COMMON FIXED POINTS FOR LIPSCHITZIAN SEMIGROUPS

SAMIR LAHRECH, ABDERRAHIM MBARKI, ABDELMALIK OUAHAB

Abstract. Lim and Xu [4] established a fixed point theorem for uniformly Lipschitzian mappings in metric spaces with uniform normal structure. Recently, Huang and Hong [1] extended hyperconvex metric space version of this theorem, by showing a common fixed point theorem for left reversible uniformly \( k \)-Lipschitzian semigroups. In this paper, we extend Huang and Hong’s theorem to metric spaces with uniform normal structure.

1. Introduction and main results

Throughout this paper, \((X, d)\) stands for a metric space, a nonempty family \(\mathcal{F}\) of subsets of \(X\) is said to define a convexity structure on \(X\) if it is stable by intersection. Recall that a subset of \(X\) is said admissible if it is an intersection of closed balls. We denote, by \(\mathcal{A}(X)\) the family of all admissible subsets of \(X\). Obviously, \(\mathcal{A}(X)\) defines a convexity structure on \(X\). In this paper any convexity structure \(\mathcal{F}\) on \(X\) is always assumed to contain \(\mathcal{A}(X)\). For \(r \geq 0\) and \(x\) in \(X\) and a bounded subset \(M\) of \(X\), we adopt the following notation:

- \(B(x, r)\) is the closed ball centered at \(x\) with radius \(r\),
- \(r(x, M) = \sup\{d(x, y) : y \in M\}\),
- \(\delta(M) = \sup\{r(x, M) : x \in M\}\),
- \(R(M) = \inf\{r(x, M) : x \in M\}\).

\textbf{Definition 1.1 (2).} A metric space \((X, d)\) is said to have normal (resp. uniform normal) structure if there exists a convexity structure \(\mathcal{F}\) on \(X\) such that \(R(A) < \delta(A)\) (resp. \(R(A) \leq c\delta(A)\) for some constant \(c \in (0, 1)\)) for all \(A\) in \(\mathcal{F}\) which is bounded and \(\delta(A) > 0\). It is also said that \(\mathcal{F}\) is normal and (resp. uniformly normal).

The uniform normal structure coefficient \(N(X)\) of \(X\) relative to \(\mathcal{F}\) is the number

\[
\sup\{\frac{R(A)}{\delta(A)} : A \in \mathcal{F} \text{ is bounded and } \delta(A) > 0\}.
\]

2000 Mathematics Subject Classification. 47H09, 47H10.
Key words and phrases. Left reversible uniformly \( k \)-Lipschitzian semigroups; common fixed point; uniform structure; convexity structure; metric space.
©2006 Texas State University - San Marcos.
Published September 20, 2006.

223
Definition 1.2 ([3]). Let \((X, d)\) a metric space and \(\mathcal{T}\) is the family of subsets of \(X\) consisting of \(X\) and sets which are complements of closed balls of \(X\). The weak topology (also called ball topology) on \(X\) is the topology whose open sets are generated by \(\mathcal{T}\).

It is clear that \(X\) is compact in the weak topology if and only if every subfamily of \(\mathcal{A}(X)\) with the finite intersection property has nonempty intersection.

Kulesza and Lim proved the following result.

Lemma 1.3 ([3]). Every complete metric space with uniform normal structure is compact in the weak topology.

For a bounded subset \(A\) of \(X\), the admissible hull of \(A\), denoted by \(ad(A)\), is the set
\[
\cap \{B : A \subseteq B \subseteq X \text{ with } B \text{ admissible}\}.
\]
The following definition is a net version of [4] def. 5.

Definition 1.4 ([4]). A metric space \((X, d)\) is said to have the property (P) if given any two bounded nets \(\{x_i\}_{i \in I}\) and \(\{z_i\}_{i \in I}\) in \(X\), one can find some \(z \in \cap_{i \in I} ad\{z_j : j \geq i\}\) such that
\[
\liminf_{i \in I} d(z, x_i) \leq \liminf_{j \in I} \limsup_{i \in I} d(z_j, x_i),
\]
where \(\liminf_{i \in I} d(z, x_i) = \inf_{\beta \in I} \sup_{i \geq \beta} d(z, x_i)\).

Remark 1.5. If \(X\) has uniform normal structure, then \(\cap_{i \in I} ad\{z_j : j \geq i\} \neq \emptyset\) (by Lemma 1.3). Also, if \(X\) is a weakly compact convex subset of a normed linear space, then admissible hulls are closed convex and \(\cap_{i \in I} ad\{z_j : j \geq i\} \neq \emptyset\) by weak compactness of \(X\) and that \(X\) possesses property (P) follows directly from the weak lower semicontinuity of the function \(x \mapsto \liminf_{i \in I} \|x_i - x\|\).

The following Lemma is a net version of [4] lemma. 5.

Lemma 1.6. Let \((X, d)\) be a complete bounded metric space with both property (P) and uniform normal structure. Then for any net \(\{x_i\}_{i \in I}\) in \(X\) and any \(\tau > N(X)\), the normal structure coefficient with respect to the given convexity structure \(\mathcal{F}\), there exists a point \(z \in X\) satisfying the properties:
\[
\begin{align*}
(i) & \quad \liminf_{i \in I} d(z, x_i) \leq \tau d(\{x_i\}_{i \in I}); \\
(ii) & \quad d(z, y) \leq \liminf_{i \in I} d(x_i, y) \text{ for all } y \in X.
\end{align*}
\]

Proof. Using the Lemma 1.3 to conclude that \(\cap_{i \in I} A_i \neq \emptyset\) for any deceasing net \(\{A_i\}_{i \in I}\) of admissible subsets of \(X\), the rest of the proof of lemma is the same as that in Lim et al. [4].

Let \(S\) be a semigroup of selfmaps on a metric space \((X, d)\). For any \(x \in X\) (resp. \(b \in S\)), we denote by \(Sx\) (resp. \(bS\)) the subset \(\{gx : g \in S\}\) (resp. \(\{bg : g \in S\}\)) of \(X\) (resp. of \(S\)). Recall that a semigroup \(S\) is said to be left reversible if, for any \(f, g\) in \(S\), there are \(a, b\) in \(S\) such that \(fa = gb\). Examples of left reversible semigroups include all commutative semigroups and all groups.

Let \(S\) be a left semigroup. For \(a, b\) in \(S\) we say that \(a \geq b\) if \(a \in bS \cup \{b\}\). Then \((S, \geq)\) is a directed set. In what follows in this paper, we deal only with “≥”.

Definition 1.7 ([3]). A semigroup \(S\) acting on a metric space \((X, d)\) is said to be a uniformly \(k\)-Lipschitzian semigroup if
\[
d(tx, ty) \leq kd(x, y)
\]
for all \( t \) in \( S \) and all \( x, y \) in \( X \).

If \( S \) is a left reversible semigroup, then \((S, \geq)\) is a linearly directed set if any \( a, b \) in \( S \) satisfy either \( a \geq b \) or \( b \geq a \). For example, if \( \Delta = \{ T_s : s \in [0, \infty) \} \) is a family of selfmaps on \( \mathbb{R} \) such that \( T_{h+t}(x) = T_h T_t(x) \) for all \( h, t \) in \([0, \infty)\) and \( x \in \mathbb{R} \), then \((\Delta, \geq)\) is a linearly directed left reversible semigroup.

Our main result is as follows.

**Theorem 1.8.** Let \((X, d)\) be a complete bounded metric space with both property (P) and uniform normal structure and let \( S \) be a left reversible uniformly \( k \)-Lipschitzian semigroup of selfmaps on \( X \) such that \( k < N(X)^{-1/2} \) and \((S, \geq)\) is a linearly directed set. Then \( S \) has a common fixed point in \( X \).

**Proof.** Choose a constant \( \tau \), \( 1 > \tau > N(X) \), such that \( k < \tau^{-1/2} \). Fix an \( x_0 \in X \). For \( t \in S \), denote \( t x_0 \) by \( x_{0,t} \). Then \( \{ x_{0,t} \} \) is a net in \( X \). By Lemma 1.6, we can inductively construct a sequence \( \{ x_j \} \subset X \) such that for each integer \( j \leq 0 \),

- \( \lim_{t \in S} d(x_{j+1}, x_{j,t}) \leq \tau d(Sx_j) \);
- \( d(x_{j+1}, y) \leq \lim_{t \in S} d(x_{j,t}, y) \) for all \( y \) in \( X \).

Write \( D_j = \lim_{t \in S} d(x_{j+1}, x_{j,t}) \) and \( h = \tau k^2 < 1 \).

The rest of the proof of Theorem is the same as that in Huang and Hong [1]. \( \Box \)

**Remark 1.9.** It can be seen from the above that the conclusion of main theorem is still valid if we only assume that \( A(X) \), the family of all admissible subsets of \( X \), is uniformly normal.

The following corollary follows immediately from the main Theorem.

**Corollary 1.10** (Huang and Hong [1]). Let \((X, d)\) be a bounded hyperconvex metric space with both property (P) and let \( S \) be a left reversible uniformly \( k \)-Lipschitzian semigroup of selfmaps on \( X \) such that \( k < \sqrt{2} \) and \((S, \geq)\) is a linearly directed set. Then \( S \) has a common fixed point in \( X \).

**References**


Samir Lahrech
DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ OUJDA, 60000 OUJDA, MOROCCO
E-mail address: lahrech@sciences.univ-oujda.ac.ma

Abderrahim Mbarki
Current address: NATIONAL SCHOOL OF APPLIED SCIENCES, P.O. BOX 669, OUJDA UNIVERSITY, MOROCCO
E-mail address: ambarki@ensa.univ-oujda.ac.ma

Abdelmalek Ouahab
DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ OUJDA, 60000 OUJDA, MOROCCO
E-mail address: ouahab@sciences.univ-oujda.ac.ma