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SOLVING OSCILLATORY PROBLEMS USING A BLOCK HYBRID TRIGONOMETRICALLY FITTED METHOD WITH TWO OFF-STEP POINTS

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ABSTRACT. A continuous hybrid method using trigonometric basis (CHMTB) with two 'off-step' points is developed and used to produce three discrete hybrid methods which are simultaneously applied as numerical integrators by assembling them into a block hybrid trigonometrically fitted method (BHTFM) for solving oscillatory initial value problems (IVPs). The stability property of the BHTFM is discussed and the performance of the method is demonstrated on some numerical examples to show accuracy and efficiency advantages.

1. INTRODUCTION

In this article, we consider the first-order differential equation

$$y' = f(x, y), \quad y(a) = y_0, \quad x \in [a, b], \tag{1.1}$$

with oscillating solutions where $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $y, y_0 \in \mathbb{R}^m$. Oscillatory IVPs frequently arise in areas such as classical mechanics, celestial mechanics, quantum mechanics, and biological sciences. Several numerical methods based on the use of polynomial basis functions have been developed for solving this class of important problems (see Lambert [14, 15], Hairer et al in [9], Hairer [10], and Sommeijer [19]). Other methods based on exponential fitting techniques which take advantage of the special properties of the solution that may be known in advance have been proposed (see Simos [18], Vanden et al [20], Vigo-Aguiar et al [21], Franco [5], Fang et al [3], Nguyen et al [16], and Jator et al [12]). The motivation governing the exponentially-fitted methods is inherent in the fact that if the frequency or a reasonable estimate of it is known in advance, these methods will be more advantageous than the polynomial based methods.

The goal of this article is to construct a CHMTB which provides three discrete methods that are combined and applied as a BHTFM which takes the frequency of the solution as a priori knowledge. The coefficients of the BHTFM are functions of the frequency and the stepsize, hence the solutions provided by the proposed method are exact if (1.1) has periodic solutions with known frequencies. We are

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motivated to use the hybrid method in order to increase the order of the method, while preserving good stability properties. Hybrid methods were first proposed to overcome the Dahlquist [2] barrier theorem whereby the conventional linear multistep method was modified by incorporating off-step points in the derivation process (see Gear [6], Gragg et al in [7], Butcher [1], Gupta [8], Lambert [15], and Kohfeld et al in [13]). These methods were shown to enjoy both higher order and good stability properties, but included additional off-grid functions. Gupta [8] noted that the design of algorithms for hybrid methods is more tedious due to the occurrence of off-step functions which increase the number of predictors needed to implement the methods. In order to avoid this deficiency, we adopt a different approach based on a block-by-block implementation instead of the traditional step-by-step implementation, generally performed in a predictor-corrector mode.

Hence, we adopted the approach given in Jator et al in [12], where the CHMTB is used to generate a main and two additional methods which are combined and used as a BHTFM to simultaneously produce approximations

$$\{y_{n+\mu}, y_{n+v}, y_{n+1}\} \text{ at a block of points } \{x_{n+\mu}, x_{n+v}, x_{n+1}\},$$

$h = x_{n+1} - x_n$, $n = 0, \dots, N-1$, on a partition $[a, b]$, where $\mu, v \in (0, 1)$, $a, b \in \mathbb{R}$, h is the constant stepsize, n is a grid index and $N > 0$ is the number of steps. Block methods have also been considered by Shampine and Watts [17]. We emphasize that the BHTFM simultaneously generates approximations $\{y_{n+\mu}, y_{n+v}, y_{n+1}\}$ to the exact solutions $\{y(x_{n+\mu}), y(x_{n+v}), y(x_{n+1})\}$.

To apply the block method at the next block to obtain y_{n+2}, the only necessary starting value is y_{n+1}, and the loss of accuracy in y_{n+1} does not affect subsequent points, thus the order of the algorithm is maintained. It is unnecessary to make a function evaluation at the initial part of the new block. Thus, at all blocks except the first, the first function evaluation is already available from the previous block.

The organization of this article is as follows. In Section 2, we obtain a trigonometric basis representation $U(x)$ for the exact solution $y(x)$ which is used to generate two discrete methods which are combined into a BHTFM for solving (1.1). The analysis and implementation of the BHTFM are discussed in Section 3. Numerical examples are given in Section 4 to show the accuracy and efficiency of the BHTFM. Finally, we give some concluding remarks in Section 5.

2. Development of method

In this section, our objective is to construct a CHMTB which produces three discrete methods as by-products. The main method has the form

$$y_{n+1} = y_n + h(\beta_0(u) f_n + \beta_1(u) f_{n+1} + \beta_v(u) f_{n+v} + \beta_\mu(u) f_{n+\mu}), \tag{2.1}$$

and the additional methods are given by

$$\begin{aligned} y_{n+v} &= y_n + h(\hat{\beta}_0(u) f_n + \hat{\beta}_1(u) f_{n+1} + \hat{\beta}_v(u) f_{n+v} + \hat{\beta}_\mu(u) f_{n+\mu}), \\ y_{n+\mu} &= y_n + h(\check{\beta}_0(u) f_n + \check{\beta}_1(u) f_{n+1} + \check{\beta}_v(u) f_{n+v} + \check{\beta}_\mu(u) f_{n+\mu}), \end{aligned} \tag{2.2}$$

where $u = wh$, $\beta_j(u)$, $\hat{\beta}_j(u)$, $\check{\beta}_j(u)$, $\beta_v(u)$, $\hat{\beta}_v(u)$, $\check{\beta}_v(u)$, $\beta_\mu(u)$, $\hat{\beta}_\mu(u)$ and $\check{\beta}_\mu(u)$, $j = 0, 1$, are coefficients that depend on the stepsize and frequency. We note that y_{n+v} and $y_{n+\mu}$ are the numerical approximation to the analytical solutions $y(x_{n+v})$, and $y(x_{n+\mu})$ respectively and $f_{n+v} = f(x_{n+v}, y_{n+v})$, $f_{n+\mu} = f(x_{n+\mu}, y_{n+\mu})$, $f_{n+j} = f(x_{n+j}, y_{n+j})$ with $j = 0, 1$. To obtain equations (2.1) and (2.2), we proceed by seeking to approximate the exact solution $y(x)$ on the interval $[x_n, x_n + h]$ by the interpolating function $U(x)$ of the form

$$U(x) = a_0 + a_1 x + a_2 x^2 + a_3 \sin(wx) + a_4 \cos(wx), \tag{2.3}$$

where a_0, a_1, a_2, a_3 and a_4 are coefficients that must be uniquely determined. We then impose that the interpolating function equation (2.3) coincides with the analytical solution at the end point x_n to obtain the equation

$$U(x_n) = y_n. \tag{2.4}$$

We also require that the function (2.3) satisfy the differential equation (1.1) at the points $x_{n+\mu}, x_{n+v}, x_{n+j}$, $j = 0, 1$ to obtain the following set of three equations:

$$U'(x_{n+\mu}) = f_{n+\mu}, \quad U'(x_{n+v}) = f_{n+v}, \quad U'(x_{n+j}) = f_{n+j}, \quad j = 0, 1. \tag{2.5}$$

Equations (2.4) and (2.5) lead to a system of five equations which is solved by Cramer's rule to obtain a_j, $j = 0, 1, 2, 3, 4$. Our continuous CHMTB is constructed by substituting the values of a_j into equation (2.3). After some algebraic manipulation, the CHMTB is expressed in the form

$$U(x) = y_n + h(\beta_0(w,x) f_n + \beta_1(w,x) f_{n+1} + \beta_v(w,x) f_{n+v} + \beta_\mu(w,x) f_{n+\mu}), \tag{2.6}$$

where w is the frequency, $\beta_0(w,x)$, $\beta_1(w,x)$, $\beta_v(w,x)$, and $\beta_\mu(w,x)$, are continuous coefficients. The continuous method (2.6) is used to generate the main method of the form (2.1) and two additional methods of the form (2.2) by choosing, $v = \frac{1}{2}$ and $\mu = \frac{1}{4}$. Thus, evaluating (2.6) at $x = \{x_{n+1}, x_{n+\frac{1}{2}}, x_{n+\frac{1}{4}}\}$ and letting $u = wh$, we obtain the coefficients of (2.1) and (2.2) as follows:

$$\begin{aligned} \beta_0 &= \frac{\cos(\frac{u}{8})(\csc(\frac{u}{4})^3)\sin(\frac{u}{8})(u - 2\sin(\frac{u}{2}))}{2u} \\ \beta_1 &= -(\frac{\cos(\frac{u}{8})\csc(\frac{u}{4})^3\sin(\frac{u}{8})(-u + 2\sin(\frac{u}{2}))}{2u}) \\ \beta_v &= -(\frac{\cos(\frac{u}{8})\csc(\frac{u}{4})^3\sin(\frac{u}{8})(u\cos(\frac{u}{2}) - 2\sin(\frac{u}{2}))}{u}), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \hat{\beta}_0 &= \frac{(\csc(\frac{u}{8})^2)(u - 4\sin(\frac{u}{4}))}{8u} \\ \hat{\beta}_v &= \frac{(\csc(\frac{u}{8})^2)(u - 4\sin(\frac{u}{4}))}{8u} \\ \hat{\beta}_\mu &= -(\frac{(\csc(\frac{u}{8})^2)(u\cos(\frac{u}{4}) - 4\sin(\frac{u}{4}))}{4u}) \\ \check{\beta}_0 &= \frac{(\csc(\frac{u}{4})^3)\sin(\frac{u}{8})(8u\cos(\frac{u}{8}) + 3u\cos(\frac{(3u)}{8}) - 8(2\sin(\frac{(3u)}{8}) + \sin(\frac{(5u)}{8})))}{16u} \\ \check{\beta}_1 &= -(\frac{(\csc(\frac{u}{4})^3)(u\cos(\frac{u}{8}) - 8\sin(\frac{u}{8}))\sin(\frac{u}{8})}{16u}) \\ \check{\beta}_v &= \frac{(3 + 3\cos(\frac{u}{4}) + \cos(\frac{u}{2}))(\csc(\frac{u}{4})^3)(u\cos(\frac{u}{8}) - 8\sin(\frac{u}{8}))\sin(\frac{u}{8})}{8u} \\ \check{\beta}_\mu &= -(\frac{(\cos(\frac{u}{8})^2)(\csc(\frac{u}{4})^3)\sin(\frac{u}{8})(3u\cos(\frac{u}{8}) + 3u\cos(\frac{(3u)}{8}) - 16\sin(\frac{(3u)}{8}))}{4u}), \end{aligned} \tag{2.8}$$

with $\hat{\beta}_1 = 0$ and $\beta_\mu = 0$.

3. Error analysis and stability

3.1. **Local truncation error.** The Taylor series is used for small values of u (see Simos [18]). Thus the coefficients in equation (2.7) can be expressed as

$$\begin{aligned}
\beta_0 &= \frac{1}{6} + \frac{u^2}{720} + \frac{u^4}{80640} + \frac{u^6}{9676800} + \frac{u^8}{1226244096} + \frac{691u^{10}}{111588212736000} + \dots \\
\beta_1 &= \frac{1}{6} + \frac{u^2}{720} + \frac{u^4}{80640} + \frac{u^6}{9676800} + \frac{u^8}{1226244096} + \frac{691u^{10}}{111588212736000} + \dots \\
\beta_v &= \frac{2}{3} - \frac{u^2}{360} - \frac{u^4}{40320} - \frac{u^6}{4838400} - \frac{u^8}{613122048} - \frac{691u^{10}}{55794106368000} + \dots
\end{aligned}$$

and the coefficients in equation (2.8) can be expressed as

$$\begin{aligned}
\hat{\beta}_0 &= \frac{1}{12} + \frac{u^2}{5760} + \frac{u^4}{2580480} + \frac{u^6}{1238630400} + \frac{u^8}{627836977152} \\
&\quad + \frac{691u^{10}}{228532659683328000} + \dots \\
\hat{\beta}_v &= \frac{1}{12} + \frac{u^2}{5760} + \frac{u^4}{2580480} + \frac{u^6}{1238630400} + \frac{u^8}{627836977152} \\
&\quad + \frac{691u^{10}}{228532659683328000} + \dots \\
\hat{\beta}_\mu &= \frac{1}{3} - \frac{u^2}{2880} - \frac{u^4}{1290240} - \frac{u^6}{619315200} - \frac{u^8}{313918488576} \\
&\quad - \frac{691u^{10}}{114266329841664000} + \dots \\
\check{\beta}_0 &= \frac{37}{384} + \frac{67u^2}{184320} + \frac{401u^4}{165150720} + \frac{1649u^6}{79272345600} + \frac{1711u^8}{9132174213120} \\
&\quad + \frac{12094087u^{10}}{7313045109866496000} + \dots \\
\check{\beta}_1 &= \frac{1}{384} + \frac{13u^2}{184320} + \frac{37u^4}{33030144} + \frac{1091u^6}{79272345600} + \frac{2087u^8}{14350559477760} \\
&\quad + \frac{10191073u^{10}}{7313045109866496000} + \dots \\
\check{\beta}_v &= -\frac{7}{192} + \frac{7u^2}{46080} - \frac{11u^4}{11796480} - \frac{29u^6}{1415577600} - \frac{12503u^8}{50226958172160} \\
&\quad - \frac{659969u^{10}}{261180182495232000} + \dots \\
\check{\beta}_\mu &= \frac{3}{16} - \frac{3u^2}{5120} - \frac{3u^4}{1146880} - \frac{31u^6}{2202009600} - \frac{13u^8}{155021475840} \\
&\quad - \frac{11747u^{10}}{22571126882304000} + \dots .
\end{aligned}$$

Thus, the Local Truncation Errors (LTEs) for methods (2.1) and (2.2) are given by

$$\begin{aligned} \mathrm{LTE}(2.1) &= -\frac{h^5}{2880}(w^2y^{(3)}(x_n)+y^{(5)}(x_n)),\\ \mathrm{LTE}(2.2) &= -\frac{h^5}{92160}(w^2y^{(3)}(x_n)+y^{(5)}(x_n)),\\ \mathrm{LTE}(2.2) &= -\frac{53h^5}{1474560}(w^2y^{(3)}(x_n)+y^{(5)}(x_n)). \end{aligned} \tag{3.1}$$

Remark 3.1. We noticed that as $u \to 0$, the method (2) reduces to the fourth-order method of Gragg and Stetter [7], which is based on polynomial basis functions.

3.2. **Stability.** The BHTFM is constructed by combining equations (2.1) and (2.2), where the coefficients of the method are explicitly given by equations (2.7) and (2.8). We then define the block-by-block method for computing vectors $Y_0, Y_1, \cdots$ in sequence (see [4]). Let the $\eta-$vector ($\eta = 3$ is the number of points within the block) $Y_\gamma, Y_{\gamma-1}, F_\gamma, F_{\gamma-1}$ be given as

$$\begin{aligned} Y_\gamma &= (y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, y_{n+1})^T, \quad Y_{\gamma-1} = (y_{n-\frac{1}{4}}, y_{n-\frac{1}{2}}, y_n)^T,\\ F_\gamma &= (f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+1})^T, \quad F_{\gamma-1} = (f_{n-\frac{1}{4}}, f_{n-\frac{1}{2}}, f_n)^T, \end{aligned}$$

then the 1-block 3-point method for equation (1.1) is given by

$$Y_\gamma = \sum_{i=1}^{1} A^{(i)}Y_{\gamma-i} + \sum_{i=0}^{1} B^{(i)}F_{\gamma-i}, \tag{3.2}$$

where $A^{(i)}, B^{(i)}$, $i = 0, 1$ are 3×3 matrices whose entries are given by the coefficients of (2.1) and (2.2).

Zero-stability.

Definition 3.2. The block method (3.2) is zero stable provided the roots R_j, $j = 1, 2, 3$ of the first characteristic polynomial $\rho(R)$ specified by

$$\rho(R) = \det\Big[\sum_{i=0}^{1} A^{(i)}R^{1-i}\Big] = 0, \quad A^{(0)} = -I, \tag{3.3}$$

satisfies $|R_j| \le 1$, $j = 1, 2, 3$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1 (see [4]).

Consistency of BHTFM. We note that the block method (3.2) is consistent as it has order $p > 1$. We see from equation (3.3) and definition (3.2) that the block method (3.2) is zero-stable since $\rho(R) = R^2(R-1) = 0$ satisfies $|R_j| \le 1$, $j = 1, 2, 3$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1. The block method (3.2) is thus convergent, as zero-stability plus consistency equals convergence.

Linear stability of the BHTFM. To analyze the linear stability of the BHTFM, we apply the method to the test equation $y' = \lambda y$, where λ is expected to run through the (negative) eigenvalues of the Jacobian matrix. Then an application of (3.2) to the test equation yields

$$Y_\gamma = M(q;u)Y_{\gamma-i}, \quad q = h\lambda, \quad u = wh, \tag{3.4}$$

where

$$M(q;u) = (A^{(0)} - qB^{(0)})^{-1}(A^{(1)} + qB^{(1)}),$$

where the matrix $M(q;u)$ is the amplification matrix which determines the stability of the method.

Definition 3.3. A region of stability is a region in the $q-u$ plane, throughout which $|\rho(q;u)| \le 1$, where $\rho(q;u)$ is the spectral radius of $M(q;u)$.

It is easily seen that the eigenvalues of $M(q;u)$ are $\lambda_1 = 0$, $\lambda_2 = 0$, and

$$\begin{aligned}
\lambda_3 = \Big(&-16(2+q)(q^2+u^2)\cos(\frac{u}{8}) + (6q^3 + 16u^2 + 6qu^2 + q^2(16+u^2))\cos(\frac{3u}{8}) \\
&+ 16q^2\cos(\frac{5u}{8}) + 10q^3\cos(\frac{5u}{8}) + 16u^2\cos(\frac{5u}{8}) + 10qu^2\cos(\frac{5u}{8}) \\
&+ 3q^2u^2\cos(\frac{5u}{8}) + q^3u\sin(\frac{3u}{8}) + 3q^3u\sin(\frac{5u}{8})\Big) \\
&\div \Big(16(-2+q)(q^2+u^2)\cos(\frac{u}{8}) + (-10q^3 + 16u^2 - 10qu^2 \\
&+ q^2(16+3u^2))\cos(\frac{3u}{8}) + 16q^2\cos(\frac{5u}{8}) - 6q^3\cos(\frac{5u}{8}) + 16u^2\cos(\frac{5u}{8}) \\
&- 6qu^2\cos(\frac{5u}{8}) + q^2u^2\cos(\frac{5u}{8}) - 3q^3u\sin(\frac{3u}{8}) - q^3u\sin(\frac{5u}{8})\Big).
\end{aligned}$$

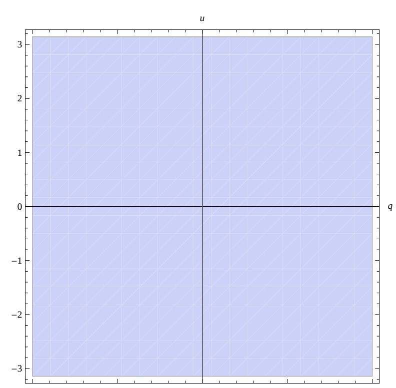

Figure 1. The shaded region represents the truncated region of absolute stability

We observed that in the $q-u$ plane the BHTFM is stable for $q \in [-100, 100]$, and $u \in [-\pi, \pi]$. Figure 1 is a plot of the stability region. Figure 2 shows the zeros and poles of λ_3.

3.3. **Implementation.** We emphasize that methods (2.1) and (2.2) are combined to form the block method (3.2), which is implemented to solve (1.1) without requiring starting values and predictors. For instance, if we let $n = 0$ and $\gamma = 1$ in (3.2), then $(y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_1)^T$ are simultaneously obtained over the sub-interval $[x_0, x_1]$, as y_0 is known from the IVP. Similarly, if $n = 1$ and $\gamma = 2$, then $(y_{\frac{5}{4}}, y_{\frac{3}{2}}, y_2)^T$ are simultaneously obtained over the sub-interval $[x_1, x_2]$, as y_1 is known from the previous block, and so on; until we reach the final sub-interval $[x_{N-1}, x_N]$. We note that for

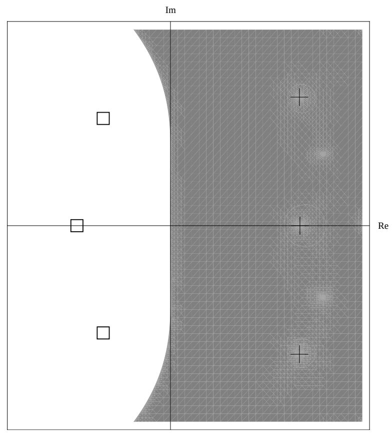

Figure 2. λ_3 has three zeros($\Box$) and no poles($+$) in $\mathbb{C}^-$, hence the BHTFM is A_0-stable.

linear problems, we solve (1.1) directly using the feature solve[ ] in Matlab, while nonlinear problems were solved by implementing the Newton's method in Matlab enhanced by the feature fsolve[ ].

4. Numerical examples

In this section, we give numerical examples to illustrate the accuracy (small errors) and efficiency (fewer number of function evaluations (FNCs)) of the BHTFM. We find the approximate solution on the partition π_N, where

$$\pi_N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b, \tag{4.1}$$

and we give the errors at the endpoints calculated as Error$=y_N - y(x_N)$. We note that the method requires only three function evaluations per step and in general requires $(3N+1)$ FNCs on the entire interval. All computations were carried out using a written code in Matlab.

Example 4.1. We consider the following inhomogeneous IVP by Simos [18].

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11, \quad x \in [0, 1000]$$

where the analytical solution is given by

$$\text{Exact: } y(x) = \cos(10x) + \sin(10x) + \sin(x).$$

The exponentially-fitted method in Simos [18] is fourth order and hence comparable to our method, BHTFM. We see from Table (1) that BHTFM is more efficient than the method in Simos [18]. We also compare the computational efficiency of the two methods by considering the FNCs over N integration steps for each method. Our method, BHTFM, requires only $3N + 1$ function evaluations in N steps compared to $4N$ function evaluations in N steps for the method in Simos [18]. Hence for this example, BHTFM performs better.

Example 4.2. We consider the IVP (see Vigo-Aguiar et al [21])

$$y'' + K^2 y = K^2 x, \quad y(0) = 10^{-5}, \quad y'(0) = 1 - K10^{-5}\cot(K), \quad x \in [0, 100]$$

	BHTFM		Simos [18]	
N	—Error—	FNCs	—Error—	FNCs
1000	1.2×10^{-3}	6002	1.4×10^{-1}	8000
2000	1.2×10^{-3}	12002	3.5×10^{-2}	16000
4000	1.4×10^{-5}	24002	1.1×10^{-3}	32000
8000	1.5×10^{-7}	48002	8.4×10^{-5}	64000
16000	8.7×10^{-9}	96002	5.5×10^{-6}	128000
32000	1.1×10^{-9}	192002	3.5×10^{-7}	256000

TABLE 1. Results, with $\omega = 10$, for example (4.1)

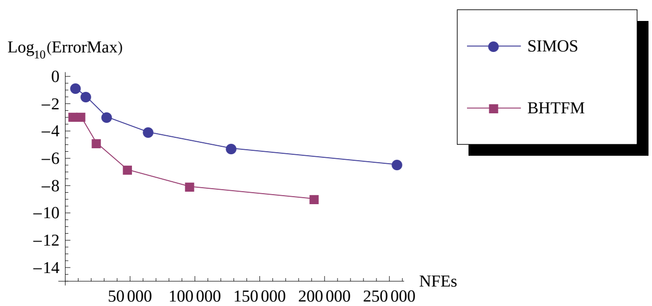

FIGURE 3. Efficiency curves for example (4.1)

where $K = 314.16$, and we choose $\omega = 314.16$. The analytical solution is given by

$$\text{Exact: } y(x) = x + 10^{-5}(\cos(Kx) - \cot(K)\sin(Kx)).$$

	BHTFM			CHEBY24	
N	—Error—	FNCs	N	—Error—	FNCs
9	5.07×10^{-11}	48	9	1.84×10^{-11}	450
20	9.17×10^{-12}	122			
40	4×10^{-15}	242			

TABLE 2. Results, with $\omega = 314.16$, for example (4.2) on $[0, 100]$.

	BHTFM			CHEBY1	
N	—Error—	FNCs	N	—Error—	FNCs
2	4.13×10^{-17}	14	1	1×10^{-16}	8

TABLE 3. Results, with $\omega = 314.16$, for example (4.2) on $[0, 1]$

This problem demonstrates the performance of BHTFM on a well-known oscillatory problem. We compare the results from BHTFM with the Dissipative Chebyshev exponential-fitted methods, CHEBY24 and CHEBY1 used in Vigo-Aguiar et al [21]. We see that BHTFM uses fewer number of function evaluations with better accuracy than CHEBY24 that is designed to use fewer number of steps. Integrating in the interval $[0, 1]$ with a stepsize equal to half the total length of the interval, we obtain an error of order 10^{-17}. Hence BHTFM is a more efficient integrator. We note that compared with the methods CHEBY24 and CHEBY1 which use stepsizes considerably larger than those used in multistep methods, BHTFM is very competitive to CHEBY1 and superior to CHEBY24.

Example 4.3. We consider the nonlinear perturbed system on the range $[0, 10]$, with $\epsilon = 10^{-3}$ (see Fang et al [3]).

$$y_1'' + 25y_1 + \epsilon(y_1^2 + y_2^2) = \epsilon\phi_1(x), \quad y_1(0) = 1, \quad y_1'(0) = 0,$$
$$y_2'' + 25y_2 + \epsilon(y_1^2 + y_2^2) = \epsilon\phi_2(x), \quad y_2(0) = \epsilon, \quad y_2'(0) = 5,$$

where

$$\phi_1(x) = 1 + \epsilon^2 + 2\epsilon\sin(5x + x^2) + 2\cos(x^2) + (25 - 4x^2)\sin(x^2),$$
$$\phi_2(x) = 1 + \epsilon^2 + 2\epsilon\sin(5x + x^2) - 2\sin(x^2) + (25 - 4x^2)\cos(x^2),$$

and the exact solution is given by

$$\text{Exact: } y_1(x) = \cos(5x) + \epsilon\sin(x^2), \quad y_2(x) = \sin(5x) + \epsilon\cos(x^2),$$

represents a periodic motion of constant frequency with small perturbation of variable frequency.

BHTFM		ARKN(5)		TFARKN(5)	
N	$-\log_{10}(\text{Err})$	N(rejected)	$-\log_{10}(\text{Err})$	N(rejected)	$-\log_{10}(\text{Err})$
50	4.04	42(15)	2.82	29(6)	2.78
90	5.04	86(7)	4.96	88(9)	5.33
170	6.07	260(5)	7.16	262(8)	7.85

Table 4. Results, with $\omega = 5$, for example (4.3)

We use this problem to demonstrate the performance of the BHTDA on a nonlinear perturbed system. This problem was also solved by Fang et al [3] using a variable stepsize fifth-order trigonometrically fitted Runge-Kutta-Nyström method TFARKN5(3) and a fifth-order Runge-Kutta-Nyström method (ARKN5(3)) which was constructed by Franco [5]. We compare the maximum global error (Err $= \max|y(x) - y|$) for the three methods in Table 4. We remark that TFARKN5(3) and ARKN5(3) are expected to perform better because they are exact when the solution involves a linear combination of trigonometric functions as well as implemented as a variable-step method. However, BHTFM which is implemented using a fixed step-size is highly competitive to them.

Example 4.4. We consider the nonlinear Duffing equation which was also solved by Simos [18] and Ixaru et al [11]

$$y'' + y + y^3 = B\cos(\Omega x), \quad y(0) = C_0, \quad y'(0) = 0.$$

The analytical solution is given by

$$\text{Exact:} y(x) = C_1\cos(\Omega x) + C_2\cos(3\Omega x) + C_3\cos(5\Omega x) + C_4\cos(7\Omega x),$$

where $\Omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$, $C_1 = 0.200179477536$, $C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-6}$, $C_4 = 0.374 \times 10^{-9}$. We choose $\omega = 1.01$

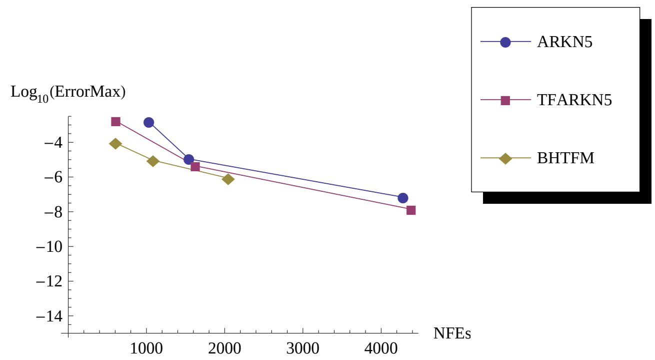

FIGURE 4. Efficiency curves for example (4.3)

BHTFM		Simos		Ixaru et al	
N	—Error—	N	—Error—	N	—Error—
150	1.3×10^{-3}	300	1.7×10^{-3}	300	1.1×10^{-3}
300	5.6×10^{-5}	600	1.9×10^{-4}	600	5.4×10^{-5}
600	3.2×10^{-6}	1200	1.4×10^{-5}	1200	1.9×10^{-6}
1200	1.7×10^{-7}	2400	8.7×10^{-7}	2400	6.2×10^{-8}

TABLE 5. Results, with $\omega = 1.01$, for example (4.4)

We compare the end-point global errors for BHTFM with the fourth order methods in Simos [18] and Ixaru et al [11]. We see from Table 5 that the results produced by BHTFM are better than those given Simos [18], as BHTFM produces better error magnitude while using only half the number of steps needed by Simos [18]. BHTFM is very competitive to the method used by Ixaru et al [11].

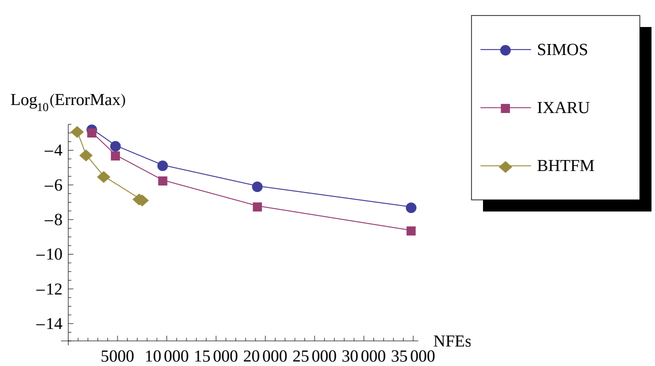

FIGURE 5. Efficiency curves for example (4.4)

Example 4.5. A nearly sinusoidal problem. We consider the following IVP on the range $0 \le t \le 10$, (see Nguyen et al [16, p. 205])

$$y_1' = -2y_1 + y_2 + \sin(t), \quad y_1(0) = 2,$$
$$y_2' = -(\beta+2)y_1 + (\beta+1)y_2 + \sin(t) - \cos(t), \quad y_2(0) = 3.$$

We choose $\beta = -3$ and $\beta = -1000$ to illustrate the phenomenon of stiffness. Given the initial conditions $y_1(0) = 2$ and $y_2(0) = 3$, the exact solution is β-independent and is given by

$$\text{Exact: } y_1(t) = 2\exp(-t) + \sin(t), \quad y_2(t) = 2\exp(-t) + \cos(t).$$

BHTFM with ($\beta = -3$)			Nguyen et al [16] with ($\beta = -3$)		
N	—Error—	FNCs	N	—Error—	FNCs
6	8.9×10^{-6}	38	10	5.4×10^{-6}	47
10	9.0×10^{-7}	62	19	8.3×10^{-8}	88
19	5.8×10^{-8}	116	23	4.5×10^{-4}	113

Table 6. Results, with $\omega = 1$, for example (4.5) with $\beta = -3$.

BHTFM with ($\beta = -1000$)			Nguyen et al [16] with ($\beta = -1000$)		
N	—Error—	FNCs	N	—Error—	FNCs
6	8.9×10^{-6}	38	13	1.0×10^{-6}	61
10	9×10^{-7}	62	16	1.6×10^{-7}	76
13	2.9×10^{-7}	80	21	7.0×10^{-8}	98
16	1.1×10^{-7}	99			
21	3.8×10^{-8}	128			

Table 7. Results, with $\omega = 1$, for example (4.5) with $\beta = -1000$.

This example is chosen to demonstrate the performance of BHTFM on stiff problems. We compute the solutions to Example (4.5) with $\beta = -3, -1000$. We obtain comparable or better absolute errors than Nguyen et al ([16]). This efficiency is achieved using fewer number of blocks and less function evaluations than Nguyen et al ([16]). For example when $\beta = -3$, our method generates a solution with error magnitude 10^{-6} involving just 6 blocks and 38 function evaluations, whereas [16] attains the same error magnitude using 10 blocks and 47 function evaluations. When $\beta = -1000$ and using 10 blocks, BHTFM generates a solution with error magnitude 10^{-7} involving 62 function evaluations, whereas [16] attains an error magnitude of 10^{-6} while using 13 blocks. We see that BHTFM performs better using fewer blocks and is competitive to Nguyen et al ([16]) which is of order six and is thus expected to do better.

Example 4.6. Linear Kramarz problem. We consider the following second-order IVP, (see Nguyen et al [16, p. 204])

$$y''(t) = \begin{pmatrix} 2498 & 4998 \\ -2499 & -4999 \end{pmatrix} y(t), \quad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0 \le t \le 100.$$

Exact: $y(t) = (2\cos(t), -\cos(t))^T$.

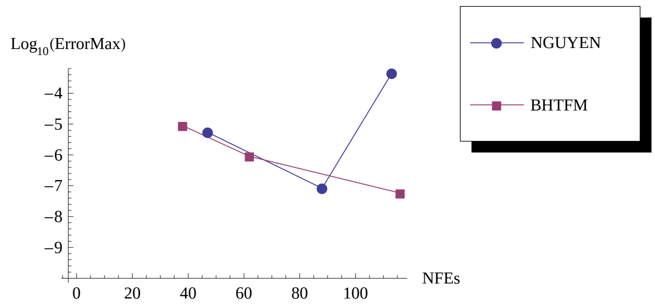

FIGURE 6. Efficiency curves for example (4.5) with $\beta = -3$

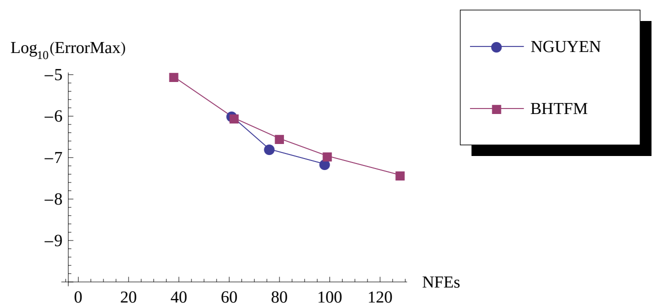

FIGURE 7. Efficiency curves for example (4.5) with $\beta = -1000$

We use this example to show the efficiency of BHTFM on linear systems. Nguyen et al [16] used the "trigonometric implicit Runke-Kutta", TIRK3, method to solve the above linear Kramarz problem. Clearly, BHTFM performs better as seen in Table 8.

	BHTFM			Nguyen et al [16]	
N	—Error—	FNCs	N	—Error—	FNCs
10	8.3×10^{-15}	144	73	3.3×10^{-12}	327
30	5×10^{-14}	364	142	0.9×10^{-11}	707
40	7.2×10^{-14}	484	170	3.7×10^{-12}	811
43	9.5×10^{-14}	520	-	-	-

TABLE 8. Results, with $\omega = 1$, for example (4.6)

4.1. **Estimating the frequency.** A preliminary testing indicates that a good estimate of the frequency can be obtained by demanding that $LTE(2) - LTE(3) = 0$, and solving for the frequency quadratically. That is, solve for ω given that

$$\Big(-\frac{h^5}{2880}(w^2y^{(3)}(x_n)+y^{(5)}(x_n))\Big)-\Big(-\frac{h^5}{92160}(w^2y^{(3)}(x_n)+y^{(5)}(x_n))\Big)=0,$$

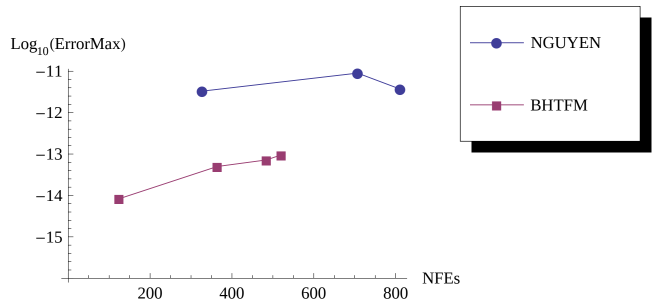

Figure 8. Efficiency curves for example (4.6)

where $y^{(j)}$, $j = 2, \dots, 5$ are derivatives that can be obtained analytically from the given differential equation or could be calculated via the Taylor series expansion. We used this procedure to calculate ω for the problem given in example (4.1) and obtained

$$\omega = \pm\sqrt{(9091/91)} \approx \pm 9.99505,$$

which approximately gives the known frequency $\omega = 10.0$. Hence, this procedure is interesting and will be seriously considered in our future research.

Conclusion. We have proposed a BHTFM for solving periodic IVPs. The BHTFM is A_0-stable and hence, an excellent candidate for solving stiff IVPs. This method has only two 'off-step' points and has the advantages of being self-starting, having good accuracy with order 4, and requiring only three functions evaluation at each integration step. We have presented representative numerical examples that are linear, non-linear, stiff and highly oscillatory. These examples show that the BHTFM is more accurate and efficient than those in Nguyen et al [16], Simos [18], Vigo-Aguiar et al [21], and Fang [3]. Details of the numerical results are displayed in Tables 1, 2, 3, 4, 5, 6, 7, 8 and the efficiency curves are presented in Figures 3, 4, 5, 6, 7, 8 . Our future research will incorporate a technique for accurately estimating the frequency as suggested in subsection 4.1 as well as implementing the method in a variable step mode.

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