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STATIONARY RADIAL SOLUTIONS FOR A QUASILINEAR CAHN-HILLIARD MODEL IN N SPACE DIMENSIONS

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Abstract. We study the Neumann boundary value problem for stationary radial solutions of a quasilinear Cahn-Hilliard model in a ball $B_R(\mathbf{0})$ in $\mathbb{R}^N$. We establish new results on the existence, uniqueness, and multiplicity (by "branching") of such solutions. We show striking differences in pattern formation produced by the Cahn-Hilliard model with the p-Laplacian and a $C^{1,\alpha}$ potential ($0 < \alpha \leq 1$) in place of the regular (linear) Laplace operator and a C^2 potential. The corresponding energy functional exhibits one-dimensional continua ("curves") of critical points as opposed to the classical case with the Laplace operator. These facts offer a different explanation of the "slow dynamics" on the attractor for the dynamical system generated by the corresponding time-dependent parabolic problem.

1. Introduction

The *Cahn-Hilliard equation* is one of the well-known models for phase transitions in a material with two phases, such as glass, metal alloys, and polymers. One observes a material in the state of melting; a binary mixture having temperature at which both phases can coexist. The model we treat in our present article is a generalization of the classical model discovered by J. W. Cahn and J. E. Hilliard [7] half a century ago. This model, in its full generality, may be written as

$$u_t = \Delta\left[-\varepsilon^p\,\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) + W'(u)\right] \quad \text{for } (x,t)\in\Omega\times(0,\infty)\,, \tag{1.1}$$

subject to the Neumann (i.e., no-flux) boundary conditions

$$\begin{gathered} |\nabla u|^{p-2}(\boldsymbol{\nu}\cdot\nabla)u = (\boldsymbol{\nu}\cdot\nabla)\left[-\varepsilon^p\,\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) + W'(u)\right] = 0 \\ \text{at } x\in\partial\Omega \text{ for } t>0\,, \end{gathered} \tag{1.2}$$

where $1 < p < \infty$, $\varepsilon > 0$, and $W : \mathbb{R} \to \mathbb{R}$ is a given potential function of class C^1 whose first derivative W' might be *only* continuous (or Hölder-continuous at most). The material occupies a bounded domain $\Omega \subset \mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$. As usual, the vector field $\boldsymbol{\nu} \in \partial\Omega \to \mathbb{R}^N$ denotes the unit outer normal to the boundary of Ω. We refer to the monograph by Temam [19], Chapt. III, §4.2, pp. 147–158, for a weak formulation of this initial-boundary value

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problem in the semilinear case $p = 2$. The novelty in the work reported here is that we allow $p \neq 2$ and W does not have to be of class C^2 or even smoother (of class C^3 or C^4 assumed in [1, 8, 12]). This means that we consider also singular or degenerate diffusion which corresponds to $1 < p < 2$ or $2 < p < \infty$, respectively. We abbreviate by $\Delta_p u \stackrel{\text{def}}{=} \nabla \cdot \big(|\nabla u|^{p-2}\nabla u\big)$ the well-known p-Laplace operator; of course, $\Delta_2 \equiv \Delta$ is the (linear) Laplace operator. We will consider Δ_p with the (homogeneous) Neumann boundary conditions $(\boldsymbol{\nu}\cdot\nabla)u = 0$ on $\partial\Omega$ throughout this article.

Clearly, if W is of class C^2 then the boundary conditions (1.2) are equivalent with the Navier boundary conditions

$$(\boldsymbol{\nu}\cdot\nabla)u = (\boldsymbol{\nu}\cdot\nabla)(\Delta_p u) = 0 \quad \text{at } x \in \partial\Omega \text{ for } t > 0\,. \tag{1.3}$$

The classical choice of W is the double-well potential $W(s) = (1-s^2)^2$ for $s \in \mathbb{R}$ which attains global minimum at two points, $s_1 = -1$ and $s_2 = 1$ (see Cahn and Hilliard [7], Gunton and Droz [14], and Langer [15]). These points of minimum are nondegenerate, with $W'(\pm 1) = 0$ and $W''(\pm 1) = 8 > 0$. This hypothesis gives us an entirely different behavior of the stationary solutions satisfying

$$-\varepsilon^p\,\Delta_p u + W'(u) = 0, \quad x \in \Omega; \tag{1.4}$$

$$(\boldsymbol{\nu}\cdot\nabla)u = 0, \quad x \in \partial\Omega, \tag{1.5}$$

for the classical linear diffusion ($p = 2$) and the degenerate nonlinear diffusion ($p > 2$). The latter case exhibits a much greater variety of these stationary solutions. Notice that, in this case, $W'(s) = 4s(s^2-1)$ for $s \in \mathbb{R}$. On the other hand, one can observe the same phenomenon for the classical linear diffusion if the potential W is modified to $W(s) = |1-s^2|^\alpha$ for $s \in \mathbb{R}$, where α is a constant, $1 < \alpha < 2$. In the work reported here we focus on problem (1.4), (1.5) with arbitrary $p, \alpha > 1$. Note that this is the boundary value problem for all *stationary solutions* of the so-called (generalized) *bi-stable equation*

$$u_t = \varepsilon^p\,\Delta_p u - W'(u) \quad \text{for } (x,t) \in \Omega \times (0,\infty)\,, \tag{1.6}$$

subject to the boundary conditions

$$(\boldsymbol{\nu}\cdot\nabla)u = 0 \quad \text{at } x \in \partial\Omega \text{ for } t > 0\,. \tag{1.7}$$

The term "generalized" refers to allowing $p \in (1,\infty)$ rather than setting $p = 2$ (the classical semilinear equation with the linear Laplace operator).

The stationary problem (1.4), (1.5) is rather difficult to solve in an arbitrary bounded domain $\Omega \subset \mathbb{R}^N$ even for $p = \alpha = 2$. Besides the two constant solutions $u \equiv \pm 1$ in Ω, it may exhibit various other nonconstant solutions describing transition layers between the two phases; see, e.g., Alikakos and Fusco [2, 3] and Bates and Fusco [4]. Therefore, throughout this article, we restrict ourselves to the case of *radially symmetric* solutions of the Neumann boundary value problem (1.4), (1.5) in a ball of radius R ($0 < R < \infty$),

$$\Omega = B_R(\mathbf{0}) = \{x \in \mathbb{R}^N : |x| < R\}\,,$$

but with any $p \in (1,\infty)$. Notice that, after replacing ε ($\varepsilon > 0$) by ε/R, we may (and sometimes will) assume $R = 1$ without loss of generality. Equivalently, setting $u(x) = u(|x|)$ with $r = |x|$ for

$$x \in \overline{B_R(\mathbf{0})} = \{x \in \mathbb{R}^N : |x| \le R\}\,,$$

we consider the (one-dimensional) two-point boundary value problem for the ordinary differential equation

$$-\varepsilon^p r^{-(N-1)} \left(r^{N-1} |u_r|^{p-2} u_r\right)_r + W'(u) = 0, \quad 0 < r < R, \tag{1.8}$$

subject to the Neumann boundary conditions

$$u_r(0) = u_r(R) = 0. \tag{1.9}$$

In the work reported here we focus on problem (1.8), (1.9) with arbitrary $p, \alpha > 1$ and even with a potential W having a more general form then just $W(s) = (1-s^2)^\alpha$ for $s \in \mathbb{R}$. We will see that for $N \geq 2$, problem (1.8), (1.9) is *quite different* from the one-dimensional case ($N = 1$) treated in the recent work of Drábek, Manásevich, and Takáč [11].

In one space dimension, i.e., when $\Omega \subset \mathbb{R}$ is a bounded open interval, the semilinear case $p = 2$ with a sufficiently smooth potential W (of class at least C^3, but mostly C^4) has been extensively investigated in the works of Alikakos, Bates, and Fusco [1], Carr and Pego [8], Fusco and Hale [12], and many others, mostly in the context of the gradient flow determined by the initial-boundary value problem for the bi-stable equation (1.6) subject to the Neumann boundary conditions (1.7).

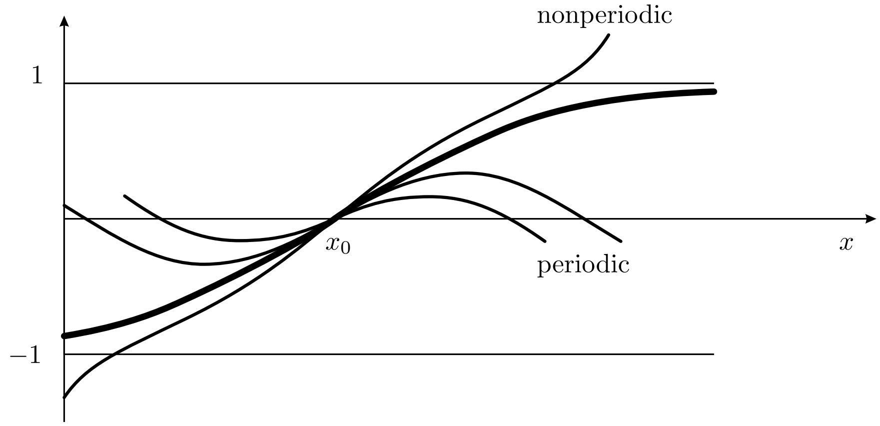

FIGURE 1. $1 < p \leq \alpha < +\infty$, $W(s) = (1 - s^2)^2$.

The following facts are known about the case $N = 1$ and $\Omega = (0, 1)$, among other numerous interesting results. If $p = \alpha = 2$, the only solutions of problem (1.4), (1.5) (i.e., problem (1.8), (1.9) where $N = 1$) are the constant solutions $u \equiv -1$, $u \equiv 0$, and $u \equiv 1$, and nonconstant solutions that can be extended to periodic functions on $\mathbb{R}$ (depending on the size of $\varepsilon > 0$) which always satisfy $-1 < u(x) < 1$ for all $x \in [0, 1]$; see [8] and [12]. A later work in [1] contains a more detailed analysis of these solutions, including numerical simulations. In a recent work, Drábek, Manásevich, and Takáč [11] have shown that the set of all solutions is qualitatively the same whenever $1 < p \leq \alpha < \infty$, cf. Figure 1. In contrast, if $1 < \alpha < p < \infty$, the structure of this set is much richer and becomes more complicated as $\varepsilon \searrow 0$, cf. Figure 2. As shown in [11], this phenomenon is a result of the loss of uniqueness in the initial value problem for the first integral

$$\frac{p-1}{p} \varepsilon^p |u_x(x)|^p - W(u(x)) = \text{const}, \quad 0 \leq x \leq 1, \tag{1.10}$$

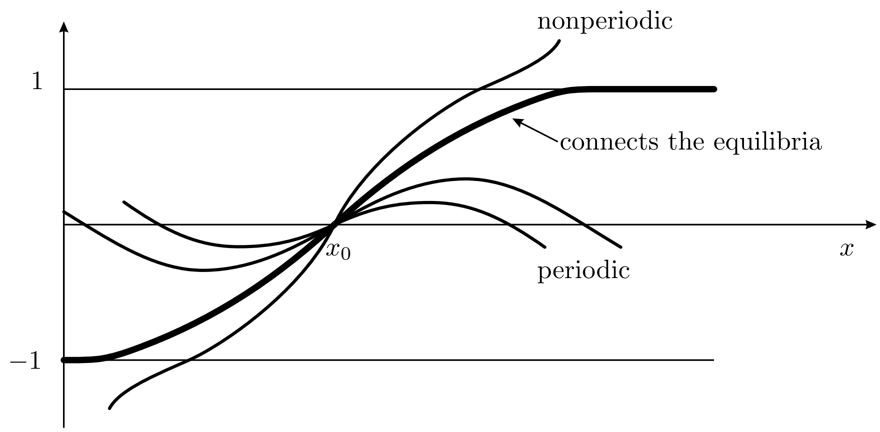

FIGURE 2. $1 < \alpha < p < +\infty$, $W(s) = (1 - s^2)^2$.

of (1.4). If $p = \alpha = 2$, functions similar to the solutions for the case $1 < \alpha < p < \infty$ have been used to explain the "slow dynamics" on the attractor for the time-dependent problem (1.6), (1.7); see e.g. [1, 2, 3, 8, 12]. One of the main contributions of the work in [11] is the fact that for $1 < \alpha < p < \infty$, the simple form of all stationary solutions to problem (1.6), (1.7) provides a rather simple explanation for the *slow dynamics* on the attractor in this time-dependent problem. This result suggests that one should consider a more general type of (nonlinear) diffusion and/or more general behavior of the potential W near its points of minimum. Such a model seems to have a somewhat different dynamical behavior on the attractor than classical semilinear models studied so far which are typically represented by the Cahn-Hilliard or bi-stable equation. It also has the following interesting features:

- The initial-boundary value problem (1.6), (1.7), with prescribed initial values in $W^{1,p}(0,1)$ at $t = 0$, has a unique weak solution for $\alpha \geq 2$ and $p > 1$.
- The boundary value problem (1.4), (1.5) exhibits continua of (multiple) nonconstant solutions for $1 < \alpha < p < \infty$ and $\varepsilon > 0$ small enough. Consequently, the functional
$$\mathcal{J}_\varepsilon(u) \stackrel{\text{def}}{=} \int_0^1 \Big(\frac{\varepsilon^p}{p} \, |u_x|^p + W(u) \Big) \, \mathrm{d}x, \quad u \in W^{1,p}(0,1), \tag{1.11}$$
representing the total free energy, has a much richer structure of the set of critical points than for $1 < p \leq \alpha < \infty$.

For $N \geq 2$ we investigate the solutions of the boundary value problem (1.8), (1.9) in the phase plane (ξ, η) where $\xi = u$ and $\eta = |u_r|^{p-2} u_r$. As $N \geq 2$, we cannot take advantage of the first integral (1.10) anymore, because the function
$$r \longmapsto \frac{p-1}{p} \, \varepsilon^p \, |u_x(r)|^p - W(u(r)) : (0, R) \to \mathbb{R}$$
is no longer independent of the variable $r \in (0, R)$. Nevertheless, we will take advantage of the fact that this function is *monotonically decreasing*, cf. eqs. (2.7) and (2.8) in the next section (Section 2). We will be able to provide the phase plane portrait and the description of the set of all solutions to problem (1.8), (1.9). A

typical feature of equation (1.8), when considered for $r \in \mathbb{R}_+$ with prescribed initial data $u(r_0) = \pm 1$ and $u_r(r_0) = 0$ at some point $r_0 \in \mathbb{R}_+$, is the nonuniqueness of solutions to this initial value problem for $1 < \alpha < p < \infty$; see Part (IV) of Theorems 3.4 (for $W(s) = |1 - s^2|^\alpha$) and 3.7 (for $W(s)$ general) in Section 3.

This article is organized as follows. The main hypotheses, notation, and some preliminaries are given in Section 2. Our main results (for $N \geq 2$) are stated in Section 3: in §3.1 for the special potential $W(s) = |1-s^2|^\alpha$ (Theorem 3.1 for $p \leq \alpha$ and Theorem 3.4 for $p > \alpha$) and in §3.2 for a general potential $W(s)$ (Theorem 3.5 for $p \leq \alpha$ and Theorem 3.7 for $p > \alpha$). In order to prove these theorems, we need rather technical results on local uniqueness and nonuniqueness (and existence, as well) of solutions to the initial value problem for the ordinary differential equation (1.8) starting from an arbitrary initial point $r_0 \in \mathbb{R}_+ = [0, \infty)$. We prove these results in Section 4. The proofs of our theorems need also some global existence (and uniqueness) results for this initial value problem, which are proved in Section 5. Finally, the proofs of our main results are completed in Section 6. The appendix (Appendix 7) contains an auxiliary lemma on a comparison of weighted averages.

2. Hypotheses, notation, and preliminaries

Throughout this article we assume that $W : \mathbb{R} \to \mathbb{R}$ is a C^1 function with $W(s) \to +\infty$ as $|s| \to \infty$. Furthermore, we assume

Hypotheses.

(H1) If $s_0 \in \mathbb{R}$ is a critical point of W (i.e., $W'(s_0) = 0$), than either
 (a) W attains a local maximum at s_0, or else
 (b) W attains a local minimum at s_0 and, moreover, W is convex in an open interval containing s_0 and there exist constants $\alpha > 1$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $\zeta > 0$, such that

$$\gamma_1 |s - s_0|^\alpha \leq W(s) - W(s_0) \leq \gamma_2 |s - s_0|^\alpha \quad \text{for all } s \in (s_0 - \zeta, s_0 + \zeta)\,. \tag{2.1}$$

(H2) If $1 < \alpha < p$ in (H1), Part (b) above, then we require that both limits

$$c_{s_0+} \stackrel{\text{def}}{=} \lim_{s \to s_0+} \Big(|s - s_0|^{1-(\alpha/p)} \, \frac{\mathrm{d}}{\mathrm{d}s} \big[(W(s) - W(s_0))^{1/p} \big] \Big)\,, \tag{2.2}$$

$$c_{s_0-} \stackrel{\text{def}}{=} - \lim_{s \to s_0-} \Big(|s - s_0|^{1-(\alpha/p)} \, \frac{\mathrm{d}}{\mathrm{d}s} \big[(W(s) - W(s_0))^{1/p} \big] \Big) \tag{2.3}$$

exist and satisfy $c_{s_0+}, c_{s_0-} \in (0, \infty)$.

To simplify our notation, we begin with a normalization of the (radial) stationary equation (1.8). Replacing the variable r by $\tilde{r} = \varepsilon^{-1} r$ and dropping the tilde in $\tilde{r}$ we arrive at

$$-r^{-(N-1)} \left(r^{N-1} \, |u'|^{p-2} u' \right)' + W'(u) = 0 \quad \text{for } 0 < r < \infty, \tag{2.4}$$

where $' \equiv \frac{\mathrm{d}}{\mathrm{d}r}$ stands for the radial space derivative. This equation is equivalent to the first-order system

$$u' = |v|^{p'-2} v, \quad v' = -\frac{N-1}{r}\, v + W'(u) \quad \text{in } (0, \infty), \tag{2.5}$$

where $p' = p/(p-1)$ denotes the conjugate exponent of p. *Trajectories* for the differential equation (2.4) in the phase plane (ξ, η) are (continuous) parametric curves $(\xi, \eta) = (u(r), v(r))$, which are parametrized by $r \in J$ from a nondegenerate interval $J \subset \mathbb{R}_+$, such that (u, v) is a solution of (2.5) in J. As usual, we have denoted $\mathbb{R}_+ = [0, \infty)$.

Notice that if $N = 1$ then system (2.5) has the first integral (conservation law)

$$\frac{1}{p'} |v|^{p'} = W(u) - C \quad \text{in } \mathbb{R}, \tag{2.6}$$

where $C \in \mathbb{R}$ is a constant. This fact was exploited in the work of Drábek, Manásevich, and Takáč [11] in an essential way. For $N \geq 2$ we need to replace the first integral by the equation

$$\frac{1}{p'} |v|^{p'} = W(u) - Z(r) \quad \text{for } r \in \mathbb{R}_+, \tag{2.7}$$

where $Z : \mathbb{R}_+ \to \mathbb{R}$ is a C^1 function that satisfies

$$Z'(r) = \frac{N-1}{r} |v|^{p'} \quad \text{for } r > 0. \tag{2.8}$$

The last equation for Z' is easily obtained by first differentiating (2.7) with respect to the variable r and then applying (2.5). We will make essential use of the fact that Z is a monotonically increasing function. The shifts of the potential W by a constant C in Figure 3 suggest the behavior of a trajectory $(\xi, \eta) = (u(r), v(r))$ parametrized by $r \in J$, such that (u, v) is a solution of (2.5) in J.

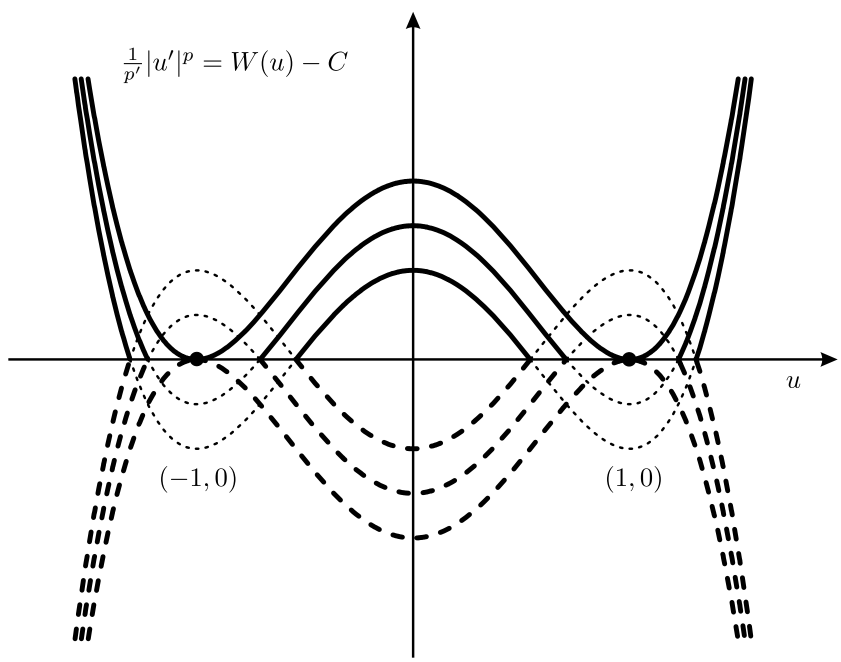

FIGURE 3. Shifts of the potential $W(s) = |1 - s^2|^\alpha$ by a constant C.

Moreover, if $u(0) = s_0$ is a local minimizer for W and $v(0) = 0$, then also the functions

$$r \mapsto r^{-1} |v(r)| \quad \text{and} \quad r \mapsto r^{-p'} (Z(r) - Z(0)) : (0, \delta) \to \mathbb{R}_+$$

are monotonically increasing, for some $\delta > 0$ small enough, provided (2.7) and (2.8) hold for $0 < r < \delta$. (Here, we take advantage of W being convex in an open interval containing s_0.) In particular, from these facts we will derive the following important inequalities,

$$\frac{1}{N}\left[W(u(r)) - W(u(0))\right] \le W(u(r)) - Z(r) \le W(u(r)) - W(u(0)) \tag{2.9}$$

for all $r \in [0, \delta)$, where $Z(0) = W(u(0)) = W(s_0)$. These inequalities will enable us to apply the same simple method that has been used in [11, Section 3] (and even earlier in Díaz and Hernández [10]) for $N = 1$ with the first integral (2.6) where $C = W(s_0)$, owing to the following simple consequence: Applying $u' = |v|^{p'-2}v$ and (2.9) to (2.7), we arrive at

$$\frac{p'}{N}\left[W(u(r)) - W(u(0))\right] \le |u'(r)|^p \le p'\left[W(u(r)) - W(u(0))\right] \tag{2.10}$$

for all $r \in [0, \delta)$. This means nothing else than the *uniqueness* or *nonuniqueness* of a local solution u to the initial value problem for equation (2.4) with the initial conditions $u(0) = s_0$ (where s_0 is a local minimizer for W) and $u'(0) = 0$ at $r = 0$, depending on whether the integral

$$\int_{s_0}^{s_0+\zeta} |W(s) - W(s_0)|^{-1/p}\,\mathrm{d}s \tag{2.11}$$

is *infinite* (forcing uniqueness) or *finite* (forcing nonuniqueness), respectively. Now thanks to (2.1), this alternative corresponds to whether $p \le \alpha$ (infinite integral) or $p > \alpha$ (finite integral). Hence, in the former case (i.e., when the integral is infinite) one gets $u(r) = s_0$ for every $r \in [0, \delta)$ which implies uniqueness. As a canonical example for both cases, one may take $W(s) = |1 - s^2|^\alpha$ for $s \in \mathbb{R}$, $\alpha > 1$, and $s_0 = \pm 1$.

3. Main results for $N \ge 2$

We assume $N \ge 2$. (An interested reader is referred to [11] for the case $N = 1$.) We formulate our main results first for the special case $W(s) = |1 - s^2|^\alpha$, $s \in \mathbb{R}$, where $\alpha > 1$ is a constant, and then for the general case when W satisfies Hypotheses (H1) and (H2) stated at the beginning of the previous section (Section 2).

3.1. The special potential $W(s) = |1 - s^2|^\alpha$.

Throughout this paragraph we take $W(s) = |1 - s^2|^\alpha$ for $s \in \mathbb{R}$. We begin with a generalization of the semilinear case $p = \alpha = 2$.

Theorem 3.1. *Assume $1 < p \le \alpha < \infty$ and let $\varepsilon > 0$ and $\theta \in \mathbb{R}$.*

(I) *Assume $|\theta| \le 1$. Then the initial value problem*

$$-\varepsilon^p\, r^{-(N-1)}\left(r^{N-1}\,|u'|^{p-2}u'\right)' + W'(u) = 0, \quad 0 < r < \infty, \tag{3.1}$$

$$u(0) = -\theta, \quad u'(0) = 0, \tag{3.2}$$

has a unique (global) solution $u \in C^1(\mathbb{R}_+)$ with $|u'|^{p-2}u' \in C^1(\mathbb{R}_+)$. In particular, if $\theta \in \{-1, 0, 1\}$ then $u \equiv -\theta$ is a constant function. If $0 < |\theta| < 1$ then the solution u satisfies $|u(r)| < |\theta|$ for every $r > 0$.

(II) *Now assume $|\theta| > 1$. Then the initial value problem* (3.1), (3.2) *has a unique solution $u \in C^1([0, R))$ with $|u'|^{p-2}u' \in C^1([0, R))$ defined on a maximal interval of existence $[0, R)$ for some $R \equiv R(\varepsilon, \theta) > 0$. This solution satisfies $\theta\, u'(r) < 0$ for all $r \in (0, R)$.*

(III) *Finally, let* $0 < |\theta| < 1$ *and fix any* $R \in (0, \infty)$. *In addition, assume* $p \ge \frac{2N}{N+1}$. *Then the (unique global) solution* $u : \mathbb{R}_+ \to \mathbb{R}$ *of the initial value problem* (3.1), (3.2) *satisfies* $u'(R) = 0$ *if and only if* $\varepsilon = \varepsilon_n \equiv \varepsilon_n(\theta, R)$ *for some* $n \in \mathbb{N}$, *where* $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$ (> 0) *is a (strictly decreasing) sequence of "nonlinear eigenvalues" for the Neumann boundary value problem* (1.8), (1.9). *Moreover,* $\varepsilon_n \to 0$ *as* $n \to \infty$.

Of course, $\mathbb{N} = \{1, 2, 3, \dots\}$. In order to treat the case $1 < \alpha < p < \infty$, we need the following lemma. This lemma is an anlogue of [11, Lemma 3.4] which was established there for $N = 1$.

Lemma 3.2. *Assume* $1 < \alpha < p < \infty$ *and let* $\varepsilon > 0$ *and* $r_0 \in \mathbb{R}_+$. *Then the initial value problem*

$$-\varepsilon^p \, r^{-(N-1)} \left(r^{N-1} \, |u'|^{p-2} u' \right)' + W'(u) = 0, \quad 0 < r < \infty, \tag{3.3}$$

$$u(r_0) = -1, \quad u'(r_0) = 0, \tag{3.4}$$

possesses a unique pair of (local) solutions

$$U_+ : J_+ \to [-1, -1 + \zeta) \quad \text{and} \quad U_- : J_- \to [-1, -1 - \zeta)$$

with the following properties, where we use the sign symbol $\nu = \pm$ *in* U_ν, J_ν, *etc.:*

- (i) $\zeta > 0$ *is a sufficiently small number and* $J_\nu = (r_0 - \vartheta_\nu, r_0 + \vartheta_\nu) \cap \mathbb{R}_+$ *is a relatively open interval in* $\mathbb{R}_+$, *where* $\vartheta_\nu > 0$ *is some number (small enough, depending on* ζ*).*
- (ii) $-1 < U_+(r) < -1 + \zeta$ *holds for every* $r \in J_+ \setminus \{r_0\}$, *whereas* $-1 - \zeta < U_-(r) < -1$ *for every* $r \in J_- \setminus \{r_0\}$, *respectively.*
- (iii) *The function* U_ν *satisfies eq.* (3.3) *in the interval* J_ν *together with the initial conditions* (3.4).

An analogous result is valid if the first initial condition in (3.4), $u(r_0) = -1$, *is replaced by* $u(r_0) = 1$ *in which case property* (ii) *has to be replaced by*

- (ii′) $1 < U_+(r) < 1 + \zeta$ *holds for every* $r \in J_+ \setminus \{r_0\}$, *whereas* $1 - \zeta < U_-(r) < 1$ *for every* $r \in J_- \setminus \{r_0\}$, *respectively.*

Remark 3.3. The conclusion of Lemma 3.2 actually means *nonuniqueness* for the initial value problem (3.3), (3.4). The graphs of the three (local) solutions, U_+, U_-, and $u \equiv -1$ (the constant solution), touch each other only at the initial point $r = r_0$, all of them with vanishing first derivative at $r = r_0$.

Lemma 3.2 forces the following changes in Part (I) of Theorem 3.1. The "degenerate case" $\theta = \pm 1$ is singled out as Part (IV) below.

Theorem 3.4. *Assume* $1 < \alpha < p < \infty$ *and let* $\varepsilon > 0$ *and* $\theta \in \mathbb{R}$.

(I) *Assume* $|\theta| < 1$. *Then the conclusion of* Part (I) of Theorem 3.1 *remains valid: The initial value problem* (3.1), (3.2) *has a unique (global) solution* $u \in C^1(\mathbb{R}_+)$ *with* $|u'|^{p-2} u' \in C^1(\mathbb{R}_+)$. *In particular, if* $\theta = 0$ *then* $u \equiv 0$ *is a constant function. If* $\theta \ne 0$ *then the solution* u *satisfies* $|u(r)| < |\theta|$ *for every* $r > 0$ *and, moreover, both* $u(r) \to 0$ *and* $u'(r) \to 0$ *as* $r \to \infty$.

(II) Part (II) of Theorem 3.1 *is valid (with* $|\theta| > 1$ *being assumed).*

(III) *Also* Part (III) of Theorem 3.1 *remains valid (with* $0 < |\theta| < 1$ *and* $R \in (0, \infty)$*). Again, also* $p \ge \frac{2N}{N+1}$ *is assumed.*

(IV) *Finally, let* $\theta = -1$, *the case* $\theta = 1$ *being analogous. Then every solution* $u \in C^1([0, R))$, *with* $|u'|^{p-2} u' \in C^1([0, R))$, *of the initial value problem* (3.1), (3.2) *defined on a maximal interval of existence* $[0, R)$, *for some* $R \equiv R(\varepsilon) > 0$, *must take one of the following three forms: either* $u \equiv -1$ *on the whole of* $\mathbb{R}_+$, *or else for* $\nu = \pm$,

$$u(r) = \begin{cases} -1 & \text{if } 0 \le r \le r_0; \\ U_\nu(r) & \text{if } r_0 \le r < R, \end{cases} \tag{3.5}$$

where $r_0 \ge 0$ *is some number and the continuation from the interval* $[r_0, r_0 + \vartheta_\nu)$ *to* $[r_0, R)$ *of the solution* U_ν *obtained in* Lemma 3.2 *is unique. Furthermore, the solution* $u(r) = U_+(r)$ *continues to exist for all* $r \ge r_0$ *and satisfies* $|u(r)| < 1$ *for every* $r > r_0$ *(i.e.,* $R = \infty$ *also in this case); it is unique for* $r > r_0$.

3.2. **A general potential** $W(s)$**.** The results from the previous paragraph (§3.1) are valid for a wider class of potentials than just $W(s) = |1 - s^2|^\alpha$ ($s \in \mathbb{R}$) considered in §3.1. Throughout this paragraph we assume that the potential W satisfies Hypotheses (H1) and (H2) stated at the beginning of Section 2.

Theorem 3.1 can be generalized as follows.

Theorem 3.5. *Assume* $1 < p \le \alpha < \infty$ *and let* $\varepsilon > 0$ *and* $\theta \in \mathbb{R}$. *In addition, assume that* W *is even about zero (i.e.,* $W(s) = W(-s)$ *for every* $s \in \mathbb{R}$*) and satisfies* $W'(0) = W'(S) = 0$ *for some* $0 < S < \infty$, $W'(s) = -W'(-s) < 0$ *for all* $s \in (0, S)$, *and*

(a) *there exist constants* $\beta > 0$, $0 < \hat{\gamma}_1 \le \hat{\gamma}_2 < \infty$, *and* $\hat{\zeta} \in (0, S)$, *such that*

$$\hat{\gamma}_1 s^\beta \le -W'(s) = W'(-s) \le \hat{\gamma}_2 s^\beta \quad \text{whenever } 0 \le s \le \hat{\zeta}\,, \tag{3.6}$$

together with Hypothesis (H1), Part (b), *that is,*

(b) W *is convex in an open interval containing* S *and there exist constants* $\alpha > 1$, $0 < \gamma_1 \le \gamma_2 < \infty$, *and* $\zeta \in (0, S)$, *such that*

$$\gamma_1 |s - S|^\alpha \le W(s) - W(S) \le \gamma_2 |s - S|^\alpha \quad \text{for all } s \in (S - \zeta, S + \zeta)\,. \tag{3.7}$$

Then the following statements hold.

(I) *Assume* $|\theta| \le S$. *Then the initial value problem* (3.1), (3.2) *has a unique (global) solution* $u \in C^1(\mathbb{R}_+)$ *with* $|u'|^{p-2}u' \in C^1(\mathbb{R}_+)$. *In particular, if* $\theta \in \{-S, 0, S\}$ *then* $u \equiv -\theta$ *is a constant function. If* $0 < |\theta| < S$ *then the solution* u *satisfies* $|u(r)| < |\theta|$ *for every* $r > 0$.

(II) *Now assume* $S < |\theta| < S + \zeta$. *Then the initial value problem* (3.1), (3.2) *has a solution* $u \in C^1([0, R))$ *with* $|u'|^{p-2}u' \in C^1([0, R))$ *defined on a maximal interval of existence* $[0, R)$ *for some* $R \equiv R(\varepsilon, \theta) > 0$. *This solution is unique in some subinterval* $[0, \delta) \subset [0, R)$, *where* $0 < \delta \le R$, *and satisfies* $\theta\, u'(r) < 0$ *for all* $r \in (0, \delta)$. *Moreover, one can take* $\delta = R$ *if* $\theta\, u'(r) < 0$ *holds for all* $r \in (0, R)$.

(III) *Finally, let* $0 < |\theta| < S$ *and fix any* $R \in (0, \infty)$. *In addition, assume* $p \ge \frac{(1+\beta)N}{N+\beta}$. *Then the (unique global) solution* $u : \mathbb{R}_+ \to \mathbb{R}$ *of the initial value problem* (3.1), (3.2) *satisfies* $u'(R) = 0$ *if and only if* $\varepsilon = \varepsilon_n \equiv \varepsilon_n(\theta, R)$ *for some* $n \in \mathbb{N}$, *where* $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$ (> 0) *is a (strictly decreasing) sequence of "nonlinear eigenvalues" for the Neumann boundary value problem* (1.8), (1.9). *Moreover,* $\varepsilon_n \to 0$ *as* $n \to \infty$.

In order to treat the case $1 < \alpha < p < \infty$, we need the following lemma. This lemma is an analogue of [11, Lemma 3.4] which was established there for $N = 1$.

Lemma 3.6. *Let $u_0 \in \mathbb{R}$ be a local minimizer for W. Assume $1 < \alpha < p < \infty$ and let $\varepsilon > 0$ and $r_0 \in \mathbb{R}_+$. Then the initial value problem*

$$-\varepsilon^p r^{-(N-1)} \left(r^{N-1} |u'|^{p-2} u'\right)' + W'(u) = 0, \quad 0 < r < \infty, \tag{3.8}$$

$$u(r_0) = u_0, \quad u'(r_0) = 0, \tag{3.9}$$

possesses a unique pair of (local) solutions

$$U_+ : J_+ \to [u_0, u_0 + \zeta) \quad \text{and} \quad U_- : J_- \to [u_0, u_0 - \zeta)$$

with the following properties, where we use the sign symbol $\nu = \pm$ in U_ν, J_ν, etc.:

(i) *$\zeta > 0$ is a sufficiently small number and $J_\nu = (r_0 - \vartheta_\nu, r_0 + \vartheta_\nu) \cap \mathbb{R}_+$ is a relatively open interval in $\mathbb{R}_+$, where $\vartheta_\nu > 0$ is some number (small enough, depending on ζ).*

(ii) *$u_0 < U_+(r) < u_0 + \zeta$ holds for every $r \in J_+ \setminus \{r_0\}$, whereas $u_0 - \zeta < U_-(r) < u_0$ for every $r \in J_- \setminus \{r_0\}$, respectively.*

(iii) *The function U_ν satisfies eq.* (3.8) *in the interval J_ν together with the initial conditions* (3.9).

Lemma 3.6 forces the following changes in Part (I) of Theorem 3.5. The "degenerate case" $\theta = \pm S$ is singled out as Part (IV) below.

Theorem 3.7. *Assume $1 < \alpha < p < \infty$ and let $\varepsilon > 0$ and $\theta \in \mathbb{R}$. In addition, assume that W has the same properties as in* Theorem 3.5, *including* (a) *and* (b) *for some $0 < S < \infty$, together with* Hypothesis (H2) *where $s_0 = S$ is taken.*

Then the following statements hold.

(I) *Assume $|\theta| < S$. Then the conclusion of* Part (I) of Theorem 3.5 *remains valid: The initial value problem* (3.1), (3.2) *has a unique (global) solution $u \in C^1(\mathbb{R}_+)$ with $|u'|^{p-2}u' \in C^1(\mathbb{R}_+)$. In particular, if $\theta = 0$ then $u \equiv 0$ is a constant function. If $\theta \neq 0$ then the solution u satisfies $|u(r)| < |\theta|$ for every $r > 0$ and, moreover, both $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$.*

(II) Part (II) of Theorem 3.5 *is valid (with $S < |\theta| < S + \zeta$ being assumed).*

(III) *Also* Part (III) of Theorem 3.5 *remains valid (with $0 < |\theta| < S$ and $R \in (0, \infty)$). Again, also $p \geq \frac{(1+\beta)N}{N+\beta}$ is assumed.*

(IV) *Finally, let $\theta = -S$, the case $\theta = S$ being analogous. Then every solution $u \in C^1([0,R))$, with $|u'|^{p-2}u' \in C^1([0,R))$, of the initial value problem* (3.1), (3.2) *defined on a maximal interval of existence $[0,R)$, for some $R \equiv R(\varepsilon) > 0$, must take one of the following three forms: either $u \equiv -S$ on the whole of $\mathbb{R}_+$, or else for $\nu = \pm$,*

$$u(r) = \begin{cases} -S & \text{if } 0 \leq r \leq r_0; \\ U_\nu(r) & \text{if } r_0 \leq r < R, \end{cases} \tag{3.10}$$

where $r_0 \geq 0$ is some number and the continuation from the interval $[r_0, r_0 + \vartheta_\nu)$ to $[r_0, R)$ of the solution U_ν obtained in Lemma 3.6 *is unique. Furthermore, the solution $u(r) = U_+(r)$ continues to exist for all $r \geq r_0$ and satisfies $|u(r)| < S$ for every $r > r_0$ (i.e., $R = \infty$ also in this case); it is unique for $r > r_0$.*

4. Local uniqueness and nonuniqueness

The results in this section will be needed in Section 6 in order to prove our main results stated in Section 3. After the scaling $\tilde{r} = \varepsilon^{-1} r$ and dropping the tilde in $\tilde{r}$, it suffices to consider equation (2.4) or, equivalently, the first-order system (2.5). With the same effect, one may replace the potential $W(s)$ by $\varepsilon^p W(s)$ instead. As we are interested in the *local* existence and uniqueness of a solution to the corresponding initial value problems, in this section we investigate the initial value problem for equation (2.4), i.e.,

$$-r^{-(N-1)}\left(r^{N-1}\,|u'|^{p-2}u'\right)' + W'(u) = 0 \quad \text{for } r \in J_0 \setminus \{0\}\,; \tag{4.1}$$

$$u(r_0) = u_0\,, \quad u'(r_0) = u_0^\sharp\,, \tag{4.2}$$

or *equivalently* for the first-order system (2.5), i.e.,

$$u' = |v|^{p'-2}v, \quad v' = -\frac{N-1}{r}\,v + W'(u) \quad \text{for } r \in J_0 \setminus \{0\}\,; \tag{4.3}$$

$$u(r_0) = u_0\,, \quad v(r_0) = v_0 = |u_0^\sharp|^{p-2}u_0^\sharp\,. \tag{4.4}$$

Here, $r_0 \in \mathbb{R}_+$ and $J_0 \subset \mathbb{R}_+$ is an interval, such that $J_0 = [0,\delta)$ for some $\delta \in (0,\infty)$ if $r_0 = 0$, whereas $J_0 = (r_0-\delta, r_0+\delta)$ for some $\delta \in (0, r_0)$ if $r_0 > 0$. The initial values $u_0, u_0^\sharp \in \mathbb{R}$ are arbitrary, except for the case $r_0 = 0$ when we take $u_0^\sharp = 0$. We always set $v_0 = |u_0^\sharp|^{p-2}u_0^\sharp$.

Below we use system (4.3), (4.4) to state the results we need. For reader's convenience we begin with a local existence result due to Reichel and Walter [17, p. 49], Theorem 1 and its Corollary.

Proposition 4.1. *Let $1 < p < \infty$ and $r_0 \in \mathbb{R}_+$. Assume that $W : \mathbb{R} \to \mathbb{R}$ is a C^1 function. Then the initial value problem* (4.3)*,* (4.4) *has a C^1 solution pair $(u,v) : J_0 \to \mathbb{R}^2$ defined on some interval $J_0 \subset \mathbb{R}_+$ as described above, for some $\delta > 0$.*

If the number $\delta > 0$ is chosen small enough, the following local uniqueness result is valid; see [17, Theorem 4, p. 57].

Lemma 4.2. *Let $1 < p < \infty$ and $r_0 \in \mathbb{R}_+$. Assume that $W : \mathbb{R} \to \mathbb{R}$ is a C^1 function. If at least one of the following three conditions is satisfied, then the initial value problem* (4.3)*,* (4.4) *has a unique C^1 solution pair $(u,v) : J_0 \to \mathbb{R}^2$ defined on some interval $J_0 \subset \mathbb{R}_+$ provided $\delta > 0$ is small enough:*

- (i) *$u_0^\sharp \neq 0$ (hence, $r_0 > 0$).*
- (ii) *$u_0^\sharp = 0$, $W'(u_0) \neq 0$, and W' is monotonically increasing in an interval $(u_0-\zeta, u_0+\zeta)$, for some $\zeta > 0$.*
- (iii) *$u_0^\sharp = 0$, $W'(u_0) = 0$, and $(s-u_0)\,W'(s) < 0$ holds for every $s \in (u_0 - \zeta, u_0+\zeta) \setminus \{u_0\}$, for some $\zeta > 0$.*

Case (i) follows from Part (α)(i) of [17, Theorem 4, p. 57]. Case (ii) follows from Parts (β)(i) and (β)(ii), respectively, depending on whether $W'(u_0) > 0$ or $W'(u_0) < 0$. Finally, Case (iii) follows from Part (δ)(ii) of [17, Theorem 4, p. 57].

Besides the work of Reichel and Walter [17, Theorem 4, p. 57], a closely related uniqueness/nonuniqueness problem for a nonautonomous ordinary differential equation was studied also in McKenna, Reichel, and Walter [16, Appendix] and del Pino, Manásevich, and Murúa [9, Appendix]. However, our analytical tools employed in this section resemble to those used in the work of Díaz and Hernández [10] investigating the (nonnegative) "dead core" solutions to an analogous quasilinear elliptic problem in one space dimension. Such tools (the first integral (1.10) of eq. (1.4) and a subsequent separation of variables in an initial value problem for the first integral) have been applied to study also bifurcation phenomena for spectral problems with the p-Laplace operator in Guedda and Veron [13] (in one space dimension).

Now it remains to treat the most difficult case

(iv) $u_0^\sharp = 0$, $W'(u_0) = 0$, and $(s - u_0)\, W'(s) \geq 0$ for every $s \in (u_0 - \zeta, u_0 + \zeta)$, for some $\zeta > 0$.

This case occurs if the potential W attains a local minimum at u_0 and W satisfies Hypothesis (H1), Part (b), and Hypothesis (H2) from the beginning of Section 2. Then, by Part (b) of (H1), W must be convex in an open interval containing u_0, i.e., W' is monotonically increasing in this interval. We will find out that the result depends on whether $1 < p \leq \alpha < \infty$ or $1 < \alpha < p < \infty$. This fact is an immediate consequence of the following proposition.

Given $r_0 \in \mathbb{R}_+$, we denote by $I_\delta \subset \mathbb{R}_+$ an interval that takes one of the following forms,

$$I_\delta = \begin{cases} [0, \delta) & \text{if } r_0 = 0, \text{ for some } \delta \in (0, \infty); \\ [r_0, r_0 + \delta) \text{ or } (r_0 - \delta, r_0] & \text{if } r_0 > 0, \text{ for some } \delta \in (0, r_0). \end{cases} \tag{4.5}$$

Proposition 4.3. *Let $1 < p, \alpha < \infty$ and $r_0 \in \mathbb{R}_+$. Assume that $W : \mathbb{R} \to \mathbb{R}$ is a C^1 function that satisfies* Hypothesis (H1) *and assume that $s_0 \in \mathbb{R}$ is a local minimizer for W. Let $(u, v) : I_\delta \to \mathbb{R}^2$ be any C^1 solution pair for the initial value problem* (4.3), (4.4) *with the initial values $(u_0, v_0) = (s_0, 0)$ on some interval $I_\delta \subset \mathbb{R}_+$, where I_δ takes one of the forms from* (4.5)*. Then, on a suitable subinterval $I_{\delta'} \subset I_\delta$ of the same form, where $0 < \delta' \leq \delta$, we have either $(u, v) \equiv (s_0, 0)$ is constant on $I_{\delta'}$, or else the following inequalities hold:*

$$W(u(r)) > W(s_0) \quad \text{for all } r \in I_{\delta'} \setminus \{r_0\} \tag{4.6}$$

together with one of the following three statements,

$$\frac{1}{N} \leq \frac{|u'(r)|^p}{p'[W(u(r)) - W(s_0)]} \leq 1 \quad \text{for } r \in I_{\delta'} \setminus \{0\} = (0, \delta')\,; \tag{4.7}$$

$$\frac{1}{1+\eta} \leq \frac{|u'(r)|^p}{p'[W(u(r)) - W(s_0)]} \leq 1 \quad \text{for } r \in I_{\delta'} \setminus \{r_0\} = (r_0, r_0 + \delta')\,; \tag{4.8}$$

$$1 \leq \frac{|u'(r)|^p}{p'[W(u(r)) - W(s_0)]} \leq \frac{1}{1-\eta} \quad \text{for } r \in I_{\delta'} \setminus \{r_0\} = (r_0 - \delta', r_0)\,. \tag{4.9}$$

Inequalities (4.7) *hold if $r_0 = 0$, whereas* (4.8) *or else* (4.9) *apply if $r_0 > 0$, with some number $\eta = \eta(\delta') \in (0, 1)$ satisfying $\eta(\xi)/\xi \to (N - 1)p'/r_0$ as $\xi \to 0+$.*

Before giving the proof of this proposition let us observe that, when inequalities (2.1) are applied to (4.6) through (4.9), the proposition has the following simple consequence. The constants $\alpha > 1$ and $\gamma_2 \geq \gamma_1 > 0$ below come from inequalities (2.1).

Corollary 4.4. *Under the hypotheses of* Proposition 4.3*, if (u, v) is not constant on any subinterval $I_\vartheta \subset I_\delta$, where $0 < \vartheta \leq \delta$, then $p > \alpha$ and there is a suitable subinterval $I_{\delta'} \subset I_\delta$, where $0 < \delta' \leq \delta$, such that the following inequalities hold:*

$$|u(r) - s_0| > 0 \quad \text{for all } \ r \in I_{\delta'} \setminus \{r_0\} \tag{4.10}$$

together with one of the following three statements,

$$\Big(\frac{p'\gamma_1}{N}\Big)^{1/p} \le \frac{|u'(r)|}{|u(r)-s_0|^{\alpha/p}} \le (p'\gamma_2)^{1/p} \tag{4.11}$$
$$\text{for } 0<r<\delta' \text{ if } I_{\delta'}=[0,\delta'),\ r_0=0\,;$$
$$\Big(\frac{p'\gamma_1}{1+\eta}\Big)^{1/p} \le \frac{|u'(r)|}{|u(r)-s_0|^{\alpha/p}} \le (p'\gamma_2)^{1/p} \tag{4.12}$$
$$\text{for } r_0<r<r_0+\delta' \text{ if } I_{\delta'}=[r_0,r_0+\delta'),\ r_0>0\,;$$
$$(p'\gamma_1)^{1/p} \le \frac{|u'(r)|}{|u(r)-s_0|^{\alpha/p}} \le \Big(\frac{p'\gamma_2}{1-\eta}\Big)^{1/p} \tag{4.13}$$
$$\text{for } r_0-\delta'<r<r_0 \text{ if } I_{\delta'}=(r_0-\delta',r_0],\ r_0>0\,.$$

In particular, if $p\le\alpha$ then $(u,v)\equiv(s_0,0)$ is constant on I_ϑ for some $\vartheta\in(0,\delta)$.

In analogy with the abbreviation $\Delta_p u \stackrel{\text{def}}{=} \nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big)$ for the p-Laplace operator, $1<p<\infty$, from now on we employ another commonly used abbreviation, the function $\phi_p(s)\stackrel{\text{def}}{=}|s|^{p-2}s$ of the variable $s\in\mathbb{R}$. Hence, $\frac{\mathrm{d}}{\mathrm{d}s}\,(|s|^p)=p\,\phi_p(s)$ for every $s\in\mathbb{R}$. The inverse function of ϕ_p is equal to $\phi_{p'}'$, by $(p-1)(p'-1)=1$.

Proof of Proposition 4.3. To simplify our notation, without any loss of generality, we replace the function $W(s)$ of the variable $s\in\mathbb{R}$ by $\tilde W(\tilde s)=W(\tilde s+s_0)-W(s_0)$ for $\tilde s\in\mathbb{R}$. In other words, dropping the tilde in $\tilde s$ and $\tilde W$, we may assume both $s_0=0$ and $W(s_0)=0$.

Let us recall equation (2.7) with the function Z satisfying (2.8): The former one holds for $r\in I_\delta$, the latter for $r\in I_\delta\setminus\{0\}$. Hence, $Z(r_0)=W(s_0)=0$ with $s_0=0$. Clearly, the function $Z: I_\delta\to\mathbb{R}$ is continuously differentiable in $I_\delta\setminus\{0\}$. It will be shown below, by a standard application of L'Hôspital's rule for $r\to 0+$, that $Z'(0)=\lim_{r\to 0+}Z'(r)=0$ in case $I_\delta=[0,\delta)$. Thus, Z is C^1 on I_δ.

First, we claim that if a solution curve $(u,v): r\mapsto(u(r),v(r)): I_\delta\to\mathbb{R}^2$ for system (4.3), parametrized by $r\in I_\delta$, passes through the initial point $(u(r_0),v(r_0))=(0,0)$ at another parameter value $r_1\in I_\delta$, $r_1\ne r_0$, that is, $(u(r_1),v(r_1))=(0,0)$, then $(u,v)\equiv(0,0)$ is constant on J, where J denotes the closed interval with the endpoints r_0 and r_1. Notice that $J=I_{\delta'}$ where $\delta'=|r_1-r_0|>0$. To prove our claim, it suffices to use $Z(r_1)=0=Z(r_0)$. But eq. (2.8) shows that Z is monotonically increasing, thus forcing $Z(r)=Z(r_0)$ for every $r\in J$. We conclude that $v\equiv 0$ is constant on J, i.e., $(u,v)\equiv(0,0)$ on $J=I_{\delta'}$.

From now on, let us consider the opposite case, $(u(r),v(r))\ne(0,0)$ for every $r\in I_\delta\setminus\{r_0\}$. Here, we may choose $\delta>0$ small enough, such that $|u(r)|<\zeta$ for every $r\in I_\delta$, where the number $\zeta>0$ is chosen in the following way, by Part (b) of Hypothesis (H1) on W: W is convex in the interval $(-\zeta,\zeta)$ and satisfies inequalities (2.1). As a consequence, we have $s\,W'(s)>0$ whenever $0<|s|<\zeta$. We infer from Lemma 4.2, Cases (i) and (ii), that if $(\tilde u,\tilde v): J\to\mathbb{R}^2$ is another solution pair of system (4.3) in some open interval $J\subset I_\delta\setminus\{r_0\}$, such that $(\tilde u(r_1),\tilde v(r_1))=(u(r_1),v(r_1))$ at some point $r_1\in J$, then $(\tilde u,\tilde v)=(u,v)$ throughout J. In other words, if $(\tilde u,\tilde v): I_{\tilde\delta}\to\mathbb{R}^2$ is another solution pair of system (4.3) in some interval $I_{\tilde\delta}$ of the same form as I_δ, where $0<\tilde\delta<\infty$, that is not identical with (u,v) in $J=I_\delta\cap I_{\tilde\delta}$, but $(\tilde u(r_1),\tilde v(r_1))=(u(r_1),v(r_1))$ at some point $r_1\in J$, then we must have $r_1 = r_0$, $J = I_\vartheta$ with $\vartheta = \min\{\delta, \tilde{\delta}\} > 0$, and $(\tilde{u}(r), \tilde{v}(r)) \neq (u(r), v(r))$ for each $r \in I_\vartheta \setminus \{r_0\}$.

Next, we show that either $u(r) > 0$ holds for all $r \in I_\delta \setminus \{r_0\}$, or else $u(r) < 0$ for all $r \in I_\delta \setminus \{r_0\}$. On the contrary, suppose that $u(r_1) = 0$ for some $r_1 \in I_\delta \setminus \{r_0\}$. Hence, $v(r_1) \neq 0$ in the case being considered, i.e., $u'(r_1) \neq 0$. This forces $r_1 > 0$.

Case $u'(r_1) > 0$. From (4.1) we deduce that the function

$$r \mapsto r^{N-1}\, \phi_p(u'(r)) = r^{N-1}\, |u'(r)|^{p-2} u'(r) : I_\delta \to \mathbb{R} \tag{4.14}$$

is strictly monotonically increasing for $r \in [r_1, \infty) \cap I_\delta$, by $W'(u(r)) > 0$, and strictly monotonically decreasing for $r \in (-\infty, r_1] \cap I_\delta$, by $W'(u(r)) < 0$. It follows that

$$r^{N-1}\, \phi_p(u'(r)) \geq r_1^{N-1}\, \phi_p(u'(r_1)) > 0 \quad \text{holds for all } r \in I_\delta\,.$$

But this contradicts $u(r_1) = 0 = u(r_0)$ for $r_1 \neq r_0$.

Case $u'(r_1) < 0$. Again, from (4.1) we deduce that the function in (4.14) is strictly monotonically decreasing for $r \in [r_1, \infty) \cap I_\delta$, by $W'(u(r)) < 0$, and strictly monotonically increasing for $r \in (-\infty, r_1] \cap I_\delta$, by $W'(u(r)) > 0$. It follows that

$$r^{N-1}\, \phi_p(u'(r)) \leq r_1^{N-1}\, \phi_p(u'(r_1)) < 0 \quad \text{holds for all } r \in I_\delta\,.$$

This contradicts $u(r_1) = 0 = u(r_0)$ for $r_1 \neq r_0$.

We have verified that $|u(r)| > 0$ holds for all $r \in I_\delta \setminus \{r_0\}$. By inequalities (2.1), this is equivalent with (4.6). In order to prove inequalities (4.7), (4.8), and (4.9), we first notice that the inequalities

$$\begin{aligned} |u'(r)|^p &\leq p'\, W(u(r)) \quad \text{for } r \in I_\delta \setminus \{r_0\} = (r_0, r_0 + \delta)\,; \\ |u'(r)|^p &\geq p'\, W(u(r)) \quad \text{for } r \in I_\delta \setminus \{r_0\} = (r_0 - \delta, r_0)\,, \end{aligned}$$

follow immediately from (2.7) and (2.8) combined with $Z(0) = W(u(r_0)) = W(0) = 0$. It remains to prove the first inequality in (4.7) and (4.8), and the second one in (4.9), respectively:

$$|u'(r)|^p \geq \frac{p'}{N}\, W(u(r)) \quad \text{for } 0 < r < \delta' \text{ if } I_{\delta'} = [0, \delta'),\ r_0 = 0\,; \tag{4.15}$$

$$|u'(r)|^p \geq \frac{p'}{1+\eta}\, W(u(r)) \quad \text{for } r_0 < r < r_0 + \delta' \text{ if } I_{\delta'} = [r_0, r_0 + \delta'),\ r_0 > 0\,; \tag{4.16}$$

$$|u'(r)|^p \leq \frac{p'}{1-\eta}\, W(u(r)) \quad \text{for } r_0 - \delta' < r < r_0 \text{ if } I_{\delta'} = (r_0 - \delta', r_0],\ r_0 > 0\,. \tag{4.17}$$

As we have already chosen $\delta > 0$ small enough, we do not need to pass to a smaller number $\delta' \in (0, \delta]$ any more in our proofs of inequalities (4.15) and (4.16); we will get $\eta = \eta(\delta) \in (0, 1)$ and, thus, we may keep $\delta' = \delta$. Only in our proof of inequality (4.17) we need to pass to a smaller number $\delta' \in (0, \delta]$ in order to guarantee $\eta = \eta(\delta') \in (0, 1)$.

Case $r_0 = 0$. We begin with an interval I_δ of the form $I_\delta = [0, \delta)$ with $\delta > 0$ small enough. Then the initial value problem (4.1), (4.2), with $(u_0, u_0^\sharp) = (0, 0)$, is equivalent with the following initial value problem for an integro-differential equation,

$$r^{N-1}\,\phi_p(u'(r)) = \textstyle\int_0^r W'(u(\hat r))\,\hat r^{N-1}\,\mathrm{d}\hat r \quad \text{for } r\in(0,\delta)\,; \tag{4.18}$$

$$u(0)=0\,. \tag{4.19}$$

Notice that, by L'Hôspital's rule,

$$\begin{aligned} v(0) = \phi_p(u'(0)) &= \lim_{r\to 0+} \phi_p(u'(r)) \\ &= \lim_{r\to 0+}\Big(\frac{1}{r^{N-1}}\int_0^r W'(u(\hat r))\,\hat r^{N-1}\,\mathrm{d}\hat r\Big) \\ &= \frac{1}{N-1}\cdot \lim_{\hat r\to 0+}(\hat r\,W'(u(\hat r))) = 0\,. \end{aligned}$$

Making use of this result and employing L'Hôspital's rule again, we get also

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\,\phi_p(u'(r))\Big|_{r=0} &= \lim_{r\to 0+}\frac{\phi_p(u'(r))}{r} \\ &= \lim_{r\to 0+}\Big(\frac{1}{r^{N}}\int_0^r W'(u(\hat r))\,\hat r^{N-1}\,\mathrm{d}\hat r\Big) \\ &= \frac{1}{N}\cdot\lim_{\hat r\to 0+} W'(u(\hat r)) = \frac{1}{N}\cdot W'(u(0)) = \frac{1}{N}\cdot W'(0) = 0\,. \end{aligned} \tag{4.20}$$

Our next claim is that the function

$$r\mapsto r^{-1}\,v(r) = r^{-1}\,\phi_p(u'(r)) = r^{-1}\,|u'(r)|^{p-2}u'(r) : (0,\delta)\to\mathbb{R} \tag{4.21}$$

is positive and monotonically increasing if $u(r)>0$ for $0<r<\delta$, and negative and monotonically decreasing if $u(r)<0$ for $0<r<\delta$. We verify this claim in the former case and leave to the interested reader an easy modification of our proof in the latter case.

Thus, let us assume $u(r)>0$ for $0<r<\delta$. Recall that $0<u(r)<\zeta$ for $0<r<\delta$. Hence, $W'(r)>0$, by Part (b) of Hypothesis (H1) on W: W is convex in the interval $(-\zeta,\zeta)$ and satisfies inequalities (2.1). Eq. (4.18) yields $u'(r)>0$ for $0<r<\delta$. Furthermore, after the substitution $\hat r = tr$ in the integral on the right-hand side of eq. (4.18), for $0\le t\le 1$, we arrive at

$$r^{-1}\,\phi_p(u'(r)) = \textstyle\int_0^1 W'(u(tr))\,t^{N-1}\,\mathrm{d}t \quad \text{for } r\in(0,\delta)\,. \tag{4.22}$$

Since both functions $r\mapsto u(tr) : (0,\delta)\to(0,\zeta)$ and $W' : (0,\zeta)\to(0,\infty)$ are monotonically increasing, with $t\in(0,1]$ fixed in the former one, so is the integrand $r\mapsto W'(u(tr)) : (0,\delta)\to(0,\infty)$. From eq. (4.22) we thus deduce that the function in (4.21) is positive and monotonically increasing as claimed.

Now we know that the function

$$r\mapsto r^{-1}\,|v(r)| = r^{-1}\,|u'(r)|^{p-1} : (0,\delta)\to\mathbb{R} \tag{4.23}$$

is positive and monotonically increasing. The monotonicity yields

$$\big|u'(\hat r)/u'(r)\big|^{p-1} \le \hat r/r \quad \text{for } 0<\hat r\le r<\delta\,.$$

Consequently, recalling $p' = p/(p-1)$, we get also

$$\int_0^r \big|\frac{u'(\hat r)}{u'(r)}\big|^p\,\frac{\mathrm{d}\hat r}{\hat r} \le \int_0^r \big(\frac{\hat r}{r}\big)^{p/(p-1)}\,\frac{\mathrm{d}\hat r}{\hat r} = \int_0^1 t^{p'-1}\,\mathrm{d}t = 1/p'$$

or, equivalently, by (2.8),

$$Z(r) - Z(0) = \int_0^r Z'(\hat r)\,\mathrm{d}\hat r = (N-1)\int_0^r |u'(\hat r)|^p\,\frac{\mathrm{d}\hat r}{\hat r} \le \frac{N-1}{p'}\,|u'(r)|^p \tag{4.24}$$

for all $r \in [0,\delta)$. Finally, we combine (2.7) and (4.24) with $Z(0) = W(u(0)) = W(0) = 0$, thus arriving at

$$\frac{1}{p'}\,|u'(r)|^p = W(u(r)) - Z(r) \ge W(u(r)) - \frac{N-1}{p'}\,|u'(r)|^p$$

for all $r \in [0,\delta)$. This inequality yields (4.15) for $I_\delta = [0,\delta)$, as desired.

Case $r_0 > 0$ and $I_\delta = [r_0, r_0+\delta)$. This case is treated analogously. The only technical difference is that, under the integral sign $\int_0^1 \dots$ on the right-hand side in (4.22), one has to insert the "Heaviside" factor $H(tr - r_0)$,

$$r^{-1}\,\phi_p(u'(r)) = \int_0^1 H(tr - r_0)\,W'(u(tr))\,t^{N-1}\,\mathrm{d}t \quad \text{for } r \in I_\delta\,, \tag{4.25}$$

where $H : \mathbb{R} \to \mathbb{R}$ stands for the *Heaviside function* defined by

$$H(\xi) \overset{\text{def}}{=} \begin{cases} 1 & \text{for } \xi > 0\,; \\ 0 & \text{for } \xi \le 0\,. \end{cases}$$

Clearly, (4.25) is equivalent with (4.18) in I_δ. Observe that, for each fixed $t \in [0,1]$, the function $r \mapsto H(tr - r_0) : I_\delta \to [0,1]$ is nonnegative and monotonically increasing. This fact guarantees that, again, the function

$$r \mapsto r^{-1}\,v(r) = r^{-1}\,\phi_p(u'(r)) : I_\delta = [r_0, r_0+\delta) \to \mathbb{R} \tag{4.26}$$

is nonnegative and monotonically increasing if $u(r) > 0$ for $r_0 < r < r_0 + \delta$, and nonpositive and monotonically decreasing if $u(r) < 0$ for $r_0 < r < r_0 + \delta$.

In analogy with the case $I_\delta = [0,\delta)$ we obtain

$$|u'(\hat r)/u'(r)|^{p-1} \le \hat r/r \quad \text{for } r_0 < \hat r \le r < r_0 + \delta\,,$$

which yields

$$\begin{aligned} \int_{r_0}^r \Big|\frac{u'(\hat r)}{u'(r)}\Big|^p \,\frac{\mathrm{d}\hat r}{\hat r} &\le \int_{r_0}^r \Big(\frac{\hat r}{r}\Big)^{p/(p-1)}\,\frac{\mathrm{d}\hat r}{\hat r} \\ &= \int_{r_0/r}^1 t^{p'-1}\,\mathrm{d}t = (1/p')[1 - (r_0/r)^{p'}] < \frac{1}{p'}\Big[1 - \Big(\frac{r_0}{r_0+\delta}\Big)^{p'}\Big] \end{aligned}$$

or, equivalently, by (2.8),

$$\begin{aligned} Z(r) - Z(r_0) &= \int_{r_0}^r Z'(\hat r)\,\mathrm{d}\hat r = (N-1)\int_{r_0}^r |u'(\hat r)|^p\,\frac{\mathrm{d}\hat r}{\hat r} \\ &\le \frac{N-1}{p'}\Big[1 - \Big(\frac{r_0}{r_0+\delta}\Big)^{p'}\Big]|u'(r)|^p \quad \text{for all } r \in [r_0, r_0+\delta)\,. \end{aligned} \tag{4.27}$$

Finally, denoting

$$\eta = \eta(\delta) \overset{\text{def}}{=} (N-1)\Big[1 - \Big(\frac{r_0}{r_0+\delta}\Big)^{p'}\Big]\,, \quad 0 < \eta < 1\,,$$

we combine (2.7) and (4.27) with $Z(r_0) = W(u(r_0)) = W(0) = 0$, thus arriving at

$$\frac{1}{p'}\,|u'(r)|^p = W(u(r)) - Z(r) \ge W(u(r)) - \frac{\eta}{p'}\,|u'(r)|^p$$

for all $r \in [r_0, r_0+\delta)$. This inequality yields (4.16) for $I_\delta = [r_0, r_0+\delta)$, $r_0 > 0$.

Case $r_0 > 0$ and $I_\delta = (r_0 - \delta, r_0]$. In this case, the integral and the "Heaviside" factor in eq. (4.25) have to be replaced by $\int_1^\infty \dots$ and $-H(r_0 - tr)$, respectively,

$$r^{-1}\,\phi_p(u'(r)) = -\int_1^\infty H(r_0 - tr)\,W'(u(tr))\,t^{N-1}\,\mathrm{d}t \quad \text{for } r \in I_\delta\,. \tag{4.28}$$

This equation is equivalent with (4.18) in I_δ. For each fixed $t \in [0,1]$, the function $r \mapsto H(r_0 - tr) : I_\delta \to [0,1]$ is nonnegative and monotonically decreasing. In analogy with the previous two cases, we treat only the case $u(r) > 0$ for $r_0 - \delta < r < r_0$, leaving to the reader an easy modification of our proof for the other case, $u(r) < 0$ for $r_0 - \delta < r < r_0$. First, observe that the function $u : (r_0 - \delta, r_0) \to (0, \zeta)$ is strictly monotonically decreasing, by (4.28) combined with $W'(s) > 0$ for $0 < s < \zeta$. Second, the function $r \mapsto u(tr) : (r_0 - \delta, r_0) \to (0, \zeta)$ being monotonically decreasing and $W' : (0, \zeta) \to (0, \infty)$ monotonically increasing, with $t \in (0,1]$ fixed in the former one, the function $r \mapsto W'(u(tr)) : (r_0 - \delta, r_0) \to (0, \infty)$ must be monotonically decreasing. Finally, it follows that the integrand on the right-hand side in (4.28),

$$r \mapsto H(r_0 - tr)\,W'(u(tr)) : I_\delta = (r_0 - \delta, r_0] \to \mathbb{R}\,,$$

is nonnegative and monotonically decreasing, and so is the function

$$r \mapsto -r^{-1}\,v(r) = \,-\,r^{-1}\,\phi_p(u'(r)) : I_\delta = (r_0 - \delta, r_0] \to \mathbb{R}\,. \tag{4.29}$$

Notice that, if $u(r) < 0$ for $r_0 - \delta < r < r_0$ then this function is nonpositive and monotonically increasing.

In analogy with the case $I_\delta = [r_0, r_0 + \delta)$ we obtain

$$|u'(\hat{r})/u'(r)|^{p-1} \le \hat{r}/r \quad \text{for } r_0 - \delta < r \le \hat{r} < r_0\,,$$

which yields

$$\begin{aligned} \int_r^{r_0} \big|\frac{u'(\hat{r})}{u'(r)}\big|^p\,\frac{\mathrm{d}\hat{r}}{\hat{r}} &\le \int_r^{r_0} \big|\frac{\hat{r}}{r}\big|^{p/(p-1)}\,\frac{\mathrm{d}\hat{r}}{\hat{r}} = \int_1^{r_0/r} t^{p'-1}\,\mathrm{d}t \\ &= (1/p')[(r_0/r)^{p'} - 1] < \frac{1}{p'}\big[\big(\frac{r_0}{r_0 - \delta}\big)^{p'} - 1\big] \end{aligned}$$

or, equivalently, by (2.8),

$$\begin{aligned} Z(r_0) - Z(r) &= \int_r^{r_0} Z'(\hat{r})\,\mathrm{d}\hat{r} = (N-1)\int_r^{r_0} |u'(\hat{r})|^p\,\frac{\mathrm{d}\hat{r}}{\hat{r}} \\ &\le \frac{N-1}{p'}\big[\big(\frac{r_0}{r_0 - \delta}\big)^{p'} - 1\big]\,|u'(r)|^p \quad \text{for all } \ r \in (r_0 - \delta, r_0]\,. \end{aligned} \tag{4.30}$$

Finally, denoting

$$\eta = \eta(\delta) \stackrel{\text{def}}{=} (N-1)\big[\big(\frac{r_0}{r_0 - \delta}\big)^{p'} - 1\big]\,, \quad 0 < \eta < \infty\,,$$

we combine (2.7) and (4.30) with $Z(r_0) = W(u(r_0)) = W(0) = 0$, thus arriving at

$$\frac{1}{p'}\,|u'(r)|^p = W(u(r)) - Z(r) \le W(u(r)) + \frac{\eta}{p'}\,|u'(r)|^p$$

for all $r \in (r_0 - \delta, r_0]$. This inequality yields (4.17) for $I_{\delta'} = (r_0 - \delta', r_0]$ where $\delta' \in (0, \delta]$ is such that $\eta(\delta') < 1$.

Our choice of $\eta = \eta(\xi)$ for $r_0 > 0$ involves the expression $\big(1 \pm \frac{\xi}{r_0}\big)^{-p'}$ which yields the asymptotic behavior $\eta(\xi)/\xi \to (N-1)p'/r_0$ as $\xi \to 0+$. The proof of the proposition is finished. $\square$

The problem of existence and uniqueness of a nonconstant solution pair $(u,v): I_\delta \to \mathbb{R}^2$ for the initial value problem (4.3), (4.4) with the initial values $(u_0,v_0) = (s_0,0)$ on some interval $I_\delta \subset \mathbb{R}_+$, considered in Proposition 4.3, is completely answered in the next proposition.

Proposition 4.5. *Let $1<\alpha<p<\infty$, $r_0 \in \mathbb{R}_+$, and let $I_\delta \subset \mathbb{R}_+$ be an interval of type* (4.5). *Assume that W satisfies* Hypothesis (H1) *and $s_0 \in \mathbb{R}$ is a local minimizer for W. Then, in a suitable subinterval $I_{\delta'} \subset I_\delta$ of the same form, where $0<\delta' \le \delta$, the initial value problem* (4.3), (4.4), *with the initial values $(u_0,v_0) = (s_0,0)$, possesses a C^1 solution pair $(u_+,v_+) : I_{\delta'} \to \mathbb{R}^2$, such that $u_+(r) > s_0$ for every $r \in I_{\delta'} \setminus \{r_0\}$, and another C^1 solution pair $(u_-,v_-) : I_{\delta'} \to \mathbb{R}^2$, such that $u_-(r) < s_0$ for every $r \in I_{\delta'} \setminus \{r_0\}$. In particular, both these solution pairs satisfy inequalities* (4.6) *through* (4.9) *(in* Proposition 4.3*) and* (4.10) *through* (4.13) *(in* Corollary 4.4*). Finally, if W satisfies also* Hypothesis (H2) *then the solution pairs (u_+,v_+) and (u_-,v_-) characterized above are unique.*

Proof. As in the proof of Proposition 4.3 above, we may assume $s_0 = 0$ and $W(s_0) = 0$. Again, recall that every C^1 solution pair (u,v) of system (4.3) in $I_\delta \setminus \{0\}$ verifies equation (2.7) with the function Z satisfying (2.8): The former one holds for $r \in I_\delta$, the latter for $r \in I_\delta \setminus \{0\}$. Hence, $Z(r_0) = W(s_0) = 0$ with $s_0 = 0$. In the proof of Proposition 4.3 we have already shown that Z is C^1 on I_δ with $Z'(0) = \lim_{r\to 0+} Z'(r) = 0$ in case $I_\delta = [0,\delta)$.

We treat only the case $r_0 = 0$, i.e., $I_\delta = [0,\delta)$, which is the most difficult one. We leave the remaining two cases with $r_0 > 0$, i.e., $I_\delta = [r_0, r_0+\delta)$ and $I_\delta = (r_0-\delta, r_0]$ for some $\delta \in (0,r_0)$, to the interested reader. The necessary changes in the proof are analogous to those in the proof of Proposition 4.3.

We continue the *a priori* setting from the proof of Proposition 4.3, case $r_0 = 0$, with $\delta > 0$ small enough. Again, we treat only the case $u(r) > 0$ for $0<r<\delta$, leaving an easy modification of the other case, $u(r) < 0$ for $0<r<\delta$, to the reader. More precisely, we will show that a C^1 solution pair (u,v) of system (4.3) in $I_\delta \setminus \{0\}$, with the initial values $(u_0,v_0) = (s_0,0) = (0,0)$, considered in the proof of Proposition 4.3 with $u(r) > 0$ for $0<r<\delta$, *exists* and is *unique*. Consequently, we look for a pair (u,v) such that also $u'(r) > 0$ for $0<r<\delta$ with the function in (4.21) being monotonically increasing. It follows that u' is monotonically increasing in $[0,\delta)$, i.e., u is convex. In other words, we look for a C^1 solution $u : [0,\delta) \to \mathbb{R}$ of the initial value problem (4.1), (4.2) in $(0,\delta)$, with the initial values $(u_0, u_0^\sharp) = (0,0)$ at $r_0 = 0$, such that $u \in \mathcal{U}$, where $\mathcal{U}$ denotes the class of all C^1 functions $U : [0,\delta) \to \mathbb{R}$ with the following properties:

(i) $U(0) = U'(0) = 0$ and $0 < U(r) < \zeta$ for every $r \in (0,\delta)$;
(ii) U satisfies both inequalities in (4.7) for every $r \in (0,\delta)$, with u replaced by U, and $s_0 = 0$ and $W(s_0) = 0$.

We recall that for an arbitrary function $u \in \mathcal{U}$, eq. (4.1) in $(0,\delta)$ is equivalent with the integro-differential equation (4.18) in $(0,\delta)$.

We begin by constructing a pointwise orderded pair of solutions $\underline{u}, \overline{u} : [0,\delta) \to [0,\zeta)$ to the initial value problem (4.18), (4.19), such that $0 < \underline{u} \leqq u \leqq \overline{u}$ holds in $(0,\delta)$ for every solution $u : [0,\delta) \to [0,\zeta)$ to problem (4.18), (4.19) that satisfies $u(r) > 0$ for $0<r<\delta$; hence, $\underline{u}, u, \overline{u} \in \mathcal{U}$. We call $\underline{u}$ ($\overline{u}$, respectively) the *minimal* (*maximal*) positive solution to problem (4.18), (4.19). We employ a standard technique using monotone iterations.

To construct $\underline{u}$, we start with the unique positive solution $u_1 : [0,\delta) \to [0,\zeta)$ of the initial value problem

$$|u_1'(r)|^p = (p'/N)\, W(u_1(r)) \quad \text{for } r \in (0,\delta)\,; \tag{4.31}$$

$$u_1(0) = 0\,, \tag{4.32}$$

cf. (4.7) with $s_0 = 0$ and $W(s_0) = 0$. (More precise details about the existence and uniqueness of u_1 will be given below in the proof of the uniqueness for u, i.e., $\underline{u} \equiv \overline{u}$ in $(0,\delta)$.) We claim that u_1 is a strict subsolution to problem (4.18), (4.19), that is, by eq. (4.22),

$$r^{-1}\,\phi_p(u_1'(r)) < \int_0^1 W'(u_1(tr))\, t^{N-1}\, \mathrm{d}t \quad \text{for } r \in (0,\delta)\,. \tag{4.33}$$

Indeed, combining (4.18), where we first replace W by $(1/N)\, W$, then take $N = 1$, with eq. (4.31) we arrive at

$$r^{-1}\,\phi_p(u_1'(r)) = N^{-1} \int_0^1 W'(u_1(tr))\, \mathrm{d}t < \int_0^1 W'(u_1(tr))\, t^{N-1}\, \mathrm{d}t$$

for $r \in (0,\delta)$. The last inequality has been deduced from Lemma 7.1, ineq. (7.2), stated in the appendix (Appendix 7).

Similarly, to construct $\overline{u}$, we start with the unique positive solution $w_1 : [0,\delta) \to [0,\zeta)$ of the initial value problem

$$|w_1'(r)|^p = p'\, W(w_1(r)) \quad \text{for } r \in (0,\delta)\,; \tag{4.34}$$

$$w_1(0) = 0\,, \tag{4.35}$$

cf. (4.7) with $s_0 = 0$ and $W(s_0) = 0$. Note that $u_1, w_1 \in \mathcal{U}$ and $u_1(r) = w_1(N^{-1/p}\, r) < w_1(r)$ for every $r \in (0,\delta)$. We claim that w_1 is a strict supersolution to problem (4.18), (4.19), that is, by eq. (4.22),

$$r^{-1}\,\phi_p(w_1'(r)) > \int_0^1 W'(w_1(tr))\, t^{N-1}\, \mathrm{d}t \quad \text{for } r \in (0,\delta)\,. \tag{4.36}$$

Indeed, combining (4.18), where we take $N = 1$, with eq. (4.34) we arrive at

$$r^{-1}\,\phi_p(w_1'(r)) = \int_0^1 W'(w_1(tr))\, \mathrm{d}t > \int_0^1 W'(w_1(tr))\, t^{N-1}\, \mathrm{d}t$$

for $r \in (0,\delta)$. We remark that we take $\delta > 0$ small enough, such that $0 < w_1(r) < \zeta$ holds for every $r \in (0,\delta)$.

Next, we construct a sequence of pairs of functions $u_k, w_k : [0,\delta) \to [0,\zeta)$ recursively for each $k = 2, 3, 4, \dots$ by requiring

$$r^{-1}\,\phi_p(u_k'(r)) = \textstyle\int_0^1 W'(u_{k-1}(tr))\, t^{N-1}\, \mathrm{d}t \quad \text{for } r \in (0,\delta)\,;$$

$$u_k(0) = 0\,,$$

and

$$r^{-1}\,\phi_p(w_k'(r)) = \textstyle\int_0^1 W'(w_{k-1}(tr))\, t^{N-1}\, \mathrm{d}t \quad \text{for } r \in (0,\delta)\,;$$

$$w_k(0) = 0\,.$$

In particular, we have $0 < u_1 < u_2 \le w_2 < w_1$ in $(0,\delta)$, by inequalities (4.33) and (4.36). Recall that W' is monotonically increasing on the interval $[0,\zeta)$. By induction on k, from $u_1 < u_2 \le w_2 < w_1$ in $(0,\delta)$ we derive $u_k \le u_{k+1} \le w_{k+1} \le w_k$ in $(0,\delta)$ also for every $k = 2, 3, 4, \dots$.

Summarizing the properties of u_k's and w_k's, we have

$$0 < u_1 < u_2 \le \cdots \le u_k \le u_{k+1} \le \cdots \le w_{k+1} \le w_k \le \cdots \le w_2 < w_1$$

in $(0,\delta)$, for all $k \ge 2$. Standard arguments for monotone iteration schemes now guarantee that both sequences $\{u_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ must converge pointwise on the interval $[0,\delta)$, say, to $\underline{u}$ and $\overline{u}$, respectively. Furthermore, both $\underline{u}$ and $\overline{u}$ verify the initial value problem (4.18), (4.19), together with

$$\begin{aligned} 0 < u_1 < u_2 \le \cdots \le u_k \le u_{k+1} \le \cdots \le \underline{u} \\ \le \overline{u} \le \cdots \le w_{k+1} \le w_k \le \cdots \le w_2 < w_1 \end{aligned} \tag{4.37}$$

in $(0,\delta)$, for all $k \ge 2$. Hence, $\underline{u}, \overline{u} \in \mathcal{U}$. In addition, every function $u \in \mathcal{U}$ satisfies $u_1 \le U \le w_1$ in $[0,\delta)$, by the inequalities in (4.7) and our definitions of u_1 and w_1. If $u \in \mathcal{U}$ verifies also the initial value problem (4.18), (4.19) then we obtain $u_k \le u \le w_k$ in $(0,\delta)$ for every $k = 1,2,3,\dots$, by induction on k again. We conclude that $\underline{u} \le u \le \overline{u}$ in $(0,\delta)$.

Finally, from the initial value problem (4.18), (4.19) for $\underline{u}$ and $\overline{u}$ in place of u, combined with $\underline{u} \leqq \overline{u}$ in $[0,\delta)$, we deduce $\phi_p(\underline{u}')) \le \phi_p(\overline{u}')$ in $(0,\delta)$, which shows that the difference $\overline{u}(r) - \underline{u}(r)$ is a monotonically increasing function of $r \in [0,\delta)$, i.e.,

$$0 \le \overline{u}(\hat{r}) - \underline{u}(\hat{r}) \le \overline{u}(r) - \underline{u}(r) \quad \text{for all } \ 0 \le \hat{r} \le r < \delta \,. \tag{4.38}$$

Consequently, if $\overline{u}(r^*) = \underline{u}(r^*)$ for some $r^* \in (0,\delta)$, then $\overline{u} = \underline{u}$ holds on the entire interval $[0,r^*]$. But this forces also $\overline{u} = \underline{u}$ on the whole of $[0,\delta)$, by the arguments we have used at the beginning of our proof of Proposition 4.3 (cf. Lemma 4.2, Cases (i) and (ii)). We conclude that, in order to prove the *uniqueness* for problem (4.18), (4.19), that is to say, to verify $\overline{u} = \underline{u}$ in $[0,\delta)$, it suffices to prove that there is some $r^* \in (0,\delta)$ such that $\overline{u}(r^*) = \underline{u}(r^*)$.

We begin our proof of uniqueness by considering an arbitrary solution $u \in \mathcal{U}$ to problem (4.18), (4.19) in $(0,\delta)$. Recall that here we assume also Hypothesis (H2). Notice that, given any $\varepsilon > 0$, if we replace the variable r by $\tilde{r} = \varepsilon^{-1} r$ and define the function $\tilde{u}(\tilde{r}) \stackrel{\text{def}}{=} u(\varepsilon\tilde{r})$ for $0 \le \tilde{r} < \varepsilon^{-1}\delta$, then $\tilde{u}$ satisfies the integro-differential equation

$$\varepsilon^{-p}\, \tilde{r}^{-1}\, \phi_p(\tilde{u}'(\tilde{r})) = \int_0^1 W'(\tilde{u}(t\tilde{r}))\, t^{N-1}\, \mathrm{d}t \quad \text{for } \tilde{r} \in (0,\varepsilon^{-1}\delta)\,, \tag{4.39}$$

by eq. (4.22). Of course, $\tilde{u}(0) = \tilde{u}'(0) = 0$. Consequently, if $\varepsilon < 1$ ($\varepsilon > 1$, respectively) then $\tilde{u}$ is a strict subsolution (supersolution) of the initial value problem (4.18), (4.19) in the interval $(0,\varepsilon^{-1}\delta)$, by

$$\tilde{r}^{-1}\, \phi_p(\tilde{u}'(\tilde{r})) < \ (>)\ \varepsilon^{-p}\, \tilde{r}^{-1}\, \phi_p(\tilde{u}'(\tilde{r})) = \int_0^1 W'(\tilde{u}(t\tilde{r}))\, t^{N-1}\, \mathrm{d}t \tag{4.40}$$

for $\tilde{r} \in (0,\varepsilon^{-1}\delta)$. In particular, if the inequality $\overline{u}(\varepsilon\tilde{r}) \le \underline{u}(\tilde{r})$ holds for all $\tilde{r} \in (0,\delta')$, where $\varepsilon \in (0,1)$ and $\delta' \in (0,\delta]$ are some constants, then it must hold for all $\tilde{r} \in (0,\delta)$, by ineq. (4.40) with $\overline{u}$ in place of u and eq. (4.22) with $\underline{u}$ in place of u. We may take $\varepsilon \in (0,1)$ arbitrarily close to 1 if

$$\lim_{r\to 0+} \frac{\overline{u}(r)}{\underline{u}(r)} = 1\,. \tag{4.41}$$

Then $\overline{u}(\varepsilon\tilde{r}) \leq \underline{u}(\tilde{r})$ for all $\tilde{r} \in (0,\delta)$ will force also $\overline{u}(\tilde{r}) \leq \underline{u}(\tilde{r})$ for all $\tilde{r} \in (0,\delta)$, by letting $\varepsilon \to 1-$. Thus, we have reduced the proof of uniqueness, which is equivalent to $\overline{u} = \underline{u}$ in $[0,\delta)$, to verifying the limit in (4.41).

To verify (4.41), first, recall that $u \in \mathcal{U}$ is an arbitrary solution to problem (4.18), (4.19) in $(0,\delta)$. We insert eq. (2.8) into (2.7) to get

$$Z'(r) + \frac{(N-1)p'}{r}\, Z(r) - \frac{(N-1)p'}{r}\, W(u(r)) = 0 \quad \text{for } 0 < r < \delta\,. \tag{4.42}$$

This differential equation is supplemented by two initial conditions, namely, $Z(0) = W(u(0)) = W(0) = 0$ and $Z'(0) = 0$. Equation (4.42) is equivalent with

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{\nu}\, Z(r)\right) = \nu\, r^{\nu-1}\, W(u(r))\,, \quad 0 < r < \delta\,,$$

where we have abbreviated $\nu \stackrel{\text{def}}{=} (N-1)p'$; hence, $\nu \geq p' > 1$ owing to $N \geq 2$. This equation, supplemented by the initial condition $Z(0) = 0$, is equivalent with the integral equation

$$r^{\nu}\, Z(r) = \nu \int_0^r W(u(\hat{r}))\, \hat{r}^{\nu-1}\, \mathrm{d}\hat{r}\,, \quad 0 < r < \delta\,. \tag{4.43}$$

We substitute (2.8) for $Z'(r)$ and (4.43) for $Z(r)$ in (4.42) to obtain

$$\frac{1}{p'}\, |u'(r)|^p + \nu \int_0^r W(u(\hat{r}))\, \big(\frac{\hat{r}}{r}\big)^{\nu}\, \frac{\mathrm{d}\hat{r}}{\hat{r}} = W(u(r)) \tag{4.44}$$

for $0 < r < \delta$. This integro-differential equation for the unknown function $u : [0,\delta) \to [0,\zeta)$ is supplemented by the condition $u \in \mathcal{U}$. Note that

$$\nu \int_0^r \big(\frac{\hat{r}}{r}\big)^{\nu}\, \frac{\mathrm{d}\hat{r}}{\hat{r}} = \nu \int_0^1 t^{\nu-1}\, \mathrm{d}t = 1 \tag{4.45}$$

for $0 < r < \delta$, after the substitution $\hat{r} = tr$.

To solve (4.44), we introduce the function $\varrho : (-\zeta,\zeta) \to \mathbb{R}$ by

$$\varrho(s) \stackrel{\text{def}}{=} \int_0^s \frac{\mathrm{d}\hat{s}}{(p'\, W(\hat{s}))^{1/p}} \quad \text{for } |s| < \zeta\,. \tag{4.46}$$

This is a continuous, strictly monotonically increasing function which is C^2 on $(-\zeta,\zeta) \setminus \{0\}$. Let $\sigma : (-\vartheta_-,\vartheta_+) \to (-\zeta,\zeta)$ denote the inverse function for ϱ, where the number $\vartheta_\nu > 0$ is given by the formula

$$\vartheta_\nu \stackrel{\text{def}}{=} \Big| \int_0^{\nu\zeta} \frac{\mathrm{d}s}{(p'\, W(s))^{1/p}} \Big|$$

with the sign symbol $\nu = \pm$ in ϑ_ν and $\nu\zeta$. It is easy to see that also σ is continuous and strictly monotonically increasing and, moreover, it is continuously differentiable on $(-\vartheta_-,\vartheta_+)$ with the derivative $\sigma'(0) = 0$ at zero. Note that

$$\sigma'(r) = \frac{1}{\varrho'(\sigma(r))} = (p'\,(W \circ \sigma)(r))^{1/p} > 0 \quad \text{for all } \ r \in (-\vartheta_-,\vartheta_+) \setminus \{0\}\,. \tag{4.47}$$

Moreover, both functions $\sigma : (-\vartheta_-,\vartheta_+) \to (-\zeta,\zeta)$ and $W : [0,\zeta) \to \mathbb{R}_+ = [0,\infty)$ being strictly monotonically increasing, so is the function $\sigma' : [0,\vartheta_+) \to \mathbb{R}_+$. It follows that $\sigma : [0,\vartheta_+) \to [0,\zeta)$ is strictly convex with $\sigma(0) = \sigma'(0) = 0$. In addition, the asymptotic behavior as $r \to 0+$ of the logarithmic derivative of the composition $W \circ \sigma : (-\vartheta_-, \vartheta_+) \to \mathbb{R}_+$ of σ and W, that is, of the function

$$r \longmapsto \frac{\mathrm{d}}{\mathrm{d}r} (\log \circ W \circ \sigma)(r) = \frac{(W \circ \sigma)'(r)}{(W \circ \sigma)(r)} : (-\vartheta_-, \vartheta_+) \setminus \{0\} \to \mathbb{R},$$

is determined by Hypothesis (H2). This logarithmic derivative is defined on both open intervals $(-\vartheta_-, 0)$ and $(0, \vartheta_+)$, with the limit $\lim_{r\to 0}(\log \circ W \circ \sigma)(r) = -\infty$, and satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} (\log \circ W \circ \sigma)(r) &= \frac{(W \circ \sigma)'(r)}{(W \circ \sigma)(r)} = \frac{W'(\sigma(r))}{W(\sigma(r))} \sigma'(r) \\ &= \frac{W'(\sigma(r))}{W(\sigma(r))} (p' \, W(\sigma)(r))^{1/p} = p \, \frac{\mathrm{d}}{\mathrm{d}s} (p' \, W(s))^{1/p} \Big|_{s=\sigma(r)}, \end{aligned} \tag{4.48}$$

by eq. (4.47). The last derivative is defined for s in both open intervals $(-\zeta, 0)$ and $(0, \zeta)$, by Hypothesis (H2) where $s_0 = 0$ and $W(s_0) = 0$ are taken.

Dividing (4.44) by $W(u(r))$ and using (4.47), we arrive at

$$|\varrho'(u(r))|^p \, |u'(r)|^p + \nu \int_0^r \frac{(W \circ u)(\hat{r})}{(W \circ u)(r)} \big(\frac{\hat{r}}{r}\big)^\nu \frac{\mathrm{d}\hat{r}}{\hat{r}} = 1$$

for $0 < r < \delta$, or, equivalently,

$$\Big|\frac{\mathrm{d}}{\mathrm{d}r} (\varrho \circ u)(r)\Big|^p + \nu \int_0^r \frac{(W \circ u)(\hat{r})}{(W \circ u)(r)} \big(\frac{\hat{r}}{r}\big)^\nu \frac{\mathrm{d}\hat{r}}{\hat{r}} = 1$$

for $0 < r < \delta$. Next, we substitute $R \stackrel{\text{def}}{=} \varrho \circ u : [0, \delta) \to [0, \vartheta_+)$, which yields $u = \sigma \circ R : [0, \delta) \to [0, \zeta)$, thus obtaining

$$\big|\mathrm{d}R\mathrm{d}r\big|^p + \nu \int_0^r \frac{(W \circ \sigma)(R(\hat{r}))}{(W \circ \sigma)(R(r))} \big(\frac{\hat{r}}{r}\big)^\nu \frac{\mathrm{d}\hat{r}}{\hat{r}} = 1 \tag{4.49}$$

for $0 < r < \delta$. This integro-differential equation for the unknown function $R : [0, \delta) \to [0, \vartheta_+)$ is supplemented by the initial condition $R(0) = 0$. We look for a C^1 solution $R : [0, \delta) \to [0, \vartheta_+)$ to this initial value problem that satisfies

$$\frac{1}{N} \le \Big|\frac{\mathrm{d}R}{\mathrm{d}r}\Big|^p \le 1 \quad \text{in } I_\delta \setminus \{0\} = (0, \delta), \tag{4.50}$$

according to inequalities (4.7). More precisely, combining (4.47) with $u \in \mathcal{U}$, we have

$$R'(r) = \frac{\mathrm{d}}{\mathrm{d}r} (\varrho \circ u)(r) = \varrho'(u(r)) \, u'(r) = \frac{u'(r)}{(p' \, W(u(r))^{1/p}} > 0$$

for $0 < r < \delta$ and, therefore, the inequalities in (4.50) read

$$N^{-1/p} \le \frac{\mathrm{d}R}{\mathrm{d}r} \le 1 \quad \text{in } (0, \delta). \tag{4.51}$$

Let us denote by $\mathcal{R}$ the class of all continuous functions $R : [0, \delta) \to \mathbb{R}$ with the following properties:

(i) $R(0) = 0$ and $0 < R(r) < \vartheta_+$ for every $r \in (0, \delta)$;
(ii) R is C^1 in $(0, \delta)$ and satisfies both inequalities in (4.51) for every $r \in (0, \delta)$.

Now we are ready to verify (4.41). We set

$$\underline{R} \stackrel{\text{def}}{=} \varrho \circ \underline{u}, \quad \overline{R} \stackrel{\text{def}}{=} \varrho \circ \overline{u} : [0, \delta) \to [0, \vartheta_+).$$

First, by L'Hôspital's rule and (4.47), we calculate

$$\begin{aligned}
\lim_{r\to 0+} \frac{\overline{u}(r)}{\underline{u}(r)} &= \lim_{r\to 0+} \frac{\overline{u}'(r)}{\underline{u}'(r)} = \lim_{r\to 0+} \frac{\sigma'(\overline{R}(r))\,\overline{R}'(r)}{\sigma'(\underline{R}(r))\,\underline{R}'(r)} \\
&= \lim_{r\to 0+} \Big[\Big(\frac{(W\circ\sigma)(\overline{R}(r))}{(W\circ\sigma)(\underline{R}(r))}\Big)^{1/p} \frac{\overline{R}'(r)}{\underline{R}'(r)}\Big] \\
&= \Big(\lim_{r\to 0+} \frac{\frac{(}{W}\circ\sigma)(\overline{R}(r))}{(W\circ\sigma)(\underline{R}(r))}\Big)^{1/p} \cdot \lim_{r\to 0+} \frac{\overline{R}'(r)}{\underline{R}'(r)}\,,
\end{aligned} \tag{4.52}$$

where the fact that

$$\lim_{r\to 0+} \frac{(W\circ\sigma)(\overline{R}(r))}{(W\circ\sigma)(\underline{R}(r))} = 1 \tag{4.53}$$

follows immediately from the asymptotic condition (2.2) in Hypothesis (H2), where $s_0 = 0$ and $W(s_0) = 0$ are taken, combined with the following claim,

$$\lim_{r\to 0+} \frac{\overline{R}(r)}{\underline{R}(r)} = \lim_{r\to 0+} \frac{\overline{R}'(r)}{\underline{R}'(r)} = 1\,. \tag{4.54}$$

More precisely, we are going to show that

$$R'(0) \overset{\text{def}}{=} \lim_{r\to 0+} R'(r) = \Big(1 + \nu\big(\frac{1}{\alpha} - \frac{1}{p}\big)\Big)^{-1/p} \in (N^{-1/p}, 1) \tag{4.55}$$

holds for $R = \varrho \circ u : [0,\delta) \to [0,\vartheta_+)$, where $u \in \mathcal{U}$ is an arbitrary solution to problem (4.18), (4.19) in $(0,\delta)$.

Indeed, taking advantage of the asymptotic condition (2.2) in Hypothesis (H2) again, we obtain

$$\lim_{\substack{s\to 0+\\ 0<\hat{s}\le s}} \Big[\frac{W(\hat{s})}{W(s)} \Big/ \big(\frac{\hat{s}}{s}\big)^{\alpha}\Big] = 1\,,$$

$$\lim_{\substack{r\to 0+\\ 0<\hat{r}\le r}} \Big[\frac{\sigma(\hat{r})}{\sigma(r)} \Big/ \big(\frac{\hat{r}}{r}\big)^{p/(p-\alpha)}\Big] = 1\,,$$

which yields

$$\lim_{\substack{r\to 0+\\ 0<\hat{r}\le r}} \Big[\frac{(W\circ\sigma)(R(\hat{r}))}{(W\circ\sigma)(R(r))} \Big/ \Big(\frac{R(\hat{r})}{R(r)}\Big)^{\alpha p/(p-\alpha)}\Big] = 1\,, \tag{4.56}$$

$$\lim_{\substack{r\to 0+\\ 0<\hat{r}\le r}} \Big[\frac{R(\hat{r})}{R(r)} \Big/ \frac{\hat{r}}{r}\Big] = 1\,. \tag{4.57}$$

Finally, we insert the last two limits into eq. (4.49), thus arriving at

$$\begin{aligned}
&\lim_{r\to 0+} \Big[|R'(r)|^p + \nu \int_0^r \big(\frac{\hat{r}}{r}\big)^{\alpha p/(p-\alpha)} \big(\frac{\hat{r}}{r}\big)^{\nu} \frac{d\hat{r}}{\hat{r}}\Big] \\
&= \lim_{r\to 0+} \Big[|R'(r)|^p + \nu \int_0^r \frac{(W\circ\sigma)(R(\hat{r}))}{(W\circ\sigma)(R(r))} \big(\frac{\hat{r}}{r}\big)^{\nu} \frac{d\hat{r}}{\hat{r}}\Big] = 1\,.
\end{aligned} \tag{4.58}$$

This yields (4.55) as desired, by

$$\lim_{r\to 0+} |R'(r)|^p + \nu \int_0^1 t^{\alpha p/(p-\alpha)}\, t^{\nu}\, \frac{dt}{t} = 1$$

where $\nu = (N-1)p' = (N-1)p/(p-1)$. The proof of the proposition is finished. $\square$

5. Global existence and uniqueness

The results in this section will be needed in the proofs of our main results in the next section (Section 6). We continue with the problem setting from the previous section (Section 4). Throughout this section we assume that the potential W satisfies Hypotheses (H1) and (H2) stated at the beginning of Section 2. Here, we investigate the global existence of a solution to the initial value problem for equation (2.4), i.e., to (4.1) and (4.2), or *equivalently* for the first-order system (2.5), i.e., to (4.3) and (4.4). This means that we assume that $u : [r_0, R) \to \mathbb{R}$ is a C^1 solution, with $|u'|^{p-2}u' \in C^1([r_0, R))$, of the following initial value problem,

$$- r^{-(N-1)} \left(r^{N-1} \, |u'|^{p-2}u'\right)' + W'(u) = 0, \quad r_0 < r < R, \tag{5.1}$$

$$u(r_0) = -\theta, \quad u'(r_0) = 0, \tag{5.2}$$

where $\theta \in \mathbb{R}$ is a given number. Furthermore, we assume that the solution u is defined on a maximal interval of existence of type $J_{\max} = [r_0, R) \subset \mathbb{R}_+$ for some $R \in (r_0, \infty)$ or $R = \infty$.

We have the following global existence result.

Proposition 5.1. *Let $1 < p, \alpha < \infty$ and $r_0 \in \mathbb{R}_+$. Assume that $W : \mathbb{R} \to \mathbb{R}$ is a C^1 function that has the same properties as in* Theorem 3.5*, including* (a) *and* (b) *for some $0 < S < \infty$, together with* Hypothesis (H2) *where $s_0 = S$ is taken. Let $u : J_{\max} \to \mathbb{R}$ be any solution of the initial value problem* (5.1), (5.2)*, where $0 < \theta \le S$ and $J_{\max} = [r_0, R)$ is a maximal interval of existence, for some R with $r_0 < R \le \infty$. If $p \le \alpha$ then assume also $0 < \theta < S$, whereas if $p > \alpha$ and $\theta = S$, then assume that there is $\delta > 0$ such that $u(r) > u(r_0) = -S$ holds for every $r \in (r_0, r_0 + \delta)$. Then u has the following properties:*

- (i) *$R = \infty$ and $|u(r)| < \theta$ for every $r > 0$ and, moreover, both $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$.*
- (ii) $(N-1) \int_{r_0}^{\infty} |u'(\hat r)|^p \, \hat r^{-1} \, \mathrm{d}\hat r = W(0) - W(\theta)$.
- (iii) *In addition, assume $p \ge \frac{(1+\beta)N}{N+\beta}$.*

Then there exist two sequences of positive real numbers, $(0 \le r_0 <)\ r_1 < r_2 < \dots < r_n < \dots$ and $(0 \le r_0 <)\ \varrho_1 < \varrho_2 < \dots < \varrho_n < \dots$, such that $r_{n-1} < \varrho_n < r_n$ and $u'(r_n) = u(\varrho_n) = 0$ hold for each $n = 1, 2, 3, \dots$, together with $r_n \to \infty$ as $n \to \infty$. These two sequences can be chosen to be maximal in the following sense: for each $r \in (r_0, \infty)$ one has also

$$u'(r) = 0 \quad \Longrightarrow \quad r = r_n \ \textit{for some} \ n \in \mathbb{N},$$
$$u(r) = 0 \quad \Longrightarrow \quad r = \varrho_n \ \textit{for some} \ n \in \mathbb{N}.$$

Proof. To prove properties (i) and (ii), let us recall that W is assumed to be even about zero and satisfying $W'(0) = W'(\pm S) = 0$ and $W'(s) = -W'(-s) < 0$ for all $s \in (0, S)$. Here, $0 < S < \infty$ is some number. According to our hypotheses on θ and u, we may assume that there is some $\delta' \in (0, R - r_0)$ such that $(-S \le)$ $-\theta = u(r_0) < u(r) \le 0$ holds for all $r \in (r_0, r_0 + \delta']$. From (5.1) we deduce that also $u'(r) > 0$ must hold for all $r \in (r_0, r_0 + \delta']$. Now it is an easy consequence of (2.7) and (2.8) that $|u(r)| < \theta$ $(\le S)$ holds for all $r \in (r_0, R)$. Notice that $Z(\hat r)$ is a strictly monotonically increasing function of $\hat r$ near any point $r \in (r_0, R)$ at which $u'(r) \ne 0$. Combining these facts with Proposition 4.1 on local existence, we conclude that $R = \infty$, i.e., $J_{\max} = [r_0, \infty)$ is the maximal interval of existence.

Next, we claim that *either* there exists some $r_1 \in (r_0, \infty)$ such that $u'(r_1) = 0$ and $u'(r) > 0$ for all $r \in (r_0, r_1)$, *or else* the function u is strictly monotonically increasing on the entire interval $[r_0, \infty)$ with a monotone limit $u_\infty = \lim_{r\to\infty} u(r) < \theta$ $(\le S)$. The strict inequality follows from (2.7) and (2.8) again.

In the latter alternative we must have $u_\infty = 0$; the case $u_\infty \neq 0$ is excluded as it would entail $W'(u_\infty) \neq 0$ which is impossible. This proves property (i) for the latter alternative. Then property (ii) is derived easily from (2.7) and (2.8) by letting $r \to \infty$.

The additional hypothesis in property (iii), $p \ge \frac{(1+\beta)N}{N+\beta}$, excludes also the case $u_\infty = 0$ which leaves us with the former alternative only, i.e., $u'(r_1) = 0$ and $u'(r) > 0$ for all $r \in (r_0, r_1)$, where $r_1 \in (r_0, \infty)$ is some number. This can be proved by combining eq. (5.1), where $r > r_0$ is large enough, with inequalities (3.6).

Now we may repeat the two alternatives considered above in the interval $[r_1, \infty)$ in place of $[r_0, \infty)$. According to our hypotheses on θ and u, we may assume that there is some $\delta' \in (0, \infty)$ such that $0 \le u(r) < u(r_1)$ $(< \theta \le S)$ holds for all $r \in (r_1, r_1 + \delta']$. From eq. (5.1) we deduce that also $u'(r) < 0$ must hold for all $r \in (r_1, r_1 + \delta']$. Now it is an easy consequence of eqs. (2.7) and (2.8) that $|u(r)| < u(r_1)$ $(< \theta \le S)$ holds for all $r \in (r_1, \infty)$. Notice that $Z(\hat{r})$ is a strictly monotonically increasing function of $\hat{r}$ near any point $r \in (r_1, \infty)$ at which $u'(r) \neq 0$.

Next, we claim that *either* there exists some $r_2 \in (r_1, \infty)$ such that $u'(r_2) = 0$ and $u'(r) < 0$ for all $r \in (r_1, r_2)$, *or else* u is strictly monotonically decreasing on the entire interval $[r_1, \infty)$ with a monotone limit $u_\infty = \lim_{r\to\infty} u(r) > -u(r_1)$ $(> -\theta \ge -S)$. The strict inequality follows from (2.7) and (2.8) again.

As above, in the latter alternative we must have $u_\infty = 0$; the case $u_\infty \neq 0$ is excluded as it would entail $W'(u_\infty) \neq 0$ which is impossible. This proves property (i) for the latter alternative. Then property (ii) is derived easily from eqs. (2.7) and (2.8) by letting $r \to \infty$.

The additional hypothesis in property (iii), $p \ge \frac{(1+\beta)N}{N+\beta}$, excludes also the case $u_\infty = 0$ which leaves us with the former alternative only, i.e., $u'(r_2) = 0$ and $u'(r) < 0$ for all $r \in (r_1, r_2)$, where $r_2 \in (r_1, \infty)$ is some number. This can be proved by combining eq. (5.1), where $r > r_0$ is large enough, with inequalities (3.6).

This recursion process *may* stop after a finite number $n \in \mathbb{N}$ of such steps, thus leaving us with a finite collection of numbers $(0 \le)\ r_0 < r_1 < r_2 < \dots r_n < \infty$ such that $u'(r) > 0$ for every $r \in (r_{k-1}, r_k)$ if k is odd, and $u'(r) < 0$ for every $r \in (r_{k-1}, r_k)$ if k is even, and the monotone limit $u_\infty = \lim_{r\to\infty} u(r) = 0$. This proves property (i). Then property (ii) is derived easily from eqs. (2.7) and (2.8) by letting $r \to \infty$.

If the recursion process *does not* stop after a finite number of steps, we obtain a sequence $\{r_k\}_{k=1}^{\infty}$ characterized in property (iii). This is the case if $p \ge \frac{(1+\beta)N}{N+\beta}$. The other sequence, $\{\varrho_k\}_{k=1}^{\infty}$, is obtained easily using the fact that $u'(r) \gtrless 0$ for all $r \in (r_{k-1}, r_k)$; $k = 1, 2, 3, \dots$. The proposition is proved. □

6. Proofs of the main results

Now we are ready to give proofs of our main results stated in Section 3. We remark that the results stated in §3.1 (Theorems 3.1 and 3.4 and Lemma 3.2, respectively) are special cases of those stated in §3.2 (Theorems 3.5 and 3.7 and Lemma 3.6). Thus, we need to prove only the latter ones, for a general potential W.

6.1. **Proof of Theorem 3.5.** *Proof of Part (I).* It is assumed that $|\theta| \leq S$. We may take $\varepsilon = 1$ without any loss of generality. We begin with the case $\theta \in \{-S, 0, S\}$. Then the constant function $u \equiv -\theta$ on $\mathbb{R}_+$ is a (global) solution to the initial value problem (3.1), (3.2), thanks to $W'(\theta) = 0$.

For $\theta = 0$ this solution is unique by eqs. (2.7) and (2.8). Indeed, given any (local) solution $u : [r_0, r_0 + \delta) \to \mathbb{R}$ of the initial value problem (4.1), (4.2) with $u(r_0) = u'(r_0) = 0$, for some $r_0 \geq 0$ and $\delta > 0$, from eqs. (2.7) and (2.8) we deduce

$$\begin{aligned} 0 &= \frac{1}{p'} |v(r_0)|^{p'} = W(0) - Z(r_0) = W(u(r_0)) - Z(r_0) \\ &\geq W(u(r)) - Z(r) = \frac{1}{p'} |v(r)|^{p'} \geq 0 \quad \text{for } r \in [r_0, r_0 + \delta), \end{aligned}$$

provided $\delta > 0$ is small enough. Here, we have used $W(u(r)) \leq W(u(r_0)) = W(0)$ and $Z(r) \geq Z(r_0) = W(0)$ for $r \in [r_0, r_0 + \delta)$. This forces $v = |u'|^{p-2}u' = 0$ throughout $[r_0, r_0 + \delta)$, i.e., $u \equiv 0$ in $[r_0, r_0 + \delta)$.

For $\theta = \pm S$ the constant function $u \equiv -\theta$ on $\mathbb{R}_+$ is the unique solution to problem (3.1), (3.2) by Corollary 4.4. Indeed, any nonconstant solution $u : [r_0, r_0 + \delta) \to \mathbb{R}$ would have to satisfy either (4.11) (if $r_0 = 0$) or (4.12) (if $r_0 > 0$). But this is impossible for $\alpha \geq p$.

So assume $0 < |\theta| < S$. Let $u : [0, R) \to \mathbb{R}$ be a solution to problem (3.1), (3.2) defined on a maximal interval of existence. Such a number R with $0 < R \leq \infty$ exists by Proposition 4.1 on local existence. It is an easy consequence of eqs. (2.7) and (2.8) that $|u(r)| < \theta$ ($\leq S$) holds for all $r \in (0, R)$. We combine these facts with Proposition 4.1 to conclude that $R = \infty$, i.e., $J_{\max} = \mathbb{R}_+$ is the maximal interval of existence. The uniqueness follows from Lemma 4.2.

Proof of Part (II). It is assumed that $S < |\theta| < S + \zeta$. As above, we may take $\varepsilon = 1$ without any loss of generality. By the symmetry of W, it suffices to treat the case $S < \theta < S + \zeta$. Then the existence and uniqueness of a (local) solution $u : [0, \delta) \to \mathbb{R}$ to problem (3.1), (3.2) follow from Proposition 4.1 and Lemma 4.2, for some $\delta > 0$. If $\delta > 0$ is chosen small enough then we have also $u'(r) < 0$ for all $r \in (0, \delta)$. Now let u be extended to a maximal interval of existence $[0, R)$ for some $R \geq \delta$ (> 0). Clearly, by Lemma 4.2, Case (i), if $u'(r) < 0$ holds for all $r \in (0, R)$, then one can take $\delta = R$.

Proof of Part (III). Finally, let $0 < |\theta| < S$. We start with $\varepsilon = 1$. By the symmetry of W again, it suffices to treat the case $0 < \theta < S$. By Part (I), problem (3.1), (3.2) possesses a unique (global) solution $u : \mathbb{R}_+ \to \mathbb{R}$ satisfying $|u(r)| < \theta$ ($< S$) for all $r > 0$. Also the restriction $p \geq \frac{(1+\beta)N}{N+\beta}$ being assumed, this solution has all properties (i), (ii), and (iii) in Proposition 5.1.

The case of $\varepsilon > 0$ arbitrary is now treated by first renaming both, the variable $r \in \mathbb{R}_+$ and the function $u(r)$ from the case $\varepsilon = 1$, by $\tilde{r}$ and $\tilde{u}(\tilde{r})$, respectively. Then we use the dilation $r = \varepsilon\tilde{r}$ in order to obtain the unique solution $u(r) = \tilde{u}(\varepsilon^{-1}r)$ of problem (3.1), (3.2). The conclusion of Part (III) now follows immediately from property (iii) in Proposition 5.1. In particular, given $R \in (0, \infty)$, we get the (strictly decreasing) sequence of "nonlinear eigenvalues" for the Neumann boundary value problem (1.8), (1.9), i.e., the numbers $\varepsilon_n \equiv \varepsilon_n(\theta, R) > 0$ for $n \in \mathbb{N}$, from the relation $\varepsilon_n^{-1} R = r_n$ for each $n \in \mathbb{N}$. This completes the proof of Theorem 3.5.

6.2. **Proof of Lemma 3.6.** This lemma is an immediate consequence of Proposition 4.5.

6.3. **Proof of Theorem 3.7.** *Proof of Part(I).* It is assumed that $|\theta| < S$. The proof of Part (I) in Theorem 3.5 remains valid in this case also for $p > \alpha$ as we do not use Corollary 4.4 in this part.

Proof of Part (II). Similarly, for $S < |\theta| < S + \zeta$, the proof of Part (II) in Theorem 3.5 applies also to $p > \alpha$.

Proof of Part (III). Also this proof is identical with the proof of Part (III) in Theorem 3.5.

Proof of Part (IV). The proof of this part is a direct consequence of the fact that the constant functions $u \equiv \pm S$ in $\mathbb{R}_+$ are solutions of the initial value problem (3.1), (3.2), combined with Proposition 4.5. The nonuniqueness of a solution $u(r)$ occurs at the point $r = r_0$; it becomes nonconstant right after this point, for $r_0 < r < R$.

If $\theta = -S$ and $u = U_+$ then $|u(r)| < S$ must hold for all $r \in (r_0, R)$, again, by eqs. (2.7) and (2.8). We combine this fact with Proposition 4.1 to conclude that $R = \infty$, i.e., $J_{\max} = \mathbb{R}_+$ is the maximal interval of existence. The uniqueness of $u(r)$ for $r > r_0$ follows from Lemma 4.2.

We have finished the proof of Theorem 3.7.

7. Appendix

Here we prove the following auxiliary result for weighted averages.

Lemma 7.1. *Let $f : [a, b] \to \mathbb{R}$ be a monotonically increasing, continuous function, $-\infty < a < b < \infty$, and let $g : [a, b] \to \mathbb{R}$ be (once) continuously differentiable, such that $g(a) = g(b) = 0$ and $g(t) > 0$ for every $t \in (a, b)$. Then we have*

$$\int_a^b f(t)\, g'(t)\,\mathrm{d}t = -\int_a^b g(t)\,\mathrm{d}f(t) \le 0\,, \tag{7.1}$$

where the equality holds if and only if $f(a) = f(b)$, in which case f is a constant function.

In particular, given $a = 0$, $b = 1$, and any $\mu, \nu \in \mathbb{R}$ with $1 \le \mu < \nu < \infty$, we may take $g(t) = t^\mu - t^\nu$ for $t \in [0, 1]$ to get

$$\mu \int_0^1 f(t)\, t^{\mu-1}\,\mathrm{d}t \le \nu \int_0^1 f(t)\, t^{\nu-1}\,\mathrm{d}t\,, \tag{7.2}$$

where the equality holds if and only if $f(a) = f(b)$, in which case f is a constant function again.

Proof. Formula (7.1) is an easy consequence of the standard formula for integration-by-parts for the Riemann-Stieltjes integral combined with our boundary conditions at the endpoints of the interval $[a, b]$. We refer for details to the textbook by Rudin [18], Chapt. 6, Exercise 17 on p. 141. □

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