EXISTENCE OF POSITIVE SOLUTIONS FOR A SUPERLINEAR ELLIPTIC SYSTEM WITH NEUMANN BOUNDARY CONDITION

JUAN C. CARDEÑO, ALFONSO CASTRO

Abstract. We prove the existence of a positive solution for a class of nonlinear elliptic systems with Neumann boundary conditions. The proof combines extensive use of a priori estimates for elliptic problems with Neumann boundary condition and Krasnoselskii’s compression-expansion theorem.

1. Introduction

The purpose of this paper is to prove that the system

\[-\Delta u + \alpha u = \beta v + f_1(x,u,v) \quad \text{in } \Omega\]
\[-\Delta v + \delta v = \gamma u + f_2(x,u,v) \quad \text{in } \Omega\]
\[\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{in } \partial \Omega,\]

has a nontrivial positive solution. In (1.1) \(\Delta\) denotes the Laplacian operator, \(\Omega \subset \mathbb{R}^N\) is a smooth bounded domain, and \(\alpha > 0, \beta > 0, \gamma > 0, \delta > 0\) are real parameters. We also assume that \(f_1(x,u,v), f_2(x,u,v)\) are measurable in \(x\), differentiable in \((u,v)\), and bounded on bounded sets. Our main result reads as follows.

Theorem 1.1. If there exist \(b \in (1, \min\{2, (N+1)/(N-1)\})\), \(m > 0\), and \(M > 0\) such that

\[m(u+v)^b \leq f_i(x,u,v) \leq M(u+v)^b \quad \text{for } i = 1, 2, u, v \geq 0,\]  

and \(\beta \gamma < \alpha \delta\), then the problem (1.1) has a positive solution.

The main tool in our proofs is Krasnoselskii’s compression-expansion theorem (see Theorem 1.2 below) which we state for sake of completeness. For a proof of this theorem the reader is referred to [12, Theorem 13.D]. To apply Theorem 1.2 to Theorem 1.1 in Section 3 we use of a priori estimates for elliptic equation with Neumann boundary conditions, see [11].

2000 Mathematics Subject Classification. 35J20, 35J60, 35B38.

Key words and phrases. Semilinear elliptic system; a priori estimates; ordered Banach space; contraction-expansion operator; Neumann boundary condition.

©2014 Texas State University - San Marcos.

Published February 10, 2014.
Theorem 1.2. Let $X$ be a real ordered Banach space with positive cone $K$. If $\Upsilon : K \to K$ is a compact operator and there exist real numbers $0 < R < \bar{R}$ such that
\[
\Upsilon(x) \not\leq x, \text{ for } x \in K, \parallel x \parallel = R,
\]
\[
\Upsilon(x) \not\geq x, \text{ for } x \in K, \parallel x \parallel = \bar{R}.
\]
then $\Upsilon$ has a fixed point with $\parallel x \parallel \in (R, \bar{R})$.

There is rich literature on systems like (1.1) in the presence of variational structure and Dirichlet boundary condition, see [2, 3, 4, 6, 7, 8]. Costa and Magalhaes [3] study system (1.1) for nonlinearities with subcritical growth. The reader may consult [2] for applications of the Mountain Pass Lemma to the study of fourth order systems. In [8], (1.1) is studied for Lipschitzian nonlinearities and $\alpha = \delta = \lambda_1$, where $\lambda_1$ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in $\Omega$.

For a survey on the study of elliptic systems the reader is referred to [4].

Throughout this paper we denote by $\parallel \cdot \parallel_p$ the norm in $L^p(\Omega)$ and by $\parallel \cdot \parallel_{k,p}$ the norm in the Sobolev space $W^{k,p}(\Omega)$ (see [1]).

2. Linear Analysis

In this section we study the linear problem
\[
-\Delta u + \alpha u - \beta v = P_1(x) \quad \text{in } \Omega
\]
\[
-\Delta v - \gamma u + \delta v = P_2(x) \quad \text{in } \Omega
\]
\[
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega,
\]
where $P_1(x) \geq 0, P_2(x) \geq 0, \alpha > 0, \beta > 0, \gamma > 0,$ and $\delta > 0$.

Lemma 2.1. For each $P_1, v \in L^2(\Omega)$, then the equation
\[
-\Delta u + \alpha u = P_1(x) + \beta v \quad \text{in } \Omega
\]
\[
\frac{\partial u}{\partial n} = 0 \quad \text{in } \partial \Omega,
\]
has a unique solution. Moreover, there exists $c > 0$, independent of $(P_1, v)$, such that
\[
\parallel u \parallel_{1,2} \leq c \parallel P_1 + \beta v \parallel_2.
\]

Proof. Let $H$ be the Sobolev space $H^1(\Omega)$, and $B : H \times H \to \mathbb{R}$ defined by $B[u, v] = \int_{\Omega} \nabla u \nabla v + \alpha u v$. Since $\alpha > 0$, $B[u, u] \geq \min\{1, \alpha\} \parallel u \parallel^2$. By the Lax-Milgram theorem (see [5]) there exists $u \in H$ such that
\[
B[u, z] = \int_{\Omega} \nabla u \nabla z + \alpha \int_{\Omega} uz = \int_{\Omega} z(x)(P_1(x) + \beta v(x))dx.
\]
Hence $u$ is a weak solution to (2.2). Taking $z = u$ and $c^{-1} = \min\{1, \alpha\}$ the lemma is proved.

Lemma 2.2. Let $P_1$, $v$, and $u$ be as in Lemma 2.1. If $v \geq 0$ then $u \geq 0$.

Proof. Suppose $u$ is not positive. Let $A = \{ x \in \Omega, u(x) < 0 \}$, and $z = u \chi_A$. By the definition of weak solution
\[
\int_{\Omega} z(P_1 + \beta v) = \int_{\Omega} \nabla u \nabla z + \alpha \int_{\Omega} uz = \int_{A} \nabla u \nabla u + \alpha \int_{A} uz.
\]
This is a contradiction since \( \int_A \nabla u \nabla u + \alpha(\int_A u^2) > 0 \), while \( \int_A z(P_1 + \beta v) < 0 \). This proves the lemma. \( \square \)

**Lemma 2.3.** For each \( v \in L^2 \), let \( u(v) \equiv u \in H^1(\Omega) \) be the solution to (2.2) given by Lemma 2.1. If \( w \in H^1(\Omega) \) is the weak solution to

\[
-\Delta w + \delta w = P_2(x) + \gamma u(v) \quad \text{in } \Omega
\]
\[
\frac{\partial w}{\partial n} = 0 \quad \text{in } \partial \Omega,
\]

then

\[
\|w\|_2 \leq \frac{1}{\alpha} \|P_2\|_2 + \frac{\delta}{\alpha \gamma} \|P_1\|_2 + \frac{\beta \gamma}{\delta \alpha} \|v\|_2.
\]

**Proof.** Multiplying (2.6) by \( w \) and using the Cauchy-Schwartz inequality we have

\[
\int_{\Omega} \nabla w \nabla w + \delta \int_{\Omega} w^2 = \int_{\Omega} P_2(x) \cdot w + \gamma u(v) \cdot w
\]
\[
\leq \|P_2\|_2 \cdot \|w\|_2 + \gamma \|u(v)\|_2 \cdot \|w\|_2
\]
\[
\leq (\|P_2\|_2 + \delta \|u(v)\|_2) \cdot \|w\|_2.
\]

Hence

\[
\|w\|_2 \leq \frac{1}{\delta} \|P_2\|_2 + \frac{\gamma}{\delta} \|u(v)\|_2.
\]

Similarly,

\[
\|u\|_2 \leq \frac{1}{\alpha} \|P_1\|_2 + \frac{\beta}{\alpha} \|v\|_2.
\]

Replacing (2.9) in (2.10),

\[
\|w\|_2 \leq \frac{1}{\gamma} \|P_2\|_2 + \frac{\gamma}{\delta} \|u(v)\|_2
\]
\[
\leq \frac{1}{\gamma} \|P_2\|_2 + \frac{\delta}{\gamma} \left( \frac{1}{\alpha} \|P_1\|_2 + \frac{\beta}{\alpha} \|v\|_2 \right)
\]
\[
\leq \frac{1}{\alpha} \|P_2\|_2 + \frac{\delta}{\alpha \gamma} \|P_1\|_2 + \frac{\beta \gamma}{\delta \alpha} \|v\|_2,
\]

which proves the lemma. \( \square \)

**Theorem 2.4.** Given \((P_1,P_2) \in L^2(\Omega) \times L^2(\Omega)\), there exists a unique pair \((u,v) \in H \times H\) satisfying (2.1). In addition, \((u,v)\) depends continuously on \((P_1,P_2)\).

**Proof.** Let \( v_1, v_2 \in L^2(\Omega) \). Let \( u(v_1) \) and \( u(v_2) \) be given by Lemma 2.1 and \( w_1, w_2 \) as given by Lemma 2.3. Hence

\[
\int_{\Omega} |\nabla (w_1 - w_2)|^2 + \delta \int_{\Omega} |(w_1 - w_2)|^2
\]
\[
= \gamma \int_{\Omega} u(v_1) - u(v_2))(w_1 - w_2)
\]
\[
\leq \gamma (\|u(v_1) - u(v_2)\|_{L_2}) \|w_1 - w_2\|_2.
\]

Therefore,

\[
\|w_1 - w_2\| \leq \frac{\gamma}{\delta} (\|u(v_1) - u(v_2)\|_{L_2}.
\]
Multiplying (2.2) by \( u(v_1) - u(v_2) \) and subtracting we have
\[
\int_{\Omega} |\nabla (u_1 - u_2)|^2 + \alpha \int_{\Omega} (u(v_1) - u(v_2))^2 \\
= \beta \int_{\Omega} ((v_1 - v_2)(u(v_1) - u(v_2))) \\
\leq \beta \|v_1 - v_2\|_2 \|u(v_1) - u(v_2)\|_2.
\]
(2.14)
Thus
\[
\|u(v_1) - u(v_2)\|_2 \leq \frac{\beta}{\alpha} \|v_1 - v_2\|_2.
\]
(2.15)
Replacing this in (2.13) yields \( \|w_1 - w_2\|_2 \leq \gamma \beta \alpha \delta \|v_1 - v_2\|_2 \). Hence by the contraction mapping principle there exists a unique \( w \) such that \( w = v \). That is \((u,v)\) satisfies
\[
-\Delta u + \alpha u = \beta v + P_1(x) \text{ in } \Omega \\
-\Delta v + \delta v = \gamma u + P_2(x) \text{ in } \Omega \\
\frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \text{ on } \partial \Omega.
\]
(2.16)
By Lemma 2.1, \( u \) depends continuously on \((P_1,v)\). Also, by Lemma 2.3, \( v \) depends continuously on \((P_1,P_2)\). Hence \((u,v)\) depends continuously on \((P_1,P_2)\), which proves the theorem. \( \square \)

**Lemma 2.5.** Let \( h_1, h_2 \in L^\infty(\Omega) \). For each \( p > 1 \) there exist \( C_2(p) = C_2 > 0 \) such that if \((y,z)\) satisfies
\[
-\Delta y + \alpha y = \beta z + h_1, \\
-\Delta z + \delta z = \gamma y + h_2, \text{ in } \Omega \\
\frac{\partial y}{\partial n} = \frac{\partial z}{\partial n} = 0 \text{ in } \partial \Omega,
\]
(2.17)
then
\[
\|y\|_{2,p} + \|z\|_{2,p} \leq C_2(\|h_1\|_\infty + \|h_2\|_\infty)
\]
(2.18)
(see [5]). In particular, by the Sobolev imbedding theorem, taking \( p > N/2 \) we may assume that
\[
\|y\|_\infty + \sup_{\Omega} \left\{ \frac{|y(\zeta) - y(\eta)|}{|\zeta - \eta|} \right\} + \|z\|_\infty + \sup_{\Omega} \left\{ \frac{|z(\zeta) - z(\eta)|}{|\zeta - \eta|} \right\} \leq C_2(\|h_1\|_p + \|h_2\|_p).
\]
(2.19)
**Proof.** Multiplying the first equation in (2.17) by \( y \) we have
\[
\int_{\Omega} |\nabla y|^2 + \alpha \int_{\Omega} y^2 = \beta \int_{\Omega} (yz) + \int_\Omega h_1 y \\
\leq \beta \int_{\Omega} (yz) + \|h_1\|_\infty |\Omega|^{1/2} \|y\|_2.
\]
(2.20)
Similarly,
\[
\int_{\Omega} |\nabla z|^2 + \delta \int_{\Omega} z^2 = \gamma \int_{\Omega} (yz) + \int_\Omega h_2 z \\
\leq \gamma \int_{\Omega} (yz) + \|h_2\|_\infty |\Omega|^{1/2} \|z\|_2.
\]
(2.21)
Since $\alpha > 0$ and $\alpha \delta - \beta \gamma > 0$, the quadratic form $G(s, t) = \alpha s^2 - (\beta + \gamma)st + \delta t^2$ positive definite. That is, there exists $C > 0$ such that $G(s, t) \geq C(s^2 + t^2)$ for all $s, t \in \mathbb{R}$. This, (2.20) and (2.21) imply
\[ C(\|y\|_2 + \|z\|_2) \leq 2\Omega^{1/2}(\|h_1\|_\infty + \|h_2\|_\infty). \quad (2.22) \]

By (2.20) and (2.22),
\[ \tilde{a}\|y\|_{1,2}^2 \leq \|y\|_2(\beta\|z\|_2 + \|\tilde{G}\|_2(\|h_1\|_\infty + \|h_2\|_\infty)) \]
\[ \leq \left( \frac{2\beta}{C} + 1 \right)\Omega^{1/2}\|y\|_2(\|h_1\|_\infty + \|h_2\|_\infty) \]
\[ \equiv C_3\|y\|_2(\|h_1\|_\infty + \|h_2\|_\infty) \]
\[ \leq C_3\|y\|_{1,2}(\|h_1\|_\infty + \|h_2\|_\infty). \]

Hence
\[ \|y\|_{1,2} \leq \frac{C_3}{\tilde{a}}(\|h_1\|_\infty + \|h_2\|_\infty). \quad (2.24) \]

Similarly,
\[ \|z\|_{1,2} \leq \frac{C_3}{\delta}(\|h_1\|_\infty + \|h_2\|_\infty). \quad (2.25) \]

From (2.24), (2.25) and the Sobolev imbedding theorem (see [5, Theorem ??]) we see that
\[ \|y\|_{2N/(N-2)} + \|z\|_{2N/(N-2)} \leq S(1, 2)(\|y\|_{1,2} + \|z\|_{1,2}) \]
\[ \leq S(1, 2)\left( \frac{C_3}{\tilde{a}} + \frac{C_3}{\delta} \right)(\|h_1\|_\infty + \|h_2\|_\infty) \]
\[ \equiv C_4(\|h_1\|_\infty + \|h_2\|_\infty). \quad (2.26) \]

By regularity properties for elliptic boundary value problems there exists a positive real number $C_2$ such that if $-\Delta u + \tau u = f$ en $\Omega$ and $(\partial u) / (\partial \eta) = 0$ in $\partial \Omega$ $\|u\|_{2,p}$ when $p \in (1, (N/2) + 1)$. This and (2.26) imply
\[ \|y\|_{2, \frac{N}{2}} + \|z\|_{2, \frac{N}{2}} \leq C_2(C_4 + \|\Omega\|_{\frac{N}{2}})(\|h_1\|_\infty + \|h_2\|_\infty). \quad (2.27) \]

Iterating this argument finitely many times we see that there exist $p > N/2$ and $C_3 > 0$ such that
\[ \|y\|_{2,p} + \|z\|_{2,p} \leq C_3(\|h_1\|_\infty + \|h_2\|_\infty), \quad (2.28) \]

which proves the lemma.

3. PROOF OF THEOREM 1.1

Let $\rho = \max\{\alpha/m, \delta/m\}$ and $\bar{R} = 2(2M\rho\|\Omega\|)^{1/(2-b)}$ (see (1.2)). For $i = 1, 2$, let
\[ g_i(x, u, v) = \begin{cases} f_i(x, u, v) & \text{for } 0 \leq u + v \leq \bar{R}, \\ f_i(x, \bar{R}u/(u + v), \bar{R}v/(u + v)) & \text{for } u + v \geq \bar{R}. \end{cases} \]

Let $X$ be the ordered Banach space $C(\bar{\Omega}) \times C(\Omega)$ with positive cone
\[ K = \left\{ (u, v) \in X : u \geq 0, v \geq 0, \|u - \frac{1}{\|\Omega\|} \int_{\Omega} u\|_\infty \leq bM\bar{R}^{b-1}\int_{\Omega} u, \right. \]
\[ \left. \|v - \frac{1}{\|\Omega\|} \int_{\Omega} v\|_\infty \leq bM\bar{R}^{b-1}\int_{\Omega} v \right\}. \quad (3.1) \]
Let (see (1.2) and Lemma 2.5)
\[ R \in \{0, \min\{\bar{R}, (2C_2M)^{1-b}\}\}. \quad (3.2) \]

For \((u, v) \in K, \|(u, v)\|_X \geq R\), we define \(\Upsilon(u, v) = (U, V)\) as the only solution to
\[ -\Delta U + \alpha U = \beta V + g_1(x, u, v) \quad \text{in} \ \Omega, \]
\[ -\Delta V + \delta V = \gamma U + g_2(x, u, v) \quad \text{in} \ \Omega, \]
\[ \frac{\partial u}{\partial n} = 0 = \frac{\partial v}{\partial n} \quad \text{in} \ \partial \Omega. \quad (3.3) \]

If \((u, v) \in K\) and \(\|(u, v)\|_X \leq R\) we define
\[ \Upsilon(u, v) = \|(u, v)\|_X \Upsilon\left(\frac{R}{\|(u, v)\|_X}\right)(u, v), \quad \Upsilon(0, 0) = (0, 0). \quad (3.4) \]

Since \(g_1, g_2\) are nonnegative continuous functions, \(\Upsilon(u, v) = (U, V)\) satisfies \(U \geq 0\) \(y V \geq 0\) for \((u, v) \in K\) (see Lemma 2.2).

Suppose that for some \((U, V) = \Upsilon(u, v)\) we have
\[ \|U - \frac{1}{|\Omega|} \int_{\Omega} U\|_\infty > bM\bar{R}^{b-1} \int_{\Omega} U, \quad (3.5) \]
with \(\|(u, v)\|_X \geq R\). Hence \(\|U\|_\infty > bM\bar{R}^{b-1} \int_{\Omega} U\), which implies that if \(\|U\|_\infty = U(x), x \in \Omega\), then there exists \(y \in \Omega\) such that \(\|y - x\| \leq m_1 \bar{R}^{(1-b)/\alpha}\) and \(U(y) \leq U(x)/2\), with \(m_1\) a constant depending only on \(\Omega\). Hence
\[ \frac{U(x) - U(y)}{\|x - y\|} \geq \frac{\|U\|_\infty}{2m_1 \bar{R}^{(b-1)/N}}. \quad (3.6) \]

Let now \(p > N\) be such that
\[ \frac{N + p - b(p - 1)}{(p - 1)N} + \frac{b}{p} > 0. \quad (3.7) \]

This and Lemma 2.5 imply
\[ \|U\|_\infty \bar{R}^{(b-1)/n} \leq C_2 \|g_1(\cdot, u, v)\|_p \]
\[ \leq C_2M \left( \int_{\Omega} (u + v)^{bp} \right)^{1/p} \]
\[ \leq C_2M \left( \int_{\Omega} (u + v)^b (u + v)^{b(p-1)} \right)^{1/p} \]
\[ \leq C_2M \|u + v\|_{\infty}^{(p(b-1))/p} \left( \int_{\Omega} (u + v)^b \right)^{1/p}. \quad (3.8) \]

Integrating the first equation in (3.3) on \(\Omega\),
\[ \alpha \int_{\Omega} U \geq m \int_{\Omega} (u + v)^b, \quad (3.9) \]

(see (1.2)). From (3.8) and (3.9),
\[ \|U\|_\infty \bar{R}^{b-1} \leq C_2M \|u + v\|_{\infty}^{(p(b-1))/p} \left( \frac{\alpha}{m} \int_{\Omega} U \right)^{1/p} \]
\[ \leq C_2M \|u + v\|_{\infty}^{(p-1)/p} \left( \frac{\alpha}{2mM} \bar{R}^{1-b} \|U\|_\infty \right)^{1/p} \]
\[ \leq C_2M \left( 2M \bar{R}^{b-1} \int_{\Omega} (u + v)^b \right)^{\frac{b(p-1)}{p}} \left( \frac{\alpha}{2mM} \bar{R}^{1-b} \int_{\Omega} \|U\|_\infty \right)^{1/p}. \quad (3.10) \]
Therefore
\[
\|U\|^{(p-1)/p} \leq m_2 \tilde{R}^{(b-1)\left(\frac{b(p-1)-\frac{b}{p} - \frac{1}{n}}{p}\right)} \left(\int_{\Omega} (u + v)^b\right)^{(p-1)/p} \\
\leq m_3 \tilde{R}^{(b-1)\left(\frac{b(p-1)-\frac{b}{p} - \frac{1}{n}}{p}\right)} \left(\int_{\Omega} (u + v)^b\right)^{(p-1)/p} \\
\leq m_3 \tilde{R}^{(b-1)\left(\frac{b(p-1)-\frac{b}{p} - \frac{1}{n}}{p}\right)} \left(\frac{\alpha}{m} \int_{\Omega} U\right)^{(p-1)/p} \\
\leq m_4 \tilde{R}^{(b-1)\left(\frac{b(p-1)-\frac{b}{p} - \frac{1}{n}}{p}\right)} \left(\tilde{R}^{1-b}\|U\|_\infty\right)^{(p-1)/p}.
\] (3.11)

Since \(m_2, m_3, m_4\) are independent of \(U\),
\[
1 \leq m_4 \tilde{R}^{(b-1)\left(\frac{b(p-1)-\frac{b}{p} - \frac{1}{n}}{p}\right)}. \quad (3.12)
\]
By (1.2), there exists \(p > N\) such that
\[
(b-1) \left(\frac{b(p-1)}{n} - \frac{1}{p} - \frac{1}{n} - \frac{p-1}{p}\right) < 0. \quad (3.13)
\]
Taking \(\tilde{R}\) sufficiently large we have a contradiction to (3.5). Thus \(\Upsilon(u, v) \in K\). For \(||(u, v)||_X < R\) the proof follows from the definition of \(\Upsilon\). Thus \(\Upsilon(K) \subset K\).

Let \(C_2\) be as in 2.5 and \(x \in \Omega\) be such that \(U(x) = \max\{U(y); y \in \Omega\}\). From the definition of \(C_2\) we conclude that if \(y \in \Omega\) and \(||y - x|| \leq C_2 M(\|u\|_\infty + \|v\|_\infty)\) then by the definition of \(g_1, g_2\), if \(\{u_j, v_j\}_j\) is a bounded sequence in \(X\) so are \(\{g_1(x, u_j, v_j)\}_j\) and \(\{g_2(x, u_j, v_j)\}_j\) in \(C(\Omega)\). Since \(g_1, g_2\) are bounded functions, due to Lemmas 2.5 \(\{U_j, V_j\}_j\) is bounded in \(W^{2,p}(\Omega) \times W^{2,p}(\Omega)\). Taking \(p > N/2\), by the Sobolev imbedding theorem (see [3]) we see that \(\{U_j, V_j\}_j\) has a converging subsequence in the space \(X\), which proves that \(\Upsilon\) is a compact operator.

Suppose that for some \((u, v)\) such that \(||u||_\infty + ||v||_\infty = R, U \geq u, V \geq v\). By (2.18),
\[
R = ||u||_\infty + ||v||_\infty \leq ||U||_\infty + ||V||_\infty \\
\leq 2C_2 M||u + v||_\infty \\
\leq 2C_2 M R^b,
\] (3.14)
which contradicts the definition of \(R\). This proves that \(\Upsilon(u, v) \not\supset (u, v)\) for \(||(u, v)||_X = R\).

Suppose that \((U, V) = \Upsilon(u, v) \leq (u, v)\) for some \((u, v)\) with \(||(u, v)||_X = \tilde{R}\). Without loss of generality we may assume that \(||u|| \geq \tilde{R}/2\). Hence, by the definition of \(K\),
\[
\int_{\Omega} u \geq \frac{\tilde{R}}{2(||\Omega||^{-1/2} + bM R^{b-1})} \geq C_3 \tilde{R}^{2-b}. \quad (3.15)
\]
Integrating the first equation in (3.3) we infer that
\[
\alpha \int_{\Omega} U = \beta \int_{\Omega} V + \int_{\Omega} g_1(u, v) \\
= \beta \int_{\Omega} V + m \int_{\Omega} (u + v)^b \geq \beta \int_{\Omega} V + m \int_{\Omega} (U + V)^b. \quad (3.16)
\]
Similarly, 
\[ \delta \int_{\Omega} V \geq \gamma \int_{\Omega} U + m \int_{\Omega} (U + V)^b. \]
By Holder inequality and the definition of \( \rho \),
\[ \int_{\Omega} (U + V)^b \leq \rho |\Omega|. \] (3.17)
Since \((U, V) \in K\),
\[ \bar{R} \leq 2\|U\|_{\infty} \leq 4MR^{b-1} \int_{\Omega} U \leq 2M\bar{R}^{b-1}\rho |\Omega|, \] (3.18)
which contradicts the definition of \( \bar{R} \). Hence \( \Upsilon \) satisfies the hypotheses of Theorem 1.2. Hence \( \Upsilon \) has a fixed point \((u, v)\) in \( \{(y, z); \|(y, z)\| \in (R, \bar{R})\} \). Therefore \((u, v)\) is a positive solution to (1.1), which proves Theorem 1.1.

Acknowledgments. Alfonso Castro was partially supported by a grant from the Simons Foundations (# 245966).

References

[6] Kwong M.K; On Krasnoselskii’s cone fixed point theorem, Lucent Technologies, Inc USA.

JUAN C. CARDEÑO
Facultad de Ciencias Naturales y Matemáticas, Universidad de Ibagué, Ibagué, Tolima, Colombia
E-mail address: juan.cardeno@unibague.edu.co

ALFONSO CASTRO
Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, USA
E-mail address: castro@math.hmc.edu