the extremal trajectory T_c. Our purpose is to characterize these solutions in variational terms. In order to do that, we express their speed in approaching 0 by means of an integrability condition:

Given $c > 0$, we define the space

$$H_c := \{u \in H^1_{\rm loc}(0, +\infty) : \int_0^{+\infty} e^{ct} u'(t)^2 \, dt < +\infty \text{ and } u(+\infty) = 0\}$$

with the norm $\|u\| = \big(\int_0^{+\infty} e^{ct} u'(t)^2 \, dt \big)^{1/2}$. This is a Hilbert space and if $u \in H_c$, u obviously tends "quickly" to 0 as $t \to +\infty$.

We introduce the functional $\mathcal{F} : H_c \to \mathbb{R}$ defined as

$$\mathcal{F}(u) = \int_0^{+\infty} e^{ct} \Big(\frac{u'(t)^2}{2} - F(u(t)) \Big) \, dt, \quad u \in H_c,$$

where $F(u) := \int_0^u f(s) \, ds$. When

(H) $f : [0, 1] \to \mathbb{R}_+$ is a Lipschitz function such that $f(0) = 0 = f(1)$ and $f(u) > 0$ if $0 < u < 1$,

one can prove that $\mathcal{F}$ is well defined, continuous and in fact differentiable in H_c.

A critical point of $\mathcal{F}$ is a solution of equation

$$(e^{ct} u')' + e^{ct} f(u) = 0, \tag{2.1}$$

or, equivalently, of (1.2). We call it *a fast solution* because of its integrability property near $+\infty$.

We prove that a potential minimizer of $\mathcal{F}$ in $\{u \in H_c : u(0) = 1\}$ has to verify $0 < u(t) < 1$, for all $t > 0$, and $u'(t) < 0$, for all $t \geq 0$, and that $\mathcal{F}$ has a minimum in $\{u \in H_c : u(0) = 1\}$ provided that there exist $0 < k < \frac{c^2}{4}$ with $F(u) \leq ku^2/2$, for all $u \in [0, 1]$. Therefore, we have the following result.

Proposition 1. *Assume (H) and there exist* $0 < k < \frac{c^2}{4}$ *so that* $F(u) \leq ku^2/2$*, for all* $u \in [0, 1]$*. Then equation* (1.2) *has a fast solution* $u \in H_c$ *defined on* $t \geq 0$ *such that* $u(0) = 1$ *and* $u'(t) < 0$*, for all* $t \geq 0$*.*

This result is particularly connected to the existence of heteroclinic solutions. Indeed, one can prove that

> *If there exists a solution of (1.2) defined on* $[0, +\infty)$*, with* $u(0) = 1$*,* $u(t) > 0$*,* $t > 0$ *and* $u(t) \to 0$ *as* $t \to +\infty$*, then equation (1.2) has an heteroclinic solution.*

So, the above proposition proves the existence of heteroclinic solutions whenever $\frac{2F(u)}{u^2} \leq \frac{c^2}{4}$, for all $u \in [0, 1]$. Consequently,

$$c^* \leq \inf\{c > 0 : \frac{2F(u)}{u^2} \leq \frac{c^2}{4}, \quad \forall u \in [0, 1]\}.$$

This upper bound generalizes the estimate in [4].

3. Fast heteroclinic solutions

After studying the fast solutions, we ask about heteroclinic connections between the two equilibria $u = 1$ and $u = 0$ of equation (1.2). As in the previous section, we begin by introducing an appropriate space to work.

Given $c > 0$, we consider the space

$$X_c := \{u \in H^1_{loc}(\mathbb{R}) : \int_{-\infty}^{+\infty} e^{ct} u'(t)^2 \, dt < +\infty \text{ and } u(+\infty) = 0\},$$

with the norm $\|u\|_c := \big(\int_{-\infty}^{+\infty} e^{ct} u'(t)^2 \, dt\big)^{1/2}$.

We will say that a solution u, of the equation

$$u'' + cu' + \lambda f(u) = 0, \tag{3.1}$$

for some $\lambda > 0$, is a *fast heteroclinic solution* if $u \in X_c$ and $u(-\infty) = 1$. Note that, under assumption (H), any heteroclinic connection $u(t)$ of (3.1) between 1 and 0 has the property $u(t) \in]0, 1[$, $u'(t) < 0$, for all $t \in \mathbb{R}$.

Our aim now is to obtain a variational characterization of the smallest value of λ for which equation (3.1) has a fast heteroclinic solution. We remark that $u(t)$ is a solution of (3.1) for some $\lambda > 0$ if and only if $v(t) := u(t/\sqrt{\lambda})$ is a solution of (1.2) with $c = c/\sqrt{\lambda}$.

To do that, we introduce two real functionals on X_c:

$$A_c(u) := \int_{-\infty}^{+\infty} e^{ct} \frac{u'(t)^2}{2} \, dt; \quad B_c(u) := \int_{-\infty}^{+\infty} e^{ct} F(u(t)) \, dt,$$

and we will look for critical points of the restriction of A_c to the set $M_c := \{u \in X_c : B_c(u) = 1\}$. (Note that M_c is non empty as a consequence of the hypothesis on f).

We define

$$\lambda(c) := \inf\{A_c(u) : u \in M_c\}.$$

It is easy to check that A_c and B_c are C^1-functionals and M_c is a C^1-manifold. By Lagrange multipliers rule, $u \in M_c$ is a critical point of the restriction of A_c to M_c if and only if $u \in M_c$ is a solution of (3.1). Playing appropriately with (3.1) we are able to prove that

> *If $\lambda(c)$ is attained, then equation (3.1) with $\lambda = \lambda(c)$ has a fast heteroclinic solution $u \in M_c$ and $A_c(u) = \lambda(c)$.*

Remark 2. Given $u \in X_c$ and $a \in \mathbb{R}$, the function $v(t) := u(t - a)$ belongs to X_c and $A_c(v) = e^{ca} A_c(u)$, $B_c(u) = e^{ca} B_c(u)$. So, if $u \in X_c$ is a critical point of A_c subject to the restriction $B_c(u) = 1$, for all $\alpha > 0$, the function $v(t) := u(t - \frac{\ln \alpha}{c})$ is a critical point of A_c subject to the restriction $B_c(u) = \alpha$. Hence, condition $B_c(u) = 1$ is a kind of normalization.

The previous result reduces the problem of the existence of fast heteroclinic solutions to prove that $\lambda(c)$ is attained. Using a convenient closed convex set, we show that $\lambda(c)$ is attained when $F(u) = o(u^2)$ as $u \to 0^+$.

Finally, working with an auxiliary functional defined on that closed convex set, we obtain our main result.

Theorem 3. *Assume (H), and also that there exists $f'(0)$ and*

$$\lambda(c) < \frac{c^2}{4f'(0)}. \tag{3.2}$$

Then, $\lambda(c)$ is attained. In particular, (2.1) with $\lambda = \lambda(c)$ has a fast heteroclinic solution.

Observe that our approach does not require differentiability except at the origin. On the other hand, if there exists $f'(0)$, working with truncations of the function εe^{-kt}, $\varepsilon \to 0$, $k \downarrow c/2$, one can prove

$$\lambda(c) \leq \frac{c^2}{4f'(0)},$$

and condition (3.2) is almost necessary.

Moreover, as a consequence of this result, if there exists $f'(0)$, $\lambda(c)$ is positive. A simple change of variable shows that $\lambda(c) = c^2\lambda(1)$. Hence, condition (3.2) is independent of c and it can be write

$$\lambda(1) < \frac{1}{4f'(0)}.$$

4. A variational characterization of c^*

Theorem 3 let us obtain a variational characterization of the *minimum propagation speed* c^*. As we have already said in the introduction,

$$c^* := \inf\{c \in \mathbb{R} : (1.2) \text{ has an heteroclinic solution.}\}$$

Mallaguti and Marcelli [8] proved that c^* is in fact a minimum and it is positive. We are going to relate this number with the function $\lambda(c)$ introduced in the previous section. In order to do that, let us define

$$\bar{c} := \frac{1}{\sqrt{\lambda(1)}}.$$

Having in mind that $\lambda(c) \leq \frac{c^2}{4f'(0)}$ and $\lambda(c) = c^2\lambda(1)$, one has that $\bar{c} \geq 2\sqrt{f'(0)}$.

From Theorem 3, if $\bar{c} > 2\sqrt{f'(0)}$, equation (1.2) with $c = \bar{c}$ has a fast heteroclinic solution and, then, $\bar{c} \geq c^*$. We can prove the following result.

Theorem 4. $\bar{c} = c^*$.

The proof of this theorem is based on the following result.

Proposition 5. *Assume that for some $c > 2\sqrt{f'(0)}$ there exists an heteroclinic solution. Then, $c = c^*$ if and only if this heteroclinic is fast.*

Remark that the previous proposition says:

> *At least when $c > 2\sqrt{f'(0)}$, c^* is the only value of the parameter for which the heteroclinic connection between the two equilibria of (1.2) is fast.*

The proof of this result follows by interpreting positive decreasing solutions of (1.2) as solutions of a suitable first order equation (as it has been done in [11, 7]).

A positive decreasing solution of (1.2) has a trajectory in the second quadrant of the (u,u')-plane. It is about looking at such a trajectory as the graph of a function ϕ, so that $u'=\phi(u)$. Putting $y(u)=\phi(u)^2$, y is a solution of

$$\frac{dy}{du}=2c\sqrt{y}-2f(u). \tag{4.1}$$

A heteroclinic solution of (1.2) corresponds to a positive solution of (4.1) on $]0,1[$ such that $y(0)=y(1)=0$.

(Note that the Cauchy problem for equation (4.1) has no uniqueness, but any solution of (4.1) can be continued as long as it remains positive.)

Summarizing, we obtain

$$c^*=\Big(\inf\Big\{\int_{-\infty}^{+\infty}e^{ct}\frac{u'(t)^2}{2}\,dt\ :\ u\in X_1,\ \int_{-\infty}^{+\infty}e^{ct}F(u(t))\,dt=1\Big\}\Big)^{-1}.$$

Moreover, when $c>c^*$ equation (1.2) has an heteroclinic connection between its equilibria though it is no fast, that is, *the extremal trajectory T_c is not an heteroclinic*, but if $c=c^*>2\sqrt{f'(0)}$, then T_c connects the two equilibria.

When $c^*=2\sqrt{f'(0)}$, (1.2) has an heteroclinic connection between its equilibria, but it is an open problem to know if it is or not a fast heteroclinic.

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