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POSITIVE SOLUTIONS FOR ELLIPTIC PROBLEMS WITH CRITICAL INDEFINITE NONLINEARITY IN BOUNDED DOMAINS

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Abstract. In this paper, we study the semilinear elliptic problem with critical nonlinearity and an indefinite weight function, namely

$$-\Delta u = \lambda u + h(x)u^{(n+2)/(n-2)}$$

in a smooth open bounded domain $\Omega \subseteq \mathbb{R}^n$, $n > 4$ with Dirichlet boundary conditions and for $\lambda \geq 0$. Under suitable assumptions on the weight function, we obtain the positive solution branch, bifurcating from the first eigenvalue $\lambda_1(\Omega)$. For $n = 2$, we get similar results for $-\Delta u = \lambda u + h(x)\phi(u)e^u$ where ϕ is bounded and superlinear near zero.

1. Introduction

In this paper, we study the following (critical exponent) semilinear elliptic problem in an open bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary

$$\begin{gathered} -\Delta u = \lambda u + h(x)u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

for dimensions $n > 4$, λ a nonnegative parameter and h a C^2 function which changes sign. If $n = 2$, we are interested in the following corresponding critical problem

$$\begin{gathered} -\Delta u = \lambda u + h(x)\phi(u)e^u \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Concerning h, we assume the following hypotheses:

- (H1) h belongs to $C^2(\bar{\Omega})$,
- (H2) h could change sign: Denoting by $\Omega^+ := \{x \in \Omega : h(x) > 0\}$ and by $\Omega^- := \{x \in \Omega : h(x) < 0\}$, we have $\Omega^+ \neq \emptyset$ and $\partial\Omega \subset (\{h > 0\} \cup \{h < 0\})$.
- (H3) $\Gamma := \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega$ with $\nabla h(x) \neq 0$ for all $x \in \Gamma$.
- (H4) $\Omega^0 := \Omega \backslash \overline{\Omega^+ \cup \Omega^-}$, possibly empty, satisfies:
  - (1) $\overline{\Omega^0} \subset \Omega^-$,
  - (2) $\partial\Omega^0 \cap \partial\Omega^+ = \emptyset$, and

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(3) $\lambda_1(\Omega^0) > \lambda_1(\Omega^+)$, where $\lambda_1(\cdot)$ is the first eigenvalue of $-\Delta$ in $\cdot$ with Dirichlet boundary conditions.

Notice that either $\partial\Omega \subset \{h > 0\}$ or $\partial\Omega \subset \{h < 0\}$ since $\Gamma \cap \partial\Omega$ is empty. We assume near each point $\bar{x} \in S_1 = \{x \in \Omega^+ | \nabla h(x) = 0\}$, either one of the following flatness conditions holds:

(H5a) $h(x) = h(\bar{x}) + \sum_{i=1}^{n} a_i |x_i - \bar{x}_i|^{\beta-1}(x - \bar{x})_i + R(x)$ with $\nabla R(x) = o(|x - \bar{x}|^{\beta})$, $a_i \neq 0$, for all i and $n - 2 < \beta < n$ if $n \geq 5$.

(H5b) $c \operatorname{dist}(x, S_1)^{\beta-1} \leq |\nabla h(x)| \leq C \operatorname{dist}(x, S_1)^{\beta-1}$ for all $x \in \Omega^+$ with $c, C > 0$ and, $c_1 |x_i|^{\beta} \leq x \cdot \nabla h(x + \bar{x})$ for $x \in B_{\sigma_0}(\bar{x})$ with $i \in (1, \dots, n)$, $c_1, \sigma_0 > 0$ and

$$n - 2 < \beta < n \quad \text{for } n \geq 6$$
$$n - 2 < \beta < n - 1 \quad \text{for } n = 5.$$

As h is C^2, using Taylor's expansion,

$$x \cdot \nabla h(x + \bar{x}) = \langle 2D^2 h(\bar{x}), x \rangle + o(|x|^2).$$

As $\beta > n - 2 \geq 2$, this condition is likely to hold at minimum points of h but not at maximum points.

Concerning ϕ, we assume, as in [1], that

(H6) ϕ is bounded, $C^1(\mathbb{R}, \mathbb{R}^+)$ and $C_1 u^{p'} \leq \phi(u) \leq C u^p$ in a neighborhood of 0, with $p' \geq p > 1$ and $C_1, C > 0$. Moreover, $\phi(u) > C_0 > 0$, for $\|u\|$ large.

(H7) ϕ' is bounded and such that $\phi + \phi' \geq 0$.

Under the above assumptions, we have the following result.

Theorem 1.1. (1) *Suppose that assumptions (H1)-(H4) and (H5a) or (H5b) are satisfied. Then, there exists a continuum of positive solutions, $\mathcal{C}$, to* (1.1) *in $\mathbb{R}^+ \times C_0^1(\Omega)$ bifurcating from $\lambda_1(\Omega)$ and satisfying*

(i) $\Pi_{\mathbb{R}}\mathcal{C} = [0, \lambda_*]$ *where $\lambda_1(\Omega) \leq \lambda_* < \lambda_1(\Omega^+)$.*

(ii) *If $\int_\Omega h {\phi_\Omega^1}^{\frac{2n}{n-2}} < 0$, then $\lambda_1(\Omega) < \lambda_*$ and there exist at least two solutions to* (1.1) *for $\lambda \in (\lambda_1(\Omega), \lambda_*)$. Here ϕ_Ω^1 is the first eigenfunction of Laplacian in Ω.*

(2) *Suppose that assumptions (H1)-(H4) and (H6)-(H7) are satisfied. Then, there exists a continuum of solutions, $\mathcal{C}$, to* (1.2) *in $\mathbb{R}^+ \times C_0^1(\Omega)$ bifurcating from $\lambda_1(\Omega)$ and satisfying*

(i) $\Pi_{\mathbb{R}}\mathcal{C} = [0, \lambda_*]$ *where $\lambda_1(\Omega) \leq \lambda_* < \lambda_1(\Omega^+)$.*

(ii) *If $\phi(u) \sim C_2 u^q$ when $u \to 0^+$ and if $\int_\Omega h {\phi_\Omega^1}^q < 0$, then $\lambda_1(\Omega) < \lambda_*$ and there exist at least two solutions to* (1.2) *for $\lambda \in (\lambda_1(\Omega), \lambda_*)$.*

Furthermore, in both cases, there exists $(0, u_0) \in \mathcal{C}$ such that $u_0 > 0$.

We remark that for $n = 2$, we can even assert that the branch extends beyond 0, as done in [1, Theorem 1.3].

If we consider star shaped domains as in [9], then we can relax the flatness condition as follows:

(H5c) $c \operatorname{dist}(x, S_1)^{\beta-1} \leq |\nabla h(x)| \leq C \operatorname{dist}(x, S_1)^{\beta-1}$ for all $x \in \Omega^+$ with $c, C > 0$ and $n - 2 < \beta < n$ for $n > 4$.

Theorem 1.2. *Suppose that assumptions (H1)-(H5c) are satisfied and Ω is star shaped. Then the same conclusions as in Theorem 1.1, (1), hold for $n > 4$.*

Notice that we get at least one positive solution for both problems for $\lambda > 0$, near zero and at least 2 solutions for some range of λ. We prove theorems 1.1 and 1.2 using the bifurcation theory of Rabinowitz (see [17]). The crucial step here is to prove the existence of uniform a priori bound in $L^\infty(\Omega)$ of solutions to (1.1) and (1.2) independent of λ in a compact set. Note that such a priori estimates are not obvious since the nonlinearity is critical and indefinite in sign in both equations (1.1) and (1.2). This problem has been studied when Ω^0 is empty and $h < 0$ near the boundary, in [1] for dimension 2 and in [14] for dimension 5 and above, using uniform bounds. Our aim here is to extend these results to the case of nonempty Ω^0 and h positive near the boundary and also to explore other possible flatness conditions.

Ouyang [21] has studied the bifurcation for equation (1.1) without the uniform bound and flatness assumptions. Here we are able to conclude that $(0, u_0)$ lies in the branch for some $u_0 > 0$ because of the bound.

Several results regarding a priori estimates are available concerning the case, h positive, for the subcritical case (i.e. u^p with $1 < p < \frac{n+2}{n-2}$ instead of $u^{\frac{n+2}{n-2}}$) using different methods: blow up methods in [6] (for $1 < p < \frac{n+1}{n-1}$), [15], moving plane methods in [13], for example.

In the indefinite case, a priori bounds are more delicate. In [3], assuming (H1), (H2), (H3) and $\Omega^0 = \emptyset$ and using a blow up method as in [15], the authors prove the existence of a priori bound for $1 < p < \frac{n+2}{n-1}$, for more general elliptic operators. The restriction of p is due to a Liouville theorem they prove. In [2], assuming the behaviour of h near $\overline{\Omega^+} \cap \overline{\Omega^-}$ instead of the nondegenerate assumption (H3), they get a priori bound for a class of problems close to (1.1). The proof is carried out using a blow up method in Ω^+ and the existence of a supersolution in Ω^0. Note that some restrictions on the exponent p remain in this work. In [18], restricting to variational and finite Morse index solutions, they prove that uniform a priori bounds exist for all p between 1 and $\frac{n+2}{n-2}$. In [11], assuming (H1)-(H3), $1 < p < \frac{n+2}{n-2}$, $\Omega^0 = \emptyset$ and $\lambda = 0$, the authors prove that solutions are uniformly bounded. In this work, they divide Ω into three regions and establish a priori estimates in each region. For this they combine the blow up method, Harnack inequality and moving plane method (in a neighborhood of Γ as in [13]). Then, the question is: is it true for $p = \frac{n+2}{n-2}$?

In [9], assuming (H1)-(H3), $n \geq 3$, $\lambda = 0$, Ω is starshaped and an assumption similar to (H5), they answer positively. They use the same approach as in [11]. The crucial step is the uniform estimates in $\Omega^+_\delta := \{x \in \Omega^+ \mid \operatorname{dist}(x, \Gamma) \geq \delta\}$ based upon on the blow up analysis in [19] for critical superlinear problems in S^n. This blow up analysis and some Pohozaev identities are also used to estimate solutions near the boundary in [9]. In [8], a similar result is established for the same equation with similar hypotheses but in S^n. But, these estimates are proved only for $\lambda = 0$. Here we show that the estimates are also true and independent of $\lambda \neq 0$ and positive in a compact interval and get the existence of solutions and multiplicity results for (1.1) via bifurcation theory, extending the earlier results of [14].

Concerning a priori estimates for solutions to (1.2), we use a similar method as for (1.1). Mainly, the difference is that to get a priori bound in Ω^+_δ, we use the blow up analysis of [5] instead of the blow up analysis of [19]. Furthermore, here we use the specificity of two-dimensional case to estimate solutions at boundary via a Kelvin transform as in [13] (see also [11]). We would like to point out that these a priori estimates are independent of λ in a compact set as done earlier in [1].

Unlike [14] and [1], here Ω^0 may be nonempty and h may be either positive or negative near the boundary of Ω. For both problems the uniform estimate in Ω^0 is obtained by constructing a supersolution and is only valid for $\lambda < \lambda_1(\Omega^0)$. When $h > 0$ near the boundary, the assumptions on the weight functions are suitably used to avoid the boundary blow up. Now, let us state the main results concerning a priori estimates for solutions to (1.1) and (1.2):

Proposition 1.3. *Let us assume $n > 4$, (H1)-(H5a) or (H5b) or for star shaped domains, (H5c). Let Λ such that $0 \leq \Lambda < \lambda_1(\Omega^+)$. Then for any solution (λ, u) to* (1.1)*, such that $0 \leq \lambda \leq \Lambda$, we have*

$$\|u\|_{L^\infty(\Omega)} \leq C(\Lambda, n, h, \Omega).$$

Similarly for (1.2), we have the estimate:

Proposition 1.4. *Let us assume $n = 2$, (H1)-(H4) and (H6)-(H7). Let Λ be such that $0 \leq \Lambda < \lambda_1(\Omega^+)$. Then for any solution (λ, u) to* (1.2)*, such that $0 \leq \lambda \leq \Lambda$, we have*

$$\|u\|_{L^\infty(\Omega)} \leq C(\Lambda, h, \Omega).$$

Finally, let us mention that using the same ideas, we can also deal with (1.1) and (1.2) in the case of $\Omega = \mathbb{R}^n$. It has been done in [1], [14] with $\Omega^0 = \emptyset$. In these works, $h(x) \to -\infty$ when $|x| \to +\infty$. Then the problem becomes sublinear at infinity. So contrary to what we do in the present paper, no careful analysis near the boundary has to be done. But uniform decay at infinity has to be worked out thanks to the behaviour of h at infinity. Now, to conclude this introduction, let us present the outline of this paper:

In the second section, we give first a priori bound in

$$\Omega_\delta^- := \{x \in \Omega^- : \operatorname{dist}(x, \Gamma \cup \Omega^0) \geq \delta\}.$$

The estimate in this region is performed by a L^q-estimate and hypoellipticity arguments. The third section, deals with a neighborhood of Γ and Ω^0. We use the moving plane method which allows us to give a priori bound in a neighborhood of Γ for both problems as in [14] and [1] (see also [4]). After that, by exhibiting a supersolution, we get an a priori estimate in a neighborhood of Ω^0. Here the fact that λ remains uniformly below to $\lambda_1(\Omega^0)$ is crucial for getting the existence of the supersolution. Section 4 is on a priori bound in

$$\Omega_{\delta,\eta}^+ := \{x \in \Omega_\delta^+ : \operatorname{dist}(x, \partial\Omega) \geq \eta\}$$

for some $(\delta, \eta > 0)$. As in [14], we extend the blow up analysis of [19] for $\lambda \geq 0$, indicating the proofs with either of the flatness assumptions (H5a), (H5b) or (H5c).

Section 5 is concerned with $\partial\Omega$. Concerning (1.1), the most difficult case is when h is positive near the boundary. In this case, we use a blow up method as in [9] to estimate solutions close to the boundary. If h is non positive, we use the maximum principle. This finalizes the proof of Propositions 1.3 and 1.4.

In the final section, we prove the main results. We would like to point out that this result holds also in the pure superlinear case (ie. $h > 0$ in $\overline{\Omega}$).

2. A priori bound in Ω_δ^-

Here we obtain a priori bounds for the solution u of (1.1) and (1.2) in the region Ω_δ^-. Note that Ω_δ^- contains a neighborhood of $\partial\Omega$ if $h < 0$ near $\partial\Omega$. The following L^p estimate is crucial, which in fact is true in both sets Ω^- and Ω^+.

Proposition 2.1. *Let $x_0 \in \Omega^{\pm}$, $\epsilon > 0$, $\Lambda \in \mathbb{R}^+$, and $B_\epsilon(x_0) \subset\subset \Omega^{\pm}$. Assume that $\lambda \le \Lambda$.*

(i) *For $n = 2$, if u is a solution of (1.2) then*
$$\int_{B_{\frac{\epsilon}{2}}(x_0)} e^u \le C(\epsilon, h, \Lambda). \tag{2.1}$$

(ii) *For $n \ge 3$, if u is a solution of (1.1), then there exists $C = C(\epsilon, \Lambda)$ such that*
$$\int_{B_{\frac{\epsilon}{2}}(x_0)} u^{\frac{n+2}{n-2}} dx \le \Big(\frac{C}{\inf_{B_\epsilon} |h|}\Big)^{\frac{n+2}{4}}. \tag{2.2}$$

This estimate follows by multiplying by the first eigenfunction of Laplacian in a ball and using integration by parts. The details are in [1] and [14] respectively.

We present the proof for 1.1 here. The proof for the other equation is similar. For a bound for u in Ω_δ^-, we need to consider 2 cases:

(i) Either $h < 0$ near $\partial\Omega$, or
(ii) $h > 0$ near $\partial\Omega$.

In case (i), let us define a δ neighborhood of $\partial\Omega$
$$G := \{x \in \Omega_\delta^- : \operatorname{dist}(x, \partial\Omega) \le \delta\}.$$
For δ small enough, by (H3) and (H4), $G \subset \Omega_\delta^-$. Let $A := \{x \in \Omega_\delta^- : -\Delta u(x) < 0\}$. We split now the domain into three sets
$$\Omega_\delta^- = (\Omega_\delta^- \setminus G) \cup (G \setminus A) \cup (G \cap A).$$
We will get the apriori estimate in the first set using the earlier integral estimate and in the second one, a pointwise estimate and then in the third set via maximum principle and the previous estimates.

For $x \in \Omega_\delta^- \setminus G$, there exists a ball $B_{\delta/2}(x) \subset \Omega_{\frac{\delta}{2}}^-$ and the integral estimate (2.2) hold for u in $B_{\delta/4}(x)$. Then we use the following Lemma [16, Lemma 9.20].

Lemma 2.2. *Let $u \in W^{2,n}(\Omega)$ with $Lu \ge f$ where L is a strictly elliptic second order operator and $f \in L^n(\Omega)$. For all $B = B_\epsilon(y) \subset \Omega$ and $q > 0$, we have*
$$\sup_{B_{\frac{\epsilon}{2}}(y)} u \le C(n, q, \epsilon)\Big(\Big(\int_B (u^+)^q\Big)^{\frac{1}{q}} + \|f\|_{L^n(B)}\Big). \tag{2.3}$$

We combine this estimate for $f = 0$, $q = \frac{n+2}{n-2}$ and $L = \Delta + \lambda$ in the ball $B_{\frac{\delta}{2}}(x) \subset \Omega_{\delta/2}^-$ (since $x \in \Omega_\delta^-$), together with the estimate (2.2) to conclude that
$$\sup_{B_{\frac{\delta}{8}}(x)} u \le C(n, \lambda, \delta)\Big\{\frac{1}{\inf_{B_{\delta/2}(x)} |h|}\Big\}^{(n-2)/4}. \tag{2.4}$$
Note that here we can take any $\lambda < \Lambda$. Thus we have if $x \in \Omega_\delta^- \setminus G$,
$$u(x) \le C(n, \Lambda, \delta)\Big\{\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\Big\}^{(n-2)/4}. \tag{2.5}$$
In the case $x \in G \setminus A$,
$$0 \le -\Delta u(x) = \lambda u(x) + h(x) u^{\frac{n+2}{n-2}}(x),$$

and hence
$$-h(x)u^{\frac{n+2}{n-2}}(x) \le \lambda u(x).$$
Since $u(x) > 0$, we have the *pointwise estimate*
$$u(x) \le \Big(\frac{\lambda}{\inf_{\Omega_\delta^-} |h|}\Big)^{(n-2)/4} \quad \text{for all } x \in G \setminus A. \tag{2.6}$$
Using the above estimates and recalling that $u = 0$ on $\partial\Omega$, we have for points on $\partial(G \cap A)$,
$$u(x) \le M, \quad M = \max\big\{C(n,\lambda,\delta)\{\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\}^{(n-2)/4}, (\frac{\lambda}{\inf_{\Omega_\delta^-} |h|})^{(n-2)/4}\big\}. \tag{2.7}$$
Define $c(x) := \lambda + h(x)u(x)^{4/n-2}$ and consider the equation
$$\Delta v + c(x)v \ge 0 \quad \text{in } (G \cap A). \tag{2.8}$$
Note that for $x \in A$, $c(x) < 0$ and that $u - M$ is a solution of (2.8). Hence, by the weak maximum principle ([16, Theorem 9.1] with $f \equiv 0$), we have
$$u(x) - M \le 0 \quad \text{in } (G \cap A).$$
Combining all the cases, we have the local estimate in this case.

Now let us consider the case (ii), $h > 0$ near $\partial\Omega$. Then, by (H3) since $\Gamma \subset \Omega$, for δ small, $\operatorname{dist}(\Omega_{\delta/2}^-, \partial\Omega) > \delta$. Therefore a straightforward application of Lemma 2.2 yields (2.4).

Thus we have the following local estimate in Ω_δ^-.

Proposition 2.3. *Let $0 \le \lambda \le \lambda_1(\Omega^+)$. Assume (H1)–(H3).*

(i) *For u, a solution of* (1.2) *and for $1 < p' < \infty$,*
$$\sup_{\Omega_\delta^-} u(x) \le C(\lambda_1(\Omega^+), \delta)\Big(\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\Big)^{1/(p'-1)}. \tag{2.9}$$
(ii) *For u, a solution of* (1.1)
$$\sup_{\Omega_\delta^-} u(x) \le C(n, \lambda_1(\Omega^+), \delta)\Big(\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\Big)^{(n-2)/4}. \tag{2.10}$$

Note that since $\inf_{\Omega_{\delta/2}^-} |h| > 0$, from Proposition 2.3, we get uniform a priori bounds in Ω_δ^-.

3. A priori bound in a neighborhood of Γ and Ω^0

Let us start by giving a priori bound in a neighborhood of Ω^0. Let Ω_δ^0 be a smooth δ-neighborhood of Ω^0. By (H4)-1 and (H4)-2 and the results in the second section (bounds in Ω_δ^-), solutions to (1.1) and (1.2) are uniformly bounded in $\partial\Omega_\delta^0$ by a constant M. Now, Let ψ the solution to
$$\begin{gathered} -\Delta\psi = \Lambda\psi \quad \text{in } \Omega_\delta^0 \\ \psi > 0 \quad \text{in } \Omega_\delta^0; \quad \psi = M \quad \text{on } \partial\Omega_\delta^0 \end{gathered} \tag{3.1}$$
which exists if $\Lambda < \lambda^1(\Omega^0)$ and δ small. By maximum principle and for $\lambda \le \Lambda$, any solution (λ, u) to (1.1) and (1.2) satisfies $u \le \psi$ in Ω_δ^0.

Now, the estimates in a δ-neighborhood of Γ, Γ_δ, are similar to step 2 in [4] (for solutions to (1.1) just replace p by $\frac{n+2}{n-2}$). Note that for solutions to (1.2), we use the assumption (H7) to carry out the moving plane method. Precisely, to show that (taking the same notations as in step 2 in [4]) $\frac{\partial f}{\partial x_1} \leq 0$ we use that $\phi(u) + \phi'(u) \geq 0$ and bounded.

4. A Priori Bound in $\Omega^+_{\delta,\eta}$

We start with (1.1). This part is done in [14] and used the blow up analysis in [19]. Precisely, let a sequence (λ_i, u_i) be solutions of (1.1), such that $0 \leq \lambda_i \leq \lambda_1(\Omega^+)$ and a sequence of local maxima $x_i \in \Omega^+_\delta$ of u_i, such that $u_i(x_i) \to \infty$ as $i \to \infty$. By the a priori bounds on Ω^-_δ, Γ and Ω^0) obtained in the earlier 2 sections, and also in view of the bound to be obtained in the next section in a neighbourhood of $\partial\Omega$, $\{u_i\}$ remains uniformly bounded on the boundary of $\Omega^+_{\delta,\eta}$. Thus x_i has to converge to some point in the interior of $\Omega^+_{\delta,\eta}$. Note that the a priori bound near the boundary is independent of the results of this section. We postpone that because we need to use some results from this section, in particular, Proposition 4.1.

In the first subsection we recall the standard blow up argument to analyse u_i, in a small neighbourhood of x_i and various local estimates required later on. In the second subsection, we use these estimates to prove that a blow up point of u_i is necessarily a critical point of h. This motivates the flatness assumptions at the critical points of h. Using this assumption, we analyse the nature of the blow up points and show that in fact u_i does not blow up, i.e the sequence $\{u_i\}$ is uniformly bounded.

4.1. **Blow up points of $\{u_i\}$.** We start by recalling several definitions and propositions which are extensions of some results in [19] to the case of $\lambda > 0$. As the details are in [14], we give here only an outline, giving details only when new ideas are involved. The following is a standard result in the blow up analysis (see for example [22]).

Proposition 4.1. *Suppose that $h \in C^1(\Omega^+_\delta)$ and there exist A_1 and A_2 such that in Ω^+_δ,*

$$h(x) \geq \frac{1}{A_1}, \quad \|\nabla h(x)\| \leq A_2.$$

Then for every $0 < \varepsilon < 1$, $R > 1$, there exist positive constants C_0 and C_1 depending on A_1, $A_2, \varepsilon, R, \lambda$ and n such that if v is a positive solution of

$$-\Delta v(x) = \lambda v(x) + h(x) v^{\frac{n+2}{n-2}}, \quad v > 0 \tag{4.1}$$

with $\max_B v > C_0$, then there exists a finite number $k = k(v)$ and a set $S(v, C_0) = \{x_1, \dots, x_k\} \subset \Omega^+_\delta$ such that

(i) *x_j are the local maxima of v and for $\mu_j = v(x_j)^{-\frac{2}{n-2}}$, $\{B_{R\mu_j}(x_j)\}_{1\leq j\leq k}$ are disjoint balls and*

$$\|v(x_j)^{-1} v(x_j + \mu_j x) - \delta_j(x)\|_{C^2(B_{2R}(0))} < \varepsilon,$$

where

$$\delta_j(x) = (1 + h_j|x|^2)^{\frac{2-n}{2}} \quad \text{with } h_j = (n(n-2))^{-1} h(x_j)$$

is the unique solution of

$$\begin{gathered} \Delta\delta_j + h_j \delta_j^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \\ \delta_j > 0 \quad \text{in } \mathbb{R}^n, \quad \delta_j(0) = 1, \end{gathered}$$

(ii) $v(x) \le C_1(\mathrm{dist}(x,S))^{-(\frac{n-2}{2})}, x \in \Omega_\delta^+$.

The above Proposition, in particular (ii), motivates the definition of an isolated blow up point.

Definition 4.2. A point $x_0 \in \Omega'$ is called an isolated blow up point of $\{u_i\}$, solutions of (1.1), if there exists $0 < \bar{r} < dist(x_0, \partial\Omega')$ and $C > 0$ and a sequence $\{x_i\}$ tending to x_0, such that x_i is a local maximum of $\{u_i\}, u_i(x_i) \to \infty$ and

$$u_i(x) \le C|x - x_i|^{-(\frac{n-2}{2})} \quad \forall x \in B_{\bar{r}}(x_0)\backslash\{x_i\}.$$

Since we will be interested in the blow up points staying away from each other, we also need to introduce the definition of a simple isolated blow up point.

Definition 4.3. A point x_0 is an isolated simple blow up point of $\{u_i\}$, solutions of (1.1), if it is an isolated blow up point such that for some $\rho > 0$ (independent of i), $\tilde{v}_i$ has precisely one critical point in $(0, \rho)$ $\forall$ large i, where

$$\tilde{v}_i(r) = r^{\frac{n-2}{2}}\overline{v}_i(r), \quad \overline{v}_i(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r(x_i)} u_i, \quad r > 0.$$

The following is a corollary of Proposition 4.1.

Corollary 4.4. *Let x_0 be an isolated blow up point of $\{u_i\}$. Then one can choose $R_i \to \infty$ first and then $\varepsilon_i \to 0^+$ depending on R_i and a subsequence $\{u_i\}$ so that*

(i) $r_i = \frac{R_i}{(u_i(x_i))^{\frac{2}{n-2}}} \to 0$ *and x_i is the only critical point of $u_i(x)$ in $|x - x_i| < r_i$.*

(ii) $\tilde{v}_i(r)$ *has a unique critical point in $0 < r < r_i$.*

Another important result, we will use in the following, is the Harnack inequality

Lemma 4.5 (A Harnack inequality)**.** *Let h satisfy*

$$\frac{1}{A_1} \le h(x) \le A_1 \quad \forall x \in \Omega_\delta^+ \tag{4.2}$$

and $\{u_i\}$ satisfy (1.1) having 0 as an isolated blow up point. Then for any $0 < r < \frac{\bar{r}}{3}$, with $\bar{r}$ as in Definition 4.2, we have the Harnack inequality

$$\max_{B_{2r}\backslash B_{r/2}} u_i(y) \le C \min_{B_{2r}\backslash B_{r/2}} u_i(y) \tag{4.3}$$

with a uniform $C = C(n, \lambda, \|h\|_{L^\infty(\Omega_\delta^+)})$.

The proof of this lemma follows on the same lines as in [19], [7].

Now, we look for lower and upper bounds for u_i, in a fixed neighbourhood of the blow up point. The arguments for the lower bound are as in [19] (section 2 there). For the upper bound, we need to exploit specifically the extra linear term in our case, as in [14].

Proposition 4.6. *Assume $B_2(0) \subset \Omega_\delta^+$ and*

$$A_1 \ge h(x) \ge \frac{1}{A_1}, \quad \|\nabla h(x)\| \le A_2 \quad \forall x \in B_2 \tag{4.4}$$

for some positive constants A_1, A_2. Let u_i be solutions of (1.1) and $x_i \to 0$ be an isolated blow up point with

$$u_i(x) \le \frac{A_3}{|x - x_i|^{\frac{n-2}{2}}} \quad \text{for all } x \in B_2\backslash\{x_i\}. \tag{4.5}$$

Then there exists a positive constant $C = C(n, \lambda_0, A_1, A_2, A_3)$, such that up to a subsequence,

$$u_i(x) \geq Cu_i(x_i)(1 + h_i u_i(x_i)^{\frac{4}{n-2}}|x - x_i|^2)^{(2-n)/2} \quad \text{for all } |x - x_i| \leq 1, \tag{4.6}$$

where h_i is as defined in Proposition 4.1. In particular, for any $e \in \mathbb{R}^n$, $|e| = 1$, we have

$$u_i(x_i + e) \geq C^{-1}u_i(x_i)^{-1}. \tag{4.7}$$

Proposition 4.7 (Upper bound)**.** *Let h and $\{u_i\}$ satisfy the conditions in Proposition 4.6. Also, assume that $x_i \to 0$ is an isolated simple blow up point as defined in Definition 4.3. Then there exists a positive constant $C = C(n, \lambda_0, A_1, A_2, A_3, \rho)$ such that*

$$u_i(x) \leq Cu_i(x_i)^{-1}|x - x_i|^{2-n} \quad \text{for all } 0 < |x - x_i| \leq 1. \tag{4.8}$$

In particular, $u_i(x_i + e)u_i(x_i) = O(1)$, where e is a unit vector in $\mathbb{R}^n$.

The following lemma is crucial in our analysis.

Lemma 4.8. *Under the assumptions of Proposition 4.7, let $\{(\lambda_i, u_i)\}$ be a sequence of solutions of (1.1) and $x_i \to 0$ be an isolated simple blow up point. Then upto a subsequence, we have*

(i) *if λ_i goes to zero, then*

$$u_i(x)u_i(x_i) \to w(x) = \frac{a}{|x|^{n-2}} + g(x)$$

where $a = h_i^{-(n-2)/2}$ and g is some harmonic function.

(ii) *if λ_i goes to $\tilde{\lambda} > 0$, then*

$$u_i(x)u_i(x_i) \to w(x) = \frac{\alpha C_n}{|x|^{n-2}} + E(x) + \varphi(x)$$

where $G(x) = \alpha C_n|x|^{2-n} + E(x)$ is the unique solution in the sense of distribution for the equation

$$\begin{gathered} -\Delta G = \lambda G + \alpha\delta_0 \quad \text{in } B_{\sigma_1}(0) \\ G = 0 \quad \text{on } \partial B_{\sigma_1}(0) \end{gathered}$$

and φ is the unique C^2 solution of the boundary value problem

$$\begin{gathered} -\Delta\varphi = \lambda\varphi \quad \text{in } B_{\sigma_1}(0) \\ \varphi = w \quad \text{on } \partial B_{\sigma_1}(0). \end{gathered}$$

Here σ_1 is sufficiently small such that $\lambda < \lambda_1(B_{\sigma_1}(0))$.

The proofs are in [14]. There the following lemma is used to handle the linear term. This imposes a restriction $n > 4$. Now, we fix $e \in \mathbb{R}^n$ such that $|e| = 1$. As in [7] (see Proposition 3.5), we have the following result.

Lemma 4.9. *Let us suppose $\{(\lambda_i, u_i)\}$ is a sequence of solutions of (1.1) and $\bar{x} = 0$ is an isolated and simple blow up point. Suppose also that $n > 4$. Then, there exists a positive constant $C = C(n, h, \rho)$ such that*

$$\lambda_i u_i(x_i)^{\frac{2(n-4)}{n-2}} \leq Cu_i(x_i)^2 u_i(x_i + e)^2 + o(1)\,. \tag{4.9}$$

4.1.1. *Nature of Blow up points for solutions to* (1.1). We need the following Identity.

Lemma 4.10 (Pohozaev Identity). *Let v be a C^2 solution of* (1.1). *Then for any $B_\sigma \subset \Omega_\delta^+$,*

$$\int_{\partial B_\sigma} B(\sigma, x, v, \nabla v) = \Big\{\lambda \int_{B_\sigma} v^2 - \frac{\lambda}{2}\int_{\partial B_\sigma} \sigma v^2\Big\} + \frac{n-2}{2n}\int_{B_\sigma}(x\cdot\nabla h)v^{\frac{2n}{n-2}} - \frac{\sigma(n-2)}{2n}\int_{\partial B_\sigma} h(x)v^{\frac{2n}{n-2}},$$

where

$$B(\sigma, x, v, \nabla v) := -\frac{(n-2)}{2}v\frac{\partial v}{\partial \nu} - \frac{\sigma}{2}|\nabla v|^2 + \sigma\Big|\frac{\partial v}{\partial \nu}\Big|^2$$

and ν denotes the unit outer normal vector field on ∂B_σ.

For a proof see for example [19], [14]. The earlier estimates and Pohozaev identity can be used to derive various conclusions about the possible blow up points of $\{u_i\}$. In particular, the following result is often used.

Lemma 4.11. *For $u(x) = \frac{a}{|x|^{n-2}} + b(x)$ where $a > 0$ and $b(x)$ is a nonnegative differentiable function, with $b(0) > 0$, we have*

$$B(\sigma, x, u, \nabla u) < 0,$$

on ∂B_σ for all σ small.

The proof follows from direct computations. After suitable rescaling of the solutions, using the above identities, one can prove that in fact that the blow up points are simple, isolated and they have to be the critical points of h, as in [19, 14].

Proposition 4.12. *Suppose that (H1)-(H5) are satisfied. Then any isolated blow up point is simple and is a critical point of h.*

Proposition 4.13. *Suppose that (H1)-(H5) are satisfied.The blow up points of $\{u_i\}_i$ are isolated: More precisely, for $\varepsilon > 0$ and $R > 1$, there exists some positive constant $r^* = r^*(n, \varepsilon, R, A_1, c_1, c_2, d,$ modulus of continuity of $\nabla h)$ such that for any solution u_i with $\max_{\Omega'} u_i > C^*$, we have*

$$|q_l - q_j| \geq r^* \quad \forall 1 \leq l \neq j \leq k,$$

where $q_l = q_l(u_i)$, $k = k(u_i)$ are as in Proposition 4.1.

4.1.2. *A priori estimates for solutions of* (1.1).

Proposition 4.14. *Assume (H1)-(H4). Further either (H5a) or (H5b) holds or (H5c) holds for Ω star shaped. Let $\{(\lambda_i, u_i)\}_i$ be a sequence of solutions of* (1.1) *with $0 \leq \lambda_i \to \tilde\lambda$. Then, $\{u_i\}_i$ is uniformly bounded in $L^\infty(\Omega^+_{\delta,\eta})$.*

Proof. Let us first consider the case when (H5a) holds.
(i) If $\tilde\lambda > 0$, the analysis follows from Proposition 5.5 (iii) of [14] . Using Pohozaev identity in a ball around a blow up point, a contradiction is derived there. So $\tilde\lambda$ cannot be positive.
(ii) If $\tilde\lambda = 0$ and $n \geq 5$, we get a contradiction by using Pohozaev identity as in the Appendix in [14]. Combining both, we conclude that the solutions are uniformly bounded, if (H5a) holds.

Below we recall the proof for $\tilde{\lambda} = 0$ and $n > 4$ when (H5a) holds because some estimates will be needed for the other cases. We will follow the arguments in [19, Theorem 4.4], adapted here for the bounded domain $\Omega^+_{\delta,\eta}$. In the proof, we need the following Pohozaev identity: If u is a C^2-solution to $-\Delta u = \lambda u + h(x)u^{\frac{n+2}{n-2}}$ in B_σ for some $\sigma > 0$, then

$$\int_{B_\sigma} (\nabla h)u^{\frac{2n}{n-2}} = \frac{2n}{n-2}\int_{\partial B_\sigma}(\frac{\partial u}{\partial \nu}\nabla u - \frac{1}{2}|\nabla u|^2\nu) + \int_{\partial B_\sigma}\frac{\lambda}{2}u^2\nu + \int_{\partial B_\sigma} hu^{\frac{2n}{n-2}}\nu \tag{4.10}$$

where ν is the unit outward normal. Let $\{y_i\} \in \Omega^+_{\delta,\eta}$. Without loss of generality, we can assume that $y_i \to 0$. Then, for a small ball B_σ around y_i, by the above Pohozaev identity, we have

$$\int_{B_\sigma} \nabla h(x)(u_i(x))^{\frac{2n}{n-2}} = I_1 + I_2 + I_3$$

where

$$I_1 = \frac{2n}{n-2}\int_{\partial B_\sigma}(\frac{\partial u_i}{\partial \nu}\nabla u_i - \frac{1}{2}|\nabla u_i|^2\nu),$$
$$I_2 = \frac{\lambda_i}{2}\int_{\partial B_\sigma} u_i^2\nu, \quad I_3 = \int_{\partial B_\sigma} h(x)(u_i(x))^{\frac{2n}{n-2}}\nu.$$

We will estimate I_k's as follows: First, we observe that

$$|I_1| \le C\int_{\partial B_\sigma} |\nabla u_i|^2.$$

Then, we need to evaluate $\int_{\partial B_\sigma} |\nabla u_i|^2$ for a suitable value of σ. For this, let $A := B_{\frac{1}{2}} \backslash B_{\sigma_1}$ such that $\sigma_1 < \frac{1}{2}$. Let η_i a cut off function such that

$$\eta_i(x) = \begin{cases} 1 & \text{if } x \in A_i = \{x, \sigma_1 + \epsilon_i \le |x| \le \frac{1}{2} - \epsilon_i\} \\ 1 - \frac{|x|}{\epsilon_i} & \text{if } x \in A\backslash A_i \end{cases}$$

where $\epsilon_i := u_i(y_i)^{-1}$. Now, multiplying the equation (1.1) by $\eta_i u_i$, we get

$$\int_A (\nabla u_i \cdot \nabla(\eta_i u_i)) = \lambda_i \int_A u_i^2\eta_i + \int_A hu_i^{\frac{2n}{n-2}}\eta_i.$$

It follows that

$$\begin{aligned} \int_{A_i} |\nabla u_i|^2 &\le \int_A |\nabla u_i|^2\eta_i \\ &\le \lambda_i \int_A u_i^2 + \int_A hu_i^{\frac{2n}{n-2}} + \frac{1}{2}|\int_{A\backslash A_i} \nabla(u_i^2)\cdot\nabla\eta_i|, \end{aligned} \tag{4.11}$$

$$\int_A hu_i^{\frac{2n}{n-2}} \le \frac{C}{(u_i(y_i))^{\frac{2n}{n-2}}}, \tag{4.12}$$

$$\begin{aligned} |\int_{A\backslash A_i} \nabla(u_i^2)\cdot\nabla\eta_i| &\le C\Big(\int_{A\backslash A_i} |\nabla(u_i^2)|^2\Big)^{1/2} \times \Big(\int_{A\backslash A_i} |\nabla\eta_i|^2\Big)^{1/2} \\ &\le \frac{C}{(u_i(y_i))^2} \times \frac{1}{(u_i(y_i))^{n-1}} = \frac{C}{(u_i(y_i))^{n+1}}. \end{aligned} \tag{4.13}$$

Hence, from (4.11), (4.12) and (4.13), we get

$$\int_{A_i} |\nabla u_i|^2 \leq \begin{cases} \frac{C}{(u_i(y_i))^{\frac{2n}{n-2}}} & \text{for } n \geq 6, \\ \frac{C}{(u_i(y_i))^{\frac{8}{3}}} & \text{for } n = 5. \end{cases}$$

Now, taking $\sigma_i \in [\sigma_1 + \epsilon_1, \frac{1}{2} - \epsilon_1]$ such that

$$\int_{\partial B_{\sigma_i}} |\nabla u_i|^2 = \min_{\sigma \in [\sigma_1+\epsilon_1, \frac{1}{2}-\epsilon_1]} \int_{\partial B_\sigma} |\nabla u_i|^2,$$

we have

$$\int_{\partial B_{\sigma_i}} |\nabla u_i|^2 \leq \begin{cases} \frac{C}{(u_i(y_i))^{\frac{2n}{n-2}}} \times \frac{1}{(\frac{1}{2}-\sigma_1-2\epsilon_i)} & \text{for } n \geq 6, \\ \frac{C}{(u_i(y_i))^{\frac{8}{3}}} \times \frac{1}{(\frac{1}{2}-\sigma_1-2\epsilon_i)} & \text{for } n = 5 \end{cases}$$

Therefore,

$$|I_1|, |I_3| \leq \begin{cases} \frac{C}{(u_i(y_i))^{\frac{2n}{n-2}}} & \text{for } n \geq 6, \\ \frac{C}{(u_i(y_i))^{\frac{8}{3}}} & \text{for } n = 5 \end{cases} \tag{4.14}$$

Also we have for all $n > 4$,

$$|I_3| \leq \frac{C}{(u_i(y_i))^{2n/(n-2)}}.$$

Then

$$\int_{B_\sigma} \nabla h(x)(u_i(x))^{\frac{2n}{n-2}} \leq \begin{cases} \frac{C}{u_i(y_i)^{\frac{2n}{n-2}}} & \text{for } n \geq 6, \\ \frac{C}{(u_i(y_i))^{\frac{8}{3}}} & \text{for } n = 5 \end{cases}$$

Now, we follow the arguments in [19, Theorem 4.4 and Corollary 4.1] to prove

Step 1: $|y_i| = O(\frac{1}{u_i(y_i)^{\frac{2}{n-2}}})$ so that $y_i u_i(y_i)^{\frac{2}{n-2}} = \xi_i \to \xi$,

Step 2: Multiplying (4.10) by $u_i(y_i)^{\frac{2(\beta-1)}{n-2}}$), and using estimates on I_i and rescaling arguments, we get $\int_{\mathbb{R}^n} \nabla h(z+\xi) \frac{dz}{(1+k^2|z|^2)^n} = 0$ which contradicts (H5a).

Let us consider the case when (H5b) holds. In this case, we use Lemma 4.10,

$$\begin{aligned} \int_{\partial B_\sigma} B(\sigma, x, v, \nabla u_i) &= \Big\{ \lambda_i \int_{B_\sigma} u_i^2 - \frac{\lambda_i}{2} \int_{\partial B_\sigma} \sigma u_i^2 \Big\} + \frac{n-2}{2n} \int_{B_\sigma} (x \cdot \nabla h) u_i^{\frac{2n}{n-2}} \\ &\quad - \frac{\sigma(n-2)}{2n} \int_{\partial B_\sigma} h(x) u_i^{\frac{2n}{n-2}}, \end{aligned}$$

from which together with (H5b), $\sigma < \sigma_0$ and $\lambda_i \geq 0$ it follows that

$$\begin{aligned} \int_{B_\sigma} (x \cdot \nabla h) u_i^{\frac{2n}{n-2}} &\leq -\frac{2n}{n-2} \int_{\partial B_\sigma} B(\sigma, x, v, \nabla u_i) + \sigma \int_{\partial B_\sigma} h(x) u_i^{\frac{2n}{n-2}} \\ &\quad + \frac{\lambda_i}{2} \int_{\partial B_\sigma} \sigma u_i^2. \end{aligned} \tag{4.15}$$

Then, from (4.12) and (4.15) we get

$$\begin{aligned} \int_{B_\sigma} |x|^\beta u_i^{\frac{2n}{n-2}} &\leq \int_{B_\sigma} (x \cdot \nabla h) u_i^{\frac{2n}{n-2}} \\ &\leq K_1 \int_{\partial B_\sigma} u_i |\nabla u_i| + K_2 \int_{\partial B_\sigma} |\nabla u_i|^2 + O(u_i(y_i)^{-\frac{2n}{n-2}}). \end{aligned} \tag{4.16}$$

The two boundary integrals can be estimated like I_1 in (4.14). Hence, we get for $n > 5$,

$$\int_{B_{r_i}} |x|^{\beta} u_i^{\frac{2n}{n-2}} \le O(u_i(y_i)^{-\frac{2n-2}{n-2}}). \tag{4.17}$$

Now using a change of variable and using Proposition (4.1), (i), we see that the left hand side integral is positive but the right hand side goes to 0, since $\beta < n$, a contradiction. Similarly the case $n = 5$ can be handled.

Now let (H5c) hold for a star shaped domain Ω. The results of the previous sections imply that the blow up points in Ω^+ are finite in number, say $\{p_1, p_2, \dots, p_m\}$. Using again the analysis from [14, Proposition 5.5 (iii)], $\tilde{\lambda}$ cannot be positive. Let $\tilde{\lambda} = 0$. Now assuming (H5c) and Ω is star shaped, we show that there cannot be any blow up points, using the ideas in [9] (see section 3 there).

We apply Pohozaev identity to

$$\Omega_\varepsilon = \Omega \setminus \cup_{i=1}^m B_\varepsilon(p_i)$$

for a fixed ε, to obtain

$$\begin{aligned}
&\Big\{\lambda_i \int_{\Omega_\varepsilon} u_i^2 - \frac{\lambda_i}{2} \int_{\partial\Omega_\varepsilon} (x\cdot\nu) u_i^2\Big\} \\
&+ \frac{n-2}{2n} \int_{\Omega_\varepsilon} (x\cdot\nabla h) u_i^{\frac{2n}{n-2}} - \frac{(n-2)}{2n} \int_{\partial\Omega_\varepsilon} h(x) u_i^{\frac{2n}{n-2}} (x\cdot\nu) \\
&= \int_{\partial\Omega_\varepsilon} \Big(\frac{n-2}{2}\frac{\partial u_i}{\partial\nu} u_i - \frac{1}{2}|\nabla u|^2 (x\cdot\nu) + \frac{\partial u_i}{\partial \nu}(x\cdot\nabla u_i)\Big)
\end{aligned}$$

Let y_i tend to p_1. Now multiply both sides by $u_i(y_i)^2$ and take the limit as $i\to\infty$. From Lemma 4.8, since $\lambda_i \to 0$, we have

$$u_i(x)u_i(y_i) \to w(x) = \frac{a_1}{|x-p_1|^{n-2}} + \dots + \frac{a_m}{|x-p_m|^{n-2}} + g(x)$$

with some harmonic $g(x)$. Since w is smooth outside each $B_\varepsilon(p_i)$, we have

$$u_i^2(y_i)\int_{\Omega_\varepsilon} (x\cdot\nabla h) u_i^{\frac{2n}{n-2}} = \frac{1}{u_i^{\frac{2n}{n-2}-2}} \int_{\Omega_\varepsilon} (x\cdot\nabla h)(u_i(y_i)u_i(x))^{\frac{2n}{n-2}} \to 0.$$

Also using $\lambda_i \to 0$,

$$\lambda_i \int_{\Omega_\varepsilon} (u_i(y_i)u_i(x))^2 \to 0.$$

Thus in the limit

$$\int_{\partial\Omega_\varepsilon} \frac{1}{2}|\nabla w|^2 (x\cdot\nu) = \sum_{i=}^{m} \int_{\partial B_\sigma(p_i)} \Big(\frac{1}{2}|\nabla w|^2(x\cdot\nu) - \frac{n-2}{2}\frac{\partial w}{\partial\nu} w - \frac{\partial w}{\partial\nu}(x\cdot\nabla w)\Big)$$

Since Ω is star shaped, $(x\cdot\nu) > 0$ on $\partial\Omega$. Furthermore $|\nabla w|^2 > 0$ on $\partial\Omega$. Hence at least for some i,

$$I = \int_{\partial B_\sigma(p_i)} \Big(\frac{1}{2}|\nabla w|^2(x\cdot\nu) - \frac{n-2}{2}\frac{\partial w}{\partial\nu} w - \frac{\partial w}{\partial\nu}(x\cdot\nabla w)\Big) > 0.$$

Now we apply Pohozaev identity on $B_\sigma(p_i)$ and multiply by $u_i(y_i)^2$ and pass to the limit to get

$$I = \lim (u_i^2(y_i) \int_{B_\sigma} (x\cdot\nabla h) u_i^{\frac{2n}{n-2}})$$

The same calculations as in the proof of [14, Proposition 5.5 (iii)] show that

$$\begin{aligned}u_i^2\int_{B_\sigma}(x\cdot\nabla h)u_i^{\frac{2n}{n-2}} &\le u_i^2\int_{B_\sigma}|x|^\beta u_i^{\frac{2n}{n-2}}\\ &= \frac{1}{u_i^{\frac{2\beta}{n-2}-2}}\Big(\int_0^{R_i}\frac{r^{\beta+n-1}}{(1+(h_i)^2r^2)^{2n}}dr+o(1)\Big)\to 0\end{aligned}$$

if $n-2<\beta$. Thus for λ_i approaching to 0, the solutions cannot blow up and hence are uniformly bounded. These prove the proposition 1.3.

4.2. **A priori bound for solutions to** (1.2) **in** $\Omega^+_{\delta,\eta}$**.** In the following, we assume (H1)-(H4), (H6), $n=2$, $0\le\lambda\le\Lambda$. We suppose also that *the solutions to* (1.2) *are uniformly bounded on* $\partial\Omega^+_{\delta,\eta}$. These bounds are to be proved in the next section independent of the bounds obtained in this section. We recall the following result from [5].

Theorem 4.15 (Blow-up analysis (Brezis-Merle))**.** *Assume u_n a sequence of solutions to*

$$-\Delta u_n = V_n(x)e^{u_n}\quad \text{in } \Omega \tag{4.18}$$

where Ω is a bounded domain and V_n, u_n satisfy

- (i) $V_n\ge 0$,
- (ii) $\|V_n\|_{L^p(\Omega)}\le C_1$, $\|e^{u_n}\|_{L^{p'}(\Omega)}\le C_2$ *with p ($1<p\le\infty$) and p' conjugate and C_1, C_2 positive.*

Then, there exists a subsequence (u_{n_k}) satisfying the following alternative:

- (i) *Either u_{n_k} is bounded in $L^\infty_{loc}(\Omega)$,*
- (ii) *Or $u_{n_k}\to-\infty$ uniformly in any compact subset of Ω.*
- (iii) *Or the blow up set S (relative to u_{n_k}) is finite non empty and $u_{n_k}\to-\infty$ in Ω/S. In addition, $V_{n_k}e^{u_{n_k}}$ converges in the sense of measures in Ω to $\sum\alpha_i\delta(a_i)$ with $\alpha_i\ge\frac{4\pi}{p'}$, $\forall i$ and $S=\cup_i\{a_i\}$.*

Indeed, arguing by contradiction: Let (λ_n,u_n) a sequence of solutions to (1.2) such that $|\lambda_n|\le\Lambda$ and $\|u_n\|_{L^\infty(\Omega^+_{\delta,\eta})}\to+\infty$. let $V_n:=(\lambda_n u_n e^{-u_n}+\phi(u_n))$ which is clearly bounded in $L^\infty(\Omega^+_{\delta,\eta})$ independently of n. Since $\lambda_n\ge 0$ and using theorem 4.15 and Lemma 2.1, we get the contradiction.

5. A priori bound in a neighborhood of $\partial\Omega$

By assumptions (H2), (H3) and (H4), either $h<0$ or $h>0$ near the boundary of Ω. We distinguish these two cases in the proof.

Consider first that $h<0$ in a neighborhood of $\partial\Omega$. Since a δ neighbourhood of the boundary, $N_\delta(\partial\Omega)\subset\Omega^-_\delta$ for $\delta>0$ small enough and since solutions to (1.1) and (1.2) are uniformly bounded in Ω^-_δ by the results of section 2, solutions to (1.1) and (1.2) are uniformly bounded in a neighborhood of $\partial\Omega$.

Now, let us see the more delicate case: $h>0$ near $\partial\Omega$. We apply different arguments for (1.1) and (1.2). Concerning (1.1), we use a blow up analysis as in [9]. For (1.2), we use the moving plane method as in [13].

5.1. **Blow up analysis near the boundary for** (1.1)**.** We will follow the main steps in [9] to analyse the behaviour of a possible blowing up sequence at boundary $\partial\Omega$. Let $\{\lambda_i, u_i\}$ a sequence of solutions to (1.1). If $\{u_i\}$ is not bounded, then from Proposition 4.1, we can assert the existence of the sets

$$S_i := \{x : x \text{ is a local maximum of } u_i\}$$

and for every C, define $\tilde{S}_i(C)$, a subset of S_i, consisting of points satisfying: $u_i(x) \geq C$ and for any two points $p, q \in \tilde{S}_i(C)$,

$$u_i(p)d(p,q)^{\frac{n-2}{2}} \geq C.$$

Furthermore, we have

$$u_i(x)d(x,\, \tilde{S}_i(C))^{\frac{n-2}{2}} \leq K(C).$$

for a constant $K(C)$ depending on C. We will show that for C large enough,

$$N_\eta(\partial\Omega) \cap \tilde{S}_i(C) = \emptyset, \tag{5.1}$$

where $N_\eta(\partial\Omega)$ is a η-neighborhood of $\partial\Omega$. This will complete the proof in case of (1.1). The proof is carried out as follows:

Step 1 We show that $\tilde{S}_i(C)$ is discrete.

Step 2 $N_\eta(\partial\Omega) \cap \tilde{S}_i(C) = \emptyset$ for $\eta > 0$ small enough and C large enough.

Step 1: Let us define for a fixed constant C, p_i, $q_i \in \tilde{S}_i(C)$ such that (thanks to Proposition 4.1)

$$\begin{gathered} \sigma_i := d(p_i, q_i) = \inf_{p,q \in \tilde{S}_i(C)} d(p,q) \\ d_i := d(p_i, \partial\Omega) \leq d(q_i, \partial\Omega) \\ m_i := u(p_i)^{-\frac{2}{n-2}}. \end{gathered}$$

Arguing by contradiction, we assume that σ_i ,$m_i \to 0$ and p_i tend to p_0, a point on the boundary, when $i \to +\infty$. As in [9], we distinguish three main cases:

(1) $m_i = o(d_i)$ and $\sigma_i = O(d_i)$,
(2) $m_i = o(d_i)$ and $d_i = o(\sigma_i)$,
(3) $d_i = O(m_i)$.

In case 1, we use the rescaling function

$$v_i(x) := \sigma_i^{\frac{n-2}{2}} u_i(\sigma_i x + p_i).$$

By definition of σ_i, v_i has only isolated blow-up points. Moreover, since $\sigma_i = o(d_i)$, we can argue similarly as in subsection 3.1 (see in particular Proposition 4.12) to get that $\{v_i\}$ is bounded. For this, note that if the sequence $\{v_i\}$ is not bounded, then there exists at least two blow up points with finite distance between them.

In case 2, we use the rescaling function

$$v_i(x) := d_i^{\frac{n-2}{2}} u_i(d_i x + p_i).$$

Since $d_i = o(\sigma_i)$ and $m_i = o(d_i)$, we are in the situation of one isolated and simple blow up point in a half space (because $D_i = \frac{\Omega - p_i}{d_i} \to H := \{x, x_n \geq -1\}$ up to some standard geometric transformations). So we cannot use directly the blow up analysis from Section 3.1. Since the proof is similar to section 2.2 in [9], we just sketch the proof: From

$$-\Delta v_i = \lambda_i d_i^2 v_i + \tilde{h} v_i^{\frac{n+2}{n-2}}$$

where $\tilde{h}(x) = h(d_i x + p_i)$, we prove that

$$v_i(x)v_i(0) \to w(x) := a\Big(\frac{1}{|x|^{n-2}} - \frac{1}{|x+e|^{n-2}} + \frac{x_n+1}{2^{n-1}}\Big) \tag{5.2}$$

with $a = \lim \big[\frac{n(n-2)}{h(p_i)}\big]$ and $e = (0, \dots, 2)$.

The contradiction is based on Lemma 4.10 (Pohozaev identity) in $D_i \cap B_R$ where R is large. Indeed, multiplying this pohozaev identity by $v_i(0)^2$, we have

$$\begin{aligned}
&v_i(0)^2 \int_{\partial(B_R\cap D_i)} B(R,x,v_i,\nabla v_i)\\
&= v_i(0)^2\Big\{\lambda_i d_i^2 \int_{B_R\cap D_i} |v_i|^2 - \frac{\lambda_i}{2} d_i^2 \int_{\partial(B_R\cap D_i)} x.\nu v_i^2\Big\}\\
&\quad + v_i(0)^2 d_i \frac{n-2}{2n} \int_{B_R\cap D_i} (x\cdot\nabla h) v^{\frac{2n}{n-2}}\\
&\quad - v_i(0)^2 \frac{n-2}{2n} \int_{\partial(B_R\cap D_i)} x.\nu h(d_i x + p_i) v_i^{\frac{2n}{n-2}},
\end{aligned}$$

where ν is the unit outward normal. Now, when $i \to +\infty$, we have

$$v_i(0)^2 \int_{\partial(B_R\cap D_i)} B(R,x,v_i,\nabla v_i) \to I_R = \int_{\partial(B_R\cap H)} B(R,x,w,\nabla w), \tag{5.3}$$

Moreover, I_R tends to $-\infty$ when $R \to +\infty$ (see [9] p.76). Furthermore,

$$v_i(0)^2 \frac{\lambda_i}{2} d_i^2 \int_{\partial(B_R\cap D_i)} v_i^2 x.\nu \to 0 \tag{5.4}$$

which follows from (5.2) for any R fixed. Now we claim that

$$v_i(0)^2 d_i \frac{n-2}{2n} \int_{B_R\cap D_i} (x\cdot\nabla h) v^{\frac{2n}{n-2}} \to 0. \tag{5.5}$$

We need to consider 2 cases: $\nabla h(p_0) \neq 0$ or $\nabla h(p_0) = 0$. Suppose that $p_i \to p_0$ satisfying $\nabla h(p_0) \neq 0$. First,

$$\begin{aligned}
&d_i \int_{B_{\frac12}} \nabla h(d_i x + p_i) v_i^{\frac{2n}{n-2}}\\
&= \frac{2n}{n-2} \int_{\partial B_{\frac12}} (\frac{\partial v_i}{\partial\nu}\nabla v_i - \frac12|\nabla v_i|^2)\nu + \frac{2n}{n-2}\int_{\partial B_{\frac12}} \frac{\lambda_i}{2} v_i^2 \nu + \int_{\partial B_{\frac12}} h v_i^{\frac{2n}{n-2}} \nu.
\end{aligned} \tag{5.6}$$

Multiplying both sides of (5.6) by $v_i(0)^2$ and using (4.9) to estimate λ_i terms, we get

$$d_i v_i(0)^2 \le C.$$

from which (5.5) follows. If $\nabla h(p_0) = 0$, we argue as in [9] (see p. 75).(5.5) now follows. From (5.3), (5.4), (5.5) and $\lambda_i \ge 0$ we get the contradiction in *case 2*.

Finally, in case 3, we define

$$v_i(x) := m_i^{\frac{n-2}{2}} u_i(m_i x + p_i).$$

Using Proposition 4.1, we see that $\{v_i\}$ is bounded in $B_r(0)$ with r small enough. Then, using results from subsection 3.1 we can show that

(i) either $\{v_i\}$ is weakly convergent to v_0 satisfying

$$\begin{aligned} -\Delta v_0 &= h(p_0) v_0^{\frac{n+2}{n-2}} \\ v_0(0) = 1 \quad & v_0\big|_{\partial H} \equiv 0, \end{aligned} \tag{5.7}$$

where $H := \{x : x_n \geq -b\}$ for some $b > 0$ (since $d_i = O(m_i)$),

(ii) or there exists $\tilde{z}_i \in D_i$ such that $d(\tilde{z}_i, \partial D_i) \to 0$ and $v_i(\tilde{z}_i) \to +\infty$ when $i \to +\infty$.

In the first case, we get the contradiction since there is no solution to (5.7). In the second case, let z_i the preimage of $\tilde{z}_i$ and $r_i = d(z_i, \partial\Omega)$. Using the definition of σ_i, $w_i(x) := u_i(r_i x + p_i)$ has only one possible blow up point in a half space. In this case, we argue as in *case 2* to get the contradiction. If $\{w_i\}$ is bounded, we use again (5.7) to get the contradiction. Thus, we have proved that $\tilde{S}_i$ is discrete in a neighborhood of $\partial\Omega$ for C large.

Step 2: To show that $\tilde{S}_i \cap N_\eta(\partial\Omega) = \emptyset$, we have just to repeat the arguments in case 2 and case 3. This completes the proof of a priori bound in a neighborhood of $\partial\Omega$ for (1.1).

5.2. **A priori bound in a neighborhood of $\partial\Omega$ for** (1.2)**.** Here we deal with (1.2). We get the a priori bound using the moving plane method which implies that solutions to (1.2) are nonincreasing near the boundary. Precisely, we proceed as in [13]. Let (λ, u) be a solution to (1.2). We prove

(1) First an integral estimate:

$$\int_\Omega e^u \phi_1 \leq C \tag{5.8}$$

where ϕ_1 is the first eigenfunction of $-\Delta$ with Dirichlet boundary conditions satisfying $\|\phi_1\|_\infty = 1$.

(2) For any x_0 in $\partial\Omega$,

$$u \leq C \quad \text{in } B_\delta(x_0) \cap \Omega \tag{5.9}$$

where C and δ do not depend on x_0.

Proof of (5.8)*:* Multiplying (1.2) by ϕ_1 and integrating by parts, we get

$$\lambda_1 \int_\Omega u\phi_1 = \lambda \int_\Omega u\phi_1 + \int_\Omega h(x)\phi(u) e^u \phi_1. \tag{5.10}$$

Using that $\{x : h(x) \geq \delta\} = \Omega_\delta^+ \supset N_\epsilon(\partial\Omega)$, where $N_\epsilon(\partial\Omega)$ is a ϵ-neighborhood of $\partial\Omega$, and that

$$u(x) \leq C \quad \text{in } \Omega \backslash (N_\epsilon(\partial\Omega)) \tag{5.11}$$

with C not depending on u, we have

$$\lambda \int_\Omega u\phi_1 + \int_\Omega h(x)\phi(u) e^u \phi_1 \geq (\lambda_1(\Omega) + a) \int_\Omega u\phi_1 - C_0 \tag{5.12}$$

where C_0 and $a > 0$ do not depend on u. From (5.10) and (5.12), we get $\int_\Omega u\phi_1 \leq C$ and using in addition (5.11), we get (5.8).

Proof of (5.9)*:* This is done by moving plane method as in section 3 for points in Γ_δ. We give the details here as some arguments are different. To apply moving plane method in a neighbourhood of x_0, we need that $\partial\Omega \cap B_\delta(x_0)$ is convex. If it is not convex, we make a kelvin transform. Precisely, without generality, we can assume $x_0 = 0$ and the unit outward normal belongs to the x_1-axis. Then, taking

y_0 in the x_1-axis such that $B_{\epsilon_0}(y_0) \cap \Omega = \emptyset$ and $\partial B_{\epsilon_0}(y_0)$ tangent in x_0 to $\partial\Omega$, we define the inversion $I(y_0)$ defining in $\mathbb{R}^2 \backslash \{y_0\}$ as

$$I(y_0) : x \to y = y_0 + \frac{|y_0|^2}{|x - y_0|^2}(x - y_0)$$

and

$$v(x) = u(y_0 + |y_0|^2 \frac{(x - y_0)}{|x - y_0|^2}) \quad \text{for } x \text{ in } \Omega^* = I(y_0)^{-1}(\Omega) \subset B_{\epsilon_0}(y_0)$$

which satisfies

$$-\Delta v = \frac{|y_0|^4}{|x - y_0|^4}(\lambda v + h(y_0 + |y_0|^2 \frac{(x - y_0)}{|x - y_0|^2})\phi(v)e^v).$$

Now, we define

$$f(x, v) = \frac{|y_0|^4}{|x - y_0|^4}(\lambda v + h(y_0 + |y_0|^2 \frac{(x - y_0)}{|x - y_0|^2})\phi(v)e^v).$$

Therefore, since $v = 0$ on $I(y_0)^{-1}(\partial\Omega)$ which is strictly convex near x_0, the moving plane method can be carried out if

$$\frac{\partial f(x, v)}{\partial \nu} \le 0 \tag{5.13}$$

Let

$$\Sigma_\epsilon = \{x \in \Omega^* \mid \operatorname{dist}(x, \partial\Omega^*) \le \epsilon\}.$$

Then, we prove (5.13) in $\Sigma_\epsilon \cap B_{\epsilon_1}(x_0)$ as follows:

$$\begin{aligned}
\frac{\partial f(x, v)}{\partial \nu} &= |y_0|^4(\lambda v + h\phi(v)e^v)\partial_\nu \frac{1}{|x - y_0|^4} \\
&\quad + \frac{|y_0|^4}{|x - y_0|^4}\partial_\nu h(y_0 + |y_0|^2 \frac{(x - y_0)}{|x - y_0|^2})\phi(v)e^v \\
&\le -\frac{C_1}{|x - y_0|^5}|y_0|^4(h\phi(v)e^v) + \frac{C_2\|\nabla h\|_\infty}{|x - y_0|^6}|y_0|^6\phi(v)e^v \\
&= \frac{|y_0|^4}{|x - y_0|^5}(-C_1 h + C_2\|\nabla h\|_\infty \frac{|y_0|^2}{|x - y_0|})\phi(v)e^v \\
&\le \frac{|y_0|^4}{|x - y_0|^5}(-C_1\delta + C(|I(y_0)x - y_0|)\phi(v)e^v \\
&\le \frac{|y_0|^4}{|x - y_0|^5}(-C_1\delta + C(|y_0| + \epsilon + \epsilon_1))\phi(v)e^v \le 0
\end{aligned} \tag{5.14}$$

since $h \ge \delta > 0$, $C_1 > 0$ and for $|y_0|$, ϵ_1, ϵ small enough. Now, we proceed as in [13] (see also [11]): the unit outward directions in $x \in B_{\epsilon_1}(x_0) \cap \partial\Omega^*$ forms a cone centered at $(1, 0)$ with a positive angle θ. Let

$$I = \{\nu \in \mathbb{R}^2 : \nu.(1, 0) \ge |\nu| \cos\theta, |\nu| \le \frac{1}{2}\epsilon_1\}$$

be a piece of cone and $I_x = \{x - \nu \mid \nu \in I\}$. Then by above arguments, we have

$$v(y) \ge v(x) \quad \text{for } x \in B_{\frac{\epsilon_1}{2}}(x_0) \cap \Omega^*, \, y \in I_x \cap \Omega^*.$$

Using (5.8), we get

$$\int_{\Omega^*} f(x, v) \operatorname{dist}(x, \partial\Omega^*) dx \le C.$$

Therefore, for $x \in B_{\frac{\epsilon_1}{2}}(x_0) \cap \Omega^*$,

$$\begin{aligned} C &\geq \int_{\Omega^*} f(y,v)\, \mathrm{dist}(y, \partial\Omega^*) dy \\ &\geq k \int_{I_x} \phi(v(y)) e^{v(y)}\, \mathrm{dist}(y, \partial\Omega^*) dy \\ &\geq k_1 \phi(v(x)) e^{v(x)} \end{aligned}$$

since $\phi(\cdot)e^{\cdot}$ is increasing. Thus, $v(x) \leq C$ for $x \in B_{\frac{\epsilon_1}{2}}(x_0) \cap \Omega^*$ which implies (5.9) for some δ depending on $|y_0|$. The proof is now completed for solutions to (1.2).

6. Existence of solutions to (1.1) and (1.2)

Proof of Theorem 1.1 and 1.2. First, note that from previous sections, we have uniform a priori bound of solutions to (1.1) in $[0, \Lambda] \times L^\infty(\Omega)$ with $\Lambda < \lambda_1(\Omega^0)$. From assumption (H4)-3 and bootstrap arguments, we have that any solution (λ, u) to (1.1) such that $0 \leq \lambda \leq \lambda_1(\Omega^+)$ satisfies:

$$\|u\|_{C_0^1(\Omega)} \leq C. \tag{6.1}$$

Multiplying (1.1) by $\phi^1_{\Omega^+}$ and integrating by parts, we get that any non trivial solution (λ, u) satisfies $\lambda \leq \lambda_1(\Omega^+)$. Thus, (6.1) is also true for any nontrivial solution (λ, u) with $\lambda \geq 0$.

Now, the existence of $\mathcal{C}$ follows from global bifurcation theory of Rabinowitz (see [17]). Furthermore, from (6.1), $\mathcal{C}$ reaches $\{\lambda = 0\}$ and $\Pi_{\mathbb{R}}\mathcal{C} = [0, \lambda*]$ with $\lambda* < \lambda_1(\Omega^+)$. This completes the proof of 1-(i) and 2-(i). To prove 1-(ii) and 2-(ii), we have just to prove that the bifurcation from $\lambda_1(\Omega)$ is supercritical (i.e. goes towards the right). It implies that $\lambda* > \lambda_1(\Omega)$. To get the supercritical branching, we use standard arguments: From $\int_\Omega h {\phi^1_\Omega}^{\frac{2n}{n-2}} < 0$ and the main result from [12], we show that the unique curve of solutions to (1.1) emanating from $\lambda_1(\Omega)$ is defined only for $\lambda > \lambda_1(\Omega)$ (see [21] for more details). To prove that the solution $(0, u_0)$ is such that $u_0 > 0$, we can follow the arguments from the proof of Theorem 1.2, section 6 of [14]. This completes the proof of Theorem 1.1 and 1.2. $\square$

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