MULTIPICITY RESULTS FOR NONLINEAR ELLIPTIC EQUATIONS

SAMIRA BENMOULOUD, MOSTAFA KHIDDI, SIMOHAMMED SBAI

Abstract. Let Ω be a bounded domain in \( \mathbb{R}^N \), \( N \geq 3 \), and \( p = \frac{2N}{N-2} \) the limiting Sobolev exponent. We show that for \( f \in H^0_{1,0}(\Omega)^* \), satisfying suitable conditions, the nonlinear elliptic problem

\[
-\Delta u = |u|^{p-2}u + f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\]

has at least three solutions in \( H^0_{1,0}(\Omega) \).

1. Introduction

It is well known [6, Theorems 1 and 2] that for \( f \neq 0 \) and \( \|f\| \) sufficiently small, the problem

\[
-\Delta u = |u|^{p-2}u + f \quad \text{on } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\]  

has at least two distinct solutions \( u_0 \) and \( u_1 \) which are critical points of the functional

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p} \int_{\Omega} |u|^p - \int_{\Omega} fu,
\]

such that \( I(u_1) > I(u_0) \). In this note we suppose \( f \geq 0 \) and satisfies

\[
\|f\| < S \S_\frac{N}{2},
\]

where

\[
\frac{1}{2} < \alpha < \left( \frac{N-2}{N+2} \right)^{\frac{N+2}{N-2}}, \quad \text{and} \quad S = \inf_{u \in H^0_{1,0}(\Omega), \|u\|_p = 1} \|\nabla u\|_2^2,
\]

which corresponds to the best constant for the Sobolev embedding \( H^0_{1,0}(\Omega) \hookrightarrow L^p(\Omega) \).

We determine a special \( \omega_\varepsilon \), from the extremal functions for the Sobolev inequality in \( \mathbb{R}^N \), and consider \( \Gamma \) the class of continuous paths joining 0 to \( \omega_\varepsilon \).

Proposition 1.1. Let

\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)).
\]
Then there is a sequence \((u_j) \subset H^1_0(\Omega)\) such that

\[
I(u_j) \to c,
I'(u_j) \to 0 \quad \text{in \((H^1_0(\Omega))^*\)},
I(u_0) < I(u_1) < c.
\]

Let \(u\) denotes the weak limit in \(H^1_0(\Omega)\) of (a subsequence of) \((u_n)\), our principal result is as follows.

**Theorem 1.2.** Let \(f \in H^1_0(\Omega)^*\), \(f \geq 0\) satisfies (1.2). Then either

1. \(I(u) = c\) and Problem (1.1) has at least three solutions. Or
2. \(I(u) \leq c - \frac{1}{N} S^{N/2}\).

Note that the existence results of biharmonic analogue of Problem (1.1) have been studied in \([2]\), so a result similar to that of Theorem 1.2 may be established for the bilaplacian operator.

2. The proof of Proposition 1.1

We start with a variant of the mountain pass theorem of Ambrosetti-Rabinowitz without the Palais-Smale condition

**Theorem 2.1.** Let \(E\) be a real Banach space and \(I \in C^1(E, \mathbb{R})\). Suppose there exists a neighborhood \(U\) of 0 in \(E\) and a constant \(\rho > 0\) such that

- \((H1)\) \(I(u) \geq \rho\), for all \(u \in \partial U\).
- \((H2)\) \(I(0) < \rho\) and, \(I(v) < \rho\) for some \(v \in E \setminus U\).

Let \(c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))\), where \(\Gamma = \{\gamma : [0,1] \to E, \ \text{is continuous}, \ \gamma(0) = 0, \ \gamma(1) = v\}\). Then there is a sequence \((u_n)\) in \(E\) such that

\[
I(u_n) \to c,
I'(u_n) \to 0 \quad \text{in \(E^*\)}.
\]

On \(H^1_0(\Omega)\) we define a variational functional \(I : H^1_0(\Omega) \to \mathbb{R}\) for problem (1.1), by

\[
I(u) = \frac{1}{2} \|\nabla u\|^2 \|2 - \frac{1}{p} \|u\|^p - \int_{\Omega} fu.
\]

Clearly \(I\) is \(C^1\) on \(E\) and \(I(0) = 0\). We shall verify the assumptions of Theorem 2.1

**Verification of \(H1\).** Let \(r \in [0, \alpha S^{N/4}]\) and \(u \in H^1_0(\Omega)\) be such that \(\|\nabla u\|^2 = r\). We have

\[
I(u) \geq \frac{1}{2} \alpha^2 S^{N/2} - \frac{1}{p} r^p S^{-p/2} - \|f\| r.
\]

Letting \(r \to \alpha S^{N/4}\), we obtain

\[
I(u) \geq \frac{1}{2} \alpha^2 S^{N/2} - \frac{1}{p} \alpha^p S^{N/2} - \frac{1}{4N} \alpha^2 S^{N/2}.
\]

Set

\[
\rho = \frac{\alpha^p S^{N/2}}{2N},
\]

hence \(I(u) > \rho\) for all \(u \in \partial B(0, r)\).
For every $\phi \in C_0^\infty(\Omega)$ be a fixed function such that $\phi \equiv 1$ for $x$ in some neighborhood of 0. For $\varepsilon > 0$, define
\[ u_\varepsilon(x) = \frac{\phi(x)}{\varepsilon + |x|^2} \frac{N - 2}{2}, \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|^p}. \]
Hence, from [1],
\[ \|\nabla v_\varepsilon\|^2_2 = S + O(\varepsilon^{\frac{N-2}{2}}). \tag{2.1} \]
\textbf{Verification of (H2).} Assume $0 < \mu \neq 0$, [4] Lemma 2.1, gives a real $t^+ > 0$ such that
\[ t^+ > (\frac{\|\nabla \mu v_\varepsilon\|^2_2}{(p-1)} \|\mu v_\varepsilon\|^p) = \frac{1}{\mu} (\frac{N - 2}{N + 2}) \frac{N - 2}{4} \|\nabla v_\varepsilon\|^\frac{N-2}{2} \tag{2.2} \]
and
\[ t^+ < \frac{1}{\mu} \|\nabla v_\varepsilon\|^\frac{N-2}{2}. \tag{2.3} \]
Set $\omega = t^+ \mu v_\varepsilon$. We have
\[ \|\nabla \omega\|^2_2 = t^+ \mu \|\nabla v_\varepsilon\|^2_2 > (\frac{N - 2}{N + 2}) \frac{N - 2}{4} \|\nabla v_\varepsilon\|^\frac{N-2}{2} > (\frac{N - 2}{N + 2}) \frac{N - 2}{4} S^\frac{N-2}{2} > \alpha S^\frac{N-2}{2} > r. \]
On the other hand, from (2.2) and (2.3), we get
\begin{align*}
I(\omega) &< \frac{1}{2} (t^+)^2 \|\nabla \omega\|^2_2 - \frac{1}{p} (t^+)^p \\
&< \frac{1}{2\mu^2} \|\nabla v_\varepsilon\|^N_2 - \frac{1}{\mu^p} (\frac{N - 2}{N + 2}) \frac{(N - 2)}{4} \|\nabla v_\varepsilon\|^N_2.
\end{align*}
Using (2.1), we deduce
\[ I(\omega) < (\frac{1}{2\mu^2} - \frac{1}{\mu^p} \frac{N - 2}{N + 2} (\frac{N - 2}{N + 2}) \frac{N - 2}{4}) (S + O(\varepsilon^{\frac{N-2}{2}}))^{N/2} \frac{c p S^{N/2}}{2N}, \]
for $\mu$ large enough. Then $c \geq \rho > I(\omega)$. Recall that $\omega \in \Lambda^-$ ( [3] Lemma 2.1) with
\[ \Lambda^- = \{ u \in H_0^1(\Omega) / < I'(u), u >= 0, \|\nabla u\|^2_2 - (p-1)\|u\|^p_\mu < 0 \}, \]
and that $\inf_{\Lambda^-} I$ is attained by $u_0$ [3] Theorem 2. We conclude that
\[ c \geq \rho > I(\omega) \geq I(u_0) > I(u_0). \]
\section{Proof of the Theorem 1.2}
Applying Proposition 1.1, we obtain a sequence $(u_j) \subset H_0^1(\Omega)$ such that
\begin{align*}
I(u_0) &\to c, \tag{3.1} \\
I'(u_0) &\to 0 \quad \text{in } H_0^1(\Omega)^*. \tag{3.2}
\end{align*}
This implies that $\|\nabla u_0\|^2_2$ is uniformly bounded. Hence for a subsequence of $u_j$, still denoted by $u_j$, we can find $u \in H_0^1(\Omega)$ such that
\begin{align*}
u_j &\to u \quad \text{weakly in } H_0^1(\Omega), \\
u_j &\to u \quad \text{strongly in } L^q, \quad q < p, \\
u_j &\to u \quad \text{a.e. on } \Omega.
\end{align*}
From (3.2), we deduce that $u$ is a (weak) solution of Problem (1.1). In particular $u$ satisfies
\[ \|u\|^2_2 - \|u\|^p_\mu = \int u \quad \text{for } \mu > 0. \tag{3.3} \]
Let \( u_j = u + v_j \), where \( v_j \to 0 \) weakly in \( H^1_0(\Omega) \) and \( v_j \to 0 \) a.e on \( \Omega \). We have
\[
\| \nabla u_j \|_2^2 = \| \nabla u \|_2^2 + \| \nabla v_j \|_2^2 + o(1).
\]
and by (3.1),
\[
I(u) + \frac{1}{2} \| \nabla v_j \|_2^2 - \frac{1}{p} \| v_j \|_p^p = c + o(1),
\]
thanks to Brezis-Lieb Lemma [5]. By (3.2) and (3.3), \( \| \nabla v_j \|_2^2 - \| v_j \|_p^p = o(1) \), which gives
\[
I(u) + \frac{1}{N} \| \nabla v_j \|_2^2 = c + o(1).
\]
Set \( l = \lim_{j \to +\infty} \| \nabla v_j \|_2^2 \), then \( \lim_{j \to +\infty} \| v_j \|_p^p = l \). Using Sobolev inequality one see that \( l \geq S^2/p \). Then \( l = 0 \), or \( l \geq S^2/N \). We get, either
\[
I(u) = c,
\]
and since
\[
I(u) > I(u_1) > I(u_0),
\]
\( u \) is a solution of Problem (1.1) distinct from \( u_0 \) and \( u_1 \), or
\[
I(u) \leq c - \frac{1}{N} S^{2/N}.
\]

**Remark 3.1.** One can show that \( c < \frac{1}{N} S^{2/N} \), consequently \( I(u) < 0 \) in the second case

4. **SEMILINEAR BIHARMONIC EQUATION**

In [2], Benmouloud considered the problem
\[
\Delta^2 u = |u|^{p-2} u + f \quad \text{in } \Omega
\]
\[
\Delta u = u = 0 \quad \text{on } \partial \Omega
\]
where \( \Omega \) is a bonded domain in \( \mathbb{R}^N \), \( N \geq 5 \) \( p = \frac{2N}{N-4} \) and \( \Delta^2 \) denotes the biharmonic operator. She proved that for \( f \in H^{-1} \) subject to a suitable condition, this problem has at least two distinct solutions in \( H^2(\Omega) \cap H^1_0(\Omega) \). The existence of on solution follows from the mountain-pass theorem, with Palais-Smale condition, and a second is obtained by a constrained minimization (see also [3]).

It follows from this study that an analog result of Theorem 1.2 may be established by a similar argument with suitable smallness condition on \( f \).

**References**


Samira Benmouloud  
E.G.A.L, DÉPT. MATHS, FAC. SCIENCES, Université Ibn Tofail, BP. 133, Kénitra, Maroc  
E-mail address: ben.sam@netcourrier.com

Mostafa Khiddi  
E.G.A.L, DÉPT. MATHS, FAC. SCIENCES, Université Ibn Tofail, BP. 133, Kénitra, Maroc

Simohammed Sbai  
E.G.A.L, DÉPT. MATHS, FAC. SCIENCES, Université Ibn Tofail, BP. 133, Kénitra, Maroc  
E-mail address: sbaisi@netcourrier.com