ON A PROBLEM OF LOWER LIMIT IN THE STUDY OF NONRESONANCE WITH LERAY-LIONS OPERATOR

AOMAR ANANE, OMAR CHAKRONE, MOHAMMED CHEHABI

Abstract. We prove the solvability of the Dirichlet problem

\[ Au = f(u) + h \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega \]

for a given \( h \), under a condition involving only the asymptotic behaviour of the potential \( F \) of \( f \), where \( A \) is a Leray-Lions operator.

1. Introduction and statement of results

This paper concerns the existence of solutions to the problem

\[ Au = f(u) + h \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega \quad (1.1) \]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 1 \), \( A \) is an operator of the form \( A(u) = -\sum_{i=1}^{N} \frac{d}{dx_i} A_i(\nabla u) \), \( f \) is a continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) and \( h \) is a given function on \( \Omega \). Also we consider the problem

\[ -\Delta_p u = f(u) + h \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega \quad (1.2) \]

where \( \Delta_p \) denotes the \( p \)-Laplacian \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \), \( 1 < p < \infty \).

A classical result, essentially due to Hammerstein [9] asserts that if \( f \) satisfies a suitable polynomial growth restriction connected with the Sobolev imbeddings and if

\[ \limsup_{x \to \pm \infty} \frac{2F(s)}{|s|^2} < \lambda_1, \quad (1.3) \]

then problem (1.2) with \( p = 2 \) is solvable for any \( h \). Here \( F \) denotes the primitive \( F(s) = \int_0^s f(t)dt \) and \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) on \( H_0^1(\Omega) \). Several improvements of this result have been considered in the recent years.

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In 1989, the case $N = 1$ and $p = 2$ was considered in [7]. It was shown there that \[(1.2)\] with $p = 2$ is solvable for any $h \in L^\infty(\Omega)$ if
\[
\lim_{s \to \pm\infty} \frac{2F(s)}{|s|^2} < \lambda_1. \tag{1.4}
\]
When $N \geq 1$ and $p = 2$, showed later in [8] that \[(1.2)\] is solvable for any $h \in L^\infty(\Omega)$ if
\[
\lim_{s \to \pm\infty} \frac{2F(s)}{|s|^2} < \left(\frac{\pi}{2R(\Omega)}\right)^2, \tag{1.5}
\]
where $R(\Omega)$ denotes the radius of the smallest open ball $B(\Omega)$ containing $\Omega$. This result was extended to the $p$-Laplacian case in [5] where solvability of \[(1.2)\] was derived under the condition
\[
\lim_{s \to \pm\infty} \frac{pF(s)}{|s|^p} < (p-1) \left\{ \frac{1}{\pi R(\Omega)} \int_0^1 \frac{dt}{(1-t^p)^{1/p}} \right\}^p. \tag{1.6}
\]
Note that this condition reduces to \[(1.5)\] when $p = 2$.

The question now naturally arises whether $(p-1)\left\{ \frac{1}{\pi R(\Omega)} \int_0^1 \frac{dt}{(1-t^p)^{1/p}} \right\}^p$ can be replaced by $\lambda_1$ in \[(1.6)\], where $\lambda_1$ denotes the first eigenvalue of $-\Delta_p$ on $W^{1,p}_0(\Omega)$ (cf [11]).

Observe that for $N > 1$ and $p = 2$, $(\frac{\pi}{2R(\Omega)})^2 < \lambda_1$, and a similar strict inequality holds when $1 < p < \infty$. In [2], it was showed that the constants in \[(1.5)\] and \[(1.6)\] can be improved a little bit.

Denote by $l(\Omega)$ the length of the smallest edge of an arbitrary parallelepiped containing $\Omega$. If
\[
\lim_{s \to \pm\infty} \frac{pF(s)}{|s|^p} < C_p(l) \tag{1.7}
\]
where $C_p(l) = (p-1)\left\{ \frac{2}{\pi R(\Omega)} \int_0^1 \frac{dt}{(1-t^p)^{1/p}} \right\}^p$ then for any $h \in L^\infty(\Omega)$ the problem \[(1.2)\] has a solution $u \in W^{1,p}_0(\Omega) \cap C^1(\Omega)$.

Observe that for $N = 1$, $C_p = \lambda_1$ the first eigenvalue of $-\Delta$ on $\Omega = [0, l(\Omega)]$.

In particular, $C_2 = \left(\frac{\pi}{2}\right)^2$, and it recovers the result of [7]. It is clear that \[(1.7)\] is a weaker hypothesis than \[(1.6)\]. The difference between \[(1.7)\] and \[(1.6)\] is particularly important when $\Omega$ is a rectangle or a triangle. However $C_p(l) < \lambda_1$ when $N > 1$, and the question raised above remains open.

In this paper we investigate the question of replacing $\Delta_p$ by the operator of the form
\[
A(u) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} A_i(\nabla u).
\]
We assume the following hypotheses:

(A0) For all $i \in \{1, 2, \ldots, N\}$, $A_i : \mathbb{R}^N \to \mathbb{R}$ is continuous.

(A1) there exists $(c, k) \in [0, +\infty)^2$ such that $|A_i(\xi)| \leq c|\xi|^{p-1} + K$ for all $\xi \in \mathbb{R}^N$, and all $i \in \{1, 2, \ldots, N\}$.

(A2) (a) $\sum_{i=1}^N (A_i(\xi) - A_i(\xi'))(\xi - \xi') > 0$ for all $\xi \neq \xi' \in \mathbb{R}^N$;
(b) for all $i \in \{1, 2, \ldots, N\}$, the function defined by $r_i(s) = A_i(0, \ldots, 0, s, 0, \ldots, 0)$ for $s \in \mathbb{R}$ is odd;
(c) for each $i \in \{1, 2, \ldots, N\}$, there exists $a_i \in [0, +\infty]$ such that $\lim_{s \to +\infty} r_i(s)/s^{p-1} = a_i$;
(d) for each $i \in \{1, 2, \ldots, N\}$, $r_i \in C^1(\mathbb{R}^+)$ and $\lim_{s \to 0} sr_i'(s) = 0$;
(e) for all $i \in \{1, 2, \ldots, N\}$, $A_i(\xi) = 0$ for all $\xi \in \mathbb{R}^N$ such that $\xi_i = 0$.

**Remark 1.1.** (1) The hypothesis (A2)(d) is in particular satisfied if we suppose that for $i \in \{1, \ldots, N\}$, $r_i \in C^1(\mathbb{R}^*)$ and there exists $q_i$, $1 < q_i < \infty$, there exists $\eta_i > 0$, there exists $(a, b) \in \mathbb{R}^2$, such that for all $|s| < \eta_i$, $|r_i'(s)| \leq a|s|^{q_i-2} + b$.

(2) The assumption (A2)(d) is an hypothesis of homogenization at infinity for the operator $A$.

**Definition 1.2.** For $i \in \{1, 2, \ldots, N\}$, we define

$$l_i(s) = \frac{1}{p-1}[sr_i(s) - \int_0^s r_i(t)dt] \quad \forall s \in \mathbb{R}.$$ 

**Proposition 1.3.** Assume (A0), (A1) and (A2). Then: (1) The operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined, strictly monotone and

$$\langle Au, v \rangle = \sum_{i=1}^N \int_{\Omega} A_i(\nabla u) \frac{\partial v}{\partial x_i} dx \quad \forall u, v \in W_0^{1,p}(\Omega).$$

(2) For each $i \in \{1, 2, \ldots, N\}$, the function $r_i : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and $r_i(0) = 0$.

(3) For each $i \in \{1, 2, \ldots, N\}$, the function $l_i$ satisfies

(i) $l_i$ is even, continuous and $l_i(0) = 0$;

(ii) $\lim_{s \to +\infty} \frac{l_i(s)}{sp} = \frac{a_i}{p}$

(iii) $l_i \in C^1(\mathbb{R})$ and $l'_i(s) = \begin{cases} \frac{sr_i(s)}{p} & \text{if } s \neq 0 \\ 0 & \text{if } s = 0. \end{cases}$

(iv) $l_i$ is strictly increasing in $\mathbb{R}^+$.

**Proof.** (1) By (A0), (A1), it is clear that the operator $A$ is defined from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, we have

$$\langle Au, v \rangle = \sum_{i=1}^N \int_{\Omega} A_i(\nabla u) \frac{\partial v}{\partial x_i} dx \quad \forall u, v \in W_0^{1,p}(\Omega)$$

and by (A1)(a), we verify easily that $A$ is strictly monotone.

(2) Let $i \in \{1, \ldots, N\}$. By (A0) and (A2)-(b), $r_i$ is continuous and $r_i(0) = 0$, in the end $r_i$ is strictly increasing. Indeed, let $(s, s') \in \mathbb{R}^2$ such that $s \neq s'$, we have

$$(r_i(s) - r_i(s'))(s - s') = \sum_{i=1}^N (A_i(\xi) - A_i(\xi'))(\xi_i - \xi'_i) > 0$$

where $\xi = (0, \ldots, s, \ldots, 0)$ and $\xi' = (0, \ldots, s', \ldots, 0)$

(3)(i) By the foregoing, the function $l_i$ is even, continuous and $l_i(0) = 0$ for every $i \in \{1, \ldots, N\}$

(3)(ii) We show first that

$$\lim_{s \to +\infty} \frac{1}{sp} \int_0^s r_i(t)dt = \frac{a_i}{p}. \quad (1.8)$$

Let $\varepsilon > 0$, by (A2)(c), there exists $\eta_e = \eta$ such that $|r_i(s) - a_is^{p-1}| \leq \varepsilon s^{p-1}$ for all $s \geq \eta$.

Integrating from $\eta$ to $s$, we obtain

$$\left| \int_0^s r_i(t)dt - \int_0^s r_i(t)dt - \frac{a_i}{p}[s^p - \eta^p]\right| \leq \frac{\varepsilon}{p}[s^p - \eta^p].$$
Dividing by $s^p$ and letting $n \to +\infty$, we obtain
\[
\lim_{s \to +\infty} \left| \frac{1}{s^p} \int_0^s r_i(t) \, dt - \frac{a_i}{p} \right| = 0
\]
i.e (1.8) holds. Writing
\[
\frac{l_i(s)}{s^p} = \frac{1}{p-1} \left\{ \frac{r_i(s)}{s^{p-1}} - \frac{1}{s^p} \int_0^s r_i(t) \, dt \right\}.
\]
By (1.8) and (A2)(c), we have \( \lim_{s \to +\infty} \frac{l_i(s)}{s^p} = \frac{a_i}{p} \).

(3)(iii) Since \( r_i \in C^1(\mathbb{R}^+) \), we have \( l_i \in C^1(\mathbb{R}) \) and \( l_i'(s) = \frac{1}{s-1} sr_i'(s) \) for every \( s \neq 0 \). On the other hand, for \( s = 0 \), since \( r_i \) is increasing and odd, we have
\[
|l_i'(s)| = \frac{1}{p-1} |r_i(s)| - \frac{1}{s^p} \int_0^s r_i(t) \, dt \leq \frac{2}{p-1} r_i(|s|).
\]
It results that \( l_i'(0) \) exists and \( l_i'(0) = 0 \). By (A2)-(d) we obtain \( \lim_{s \to 0} l_i'(s) = \lim_{s \to 0} sr_i'(s) \). This proves that \( l_i \in C^1(\mathbb{R}) \).

(3)(iv) is a consequence of (3)(iii). \( \square \)

**Example 1.4.** We give at first some examples for operators \( A \) satisfying the hypothesis (A0), (A1) and (A2).

(1) Let
\[
Au = -\Delta_p u = -\sum_{i=1}^N \frac{\partial}{\partial x_i} (|\nabla u|^{p-2} \frac{\partial u}{\partial x_i})
\]
Then we have \( A_i(\xi) = |\xi|^{p-2} \xi \) for every \( \xi = (\xi_i) \in \mathbb{R}^N \).

\( r(s) = r_i(s) = |s|^{p-2} s \) for every \( s \in \mathbb{R} \) and every \( i \in \{1, \ldots, N\} \).

\( l(s) = l_i(s) = \frac{1}{p} |s|^p \) for every \( s \in \mathbb{R} \) and every \( i \in \{1, \ldots, N\} \).

(2) Let
\[
Au = -\Delta_p u - \Delta_q u = -\sum_{i=1}^N \frac{\partial}{\partial x_i} (|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} + |\nabla u|^{q-2} \frac{\partial u}{\partial x_i})
\]
where \( 1 < q < p < +\infty \). Then we have \( A_i(\xi) = |\xi|^{p-2} \xi + |\xi|^{q-2} \xi \) for every \( \xi = (\xi_i) \in \mathbb{R}^N \).

\( r(s) = r_i(s) = |s|^{p-2} s + |s|^{q-2} s \) for every \( s \in \mathbb{R} \) and every \( i \in \{1, \ldots, N\} \).

\( l(s) = l_i(s) = \frac{1}{p} |s|^p + \frac{q-1}{q(p-1)} |s|^q \) for every \( s \in \mathbb{R} \) and every \( i \in \{1, \ldots, N\} \).

(3) Let
\[
Au = -\Delta_{p, \varepsilon} u = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left[ \varepsilon + |\nabla u|^2 \right]^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i},
\]
where \( \varepsilon > 0 \). Then we have \( A_i(\xi) = (\varepsilon + |\xi|^2)^{\frac{p-2}{2}} \xi \) for every \( \xi = (\xi_i) \in \mathbb{R}^N \).

\( r(s) = r_i(s) = (\varepsilon + |s|^2)^{\frac{p-2}{2}} s \) for every \( s \in \mathbb{R} \) and every \( i \in \{1, \ldots, N\} \).

\( l(s) = l_i(s) = (\varepsilon + |s|^2)^{\frac{p-2}{2}} \left( \frac{p}{p-1} \xi - \frac{2}{p(p-1)} \xi \right) + \frac{1}{p(p-1)} \varepsilon^\frac{q}{2} \) for every \( s \in \mathbb{R} \) and every \( i \in \{1, \ldots, N\} \).
2. Proof of Main Theorem

We consider the Dirichlet problem \([1.1]\) where \(\Omega\) is a bounded domain of \(\mathbb{R}^N\), \(N \geq 1\), \(f\) is a continuous function from \(\mathbb{R}\) to \(\mathbb{R}\) and \(h \in L^\infty(\Omega)\).

Denote by \([AB]\) the smallest edge of an arbitrary parallelepiped containing \(\Omega\). Making an orthogonal change of variables, we can always suppose that \(AB\) is parallel to one of the axis of \(\mathbb{R}^N\). So \(\Omega \subset P = \prod_{j=1}^N [a_j, b_j] \) with, for some \(i\), \([AB] = b_i - a_i = \min_{1 \leq j \leq N} \{b_j - a_j\}\), a quantity which we denote by \(b - a\).

Denote by \(l = l_i\), \(r = r_i\), \(F\) the primitive \(F(s) = \int_0^s f(t)dt\) and

\[
C_p = (p - 1) \left\{ \frac{2}{b - a} \int_0^1 \frac{dt}{(1 - t^p)^{\frac{1}{p}}} \right\}^p.
\]

**Theorem 2.1.** Assume

\[
\lim \inf_{s \to \pm \infty} \frac{F(s)}{l(s)} < C_p. \tag{2.1}
\]

Then for any \(h \in L^\infty(\Omega)\), the problem \([1.1]\) has a solution \(u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)\) in the weak sense; i.e.

\[
\sum_{i=1}^N \int_{\Omega} A_i(\nabla u) \frac{\partial \varphi}{\partial x_i} dx = \int_{\Omega} f(u)\varphi + \int_{\Omega} h\varphi \quad \forall \varphi \in W^{1,p}_0(\Omega).
\]

**Definition 2.2.** An upper solution for \([1.1]\) is defined as a function \(\beta : \overline{\Omega} \to \mathbb{R}\) such that

- \(\beta \in C^1(\overline{\Omega})\)
- \(A(\beta) \in C(\overline{\Omega})\)
- \(A(\beta)(x) \geq f(\beta(x)) + h(x)\) a.e \(x\) in \(\Omega\).

A lower solution \(\alpha\) is defined by reversing the inequalities above.

**Lemma 2.3.** Assume that \([1.1]\) admits an upper solution \(\beta\) and a lower solution \(\alpha\) with \(\alpha(x) \leq \beta(x)\) in \(\Omega\). Then \([1.1]\) admits a solution \(u \in W^{1,p}_0(\Omega) \cap C^1(\Omega)\), with \(\alpha(x) \leq u(x) \leq \beta(x)\) in \(\Omega\).

**Proof.** Let

\[
\tilde{f}(x, s) = \begin{cases} f(\beta(x)) & \text{if } s \geq \beta(x), \\ f(s) & \text{if } \alpha(x) \leq s \leq \beta(x), \\ f(\alpha(x)) & \text{if } s \leq \alpha(x) \end{cases}
\]

for every \((x, s) \in \overline{\Omega} \times \mathbb{R}\) such that \(\tilde{f}\) is bounded and continuous in \(\overline{\Omega} \times \mathbb{R}\), then the problem

\[
Au = \tilde{f}(x, u) + h(x) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega \tag{2.2}
\]

admits a solution \(u \in W^{1,p}_0(\Omega)\) in the weak sense, indeed the operator \(A\) is strictly monotone, so we can use the result of Lions \([10]\) concerned the pseudomonotones operators.

We claim that \(\alpha(x) \leq u(x) \leq \beta(x)\) in \(\Omega\), which clearly implies the conclusion.

To prove the first inequality, one multiplies \((2.2)\) by \(w = u - u_\alpha\), where \(u_\alpha(x) = \max(u(x), \alpha(x))\), integrates by parts and uses the fact that \(\alpha\) is a lower solution we obtain \(\langle A(u) - A(u - w), w \rangle \leq 0\), which implies \(w = 0\) (since \(A\) is strictly monotone). \(\square\)
Lemma 2.4. Let $a < b$ and $M > 0$, and assume
\[
\liminf_{s \to +\infty} \frac{F(s)}{l(s)} < C_p, \tag{2.3}
\]
then there exists $\beta_1 \in C^1(I)$ such that $(r(\beta'_1(t)))' \in C(I)$ and
\[
-(r(\beta'_1(t)))' \geq f(\beta_1(t)) + M \quad \forall t \in I, \\
\beta_1(t) \geq 0 \quad \forall t \in I
\]
where $I = [a, b]$.

Lemma 2.5. Assume
\[
\liminf_{s \to -\infty} \frac{F(s)}{l(s)} < C_p, \tag{2.4}
\]
then there exists $\alpha_1 \in C^1(I)$ such that $(r(\alpha'_1(t)))' \in C(I)$ and
\[
-(r(\alpha'_1(t)))' \leq f(\alpha_1(t)) - M \quad \forall t \in I, \\
\alpha_1(t) \leq 0 \quad \forall t \in I
\]
where $I = [a, b]$.

Accepting for a moment the conclusion of these two lemmas, let us turn to the
Proof of Theorem 2.1. By Lemma 2.3, it suffices to show the existence of an upper
solution and a lower solution for (1.1). Let us describe the construction of the upper
solution (that of the lower solution is similar).

Let $M > \|h\|_{\infty}$ and $i \in \{1, 2, \ldots, N\}$ such that $b = b_i$, $a = a_i$. By Lemma 2.4
there exists $\beta_1 : I \to \mathbb{R}$ such that $\beta_1 \in C^1(I)$, $(r(\beta'_1(t)))' \in C(I)$ and
\[
-(r(\beta'_1(t)))' \geq f(\beta_1(t)) + M \quad \forall t \in I, \\
\beta_1(t) \geq 0 \quad \forall t \in I
\]

Writing $\beta(x) = \beta_1(x_i)$ for all $x = (x_i) \in \overline{\Omega}$, it is clear that $\beta \in C^1(\overline{\Omega})$, $A(\beta(x)) = A(\beta_1(x_i)) \in C(\overline{\Omega})$, and we have by (A2)(e):
\[
A(\beta(x)) = -\sum_{j=1}^n \frac{\partial}{\partial x_j} A_j(\nabla \beta(x)) = -\frac{\partial}{\partial x_i} (r_i(\beta'_1(x_i))) = -(r(\beta'_1(x_i)))' \geq f(\beta_1(x_i)) + M = f(\beta(x)) + M \geq f(\beta(x)) + h(x) \quad \text{a.e. } x \in \Omega
\]

The proof of Theorem 2.1 is thus complete.

Next, we present the proof of Lemma 2.4. The proof of Lemma 2.5 follows similarly.

First case. Suppose $\inf_{s \geq 0} f(s) = -\infty$. Then there exists $\beta \in \mathbb{R}^*+$ such that $f(\beta) < -M$, and the constant function $\beta$ provides a solution to the problem in Lemma 2.4.
Second case. Suppose now $\inf_{s \geq 0} f(s) > -\infty$. Let $K > M$ such that $\inf_{s \geq 0} f(s) > -K + 1$. Thus $f(s) + K \geq 1$ for all $s \geq 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(s) = \begin{cases} f(s) + K & \text{if } s \geq 0 \\ f(0) + K & \text{if } s < 0 \end{cases}$$

and denote $G(s) = \int_{s}^{\infty} g(t) \, dt$ for all $s$ in $\mathbb{R}$. It is easy to see that $g(s) \geq 1$ for all $s$ in $\mathbb{R}$ and that

$$0 \leq \liminf_{s \to +\infty} \frac{G(s)}{l(s)} = \liminf_{s \to +\infty} \frac{F(s)}{l(s)} < C_p.$$ 

Now it is clearly sufficient to prove the existence of a function $\beta_1 : I \to \mathbb{R}$ such that $\beta_1 \in C^1(I)$, $(r(\beta_1'(t)))' \in C(I)$ and

$$-(r(\beta_1'(t)))' = g(\beta_1(t)) \quad \forall t \in I$$

$$\beta_1(t) \geq 0 \quad \forall t \in I$$

For this purpose we will need the following four Lemmas.

Lemma 2.6. Let $0 < c < \infty$ and $t \in [0, 1[$, then

$$\lim_{\alpha \to +\infty} \frac{1}{\alpha} \frac{1}{l^{-1}(c(l(\alpha)) - l(\alpha))} = \frac{1}{c^{1/p}(1 - t^p)^{1/p}}.$$ 

In particular, by Fatou Lemma,

$$\frac{1}{c^{1/p}} \frac{1}{(1 - t^p)^{1/p}} \leq \liminf_{\alpha \to +\infty} \int_{0}^{1} \frac{\alpha \, dt}{l^{-1}(c(l(\alpha)) - l(\alpha))}$$

Proof. Denote $s(\alpha) = \frac{1}{l^{-1}(c(l(\alpha)) - l(\alpha))}$ and $\frac{\alpha}{p} = d$. By Proposition 1.3(3)(ii), we have

$$\lim_{s \to +\infty} s^{1/p} = d^{1/p}.$$ 

On the other hand,

$$\lim_{\alpha \to +\infty} c(l(\alpha)) - l(\alpha) = +\infty,$$

and more generally,

$$\lim_{\alpha \to +\infty} \frac{l(\alpha) - l(\alpha)}{\alpha^p} = d(1 - t^p) > 0.$$ 

Writing

$$s(\alpha) = \frac{1}{[c(l(\alpha)) - l(\alpha)]^{1/p} l^{-1}(c(l(\alpha)) - l(\alpha))}$$

Letting $n \to +\infty$ and by the three limits above, we have

$$\lim_{\alpha \to +\infty} s(\alpha) = \frac{1}{c^{1/p}(1 - t^p)^{1/p}}.$$ 

Lemma 2.7. For $d > 0$, define

$$\tau_G(d) = \int_{0}^{d} \frac{ds}{l^{-1}[G'(d) - G(s)]}.$$ 

Then

$$\limsup_{d \to +\infty} \tau_G(d) \geq \left( \int_{0}^{1} \frac{dt}{(1 - t^p)^{1/p}} \right) \left( \frac{1}{p - 1} \liminf_{s \to +\infty} \frac{G(s)}{l(s)} \right)^{1/p}.$$
In particular (2.3) implies \( \limsup_{d \to +\infty} \tau_G(d) > (b - a)/2 \).

Proof. Let \( \rho \) be a positive number such that \( \liminf_{s \to +\infty} \frac{G(s)}{s} < \rho < C_l \). Then \( \limsup_{s \to +\infty} [\rho(s) - G(s)] = +\infty \). Let \( w_n \) be the smallest number in \([0, n]\) such that \( \max_{0 \leq s \leq n} K(s) = K(w_n) \) where \( K(s) = \rho(s) - G(s) \); it is easily seen that \( (w_n) \) is increasing with respect to \( n \). Since \( \rho(s) - G(s) < \rho(w_n) - G(w_n) \) for all \( s \in [0, w_n] \), we have \( \frac{G(w_n) - G(s)}{p-1} < \frac{\rho}{p-1} (l(w_n) - l(s)) \) for all \( s \in [0, w_n] \), since \( l : [0, +\infty] \to [0, +\infty] \) is an increasing homeomorphism, we have

\[
\frac{1}{l^{-1} \left[ \frac{\rho}{p-1} (l(w_n) - l(s)) \right]} < \frac{1}{l^{-1} \left[ \frac{\rho}{p-1} (G(w_n) - G(s)) \right]}.
\]

Integrating from 0 to \( w_n \) and changing variable \( s = uw_n \) in the first member of inequality, we obtain

\[
\int_{0}^{1} \frac{w_n}{l^{-1} \left[ \frac{\rho}{p-1} (l(w_n) - l(w_n s)) \right]} ds \leq \tau_G(w_n).
\]

Letting \( n \to +\infty \), we obtain

\[
\liminf_{n \to +\infty} \int_{0}^{1} \frac{w_n}{l^{-1} \left[ \frac{\rho}{p-1} (l(w_n) - l(w_n s)) \right]} ds \leq \limsup_{n \to +\infty} \tau_G(w_n).
\]

By Lemma 2.6 it results

\[
\limsup_{d \to +\infty} \tau_G(d) \geq \left[ \int_{0}^{1} \frac{dt}{(1-t)^{1/p}} \right] \left[ \frac{\rho}{p-1} \right]^{\frac{1}{p-1}}.
\]

Letting \( \rho \to \liminf_{s \to +\infty} \frac{G(s)}{s} \), the Lemma is proved. \( \square \)

**Lemma 2.8.** Let \( d > 0 \) and consider the mapping \( T_d \) defined by

\[
T_d(u) = d - \int_{0}^{d} r^{-1} \left( \left[ \int_{a}^{\tau} g(u(s))ds \right]^{1/(p-1)} \right) d\tau.
\]

in the Banach space \( C(I) \). Then \( T_d \) has a fixed point.

Proof. Clearly by Ascoli’s theorem \( T_d \) is compact. The proof of Lemma 2.8 uses an homotopy argument based on the Leray Schauder topological degree. So \( T_d \) will have a fixed point if the following condition holds:

There exists \( \rho > 0 \) such that \( (I - \lambda T_d)(u) \neq 0 \) for all \( u \in \partial B(0, \rho) \) for all \( \lambda \in [0, 1] \), where \( \partial B(0, \rho) = \{ u \in C(I); ||u||_{\infty} = \rho \} \).

To prove that this holds, suppose by contradiction that for all \( n = 1, 2, \ldots \) there exists \( u_n \in \partial B(0, n) \), \( \lambda_n \in [0, 1] \) such that: \( u_n = \lambda_n T_d(u_n) \). The latter relation implies

\[
u_n = \lambda_n d - \lambda_n \int_{0}^{d} r^{-1} \left( \left[ \int_{a}^{\tau} g(u(s))ds \right]^{1/(p-1)} \right) d\tau \tag{2.5}\]

Therefore, \( u_n \in C^1(I) \) and we have successively

\[
u_n'(t) = -\lambda_n r^{-1} \left( \left[ \int_{a}^{\tau} g(u(s))ds \right]^{1/(p-1)} \right) < 0 \quad \forall t \in [a, b], \tag{2.6}\\
\]

\[u_n(a) = 0,\]
\( (r \frac{u''_n(t)}{\lambda_n})' ) \in C(I) \) and
\[ -\left( r \frac{u''_n(t)}{\lambda_n} \right)' = g(u_n(t)) \quad \forall t \in I. \quad (2.7) \]

Note that by (2.6), \( u'_n(t) < 0 \) in \([a,b]\), so that \( u_n \) is decreasing. Hence, for \( n > d \), \( u_n(b) = -n \). Multiplying the equation (2.7) by \( u'_n(t) \), we obtain
\[ -\lambda_n \left( t \left( \frac{u''_n(t)}{\lambda_n} \right) \right)' = \frac{1}{p-1} \frac{d}{dt} G(u_n(t)). \quad (2.8) \]

Indeed
\[
\left( t \left( \frac{u''_n(t)}{\lambda_n} \right) \right)' = \left( t^{-1} \left( r \left( \frac{u''_n(t)}{\lambda_n} \right) \right) \right)' = (t \circ r^{-1})' \left( r \left( \frac{u''_n(t)}{\lambda_n} \right) \right) \left( r \left( \frac{u''_n(t)}{\lambda_n} \right) \right)' = \frac{1}{p-1} \frac{d}{dt} G(u_n(t)).
\]

By (2.8), we have
\[ \lambda_n \left( t \left( \frac{u''_n(t)}{\lambda_n} \right) \right) = \frac{1}{p-1} (G(\lambda_d) - G(u_n(t))) \]
and
\[ -\frac{u''_n(t)}{\lambda_n t^{-1} \left[ \frac{G(\lambda_d) - G(u_n(t))}{(p-1)\lambda_n} \right]} = 1. \]

Integrating from \( a \) to \( b \) and changing variable \( s = u_n(t) \) (\( u_n(a) = \lambda_n d \) and \( u_n(b) = -n \)), we obtain
\[ \int_{-\lambda_n d}^{\lambda_n d} ds \lambda_n t^{-1} \left[ \frac{G(\lambda_d) - G(s)}{(p-1)\lambda_n} \right] = b - a \]
i.e.
\[ \int_{0}^{\lambda_n d} ds \lambda_n t^{-1} \left[ \frac{G(\lambda_d) - G(s)}{(p-1)\lambda_n} \right] = b - a + \int_{0}^{-n} ds \lambda_n t^{-1} \left[ \frac{G(\lambda_d) - G(s)}{(p-1)\lambda_n} \right] \geq 0 \]

Since \( G(s) = sg(0) \) for \( s \leq 0 \) and changing variable \( s = -u \), we obtain
\[ 0 \leq (b - a) - \int_{0}^{n} ds \lambda_n t^{-1} \left[ \frac{G(\lambda_d) - sg(0)}{(p-1)\lambda_n} \right] \quad (2.9) \]

Denote by \( l(u) = \frac{G(\lambda_d) - G(s)}{(p-1)\lambda_n} \) such that \( l'(u) du = \frac{g(0)}{(p-1)\lambda_n} ds \) and \( ds = \frac{\lambda_n}{g(0)} l'(u) du \) for \( u \neq 0 \) and denote \( \alpha_n = l^{-1} \left[ \frac{G(\lambda_d)}{(p-1)\lambda_n} \right] \) and \( \beta_n = l^{-1} \left[ \frac{G(\lambda_d) + ng(0)}{(p-1)\lambda_n} \right] \). By (2.9), we obtain
\[ 0 \leq (b - a) - \int_{\alpha_n}^{\beta_n} r'(u) g(0) du \]
\[ = (b - a) - \frac{1}{g(0)} \int \left( l^{-1} \left[ \frac{G(\lambda_d) - ng(0)}{(p-1)\lambda_n} \right] \right) + \frac{1}{g(0)} \int \left( l^{-1} \left[ \frac{G(\lambda_d)}{(p-1)\lambda_n} \right] \right). \]

Since
\[ \frac{G(\lambda_d) - ng(0)}{(p-1)\lambda_n} \geq \frac{ng(0)}{(p-1)\lambda_n}, \quad \frac{G(\lambda_d)}{(p-1)\lambda_n} \leq \frac{d}{p-1} \max_{0 \leq s \leq d} |g(s)| \]
and $r \circ l^{-1}$ is increasing, it results that
\[
0 \leq (b - a) - \frac{1}{g(0)} r \{ l^{-1} \left[ \frac{ng(0)}{p-1} \lambda_n \right] \} + \frac{1}{g(0)} r \{ l^{-1} \left[ \frac{d}{p-1} \max_{\delta \leq s \leq d} |g(s)| \right] \}.
\]
Letting $n \to +\infty$, we get a contradiction. Let us denote by $u_d \in C(I)$ a fixed point of the mapping $T_d$ of Lemma 2.8. □

**Lemma 2.9.** There exists $d > 0$ such that $u_d(t) \geq 0$ for all $t \in [a, \frac{a+b}{2}]$.

**Proof.** We know that $u_d$ is decreasing and that $u_d(a) = d$ for all $d > 0$. Let us distinguish two cases.

First if there exists $d > 0$ such that $u_d(b) \geq 0$, then the conclusion of Lemma 2.9 clearly follows. So we can assume that $u_d(b) < 0$ for every $d > 0$. Since $u_d(a) = d > 0$, there exists $\delta_d \in ]a, b]$ such that $u_d(\delta_d) = 0$. It is clear that $u_d(t) \geq 0$ for all $t \in [a, \delta_d]$, and so it is sufficient to show that $\limsup_{d \to +\infty} \delta_d > \frac{a+b}{2}$.

Processing as in the proof of Lemma 2.8 we obtain
\[
-u_d'(t) \{ l^{-1} \left( \frac{G(d) - G(u_d(t))}{p-1} \right) \}^{-1} = 1.
\]
Integrating from $a$ to $\delta_d$ and changing variable $s = u_d(t)$, one gets
\[
\tau_G(d) = \int_0^{\delta_d} \frac{ds}{l^{-1} \left( \frac{G(d) - G(s)}{p-1} \right)} = \delta_d - a,
\]
consequently
\[
\limsup_{d \to +\infty} \delta_d > a + \frac{b - a}{2} = \frac{a + b}{2}.
\]
□

**Proof of Lemma 2.4 continued.** Denoting $u_d(t)$ by $u(t)$, we have $u \in C^1(I)$, $(r(u'))' \in C(I)$ and
\[
-(r(u'))' = g(u(s)) \quad \forall t \in I,
\]
\[
u(t) \geq 0 \quad \forall t \in [a, \frac{a+b}{2}],
\]
\[u'(a) = 0.
\]
Define a function $\beta_1$ from $[a, b]$ to $\mathbb{R}$ by
\[
\beta_1(t) = \begin{cases} 
 u \left( \frac{3a+b}{2} - t \right) & \text{if } t \in [a, \frac{a+b}{2}], \\
u(t - \frac{b-a}{2}) & \text{if } t \in [\frac{a+b}{2}, b].
\end{cases}
\]
We will show that this function $\beta$ fulfills the conditions of Lemma 2.4. To see this it is sufficient to show that
\[(a) \quad \beta_1 \text{ is nonnegative in } [a, b],
\[(b) \quad \beta_1 \in C^1([a, b]),
\[(c) \quad (r(\beta_1(t)))' \in C([a, b]) \text{ and } -(r(\beta_1'(t)))' = g(\beta_1(t)) \text{ for all } t \in [a, b].
\]
Proof of (a). If $a \leq t \leq \frac{a+b}{2}$, then $a \leq \frac{3a+b}{2} - t \leq \frac{a+b}{2}$, and if $\frac{a+b}{2} \leq t \leq b$, then $a \leq t - \frac{b-a}{2} \leq \frac{a+b}{2}$, so that the conclusion follows from the sign of $u$ on $[a, \frac{a+b}{2}]$.

Proof of (b). $\beta_1 \in C^1([a, \frac{a+b}{2}])$, $\beta_1 \in C^1([\frac{a+b}{2}, b])$, and moreover $\frac{d}{dt} \beta_1(a+b) = u'(a) = 0$ and $\frac{d}{dt} \beta_1(\frac{a+b}{2}) = u'(a) = 0.$
Proof of (c). We know that
\[-(r(u'(t))))' = g(u(t)) \text{ for } t \in [a, b], \]
therefore
\[-(r(u'(t))) = \int_a^t g(u(s)) ds. \]
If \( a \leq t \leq \frac{a+b}{2} \) then \( a \leq \frac{3a+b}{2} - t \leq \frac{a+b}{2} \), which gives
\[\beta_1(t) = u\left(\frac{3a+b}{2} - t\right) \quad \text{and} \quad \beta_1'(t) = -u'\left(\frac{3a+b}{2} - t\right). \]
We obtain
\[-(r(u'(a+b/2)) = r(\beta_1'(t)). \]
The change of variable \( u = \frac{3a+b}{2} - s \) yields
\[\int_a^{3a+b/2-t} g(u(s)) ds = \int_t^{a+b/2} g(u(\frac{3a+b}{2} - s)) ds, \]
hence
\[r(\beta_1'(t)) = \int_t^{a+b/2} g(\beta_1(s)) ds \quad \forall t \in [a, \frac{a+b}{2}] \]
and
\[-(r(\beta_1'(t)))' = g(\beta_1(t)) \quad \forall t \in [a, \frac{a+b}{2}] \]
The proof is similar for all \( t \in [\frac{a+b}{2}, b] \). \qed

References


AOMAR ANANE
DÉPARTEMENT DE MATHÉMATIQUES ET INFORMATIQUE, FACULTÉ DES SCIENCES, UNIVERSITÉ MOHAMMED 1ER, OUJDA, MAROC
E-mail address: anane@sciences.univ-oujda.ac.ma
Omar Chakrone
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: chakrone@sciences.univ-oujda.ac.ma

Mohammed Chehabi
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: chehb_md@yahoo.fr