UNIFORMLY ERGODIC THEOREM FOR COMMUTING
MULTIOPERATORS

SAMIR LAHRECH, ABDERRAHIM MBARKI, ABDELMALEK OUAHAB, SAID RAIS

Abstract. In this paper, we established some uniformly Ergodic theorems by using multioperators satisfying the E-k condition introduce in [3]. One consequence, is that if \( I - T \) is quasi-Fredholm and satisfies E-k condition then T is uniformly ergodic. Also we give some conditions for uniform ergodicity of a commuting multioperators satisfies condition E-k. These results are of interest in view of analogous results for unvalued operators (see, for example [2]) also in view of the recent developments in the ergodic theory and its applications.

1. Introduction and main results

Throughout this paper, \( X \) is a complex Banach space, and \( L(X) \) is the algebra of linear continuous operators acting in \( X \). If there is an integer \( n \) for which \( T^{n+1}X = T^nX \), then we say that \( T \) has finite descent and the smallest integer \( d(T) \) for which equality occurs is called the descent of \( T \). If there is exists an integer \( m \) for which \( \ker T^{m+1} = \ker T^m \), then \( T \) is said to have finite ascent and the smallest integer \( a(T) \) for this equality occurs is called ascent of \( T \). If both \( a(T) \) and \( d(T) \) are finite, then they are equal [1, 38.3]. We say that \( T \) is chain-finite and that its chain length is this common minimal value. Moreover [1, 38.4], in this case there is a decomposition of the vector space

\[
X = T^{d(T)}X \oplus \ker T^{d(T)}.
\]

We now focus on the topological situation: For every \( T \in L(X) \) we set

\[
M_i(T) = i^{-1}(I + T + T^2 + \cdots + T^{i-1}), \quad i = 1, 2, 3, \ldots,
\]

i.e. the averages associated with \( T \), where \( I = id_X \) is the identity of \( X \). If \( T = (T_1, T_2, \ldots, T_n) \in L(X)^n \) is commuting multioperator (briefly, c.m.), we also set

\[
M_v(T) = M_{v_1}(T_1)M_{v_2}(T_2)\cdots M_{v_n}(T_n), \quad v \in Z^n_+, \quad v \geq e,
\]

where \( Z^n_+ \) is the family of multi-indices of length \( n \) (i.e. \( n \)-tuples of nonnegative integers) and \( e := (1, 1, \ldots, 1) \in Z^n_+ \). In other words, (1.2) defines the averages associated with \( T \).

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Definition 1.1. A commuting multioperator \( T \in L(X)^n \) is said to be uniformly ergodic if the limit
\[
\lim_v M_v(T)
\]
exists in the uniform topology of \( L(X) \).

Remark 1.2. (a) If \( n = 1 \), then (1.3) is automatically fulfilled, and therefore the above definition extends the usual concept of uniformly ergodic operator (see, for example [2]).

(b) If \( T = (I, \ldots , T_j, I, \ldots , I) \in L(X)^n \), then \( T \) is uniformly ergodic if and only the \( \lim_v M_v(T) \) exists in the uniform topology of \( L(X) \).

Definition 1.3. Let \( k = (k_1, \ldots , k_n) \in \mathbb{Z}_+^n \) and \( T \in L(X)^n \) be a c.m. We say that \( T \) satisfies condition E-k if \( \lim_v (I - T_j)^{k_j} M_v(T) = 0 \) for each \( j \in \{1, \ldots , n\} \).

It is clear that condition E-k implies condition E-n for any \( n \geq k \). Thus we see that the example \( T = (T_1, I, \ldots , I) \in \mathbb{Z}_+^n \) with
\[
T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]
This shows that E-2e is strictly weaker than E-e.

Theorem 1.4. Let \( k \in \mathbb{Z}_+^n \). Suppose \( T \in L(X)^n \) satisfies condition E-k and \( \sum_{j=1}^n (I - T_j)^{k_j} X \), \( \sum_{j=1}^n (I^* - T_j)^{k_j} X^* \) are closed in \( X \) and \( X^* \) respectively. If \( \big[ \sum_{j=1}^n (I - T_j)^{k_j} X \big] \cap \big[ \cap_{j=1}^n \ker(I - T_j)^{k_j} \big] = \{0\} \). Then \( T \) is uniformly ergodic

Proof. Arguing exactly as in [5, Theorem 1], with \( \delta_T \) and \( \gamma_T \) given by
\[
\oplus_{j=1}^n x_j \rightarrow \delta_T(\oplus_{j=1}^n x_j) = \sum_{j=1}^n (I - T_j)^{k_j} x_j \text{ and } x \rightarrow \gamma_T(x) = \oplus_{j=1}^n (I - T_j)^{k_j} x.
\]

Theorem 1.5. Let \( T \in L(X) \) satisfy condition E-r, and one of the following nine conditions:

(a) \( I - T \) has chain length at most \( r \)
(b) 1 is a pole of the resolvent of order at most \( r \)
(c) \( I - T \) is quasi-Fredholm operator
(d) \( (I - T)^r X \) is closed and \( \ker(I - T)^r \) has a closed \( T \)-invariant complement
(e) \( (I - T)^r X \bigoplus \ker(I - T)^r = (I - T)^r X + \ker(I - T)^r \)
(f) \( (I - T)^m X \) is closed for all \( m \geq r \)
(g) \( (I - T)^r X \) is closed
(h) \( (I - T)^m X \) is closed for some \( m \geq r \)
(i) \( I - T \) has finite descent.

Then \( T \) is uniformly ergodic.

Proof. Firstly, from [3, Theorem 6], the above statements (a)–(i) are equivalent. Then, take \( G = (T, I, \ldots , I) \in L(X)^n \) and \( k = (r, 1, \ldots , 1) \in \mathbb{Z}_+^n \); Therefore, we have \( \sum_{j=1}^n (I - G_j)^{k_j} X = (I - T)^r X \) is closed, it follows that \( \sum_{j=1}^n (I^* - G_j)^{k_j} X^* = (I^* - T^r)^r X^* \) is closed, which implies since \( (I - T)^r X \cap \ker(I - T)^r = \{0\} \), that \( G \) is uniformly ergodic in \( L(X)^n \). From Theorem 1.4 this means that \( T \) is uniformly ergodic in \( L(X) \).
Theorem 1.6. Let $k \in \mathbb{Z}^n$. If $T \in L(X)^n$ A c.m. satisfies condition E-$k$, such that 
$\sum_{j=1}^{n} (I-T_j)$ has chain length at most 1 and $\ker(\sum_{j=1}^{n}(I-T_j)) = \cap_{j=1}^{n} \ker(I-T_j)$.

Then $T$ is uniformly ergodic.

Proof. There are two cases

Case 1: $d(\sum_{j=1}^{n}(I-T_j)) = 0$. Then $\sum_{j=1}^{n}(I-T_j)$ is bijective then $X = \sum_{j=1}^{n}(I-T_j)X \oplus \ker(\sum_{j=1}^{n}(I-T_j))$, which implies, since $\cap_{j=1}^{n} \ker(I-T_j) \subset \ker(\sum_{j=1}^{n}(I-T_j)) = \{0\}$ that $X = \sum_{j=1}^{n}(I-T_j)X \oplus \cap_{j=1}^{n} \ker(I-T_j)$, from the [3, Theorem 10] we obtain $T$ is uniformly ergodic.

Case 2: $d(\sum_{j=1}^{n}(I-T_j)) = 1$. Then $\big((\sum_{j=1}^{n}(I-T_j))X = (\sum_{j=1}^{n}(I-T_j))^2X$, so $(\sum_{j=1}^{n}(I-T_j))X = (\sum_{j=1}^{n}(I-T_j))^r(\sum_{j=1}^{n}(I-T_j))X$, with $r = \max_{1 \leq j \leq n} k_j$, so $(\sum_{j=1}^{n}(I-T_j))^r$ is a bijection of $(\sum_{j=1}^{n}(I-T_j))X$ onto itself. Which implies, since $T$ satisfies condition E-$k$, that $M_r(T)|\big((\sum_{j=1}^{n}(I-T_j))X \to 0$ and since $M_r(T)|\cap_{j=1}^{n} \ker(I-T_j)$, it follows that $T$ is uniformly ergodic. \qed

References


Samir Lahrech
Département de Mathématiques, Université Oujda, 60000 Oujda, Morocco
E-mail address: lahrech@sciences.univ-oujda.ac.ma

Abderrahim Mbarki
Current address: National school of Applied Sciences, P.O. Box 669, Oujda University, Morocco
E-mail address: ambarki@enssa.univ-oujda.ac.ma

Abdelmalek Ouahab
Département de Mathématiques, Université Oujda, 60000 Oujda, Morocco
E-mail address: ouahab@sciences.univ-oujda.ac.ma

Said Rais
Département de Mathématiques, Université Oujda, 60000 Oujda, Morocco
E-mail address: saidrais@yahoo.fr