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ASYMMETRIC ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

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Abstract. We work on the whole $\mathbb{R}^N$ and prove the existence of a first nonprincipal eigenvalue for an asymmetric problem with weights involving the p-Laplacian (cf. (1.1) below). As an application we obtain a first nontrivial curve in the corresponding Fučik spectrum.

1. Introduction

This work is mainly concerned with the following (asymmetric) eigenvalue problem

$$-\Delta_p u = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, is the p-Laplacian, λ is a real parameter, m and n are weights whose properties will be specified later, and $u^\pm = \max\{\pm u, 0\}$. Our assumptions on the weights will guarantee the existence of a unique positive principal eigenvalue $\lambda_1(m)$ for the following (symmetric) eigenvalue problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The principal motivation for considering problem (1.1) comes from the study of the Fučik spectrum. This spectrum is defined as the set Σ of those $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$-\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} \quad \text{in } \mathbb{R}^N \tag{1.3}$$

has a nontrivial solution u. The relation between (1.3) and (1.1) is clear since the line of slope r through the origin of $\mathbb{R}^2$ meets Σ at a point $(\alpha, \beta = r\alpha)$ if and only if α is an eigenvalue of (1.1) for the weights m and rn.

Many works have been devoted to the study of the Fučik spectrum in the case of a bounded domain. But to our knowledge nothing has been done in the case of an unbounded domain, in particular $\mathbb{R}^N$, even in the linear case $p = 2$. It should be pointed out here that in $\mathbb{R}^N$, the presence of weights becomes essential since for instance (1.2) has no principal eigenvalue if $m \equiv 1$ (cf. [4, 11] when $N \le p$, [15] when $N > p$).

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A study of (1.1) together with applications to the Fučik spectrum and to nonresonance was carried out recently in [2] in the case of a bounded domain $\Omega \subset \mathbb{R}^N$:

$$-\Delta_p u = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Denoting by $\mu_1(m)$ the first positive eigenvalue of the Dirichlet p-Laplacian with weight m on Ω and by ψ_m the associated normalized positive eigenfunction, it was shown in [2] that (1.4) always admits a positive nonprincipal eigenvalue, which in addition is the first eigenvalue of (1.4) greater than $\mu_1(m)$ and $\mu_1(n)$. This distinguished eigenvalue was constructed by applying a version of the mountain pass theorem to the functional $\int_\Omega |\nabla u|^p$ restricted to the C^1 manifold $\{u \in W_0^{1,p}(\Omega) : \int_\Omega [m(u^+)^p + n(u^-)^p] = 1\}$. In this process the (PS) condition was easily verified by using the $(S)_+$ property of the p-Laplacian while the geometry of the mountain pass was derived from the observation that ψ_m and $-\psi_n$ were strict local minima.

When trying to adapt the above approach to the case of the whole $\mathbb{R}^N$, the relevant functional is

$$J(u) := \int_{\mathbb{R}^N} |\nabla u|^p \tag{1.5}$$

restricted to

$$M_{m,n} := \big\{u \in W : B_{m,n}(u) := \int_{\mathbb{R}^N} [m(u^+)^p + n(u^-)^p] = 1\big\}, \tag{1.6}$$

where the space W is a suitable weighted Sobolev space on $\mathbb{R}^N$ which will be defined later. One of the main difficulties lies, as expected, in the verification of the (PS) condition. This is carried out in Proposition 3.3, whose proof uses some technique from [1, 11] as well as a result from [9] about the compact imbedding of W into a weighted Lebesgue space. Other difficulties arise in connection with the geometry of the functional (cf. the proof of Proposition 3.1) as well as in the construction of some suitable auxiliary weights (cf. Lemma 4.2 and the proof of Theorem 4.1). One should also point out that the study of the continuous dependance of our distinguished eigenvalue of (1.1) with respect to the weights requires some special care due in particular to the fact that these weights generally do not satisfy any integrability condition on $\mathbb{R}^N$ (cf. Proposition 4.7 and Corollary 4.8).

The existence of a positive nonprincipal eigenvalue for (1.1) is derived in Section 3. In Section 4, we prove that the eigenvalue $c(m,n)$ constructed in Section 3 is the first nonprincipal eigenvalue of (1.1). We also study there some of the properties of $c(m,n)$ as a function of (m,n). Section 5 is devoted to the Fučik spectrum. We show the existence of a first nontrivial curve in $\Sigma \cap (\mathbb{R}^+ \times \mathbb{R}^+)$ whose asymptotic behaviour exhibits some similarity with what is happening for the Dirichlet problem on a bounded domain. In the preliminary Section 2, we collect some known results relative to the eigenvalue problem (1.2) and to various Sobolev imbeddings or Poincaré's type inequalities to be used later.

2. Preliminaries

Throughout this work, we write the weights m and n in the form $m = m_1 - m_2$, $n = n_1 - n_2$, and we assume the following conditions:

- (H_1) $m_1, n_1 \geq 0$, $m_1, n_1 \in L^\infty_{\rm loc}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$, where $s = N/p$ if $N > p$ and $s = N_0/p$ for some integer $N_0 > p$ if $N \leq p$;

(H_2) $m_2, n_2 \geq 0$, $m_2, n_2 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, with in addition $m_2(x), n_2(x) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ a.e. in $\mathbb{R}^N$ if $N \leq p$;

(H_3) $m^+ \not\equiv 0$, $n^+ \not\equiv 0$;

(H_4) for some $a, b > 0$: $am_2(x) \leq n_2(x) \leq bm_2(x)$ a.e. in $\mathbb{R}^N$.

Note that the decomposition $m = m_1 - m_2$ does not necessarily coincide with the decomposition $m = m^+ - m^-$.

Associated with m_2, we define a weighted Sobolev space W as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_W := \Big[\int_{\mathbb{R}^N} (|\nabla u|^p + m_2|u|^p) \Big]^{1/p}. \tag{2.1}$$

Note that by (H_4), n_2 would lead to the same space W. Note also that as observed in [9], the space W does not depend on the decomposition of m into $m_1 - m_2$. The following imbeddings hold (cf. e.g. [3]): $W \hookrightarrow D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ if $N > p$, $W \hookrightarrow W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \in [p, +\infty[$ if $N = p$ and for all $q \in [p, +\infty]$ if $N < p$. Here $D^{1,p}(\mathbb{R}^N)$ denotes when $N > p$ the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\nabla u|^p)^{1/p}$ and $p^* := Np/(N-p)$ is the critical Sobolev exponent. With s as in (H_1) above and s' its Hölder conjugate, we will denote later by A the constant of the imbedding of $D^{1,p}(\mathbb{R}^N)$ into $L^{ps'}(\mathbb{R}^N) = L^{p^*}(\mathbb{R}^N)$ when $N > p$, and by B the constant of the imbedding of $W^{1,p}(\mathbb{R}^N)$ into $L^{ps'}(\mathbb{R}^N)$ when $N \leq p$. One also has the compact imbedding of W into $L^p(m_1, \mathbb{R}^N)$, the L^p space on $\mathbb{R}^N$ with weight m_1 (cf. [9]).

By a solution u of (1.1) (or of related equations), we mean a weak solution, i.e. $u \in W$ with

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_{\mathbb{R}^N} [m(u^+)^{p-1} - n(u^-)^{p-1}]v \quad \forall v \in W. \tag{2.2}$$

Note that by the above imbeddings, every integral in (2.2) is well-defined. Regularity results from [14] and [16] on general quasilinear equations imply that such a weak solution u belongs to $C^1(\mathbb{R}^N)$. It is also known that if $N < p$, or if $N > p$ and (H_4) is replaced by (H_4') (cf. Remark 2.5 below), then a weak solution u decays to zero at infinity (cf. [3] for $N < p$, [8, 10] for $N > p$).

Let us define

$$\lambda_1(m) := \inf \big\{ \int_{\mathbb{R}^N} |\nabla u|^p : u \in W \text{ and } \int_{\mathbb{R}^N} m|u|^p = 1 \big\}.$$

It is known (cf. [1, 7, 8, 9, 10]) that this infimum is achieved and that $\lambda_1(m)$ is the unique positive principal eigenvalue of (1.2). (By a principal eigenvalue, we mean an eigenvalue associated to an eigenfunction which does not change sign). Moreover $\lambda_1(m)$ is simple and admits an eigenfunction $\varphi_m \in W \cap C^1(\mathbb{R}^N)$, with $\varphi_m(x) > 0$ in $\mathbb{R}^N$ and $\int_{\mathbb{R}^N} m\varphi_m^p = 1$. One also knows that $\lambda_1(m)$ is isolated in the spectrum, which implies

$$\lambda_2(m) := \inf \{\lambda \in \mathbb{R} : \lambda \text{ eigenvalue of (1.2) with } \lambda > \lambda_1(m)\} > \lambda_1(m). \tag{2.3}$$

As we will see later (cf. Remark 3.4 or Theorem 4.1), this infimum (2.3) is also achieved, and consequently $\lambda_2(m)$ is really the second positive eigenvalue of (1.2).

Remark 2.1. Assume that m satisfies (H_1), (H_2), (H_3) and that $m^- \not\equiv 0$. If $N > p$ and $m_2 \in L^{N/p}(\mathbb{R}^N)$, then (1.2) also has a unique negative principal eigenvalue $\lambda_{-1}(m) = -\lambda_1(-m)$. If $N \leq p$, then (1.2) generally does not admit a negative principal eigenvalue; this follows from the nonexistence results of [4] for $p = 2$ and [11] for $p \neq 2$.

The following lemma will play a role in low dimensions. It can be easily derived from the proof in [1, Theorem 3]. (The assumptions about N_0 in (H_1) and about ε_0 in (H_2) are used here).

Lemma 2.2. *Let $N \leq p$. There exists $C = C(m_1, m_2, n_1, n_2, N, p, N_0)$ such that*

$$\int_{\mathbb{R}^N} |u|^p \leq C \int_{\mathbb{R}^N} |\nabla u|^p \tag{2.4}$$

for all $u \in W$ satisfying $B_{m,n}(u) \geq 0$. Moreover the constant C in (2.4) can be chosen so as to remain bounded when m_1 and n_1 vary in a bounded subset of $L^{N_0/p}(\mathbb{R}^N)$.

Remark 2.3. It suffices in all this work to assume that the inequalities in (H_2) and (H_4) hold "at infinity". More precisely denote by $(H_2)_R$ and $(H_4)_R$ the same conditions as (H_2) and (H_4) except that the inequalities $m_2(x), n_2(x) \geq$ some ε_0 and $am_2(x) \leq n_2(x) \leq bn_2(x)$ are assumed to hold only for a.e. x with $|x| \geq R$ for some $R \geq 0$. Suppose now that $m = m_1 - m_2$, $n = n_1 - n_2$ satisfy (H_1), $(H_2)_R$, (H_3), $(H_4)_R$. By writing $m = \tilde{m}_1 - \tilde{m}_2$, $n = \tilde{n}_1 - \tilde{n}_2$ where $\tilde{m}_1$, $\tilde{m}_2$, $\tilde{n}_1$, $\tilde{n}_2$ are obtained from m_1, m_2, n_1, n_2 by adding 1_{B_R} (the characteristic function of the ball with center 0 and radius R), one easily sees that $m = \tilde{m}_1 - \tilde{m}_2$, $n = \tilde{n}_1 - \tilde{n}_2$ now satisfy (H_1), (H_2), (H_3), (H_4).

Remark 2.4. In the situation of Remark 2.3, one can also show that the space W associated to m_2 coincides with the space W associated to $\tilde{m}_2$. (The proof of this fact uses the inequality that if Ω is a smooth bounded domain and if E is a subset of Ω of positive measure, then there exists a constant c such that $\|u\|_{L^p(\Omega)} \leq c(\|u\|_{L^p(E)} + \|\nabla u\|_{L^p(\Omega)})$ for all $u \in W^{1,p}(\Omega)$). It follows from this observation that the space W does not depend on the decomposition of m into $m_1 - m_2$ when m_1, m_2 satisfy (H_1) and $(H_2)_R$.

Remark 2.5. In high dimensions, assumption (H_4) can be replaced in all this work by

(H_4') $N > p$ and $m_2, n_2 \in L^{N/p}(\mathbb{R}^N)$.

This situation is in fact much simpler (for instance W is then equal to $D^{1,p}(\mathbb{R}^N)$). Without (H_4), $(H_4)_R$ (i.e. when m_2 and n_2 are unrelated) or (H_4'), it is not clear how to deal with the asymmetric problems (1.1) and (1.3).

Let us conclude this section with some general definitions relative to the (PS) condition. Let E be a real Banach space and

$$M := \{u \in E : g(u) = 1\}, \tag{2.5}$$

where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$ and denote by $\tilde{f}$ the restriction of f to M. The differential of $\tilde{f}$ at $u \in M$ has a norm which will be denoted by $\|\tilde{f}'(u)\|_*$ and which is given by the norm of the restriction of $f'(u) \in E^*$ to the tangent space of M at u:

$$T_u M := \{v \in E : \langle g'(u), v \rangle = 0\},$$

where $\langle , \rangle$ denotes the pairing between E and its dual E^*. A critical point of $\tilde{f}$ is a point $u \in M$ such that $\|\tilde{f}'(u)\|_* = 0$; $\tilde{f}(u)$ is then called a critical value of $\tilde{f}$. We recall that $\tilde{f}$ is said to satisfy the (PS) condition if for any sequence $u_k \in M$ such that $\tilde{f}(u_k)$ is bounded and $\|\tilde{f}'(u_k)\|_* \to 0$, one has that u_k admits a convergent subsequence.

3. Construction of a nonprincipal eigenvalue

In this section and in the following one, we consider the eigenvalue problem (1.1). It will always be assumed that the weights m and n satisfy the hypothesis (H_1), (H_2), (H_3) and (H_4).

We look for eigenvalues λ of (1.1) with $\lambda > 0$. Clearly the only positive principal eigenvalues of (1.1) are $\lambda_1(m)$ and $\lambda_1(n)$. Moreover multiplying by u^+ or u^-, one easily sees that if (1.1) with $\lambda > 0$ has a solution which changes sign, then $\lambda > \max\{\lambda_1(m), \lambda_1(n)\}$. Proving the existence of such a solution which changes sign is our purpose in this section.

We will use a variational approach and consider the functionals J and $B_{m,n}$ defined in (1.5) and (1.6), which are C^1 functionals on W, and the restriction $\tilde{J}$ of J to the manifold $M_{m,n}$ defined in (1.6). In this context one easily verifies that $\lambda > 0$ is an eigenvalue of (1.1) if and only if λ is a critical value of $\tilde{J}$.

A first critical point of $\tilde{J}$ comes from global minimization. Indeed

$$\tilde{J}(u) \geq \lambda_1(m)\Big[\int_{\mathbb{R}^N} m(u^+)^p\Big]^+ + \lambda_1(n)\Big[\int_{\mathbb{R}^N} n(u^-)^p\Big]^+ \geq \min\{\lambda_1(m), \lambda_1(n)\}$$

for all $u \in M_{m,n}$, and one has $\tilde{J}(u) = \min\{\lambda_1(m), \lambda_1(n)\}$ for either $u = \varphi_m$ or $u = -\varphi_n$. Consequently either φ_m or $-\varphi_n$ is a global minimum of $\tilde{J}$ and so a critical point of $\tilde{J}$.

A second critical point of $\tilde{J}$ comes from the following proposition.

Proposition 3.1. *φ_m and $-\varphi_n$ are strict local minimum of $\tilde{J}$, with corresponding critical values $\lambda_1(m)$ and $\lambda_1(n)$.*

Proof. The present proof is partly different from that of the analogous result in [2]; the difficulty lies at the level of [2, Lemma 3]. We adapt to our situation some technique from [7].

Let us consider φ_m (similar argument for $-\varphi_n$). Assume by contradiction the existence of a sequence $u_k \in M_{m,n}$ with $u_k \neq \varphi_m$, $u_k \to \varphi_m$ in W and $\tilde{J}(u_k) \leq \lambda_1(m)$. We first observe that u_k changes sign for k sufficiently large. Indeed, since $u_k \to \varphi_m$, u_k must be > 0 somewhere. If $u_k \geq 0$ in $\mathbb{R}^N$, then

$$\tilde{J}(u_k) = \int_{\mathbb{R}^N} |\nabla u_k|^p > \lambda_1(m) \int_{\mathbb{R}^N} m u_k^p = \lambda_1(m)$$

since $u_k \neq \varphi_m$ and $u_k \in M_{m,n}$. But this contradicts $\tilde{J}(u_k) \leq \lambda_1(m)$. So u_k changes sign for k sufficiently large. Now we have

$$\begin{aligned}
\lambda_1(m) \int_{\mathbb{R}^N} [m(u_k^+)^p + n(u_k^-)^p] &= \lambda_1(m) \geq \tilde{J}(u_k) \\
&= \int_{\mathbb{R}^N} |\nabla u_k^+|^p + \int_{\mathbb{R}^N} |\nabla u_k^-|^p \\
&\geq \lambda_1(m) \int_{\mathbb{R}^N} m(u_k^+)^p + \int_{\mathbb{R}^N} |\nabla u_k^-|^p.
\end{aligned}$$

Consequently

$$\lambda_1(m)\int_{\mathbb{R}^N} n(u_k^-)^p \geq \int_{\mathbb{R}^N} |\nabla u_k^-|^p. \tag{3.1}$$

Let $v_k := u_k^-/(\int_{\mathbb{R}^N} |\nabla u_k^-|^p)^{1/p}$ and $\Omega_k^- := \{x \in \mathbb{R}^N : u_k(x) < 0\}$. We deduce from (3.1) that

$$\frac{1}{\lambda_1(m)} \leq \int_{\mathbb{R}^N} n(v_k)^p \leq \int_{\Omega_k^-} n_1(v_k)^p. \tag{3.2}$$

Consider first the case $N > p$. We deduce from (3.2), using Hölder inequality, that

$$\frac{1}{\lambda_1(m)} \leq \|n_1\|_{L^s(\Omega_k^-)}\|v_k\|^p_{L^{p^*}(\mathbb{R}^N)} \leq A^p\|n_1\|_{L^s(\Omega_k^-)},$$

where the imbedding constant A was defined in Section 2, and consequently

$$\|n_1\|_{L^s(\Omega_k^-)} \geq \frac{1}{A^p\lambda_1(m)} = \varepsilon. \tag{3.3}$$

Take now $r > 0$ sufficiently large so that $\|n_1\|^s_{L^s(B_r^c)} \leq \varepsilon^s/2$, where $B_r^c = \mathbb{R}^N \backslash B_r$ and B_r denotes the ball of radius r centred at the origin. We deduce from (3.3) that $\|n_1\|^s_{L^s(\Omega_k^- \cap B_r)} \geq \varepsilon^s/2$, and consequently

$$|\Omega_k^- \cap B_r| \geq \frac{\varepsilon^s}{2\|n_1\|^s_{L^\infty(B_r)}} > 0, \tag{3.4}$$

where $|E|$ denotes the measure of the set E. Since $u_k \to \varphi_m$ in $L^p(B_r)$ and $\varphi_m(x) > 0$ for all $x \in B_r$, one has that $|\{x \in B_r : u_k(x) < 0\}| \to 0$. But this contradicts (3.4).

In the case $N \leq p$, we have a similar situation. Indeed using Hölder inequality, the imbedding of $W^{1,p}(\mathbb{R}^N)$ into $L^{ps'}(\mathbb{R}^N)$ (with constant B, cf. Section 2) and Lemma 2.2 (with constant C), one derives from (3.2) that

$$\frac{1}{\lambda_1(m)} \leq (1+C)B^p\|n_1\|_{L^s(\Omega_k^-)}. \tag{3.5}$$

The conclusion then follows as in the case $N > p$. □

To get a third critical point of $\tilde{J}$, we will use a version of the mountain pass theorem on a C^1 manifold. Let us introduce the following family of paths in the manifold $M_{m,n}$:

$$\Gamma := \{\gamma \in C([-1,1], M_{m,n}) : \gamma(-1) = \varphi_m \text{ and } \gamma(1) = -\varphi_n\}. \tag{3.6}$$

Arguing as in [2, p. 589], one shows that Γ is nonempty, and so the minimax value

$$c(m,n) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([-1,1])} \tilde{J}(u), \tag{3.7}$$

is finite. The following is the main result in this section.

Theorem 3.2. *$c(m,n)$ is an eigenvalue of (1.1) which satisfies*

$$\max\{\lambda_1(m), \lambda_1(n)\} < c(m,n). \tag{3.8}$$

The rest of this section is devoted to the proof of the above theorem. We first consider the (PS) condition.

Proposition 3.3. *The functional $\tilde{J}$ satisfies the (PS) condition on $M_{m,n}$*

Proof. Let $u_k \in M_{m,n}$ be a (PS) sequence for $\tilde{J}$. So $\int_{\mathbb{R}^N} |\nabla u_k|^p$ remains bounded and for some $\varepsilon_k \to 0$,

$$\Big| \int_{\mathbb{R}^N} |\nabla u_k|^{p-2} \nabla u_k \nabla w \Big| \le \varepsilon_k \|w\|_W \tag{3.9}$$

for all $w \in T_{u_k}(M_{m,n})$.

We will first prove that u_k remains bounded in W. In case $N > p$, u_k clearly remains bounded in $D^{1,p}(\mathbb{R}^N)$ and consequently in $L^{p^*}(\mathbb{R}^N)$. Using $B_{m,n}(u_k) = 1$ and (H_4), one has

$$\min(a,1) \int_{\mathbb{R}^N} m_2 |u_k|^p \le -1 + \int_{\mathbb{R}^N} [m_1 (u_k^+)^p + n_1 (u_k^-)^p], \tag{3.10}$$

where the right hand side remains bounded (by (H_1) and Hölder inequality). Consequently $\int_{\mathbb{R}^N} m_2 |u_k|^p$ remains bounded. In the case $N \le p$, Lemma 2.2 implies that u_k remains bounded in $W^{1,p}(\mathbb{R}^N)$ and consequently in $L^{ps'}(\mathbb{R}^N)$. One then again deduces from (3.10) that $\int_{\mathbb{R}^N} m_2 |u_k|^p$ remains bounded. Hence in any case $N > p$ or $N \le p$, u_k remains bounded in W. It follows that for a subsequence (still denoted by u_k), $u_k \to u$ weakly in W, strongly in $L^p(m_1, \mathbb{R}^N)$ and in $L^p(n_1, \mathbb{R}^N)$, and $\int_{\mathbb{R}^N} |\nabla u_k|^p$ converges.

In the rest of the proof we will assume $N > p$ (a similar argument holds in the case $N \le p$). Observe that if $w \in W$, then $(w - a_k(w) u_k) \in T_{u_k}(M_{m,n})$ where $a_k(w) := \int_{\mathbb{R}^N} [m (u_k^+)^{p-1} - n (u_k^-)^{p-1}] w$. Putting $w = (u_k - u_l) - a_k(u_k - u_l) u_k$ in (3.9), one deduces

$$\int_{\mathbb{R}^N} |\nabla u_k|^{p-2} \nabla u_k \nabla (u_k - u_l) = t_k \int_{\mathbb{R}^N} [m (u_k^+)^{p-1} - n (u_k^-)^{p-1}] (u_k - u_l) + 0(\varepsilon_k),$$

where $t_k := \int_{\mathbb{R}^N} |\nabla u_k|^p$. This implies

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_l|^{p-2} \nabla u_l) \nabla (u_k - u_l) \\
&= t_k \int_{\mathbb{R}^N} m [(u_k^+)^{p-1} - (u_l^+)^{p-1}] (u_k - u_l) \\
&\quad + t_k \int_{\mathbb{R}^N} n [-(u_k^-)^{p-1} + (u_l^-)^{p-1}] (u_k - u_l) \\
&\quad + (t_k - t_l) \int_{\mathbb{R}^N} [m (u_l^+)^{p-1} - n (u_l^-)^{p-1}] (u_k - u_l) + 0(\varepsilon_k) + 0(\varepsilon_l) \\
&\le t_k (I_1 + I_2) + |t_k - t_l| I_3 + 0(\varepsilon_k) + 0(\varepsilon_l),
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^N} m_1 [(u_k^+)^{p-1} - (u_l^+)^{p-1}] (u_k - u_l), \\
I_2 &:= \int_{\mathbb{R}^N} n_1 [-(u_k^-)^{p-1} + (u_l^-)^{p-1}] (u_k - u_l), \\
I_3 &:= \int_{\mathbb{R}^N} |m (u_l^+)^{p-1} - n (u_l^-)^{p-1}| |u_k - u_l|.
\end{aligned}$$

We claim that the right hand side of (3.11) approaches zero when $k, l \to +\infty$. Indeed using Hölder inequality and the strong convergence of u_k in $L^p(m_1, \mathbb{R}^N)$, one sees that $I_1 \to 0$. Similarly $I_2 \to 0$. Furthermore Hölder inequality implies that I_3 remains bounded. Since $(t_k - t_l) \to 0$, we conclude that the right hand side of (3.11) goes to 0 as $k, l \to +\infty$, and the claim is proved .

To go on in the proof of Proposition 3.3, we observe that for some constant $d = d(p)$ and for any $\alpha, \beta \in \mathbb{R}^N$,

$$|\alpha - \beta|^p \le d\{(|\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta)(\alpha - \beta)\}^{r/2}(|\alpha|^p + |\beta|^p)^{1-r/2}, \tag{3.12}$$

where $r = p$ if $p \in]1, 2[$ and $r = 2$ if $p \ge 2$ (cf. [13]). Applying (3.12) and Hölder inequality, one easily derives from the claim that $\nabla u_k \to \nabla u$ in $L^p(\mathbb{R}^N)$. Moreover the calculation leading to (3.11) gives, using (H_4),

$$\begin{aligned} 0 &\le \min(a,1)t_k \int_{\mathbb{R}^N} m_2(|u_k|^{p-2}u_k - |u_l|^{p-2}u_l)(u_k - u_l) \\ &\le t_k(I_1 + I_2) + |t_k - t_l|I_3 + 0(\varepsilon_k) + 0(\varepsilon_l), \end{aligned}$$

where the right hand side goes to zero by the claim. We then deduce from the above that $\int_{\mathbb{R}^N} m_2|u_k - u_l|^p \to 0$ by applying successively (3.12), Hölder inequality and the fact that $\lim_{k\to+\infty} t_k = \int_{\mathbb{R}^N} |\nabla u|^p \ne 0$ (the latter quantity is nonzero because $u \in W$ and W does not contain any nonzero constant). Consequently $u_k \to u$ in W and Proposition 3.3 is proved. □

Remark 3.4. The arguments in the above proof can be used to show that the positive part of the spectrum of (1.2) is closed (cf. chap.2 of [12] for details).

We now have all the ingredients for the next proof.

Proof of Theorem 3.2. The conclusion follows by applying the mountain pass theorem on a C^1 manifold as given in [2, Proposition 4] or in [5, Proposition 2.1]: the (PS) condition is provided by Proposition 3.3 and the geometry comes by combining Proposition 3.1 with [2, Lemma 6]. □

4. A first nonprincipal eigenvalue

We have seen at the beginning of Section 3 that $\lambda_1(m)$ and $\lambda_1(n)$ are the first two positive eigenvalues of (1.1). The present section is mainly devoted to the proof that the eigenvalue $c(m, n)$ constructed in (3.7) is the next positive eigenvalue of (1.1).

Theorem 4.1. *Problem (1.1) does not admit any eigenvalue between the values* $\max\{\lambda_1(m), \lambda_1(n)\}$ *and* $c(m, n)$.

Proof. The present proof is partly different from that of the analogous result in [2]; the difficulty lies at the level of the construction of some auxiliary weights.

Assume by contradiction that there exists an eigenvalue λ of problem (1.1) with $\max\{\lambda_1(m), \lambda_1(n)\} < \lambda < c(m, n)$. Our goal is to construct a path in Γ on which $\tilde{J}$ remains $\le \lambda$, which yields a contradiction with the definition (3.7) of $c(m, n)$.

Let $u \in M_{m,n}$ be a critical point of $\tilde{J}$ at level λ. Since u changes sign, one obtains from the equation satisfied by u,

$$0 < \int_{\mathbb{R}^N} |\nabla u^+|^p = \lambda \int_{\mathbb{R}^N} m(u^+)^p \text{ and } 0 < \int_{\mathbb{R}^N} |\nabla u^-|^p = \lambda \int_{\mathbb{R}^N} n(u^-)^p. \tag{4.1}$$

The desired path will be constructed in several steps, using u as starting point.

First we go from u to $v := u^+/B_{m,n}(u^+)^{1/p} \equiv (u^+)_{m,n}$ by writing

$$\gamma_1(t) := \left[tu + (1-t)u^+\right]_{m,n}, t \in [0, 1]. \tag{4.2}$$

Using (4.1), it is easy to show that $\gamma_1(t)$ is well-defined, belongs to $M_{m,n}$ and satisfies $\tilde{J}(\gamma_1(t)) = \lambda\ \forall t \in [0,1]$. In a similar way we go from u to $(-u^-)_{m,n}$ in $M_{m,n}$ by staying at level λ. We now describe the construction of a path in $M_{m,n}$ from v to φ_m which stays at levels $\le \lambda$. A similar construction would yield a path in $M_{m,n}$ from $(-u^-)_{m,n}$ to $-\varphi_n$ which stays at levels $\le \lambda$. Putting everything together, we get the desired path from φ_m to $-\varphi_n$.

To construct the path from v to φ_m, first consider the manifold $M_{m,m}$. Clearly $v \in M_{m,m}$. The critical points of the restriction of J to $M_{m,m}$ are the normalized eigenfunctions of (1.2). Since v does not change sign and vanishes on a set of positive measure, v is not a critical point of this restriction of J to $M_{m,m}$. Consequently there exists a C^1 path $\nu :]-\epsilon, \epsilon[\to M_{m,m}$ with $\nu(0) = v$ and $\frac{d}{dt} J(\nu(t))\big|_{t=0} \neq 0$. Following a little portion of this path ν in the positive or negative direction (call ν_1 that portion), we move from v to a point w by a path in $M_{m,m}$ which, with the exception on its starting point v where $J(v) = \lambda$, lies at levels $< \lambda$. The path $\gamma_2(t) = |\nu_1(t)|$ then lies in $M_{m,n}$ (because it lies in $M_{m,m}$ and is made of nonnegative functions), goes from v to $v_1 := |w|$ and remains, with the exception of its starting point v where $J(v) = \lambda$, at levels $< \lambda$ (since $J(|\nu_1(t)|) = J(\nu_1(t))$).

Let now $m_{(\epsilon)}$ be defined for $0 \le \epsilon \le 1$ by

$$m_{(\epsilon)} = \begin{cases} m & \text{if } m < 0 \\ \epsilon m_1 - m_2 & \text{if } m \ge 0. \end{cases}$$

One has $m_{(1)} \equiv m$, $m^+_{(1)} \not\equiv 0$, $m_{(0)} \le 0$ and so $m^+_{(0)} \equiv 0$. Hence there exists $\epsilon_0 \in]0,1[$ such that

$$\begin{aligned} m^+_{(\epsilon)} &\not\equiv 0 \quad \text{if } \epsilon_0 < \epsilon < 1, \\ m^+_{(\epsilon)} &\equiv 0 \quad \text{if } 0 \le \epsilon \le \epsilon_0. \end{aligned}$$

Using Lemma 4.2 below, we see that for $\epsilon > \epsilon_0$ close to ϵ_0, $m_{(\epsilon)}$ will be a weight l of the form $l_1 - l_2$ such that m, l satisfy (H_1), (H_2), (H_3) and (H_4), with in addition $\lambda_1(l) > \lambda$ and $l \le m$ in $\mathbb{R}^N$. Fix such an ϵ. We then consider the manifold $M_{m,l}$ and the sublevel set

$$\mathcal{O} := \{u \in M_{m,l} : J(u) < \lambda\}.$$

Clearly v_1 and $\varphi_m \in \mathcal{O}$ (because they belong to $M_{m,m}$, are ≥ 0 and have the right levels). Moreover the only critical point in $\mathcal{O}$ of the restriction of J to $M_{m,l}$ is φ_m (because the first two critical levels $\lambda_1(m)$ and $\lambda_1(l)$ verify $\lambda_1(m) < \lambda < \lambda_1(l)$). Applying [2, Lemma 14] to the component of $\mathcal{O}$ which contains v_1, we get a path γ_3 in $\mathcal{O}$ from v_1 to φ_m. We then consider the path

$$\gamma_4(t) := \frac{|\gamma_3(t)|}{(\int_{\mathbb{R}^N} m|\gamma_3(t)|^p)^{1/p}}.$$

By the choice of l, one has

$$1 = \int_{\mathbb{R}^N} [m(\gamma_3(t)^+)^p + l(\gamma_3(t)^-)^p] \le \int_{\mathbb{R}^N} [m(\gamma_3(t)^+)^p + m(\gamma_3(t)^-)^p] = \int_{\mathbb{R}^N} m|\gamma_3(t)|^p,$$

and consequently γ_4 is well-defined. Moreover γ_4 goes from v_1 to φ_m and belongs to $M_{m,n}$. Finally

$$J(\gamma_4(t)) = \frac{\int_{\mathbb{R}^N} |\nabla\gamma_3(t)|^p}{\int_{\mathbb{R}^N} m|\gamma_3(t)|^p} \le \int_{\mathbb{R}^N} |\nabla\gamma_3(t)|^p < \lambda,$$

since $\gamma_3(t) \in \mathcal{O}$. The path γ_4 thus allows us to move from v_1 to φ_m in $M_{m,n}$ by staying at levels $< \lambda$. $\square$

Lemma 4.2. *$\lambda_1[m_{(\epsilon)}] \to \infty$ as $\epsilon \downarrow \epsilon_0$.*

Proof. One has

$$\frac{1}{\lambda_1[m_{(\epsilon)}]} = \frac{\int_{\mathbb{R}^N} m_{(\epsilon)} \varphi^p_{m_{(\epsilon)}}}{\int_{\mathbb{R}^N} |\nabla \varphi_{m_{(\epsilon)}}|^p} \le \frac{\|(\epsilon m_1 - m_2)^+\|_{L^s(\mathbb{R}^N)} \|\varphi_{m_{(\epsilon)}}\|^p_{L^{ps'}(\mathbb{R}^N)}}{\int_{\mathbb{R}^N} |\nabla \varphi_{m_{(\epsilon)}}|^p}. \tag{4.3}$$

If $N > p$, one deduces from (4.3) that $1/\lambda_1[m_{(\epsilon)}] \le A\|(\epsilon m_1 - m_2)^+\|_{L^s(\mathbb{R}^N)}$, and the conclusion follows since $(\epsilon m_1 - m_2)^+ \in L^s(\mathbb{R}^N)$ and $(\epsilon m_1 - m_2)^+ \downarrow 0$ as $\epsilon \downarrow \epsilon_0$. If $N \le p$, since $B_{m_{(\epsilon)}, n_{(\epsilon)}}(\varphi_{m_{(\epsilon)}}) = 1$, one has

$$\|\varphi_{m_{(\epsilon)}}\|^p_{L^{ps'}(\mathbb{R}^N)} \le B^p(1 + C(\epsilon)) \int_{\mathbb{R}^N} |\nabla \varphi_{m_{(\epsilon)}}|^p,$$

where $C(\epsilon) = C(m_{(\epsilon)}, N, p, N_0)$ is the constant from Lemma 2.2 (which remains bounded as ϵ varies in $]\epsilon_0, 1[)$. The conclusion then follows from (4.3) as in the case $N > p$. $\square$

Theorem 4.1 for $m \equiv n$ yields the following variational characterization of the second eigenvalue of the p-Laplacian with weight on $\mathbb{R}^N$ (cf. (2.3)).

Corollary 4.3. *One has*

$$\lambda_2(m) = \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma([-1,1])} \int_{\mathbb{R}^N} |\nabla u|^p,$$

where $\Gamma_0 := \{\gamma \in C([-1,1], M_{m,m}) : \gamma(-1) = \varphi_m$ and $\gamma(1) = -\varphi_m\}$ and $M_{m,m} := \{u \in W : \int_{\mathbb{R}^N} m|u|^p = 1\}$.

We conclude this section with some properties of the eigenvalue $c(m,n)$ as a function of the weights m, n. The following slightly different variational characterization of $c(m,n)$ will be useful for this purpose. It can be obtained by an easy adaptation of arguments in [2].

Proposition 4.4. *One has*

$$c(m,n) = \inf_{\gamma \in \Gamma_1} \max_{t \in [-1,1]} J(\gamma(t)), \tag{4.4}$$

where $\Gamma_1 := \{\gamma \in C([-1,1], M_{m,n}) : \gamma(-1) \ge 0$ and $\gamma(1) \le 0\}$.

Proposition 4.5. *Let $m = m_1 - m_2$, $n = n_1 - n_2$, $\hat{m} = \hat{m}_1 - \hat{m}_2$, $\hat{n} = \hat{n}_1 - \hat{n}_2$, and assume that hypothesis (H_1), (H_2), (H_3) and (H_4) hold for the weights m, n and also for the weights $\hat{m}, \hat{n}$. If $m_1 \le \hat{m}_1$, $n_1 \le \hat{n}_1$, $\hat{m}_2 \le m_2$ and $\hat{n}_2 \le n_2$ a.e. in $\mathbb{R}^N$, then $c(m,n) \ge c(\hat{m}, \hat{n})$. If in addition*

$$\int_{\mathbb{R}^N} (\hat{m} - m)(u^+)^p + \int_{\mathbb{R}^N} (\hat{n} - n)(u^-)^p > 0$$

for at least one eigenfunction u associated to $c(m,n)$, then $c(m,n) > c(\hat{m}, \hat{n})$.

Proof. Denote by W (resp. $\hat{W}$) the weighted Sobolev space defined in Section 2 and associated to the weight m_2 (resp. $\hat{m}_2$). One clearly has $W \subset \hat{W}$. Once this has been observed, the proof is easily adapted from that of [2, Propositions 23 and 25]. One uses in particular the fact that if the path γ is admissible in formula (3.7) for $c(m,n)$, then $\gamma(t) \in \hat{W}$ for all $t \in [-1,1]$, and so by normalization one can construct a path in $\hat{W}$ which will be admissible in formula (4.4) for $c(\hat{m}, \hat{n})$. $\square$

Let us also observe that definition (3.7) clearly implies that $c(m,n)$ is homogeneous of degree -1: $c(sm, sn) = c(m,n)/s$ for $s > 0$. Some sort of separate sub-homogeneity also holds, which will be useful later:

Proposition 4.6. *Assume that the weights m, n satisfy hypothesis (H_1), (H_2), (H_3) and (H_4). If $0 < s < \hat{s}$, then $c(\hat{s}m, n) < c(sm, n)$ and $c(m, \hat{s}n) < c(m, sn)$.*

Proof. Let us first observe that the pair of weights $\hat{s}m, n$ (as well as the other pairs of weights appearing above) also satisfy hypothesis (H_1), (H_2), (H_3) and (H_4). Moreover the "m_2 parts" of all these weights are comparable in the sense of hypothesis (H_4), which implies that a single weighted Sobolev space W can be used. Once this has been observed, the proof can be easily adapted from [2, Proposition 31]. □

We finally turn to the study of the continuous dependance of $c(m,n)$ with respect to m, n. The situation here is more involved than in [2].

Proposition 4.7. *Let $m = m_1 - m_2, n = n_1 - n_2$ satisfy (H_1), (H_2), (H_3) and (H_4), and let $m_k = m_{1k} - m_{2k}, n_k = n_{1k} - n_{2k}$ with $m_{1k}, m_{2k}, n_{1k}, n_{2k} \geq 0$, $k = 1, 2, \dots$. Assume that m_{1k}, n_{1k} belong to $L^\infty_{\rm loc}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ and converge in $L^s(\mathbb{R}^N)$ to m_1, n_1 respectively. Assume that m_{2k}, n_{2k} belong to $L^\infty_{\rm loc}(\mathbb{R}^N)$ and converge to m_2, n_2 respectively in the following sense: for some $\varepsilon_k \to 0$,*

$$|m_{2k} - m_2| \leq \varepsilon_k m_2 \text{ and } |n_{2k} - n_2| \leq \varepsilon_k n_2 \quad \text{a.e. in } \mathbb{R}^N. \tag{4.5}$$

Then $c(m_k, n_k) \to c(m,n)$.

The convergence (4.5) is unusual but Proposition 4.7 will suffice to derive later the continuity of the first curve in the Fučik spectrum.

Proof of Proposition 4.7. Observe that by our assumptions, the "m_2 parts" of all the weights m, n, m_k, n_k are comparable in the sense of hypothesis (H_4) (with constants a, b independent of k), and so a single space W is involved.

We first prove the upper semicontinuity. Let $\varepsilon > 0$ and take $\gamma \in \Gamma$ such that $\max_t J(\gamma(t)) < c(m,n) + \varepsilon$. Let $\gamma_k(t) := \gamma(t)/B_{m_k,n_k}(\gamma(t))^{1/p}$. We will show that γ_k is well defined and that

$$\max_t J(\gamma_k(t)) < c(m,n) + \varepsilon \tag{4.6}$$

for k sufficiently large. Once this is done, one deduces from Proposition 4.4 that $c(m_k, n_k) < c(m,n) + \varepsilon$ and consequently, since $\varepsilon > 0$ is arbitrary, that $\limsup c(m_k, n_k) \leq c(m,n)$.

The path γ_k is clearly well defined if $B_{m_k,n_k}(\gamma(t)) > 0 \quad \forall t \in [0,1]$. To prove that this latter condition holds for k sufficiently large, assume by contradiction that for a subsequence, $B_{m_k,n_k}(\gamma(t_k)) \leq 0$ for some $t_k \in [0,1]$. For a further subsequence, $t_k \to t_0$ and $\gamma(t_k) \to \gamma(t_0)$ in W and a.e. in $\mathbb{R}^N$. We claim that

$$B_{m,n}(\gamma(t_0)) \leq 0, \tag{4.7}$$

which is impossible since $\gamma \in \Gamma$. Deriving (4.7) is a matter of going to the limit in the expression $B_{m_k,n_k}(\gamma(t_k)) \leq 0$. The "$m_{1k}$ (or n_{1k}) terms" can be handled by Hölder inequality in a standard way. To handle the "m_{2k} (or n_{2k}) terms", we observe that $\gamma(t_k) \to \gamma(t_0)$ in $L^p(m_2, \mathbb{R}^N)$, which means that $m_2^{1/p}\gamma(t_k) \to m_2^{1/p}\gamma(t_0)$ in $L^p(\mathbb{R}^N)$, and consequently, for a further subsequence and for some $v \in L^p(\mathbb{R}^N)$,

$|m_2^{1/p}\gamma(t_k)| \le v$ a.e. in $\mathbb{R}^N$. This inequality and the fact that m_{2k} and n_{2k} are controlled by m_2 (a consequence of (H_4) and (4.5)) allow the use of the dominated convergence theorem to handle these "m_{2k} (or n_{2k}) terms". Let us now prove that (4.6) holds. Write $\max_t J(\gamma_k(t)) = J(\gamma_k(\tau_k))$ and assume by contradiction that for a subsequence

$$J(\gamma_k(\tau_k)) \ge c(m,n) + \varepsilon. \tag{4.8}$$

The preceding argument shows that for a further subsequence, one has $\tau_k \to \tau_0$ and $B_{m_k,n_k}(\gamma(\tau_k)) \to B_{m,n}(\gamma(\tau_0)) = 1$. Consequently, by (4.8), $J(\gamma(\tau_0)) \ge c(m,n)+\varepsilon$, a contradiction with the choice of γ.

To prove the lower semicontinuity, suppose by contradiction that for a subsequence, one has $c(m_k,n_k) \to c_0$ with $c_0 < c(m,n)$. Let $u_k \in M_{m_k,n_k}$ be an eigenfunction associated to $c(m_k,n_k)$. We first show that u_k remains bounded in W. One clearly has by the equation of u_k that $\int_{\mathbb{R}^N} |\nabla u_k|^p = c(m_k,n_k)$ and so $\int_{\mathbb{R}^N} |\nabla u_k|^p$ remains bounded. To get a bound on $\int_{\mathbb{R}^N} m_2|u_k|^p$, one starts from $B_{m_k,n_k}(u_k) = 1$. By using the imbeddings recalled in Section 2 (and Lemma 2.2 when $N \le p$), one easily sees that the "m_{1k} (and n_{1k}) terms" remain bounded. Replacing in the remaining terms m_{2k} (resp. n_{2k}) by m_2 (resp. n_2) and using assumption (H_4) and (4.5), one deduces the desired bound on $\int_{\mathbb{R}^N} m_2|u_k|^p$. It follows that for a subsequence, $u_k \to u$ weakly in W, strongly in $L^p(m_1,\mathbb{R}^N)$ and in $L^p(n_1,\mathbb{R}^N)$.

We will now prove that $u_k \to u$ in W. Taking $u_k - u_l$ as testing function in the equations satisfied by u_k and by u_l, and writing $c(m_k,n_k) = c_k$, one has

$$\int_{\mathbb{R}^N} |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k-u_l) = c_k \int_{\mathbb{R}^N} [m_k(u_k^+)^{p-1} - n_k(u_k^-)^{p-1}](u_k-u_l),$$

which implies

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^N} (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_l|^{p-2}\nabla u_l)\nabla(u_k-u_l)\\
&= c_k \int_{\mathbb{R}^N} m_k[(u_k^+)^{p-1} - (u_l^+)^{p-1}](u_k-u_l)\\
&\quad + c_k \int_{\mathbb{R}^N} n_k[-(u_k^-)^{p-1} + (u_l^-)^{p-1}](u_k-u_l)\\
&\quad + (c_k - c_l)\int_{\mathbb{R}^N} [m_k(u_l^+)^{p-1} - n_k(u_l^-)^{p-1}](u_k-u_l)\\
&\quad + c_l \int_{\mathbb{R}^N} [(m_k-m_l)(u_l^+)^{p-1} - (n_k-n_l)(u_l^-)^{p-1}](u_k-u_l)\\
&\le c_k(I_1+I_2) + |c_k-c_l| I_3 + c_l I_4,
\end{aligned}$$

where

$$\begin{gathered}
I_1 = \int_{\mathbb{R}^N} m_{1k}[(u_k^+)^{p-1} - (u_l^+)^{p-1}](u_k-u_l),\\
I_2 = \int_{\mathbb{R}^N} n_{1k}[-(u_k^-)^{p-1} + (u_l^-)^{p-1}](u_k-u_l),\\
I_3 = \int_{\mathbb{R}^N} [|m_k|(u_k^+)^{p-1} + |n_k|(u_l^-)^{p-1}]|u_k-u_l|,\\
I_4 = \int_{\mathbb{R}^N} [|m_k-m_l|(u_l^+)^{p-1} + |n_k-n_l|(u_l^-)^{p-1}]|u_k-u_l|.
\end{gathered}$$

Arguing as in the proof of Proposition 3.3 , one then easily proves that $u_k \to u$ in W. (The full strength of the convergence (4.5) is used here to verify that $I_4 \to 0$).

The limit $u \in M_{m,n}$ and is a solution of (1.1) for $\lambda = c_0$. Since $c_0 < c(m,n)$, Theorem 4.1 implies that $c_0 = \lambda_1(m)$ and $u = \varphi_m$, or $c_0 = \lambda_1(n)$ and $u = -\varphi_n$. Consider the first case (similar argument in the second one). Writing $v_k = u_k^- / (\int_{\mathbb{R}^N} |\nabla u_k^-|^p)^{1/p}$ and $\Omega_k^- = \{x \in \mathbb{R}^N : u_k(x) < 0\}$, we deduce from the equation satisfied by u_k that

$$\frac{1}{c_k} = \int_{\mathbb{R}^N} n_k (v_k)^p \le \int_{\Omega_k^-} n_{1k} (v_k)^p.$$

Consider the case $N > p$ (a similar argument holds in the case $N \le p$). We will argue as in the proof of Proposition 3.1. By Hölder inequality one has

$$\frac{1}{c_k} \le \|n_{1k}\|_{L^s(\Omega_k^-)} \|v_k\|^p_{L^{p^*}(\mathbb{R}^N)} \le A^p \|n_{1k}\|_{L^s(\Omega_k^-)},$$

which implies that for some $\varepsilon > 0$, $\|n_{1k}\|_{L^s(\Omega_k^-)} \ge \varepsilon$ for k sufficiently large. Moreover, since $n_{1k} \to n_1$ in $L^s(\mathbb{R}^N)$, one can choose $r > 0$ so that $\|n_{1k}\|^s_{L^s(B_r^c)} \le \varepsilon^s/2$ for k sufficiently large, and consequently $\|n_{1k}\|^s_{L^s(\Omega_{kr}^-)} \ge \varepsilon^s/2$ where $\Omega_{kr}^- := \Omega_k^- \cap B_r$. Since n_{1k} converges to n_1 in $L^\infty_{\rm loc}(\mathbb{R}^N)$ one deduces from the latter relation that for some $\zeta > 0$, $|\Omega_{kr}^-| > \zeta$ for k sufficiently large. But this is impossible since $u_k \to \varphi_m$ in $L^p(B_r)$ and $\varphi_m(x) > 0$ a.e. in B_r. ☐

Arguing as in Remark 2.3, one can deduce from Proposition 4.7 the following result.

Corollary 4.8. *Let $m = m_1 - m_2, n = n_1 - n_2$ satisfy (H_1), $(H_2)_R$, (H_3), $(H_4)_R$, and let $m_k = m_{1k} - m_{2k}, n_k = n_{1k} - n_{2k}$ with $m_{1k}, m_{2k}, n_{1k}, n_{2k} \ge 0$, $k = 1, 2, \dots$. Assume that m_{1k}, n_{1k} belong to $L^\infty_{\rm loc}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ and converge in $L^s(\mathbb{R}^N)$ to m_1, n_1 respectively. Assume that m_{2k}, n_{2k} belong to $L^\infty_{\rm loc}(\mathbb{R}^N)$ and converge in $L^\infty_{\rm loc}(\mathbb{R}^N)$ to m_2, n_2 respectively, with in addition, for some $\varepsilon_k \to 0$,*

$$|m_{2k} - m_2| \le \varepsilon_k m_2 \text{ and } |n_{2k} - n_2| \le \varepsilon_k n_2 \quad \text{for a.e. } x \text{ with } |x| \ge R. \tag{4.9}$$

Then $c(m_k, n_k) \to c(m,n)$.

Remark 4.9. If (H_4) is replaced in Proposition 4.7 by (H_4') (cf. Remark 2.5), then (4.5) can be replaced by the natural requirement that m_{2k}, n_{2k} converge in $L^{N/p}(\mathbb{R}^N)$ to m_2, n_2 respectively.

5. Fučik spectrum in $\mathbb{R}^N$

The weights m and n in this section satisfy as before assumptions (H_1), (H_2), (H_3) and (H_4) (or (H_4')). The Fučik spectrum Σ was defined in the introduction (cf. (1.3)) and we will mainly consider here its part which lies in $\mathbb{R}^+ \times \mathbb{R}^+$. This part clearly contains the half lines $\lambda_1(m) \times \mathbb{R}^+$ and $\mathbb{R}^+ \times \lambda_1(n)$. These half lines are in fact exactly made of those $(\alpha, \beta) \in \Sigma \cap (\mathbb{R}^+ \times \mathbb{R}^+)$ for which (1.3) admits a solution which does not change sign. We denote below by Σ^* the set $\Sigma \cap (\mathbb{R}^+ \times \mathbb{R}^+)$ without these 2 trivial half lines. From the properties of the first eigenvalue recalled in Section 2, it easily follows that if $(\alpha, \beta) \in \Sigma^*$, then $\alpha > \lambda_1(m)$ and $\beta > \lambda_1(n)$.

Theorem 5.1. *For any $r > 0$, the line $\beta = r\alpha$ in the (α, β) plane intersects Σ^*. Moreover the first point in this intersection is given by $\alpha(r) = c(m, rn)$, $\beta(r) = r\alpha(r)$, where $c(.,.)$ is defined in (3.7).*

The proof of the above theorem is an easy consequence of Theorem 3.2 and Theorem 4.1

Letting $r > 0$ vary, we thus get a first curve $\mathcal{C} := \{(\alpha(r), \beta(r)) : r > 0\}$ in Σ^*. Here are some properties of this curve.

Proposition 5.2. *The functions $\alpha(r)$ and $\beta(r)$ in Theorem 5.1 are continuous. Moreover $\alpha(r)$ is strictly decreasing and $\beta(r)$ is strictly increasing. One also has that $\alpha(r) \to +\infty$ if $r \to 0$ and $\beta(r) \to +\infty$ if $r \to +\infty$.*

Proof. The continuity of the functions $\alpha(r)$ and $\beta(r)$ follows directly from Proposition 4.7, and their strict monotonicity from Proposition 4.6. To show that $\alpha(r) \to +\infty$ as $r \to 0$, let us assume by contradiction that $\alpha(r)$ remains bounded as $r \to 0$. Then $\beta(r) = r\alpha(r) \to 0$ as $r \to 0$, which is impossible since $\beta(r) > \lambda_1(n)$ for all $r > 0$. Similar argument for the behaviour of $\beta(r)$ as $r \to +\infty$. □

We now investigate the asymptotic values $\alpha_\infty := \lim_{r\to+\infty} \alpha(r)$ and $\beta_\infty := \lim_{r\to 0} \beta(r)$ of the first curve in $\mathbb{R}^+ \times \mathbb{R}^+$. We will limit ourselves below to the study of α_∞; similar results can be proved for β_∞ by interchanging m and n. The following lemma will be used. It is concerned with the eigenvalue problem

$$-\Delta_p u = \lambda m|u|^{p-2}u \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R, \tag{5.1}$$

where B_R denotes the ball centred at the origin with radius R. Let λ_R denote the first positive eigenvalue of (5.1) and φ_R the associated positive eigenfunction such that $\int_{B_R} m(\varphi_R)^p = 1$ (which by (H_3) clearly exist for R sufficiently large).

Lemma 5.3. *$\lambda_R \to \lambda_1(m)$ and $\varphi_R \to \varphi_m$ in W as $R \to +\infty$.*

Proof. We will adapt some arguments from [6, Lemma 5.2]. The function φ_R (extended by 0 outside B_R) clearly belongs to W and satisfies $\int_{\mathbb{R}^N} m(\varphi_R)^p = 1$. This implies

$$\lambda_1(m) \le \int_{\mathbb{R}^N} |\nabla\varphi_R|^p = \lambda_R,$$

and so $\liminf \lambda_R \ge \lambda_1(m)$. Let now $\delta \in]0, 1[$. Since $\varphi_m \in W$, there exists $\psi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\Big|\int_{\mathbb{R}^N} (|\nabla\varphi_m|^p - |\nabla\psi|^p)\Big| \le \frac{\delta}{2}, \quad \Big|\int_{\mathbb{R}^N} m_2(\varphi_m^p - |\psi|^p)\Big| \le \frac{\delta}{2},$$
$$\Big|\int_{\mathbb{R}^N} m_1(\varphi_m^p - |\psi|^p)\Big| \le \frac{\delta}{2},$$

where we have used the imbedding of W into $L^p(m_1, \mathbb{R}^N)$. This implies that $|\int_{\mathbb{R}^N} m(\varphi_m^p - |\psi|^p)| \le \delta$ and since $\int_{\mathbb{R}^N} m|\varphi_m|^p = 1$, one deduces that $\int_{\mathbb{R}^N} m|\psi|^p > 0$. We thus have, for R sufficiently large,

$$\lambda_R \le \frac{\int_{\mathbb{R}^N} |\nabla\psi|^p}{\int_{\mathbb{R}^N} m|\psi|^p} \le \frac{\delta/2 + \int_{\mathbb{R}^N} |\nabla\varphi_m|^p}{-\delta + \int_{\mathbb{R}^N} m|\varphi_m|^p} = \frac{\delta/2 + \lambda_1(m)}{-\delta + 1},$$

and since $\delta > 0$ is arbitrary, we conclude that $\limsup \lambda_R \le \lambda_1(m)$.

Let us now prove that $\varphi_R \to \varphi_m$ in W as $R \to +\infty$. One has

$$\int_{\mathbb{R}^N} |\nabla\varphi_R|^p = \lambda_R \to \lambda_1(m) = \int_{\mathbb{R}^N} |\nabla\varphi_m|^p, \tag{5.2}$$

and so $\int_{\mathbb{R}^N} |\nabla\varphi_R|^p$ remains bounded. Moreover using the imbeddings of Section 2 as well as Lemma 2.2, one deduces from $\int_{\mathbb{R}^N} m(\varphi_R)^p = 1$ that $\int_{\mathbb{R}^N} m_2(\varphi_R)^p$ remains bounded. Consequently φ_R remains bounded in W, and for a subsequence, $\varphi_R \to v$ weakly in W and strongly in $L^p(m_1, \mathbb{R}^N)$. One has

$$\int_{\mathbb{R}^N} |\nabla v|^p \le \liminf \int_{\mathbb{R}^N} |\nabla \varphi_R|^p = \lambda_1(m),$$

and also $\int_{\mathbb{R}^N} m|v|^p \ge 1$ (where the latter inequality follows from

$$\int_{\mathbb{R}^N} m_2|v|^p \le \liminf \int_{\mathbb{R}^N} m_2(\varphi_R)^p = -1 + \int_{\mathbb{R}^N} m_1|v|^p).$$

The simplicity of $\lambda_1(m)$ then implies $v = \varphi_m$. One also has

$$\lim \int_{\mathbb{R}^N} m_2(\varphi_R)^p = \lim \Big[-1 + \int_{\mathbb{R}^N} m_1(\varphi_R)^p\Big] = -1 + \int_{\mathbb{R}^N} m_1\varphi_m^p = \int_{\mathbb{R}^N} m_2\varphi_m^p. \tag{5.3}$$

Combining (5.2) and (5.3) with the weak convergence yields that $\varphi_R \to \varphi_m$ in W. □

The following proposition describes the asymptotic behaviour of the first curve $\mathcal{C}$. Let us recall that the support of a measurable function u in $\mathbb{R}^N$ is defined as the complement in $\mathbb{R}^N$ of the largest open set on which $u = 0$ a.e.

Proposition 5.4. *If $N \ge p$, then $\alpha_\infty = \lambda_1(m)$. If $N < p$, then $\alpha_\infty = \lambda_1(m)$ if* $\operatorname{supp} n^+$ *is unbounded and $\alpha_\infty > \lambda_1(m)$ if* $\operatorname{supp} n^+$ *is bounded.*

Proof. One starts by introducing

$$\overline{\alpha} := \inf\Big\{\int_{\mathbb{R}^N} |\nabla u^+|^p : u \in W, \int_{\mathbb{R}^N} m(u^+)^p = 1 \text{ and } \int_{\mathbb{R}^N} n(u^-)^p > 0\Big\} \tag{5.4}$$

and showing that $\alpha_\infty = \overline{\alpha}$. The proof of this equality is a direct adaptation of [2]. One also clearly has $\overline{\alpha} \ge \lambda_1(m)$.

We first consider the case $N \ge p$. In this case, the arguments of [2], which are local (they essentially involve approximating φ_m by a function which vanishes on a small ball where $n^+ \not\equiv 0$), can be easily adapted to the present situation and give $\overline{\alpha} = \lambda_1(m)$.

We now consider the case where $N < p$ and the support of n^+ is unbounded. We will use the function φ_R defined in Lemma 5.3. Since $\operatorname{supp} n^+$ is unbounded, for any $R > 0$, $n^+ \not\equiv 0$ on $\mathbb{R}^N \backslash \overline{B_R}$. This allows by a regularization procedure to construct $w_R \in C_c^\infty(\mathbb{R}^N \backslash \overline{B_R})$ with $w_R \ge 0$ and $\int_{\mathbb{R}^N} n w_R^p > 0$. It follows that the function

$$u_R = \varphi_R - \frac{w_R}{R\|w_R\|_W}$$

converges to φ_m in W as $R \to +\infty$ and is admissible in the definition (5.4) of $\overline{\alpha}$. Since by Lemma 5.3, $\int_{\mathbb{R}^N} |\nabla \varphi_R|^p \to \lambda_1(m)$, we conclude that $\overline{\alpha} \le \lambda_1(m)$, and so $\overline{\alpha} = \lambda_1(m)$.

We finally consider the case where $N < p$ and the support of n^+ is bounded. Assume by contradiction $\overline{\alpha} = \lambda_1(m)$ and let u_k be a minimizing sequence for $\overline{\alpha}$. Then $\int_{\mathbb{R}^N} |\nabla u_k^+|^p$ remains bounded and since $\int_{\mathbb{R}^N} m(u_k^+) = 1$, arguing e.g. as in the proof of Lemma 5.3, one deduces that u_k^+ remains bounded in W. Hence, for a subsequence, $u_k^+ \to v$ weakly in W and strongly in $L^p(m_1, \mathbb{R}^N)$. We then argue as in the second part of the proof of Lemma 5.3 to derive $v = \varphi_m$. Since $\varphi_m \ge$ some $\varepsilon > 0$ on the compact set $\operatorname{supp} n^+$, it follows from the fact that $u_k^+ \to \varphi_m$ uniformly on

$\operatorname{supp} n^+$, that $u_k^+ \geq \varepsilon/2$ on $\operatorname{supp} n^+$ for k sufficiently large. Consequently, for those k, $u_k^- = 0$ on $\operatorname{supp} n^+$, which implies

$$\int_{\mathbb{R}^N} n(u_k^-)^p = \int_{\mathbb{R}^N} n^+(u_k^-)^p - \int_{\mathbb{R}^N} n^-(u_k^-)^p = -\int_{\mathbb{R}^N} n^-(u_k^-)^p \leq 0.$$

But this contradicts the fact that u_k is admissible in the definition (5.4) of $\overline{\alpha}$. □

Remark 5.5. The distribution of Σ in the other quadrants of $\mathbb{R}\times\mathbb{R}$ can be studied in a manner similar to that used in [2]. It follows in particular that if $N > p$, $m, n \in L^{N/p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, and m and n change sign, then Σ contains a first hyperbolic-like curve in each quadrant. The case $N \leq p$ however remains unclear (see in this respect Remark 2.1).

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